Uttar Pradesh
Rajarshi Tandon Open University

UGMM - 09
Real Analysis

Block

1

REAL NUMBERS AND FUNCTIONS

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MTE 09 REAL ANALYSIS

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FEATURES OF THE COURSE

The relevance of Mathematics can be recognized in the extensive use of its basic concepts in various areas of knowledge and in the application of its techniques to solve several problems facing the mankind. These concepts and techniques derive their strength from certain underlying principles. For example, in most of the Calculus courses, the emphasis is on learning the methods of differentiation and integration rather than on the underlying principles that provide these methods. The study of such principles has been identified as an interesting field of Mathematics in the name of Mathematical Analysis of which Real Analysis is considered to be the most beautiful branch. The reason for this is that almost all areas of Modern Mathematics have their roots in Real Analysis because many abstract notions (concepts) are explained in terms of the real number system.

The present course on Real Analysis is designed for those who have a working knowledge of the Calculus of one variable and are ready for a more systematic treatment. However, the material in this course does require a certain amount of essential knowledge of Algebra, Geometry and Trigonometry. The major objectives, therefore, of this course are the following:

i) To bridge the gap between Calculus and Advanced Calculus

ii) To have a rigorous and sophisticated knowledge of the methods and concepts related to Calculus as well as other relevant areas of Mathematics.

iii) To provide adequate knowledge of conceptual mathematics for those who wish to specialise in mathematics and pursue a career in mathematics.

The unifying theme of the course is concerned with the limiting processes on the real line which forms the heart of Mathematics. It is, therefore, an essential part of the training of any student of Mathematics. The syllabus of the course has been distributed in five blocks covered in 16 units. The logical order of development of the material is as follows:

The first topic of study is the system of numbers. Since this is the first course in Analysis, we have, therefore, kept the discussion of the numbers as simple as we can, so long it gives a firm foundation for the structure of later definitions and concepts. The first block having four units is, therefore, devoted to a study of the structure of real number system and its subsystems of natural numbers, integers and rational numbers. In this block, we discuss the Arithmetical, Geometrical, Algebraical and Topological structure of the system of real numbers. Also, we study some functions and discuss a few special functions which we need at a later stage.

Sequences and Series of numbers are introduced and discussed in Block 2 which consists of three units. The main aim of this block is to unify the presentation of the course material. One of the most fundamental concepts in Real Analysis is that of a Cauchy Sequence. We shall explain the meaning and significance of a Cauchy Sequence. In fact, in this block, we begin to study the limit process as applied to sequences which are, in fact, a special class of real functions.

In Block 3, we discuss the general notion of the limit of a real functions. It has 3 units. The notion of limit of a real function is fundamental to all further ideas in Real Analysis. We shall develop this notion in this block and then use it to define the continuous functions. We also discuss some important properties of continuous functions as well as Uniform continuity.

Block 4 also contains three units. It is devoted to the notion of differentiability of a function. The mean value theorems and higher order derivatives have been discussed in this block.

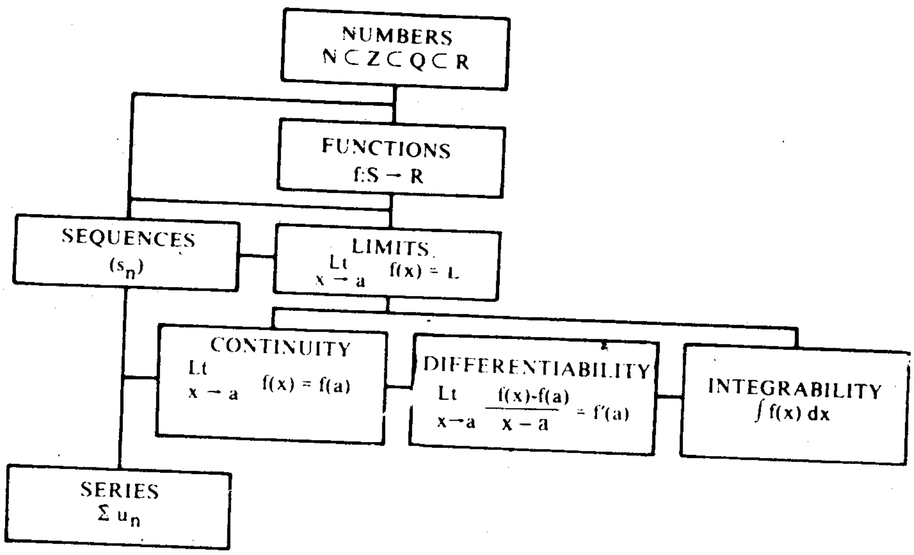
Block 5 deals with the integrability of a function. We introduce the integral of a function in a formal way and discuss some of its important properties. Integral of a function is introduced as a limit of a sum and thus removes the common misconception that integration is always a reverse process of differentiation. The Fundamental Theorem of Integral Calculus is, then, established. Finally, we discuss the sequences and series of functions in the last unit.

The block-wise description of the course is as follows:

Real Analysis

1.	Block 1	:	Real Numbers and Functions	(4 units)
2.	Block 2	:	Sequences and Series	(3 units)
3.	Block 3	:	Limit and Continuity	(3 units)
4.	Block 4	:	Differentiability	(3 units)
5.	Block 5	:	Integrability	(3 units)

The unit-wise distribution has been given in the programme guide. Also, it will be given in each relevant block. The inter-relationship of the blocks can be seen from the following flow chart:



. TO THE COUNSELLOR

This course is mainly designed for those who have some working knowledge of Calculus and who wish to take more advanced courses in Mathematics. It provides a gradual transition from the mechanical type of Mathematics to a rigorous and sophisticated material in the post Calculus courses. The main concern is to acquaint our students with the language and conceptual aspects of Mathematics. Since this cannot be done in vacuum, we have chosen to present this material in the shape of Real Analysis. There is no prerequisite for this course except the competence in precalculus Mathematics and a computational skill of the basic techniques of differentiation and integration. Therefore, it is desirable that the students have completed our course on Calculus (MTE-01).

In addition to the general guidance and counselling which will be available to our learners at the Study Centres, we have to request you for something more in Mathematics. The counselling in an open learning system is an important component of distance teaching. It is challenging when we come to counselling in a subject like Mathematics. There are certain dogmas and misconceptions about Mathematics which can be eliminated if it is presented to the students in a simple and interesting way. Therefore, the main emphasis in the counselling sessions in Mathematics will have to be on the clarity of the basic concepts and definitions as well as on the removal of the difficulties of the learners in problem solving. We have tried to present the course material with this view in mind and recalled, wherever necessary, the concepts which are essential for the discussion of the main text. The first unit has been mainly devoted to the review of the basic concepts of sets, functions and numbers. The rigour starts in Unit 2 and is slowly built through Unit 3 and Unit 4. It is continued throughout the subsequent units.

There are concepts in Mathematics which can be clearly understood just by reading the printed material. Yet there are few which require some face-to-face conversation with the learner. It is, therefore, desirable that such concepts and situations should be identified and discussed in the counselling sessions. Wherever necessary, we have given brief historical references. The purpose for this is to break the monotony and generate interest in the learning of the material.

Finally, we would like to invite your expert comments and suggestions for improvement, particularly with regard to the course contents, designing of the syllabus and the presentation of the test of the course material. For this, you may also seek the reactions and the opinions of the learners. This feedback from you as well as from the learners through you will help us a great deal to improve upon the learning materials.

TO THE LEARNER

Mathematics is generally viewed as the study of numbers (Algebra) and shapes (Geometry). However, this perception about Mathematics is somewhat insufficient. The study of Algebra and Geometry is only a part of the mathematical enterprise. The true concern of Mathematics is the study of its abstract nature in general. Your previous experience with Mathematics may have been with the courses like Calculus, Algebra, Geometry and Trigonometry. In these courses, you are required to memorize formulas and methods and then apply them to solving problems. The material in the present course, however, is a departure from this approach. You will find in this course that it has the rigour of Mathematics. Indeed you will be learning the language and grammar used by the mathematicians in communicating their ideas and discussions. To appreciate these ideas and the rigour in Mathematics, you will do well if you adopt the following guidelines:

1. Try to understand every word in every sentence in every paragraph in every section of the units and blocks. New ideas usually depend upon the previously known concepts. Therefore, a lack of understanding of the earlier material often interferes with the learning of the later ideas. Since you may not be able to grasp everything at the first reading, we recommend that you repeatedly recall earlier ideas and acquire a better notion of where they lead to. Then apply these notions to the new ideas.
2. In reading the materials (in fact for that matter in studying Mathematics in general), it is imperative that you have a pencil (pen) and paper at hand. You

must write out the definitions and the concepts, the theorems and key ideas that you encounter as you go along. This way doing Mathematics by writing along will focus your attention. Considerable effort is required to learn Mathematics and it is not proper to do so by sitting in an easy chair and reading your block as if it were a novel or a popular film magazine or a newspaper.

3. Draw pictures and diagrams wherever possible to illustrate the concepts under consideration. This will make you visualise the concepts clearly.

4. The exercises given after definitions and examples are closely related to these concepts. Working them out will enhance your grasp of the material. Answers, hints and solutions provided at the end of the unit should be seen only if you fail to do the exercise even after several attempts. You should verify your solutions with the ones given in the unit. Also, there are some short intext questions. You should be able to provide their answers if you have thoroughly grasped the key ideas in the discussion. In each block, there is a review containing some self-test questions. This will enable you to assess for yourself the conceptual knowledge of the material you have acquired in the block.

5. In working out the problems (in fact in writing Mathematics in general), cultivate the habit of writing complete sentences and full solutions. As isolated expression such as $x^2 + 4x + 3$ means very little. What about it? Do you want to find the values of x for which the expression is zero? If so, then say so in clear words. Are you writing this as an example of a polynomial with integer coefficients? If so, then say so.

6. Every mathematical concept has to be understood in two interrelated ways; the intuitive way and the formal way. An intuitive grasp of a mathematical concept generally develops only after you have dealt with the concept over a period of time. To build your intuition, you may collect a few relevant examples of the concept. Use the concept in different examples and see how it is related to other ideas and see how it helps to clarify the ideas that otherwise might be hazy. To understand the concept in the formal way, you need to know its mathematical definition and to connect it with the intuitive idea.

7. Sometimes you will find it difficult to grasp the mathematical concept as given in the text. Go to your Study Centre, consult the relevant books available, try to discuss your problem with the Counsellor or with your other colleagues who have probably understood it clearly. Once you have understood the idea, try to explain it to others so that you can reassure yourself that what you have understood is correct. If need be, collect pertinent examples, exercises, illustrations, remarks, historical notes etc. and make a note of all that is relevant to the idea. In this way you will automatically learn considerably more Mathematics than you might expect.

8. The course material is divided into 5 blocks and 16 units. Each unit is further divided into sections and subsections. The number 2.3.4, for instance refers to subsection 4 of section 3 of Unit 2. Thus, the first digit from the left indicates the unit number, the second digit tells the number of the section and the last digit stands for the subsections, if any. All the definitions, properties, theorems, examples, exercises, figures etc. have been serially numbered throughout a unit and their statements have been printed in bold face letters. The parts of a definition or of an example or of an exercise etc. have been labelled with small Roman numbers (i), (ii), (iii), etc.

NOTATIONS AND SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	is a member of (belongs to)
$\notin$	is not a member of (does not belong to)
$\subset$	is a subset of (is contained in)
$\not\subset$	is not a subset of (is not contained in)
$\supset$	is a superset
$\cup$	Union
$\cap$	intersection
ϕ	empty set
$\Rightarrow$	implies
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\sim$	equivalence relation
$\forall$	for all
$\exists$	there exists
$.$	multiplication
$+$	addition
$-$	subtraction
sup	supremum
inf	infimum
min	minimum
max	maximum
$\circ$	composition
f'	derivative of f
f^{-1}	inverse of a function f
exp	exponential
log	logarithm
ln	natural logarithm
sgn	signum
$[x]$	greatest integer not exceeding x
$\|x\|$	absolute value of x or Modulus of x
$\mathbf{R}^+$	set of positive real numbers
$\mathbf{R}$	set of real numbers
$\mathbf{I}$	Set of irrational numbers
$\mathbf{Q}$	set of rational numbers
$\mathbf{Z}$	set of integers
$\mathbf{N}$	set of natural numbers
$\mathbf{F}$	field
$\mathbf{C}$	set of complex numbers
$[a, b]$	closed interval
$]a, b[$	open interval
$]a, b]$	semi-open interval (open at left)—semi-closed interval
$[a, b[$	semi-open interval (open at right)—semi-closed interval
$+\infty$	infinity
$-\infty$	minus infinity
Σ	sum
$\sum_{n=1}^{\infty} u_n$	infinite series
(s_n)	sequence
S^c	complement of S
S'	derived set of S
$\overline{S}$	closure of S

Greek Alphabets

α	Alpha
β	Beta
γ	Gama
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
i	Iota
λ	Lambda
μ	Mu
ν	Nu
ξ	exi
π	Pi
Π	(capital Pi)
ρ	Rho
σ (Σ)	Sigma (capital Sigma)
τ	Tou
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

REFERENCES

We have discussed the course material in the course on Real Analysis in a complete form. We believe that the discussion is quite exhaustive in each unit. Nevertheless, you may like to refer to some books for some more understanding of the concepts or may be you need some additional readings. For this, we give below a list of books which may be available at your Study Centre or in a nearby institution.

1. *Real Analysis* by S.C. Malik, Wiley Eastern Limited.
2. *Elements of Real Analysis* by Shanti Narayan, S. Chand & Company Ltd.
3. *Foundations of Analysis the Theory of Limits* by Herbert S. GasPill P.P. Narayanaswami.
4. *Introduction to Mathematical Analysis* by C.R.J. Claphan, Routldge & Kegan Paul (London).
5. *Mathematical Analysis* by M.D. Hattan, Hodder and Stonghton (London).

BLOCK 1 REAL NUMBERS AND FUNCTIONS

PREVIEW

This is the first block on Real Analysis. The purpose of this block is to lay a logical and axiomatic foundation on which Real Analysis is built. It is devoted to the study of the system of real numbers, its arithmetical development, geometrical framework and algebraic structure. Also, its basic topological features are discussed. The block contains four units.

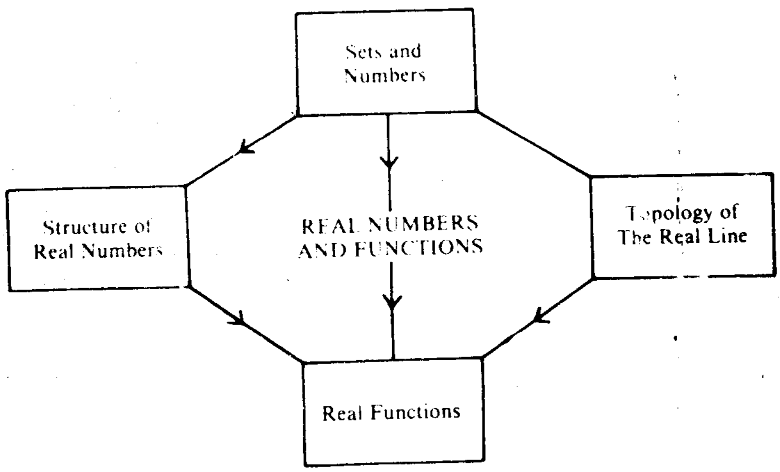
In Unit 1, we have recalled the basic concepts of sets and functions. Also, in this unit, we have discussed the development of real numbers from the rational numbers which, in turn have been constructed from the integers with the latter having been built from the natural numbers. Then, we have also described the real numbers as points on a line and conversely. In view of this, the set of real numbers is called the real line.

We continue this discussion in Unit 2, and showed that the set R of real numbers forms an algebraic structure called the Field. We shall discuss the order completeness property of the set of real numbers which distinguishes them from Q, the set of rational numbers in the sense that the set of rational numbers is not a complete ordered field.

The Unit 3 deals with the topology of the real line. In this unit, we talk of the neighbourhood of a point on the real line, open sets, limit point of a set, closed sets and compact sets. These concepts, though confined to the real line, are presented in such a way that they provide an insight into their applicability to two-dimensional (even higher dimensional) spaces.

Finally, in Unit 4, we discuss the real functions and its various classes. Algebraic functions such as polynomial functions, rational functions etc. are described. Transcendental functions viz. trigonometric, logarithmic and exponential functions have been studied. Some special functions such as monotonic functions, modulus function, signum function and bounded functions have also been discussed.

The Unitwise relationship of this block is given in the following picture:



UNIT 1 SETS AND NUMBERS

STRUCTURE

1.1 INTRODUCTION

One of the main features of Mathematics is the identification of the subject matter, its analysis and its presentation in a satisfactory manner. For this, we need a simple language— a language that admits minimum vocabulary and an easy grammar— a language that is precise and has clear meanings. In other words, the language should be a vehicle which carries ideas through the mind without affecting their meaning in any way. Set Theory comes closest to being such a language. Introduced between 1873 and 1895 by a famous German mathematician, George Cantor (1845-1918), Set Theory became the foundation of almost all the branches of Mathematics. Besides its universal appeal, it is quite amazing in its simplicity and elegance.

A rigorous presentation of Set Theory is not the purpose here because we believe that you are already familiar with it. Yet, we shall briefly recall some of its basic concepts which are essential for a systematic study of Real Analysis. Closely linked with the sets, is the notion of a function, which also you have learnt in your previous studies. In this unit, we shall review this as well as other related concepts which are necessary for our discussion.

'Real Analysis' is an important branch of Mathematics which mainly deals with the study of real numbers. What is, then, the system of the real numbers? We shall try to find an answer to this question as well as some other related questions in this unit. Also, we shall give the geometrical representation of the real numbers. This will help us in discussing the algebraic structure of real numbers in Unit 2 and some related aspects in Unit 3 and Unit 4.

OBJECTIVES

In this unit, therefore, you should be able to

→ recall the basic concepts of sets and functions,

→ discuss the development of the system of real numbers,

→ describe the geometrical representation of real numbers.

1.2 SETS AND FUNCTIONS

Most of modern Mathematics is based on the ideas that are expressed in the language of sets and functions. In this section, we shall give a brief review of certain basic concepts of Set Theory which are quite familiar to you. These concepts will also serve an important purpose of recalling certain notations and terms that will be used throughout our discussion with you. Also, this will be useful as a background material for what is going to be discussed in the subsequent units and blocks.

1.2.1 SETS

You are used to the phrases like the '**team**' of cricket players, the '**army**' of a country, the '**committee**' on the education policy, the '**panchayat**' of a village, etc. The terms

'team', 'army', 'committee', panchayat', etc. indicate the notion of a **'collection'** or **'totality'** or **'aggregate'** of objects. These are well-known examples of a **set.**

Therefore, our starting point is an informal description of the term 'set'. A set is treated as an undefined term just as a point in Geometry is undefined. However, for our purpose we say that **a set is a well-defined collection of objects.** A collection is well-defined if it is possible to say whether a given object belongs to the collection or not.

The following are some examples of sets:

i) The collection of all students registered in Indira Gandhi National Open University.

ii) The collection of the planets namely Jupiter, Saturn, Earth, Pluto, Venus, Mercury, Mars, Uranus and Neptune.

iii) The collection of all the countries in the world. (Do you know how many countries are there in the world?)

iv) The collection of numbers 1, 2, 3, 4 .... .

If we consider the collection of tall persons or beautiful ladies or popular leaders, then these collections are not well-defined and hence none of them forms a set. The reason is that the words 'tall', 'beautiful' or 'popular' are not well-defined. The objects constituting a set are called its elements or members or points of the set. Generally, sets are denoted by the capital letters **A, B, C** etc. and the elements are denoted by the small letters a, b, c, etc. If **S** is any set and x is an element of **S,** we express it by writing that $x \in S$, where the symbol $\in$ means 'belongs to' or 'is a member of'. If x is not an element of a set S, we write $x \notin S$. For example, if S is the set containing 1, 2, 3, 4 only, then $2 \in S$ and $5 \notin S$.

You know that there are two methods of describing a set. One is known as the **Tabular method** and the other is the **Set-Builder method.** In the **tabular method** we describe a set by actually listing all the elements belonging to it. For example, if **S** is the set consisting of all small letters of English alphabet, then we write

$$S = \{ a, b, c, ... x, y, z\}.$$

If **N** is the set of all natural numbers, then we write

$$N = \{ 1, 2, 3 .... \}.$$

This is also called an explicit representation of a set.

In the **set-builder method,** a set is described by stating the property which determines the set as a well-defined collection. Suppose p denotes this property and x is an element of a set **S.** Then

$S = \{ x: x \text{ satisfies } p\}$.

For example, the two sets **S** and **N** can be written as

$S = \{ x: x \text{ is a small letter of English alphabet}\}$

$N = \{n: n \text{ is a natural number}\}$.

This is also called an implicit representation of a set.

Note that in the representation of sets, the elements of a set are not repeated. Also, the elements may be listed in any manner.

EXAMPLE 1: Write the set S whose elements are all natural numbers between 7 and 12 including both 7 and 12 in the tabular as well as in the set-builder forms.

SOLUTION : Tabular form is $S = \{ 7, 8, 9, 10, 11, 12, \}$.
Set = builder form is $S = \{ n \in N: 7 \leq n \leq 12\}$.

EXERCISE 1)

(i) Write the following in the set-builder from :

$A = \{ 2, 4, 6, ...\}$.

$A = \{ 1, 3, 5, ...\}$.

A = { Dr Rajinder Prasad, Dr Radha Krishnan, Dr Zakir Hussain, Sh. V.V. Giri, Dr Fakhruddin Ali Ahmed, Dr Sanjiva Reddy, Giani Zail Singh, Sh. R. Venkataraman. }

(ii) Write the following in the tabular form :

A = { x: x is a factor of 15}.

A = { x: x is a natural number between 20 and 30}.

A = { x: x is a negative integer}.

The following standard notations are used for the sets of numbers :

N = Set of all natural numbers
= { 1, 2, 3,.... }
= {n: n is a natural number}
= Set of all positive integers.

Z = Set of all integers
= { .... —3, −2, −1, 0, 1, 2, 3, ... }
= { p: p is an integer}.

Q = Set of all rational numbers

$$= \{ x : x = \frac{p}{q} : p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0 \}.$$

R = Set of real numbers
= { x: x is a real number}.

We shall, however, discuss the development of the system of real numbers in Section 1.3.

A set is said to be **finite** if it has a finite number of elements. A set is said to be **infinite** if it is not finite. We shall, however, give a mathematical definition of finite and infinite sets in Unit 2.

Note that an element of a set must be carefully distinguished from the set consisting of this element. Thus, for instance, you must distinguish

$$x, \{x\}, \{\{x\}\}$$

from each other.

We talk of equality of numbers, equality of objects etc.

The question, therefore, arises: What is the notion of the **equality of sets?**

DEFINITION 1: EQUALITY OF SETS

Any two sets are equal if they are identical. Thus the two sets S and T are equal, written as S = T if they consist of exactly the same numbers. When the two sets S and T are unequal, we write

$$S \neq T.$$

It follows from the definition that **S = T** if and only if x ∈ **S** implies x ∈ **T** and y ∈ **T** implies y ∈ **S**. Also **S** is different from **T (S ≠ T)** if there is at least one element in one of them which is not in the other.

If every member of a given set **S** is also a member of **T**, then we say that **S** is a **subset** of **T** or **"S is contained in T"**, and write:

$$S \subset T$$

or equivalently we say that **"T contains S"** or **T** is a **superset** of **S**, and write

$$T \supset S$$

The relation

$$S \subset T$$

means that **S** is not a subset of **T** i.e. there is at least one element in **T** which is not in **S.**

Thus, you can easily see that any two sets **S** and **T** are equal if and only if **S** is a subset of **T** and **T** is a subset of **S** i.e.

$$S = T \Longleftrightarrow S \subset T \text{ and } T \subset S.$$

If S ⊂ T but T ⊄ **S,** then we say that S is a **proper subset** of **T**. Note that S ⊂ S **i.e. every set is a subset of itself**.

Another important concept is that of a set having no elements. Such a set, as you know, is called an **empty set** or a **null set** or a **void set** and is denoted by ϕ.

You can easily see that **there is only one empty set** i. e. ϕ is unique. Further ϕ **is a subset of every set.**

Now why don't you try an exercise?

EXERCISE 2)

Justify the following statements:

(i) The set N is a proper subset of Z.

(ii) The set R is not a subset of Q.

(iii) If A, B, C are any three sets such that A ⊂ B, and B ⊂ C, then A ⊂ C.

So far, we have talked about the elements and subsets of a given set. Let us now recall the method of constructing new sets from the given sets.

While studying subsets, we generally fix a set and consider the subsets of this set throughout our discussion. This set is usually called **the Universal set.** This Universal set may vary from situations to situations. For example, when we consider the subsets of **R**, then **R** is the Universal set. When we consider the set of points in the Eulidean plane, then the set of all points in the Eulidean plane is the Universal set. We shall denote the Universal set by **X**.

Now, suppose that the Universal set **X** is given as

$$X = \{1, 2, 3, 4, 5\}$$

and

$$S = \{1, 2, 3\}$$

is a subset of **X**. Consider a subset of **X** whose elements do not belong to **S**. This set is {4, 5}.

Such a set, as you know, is called the complement of S.

We define the complement of a set as follows·

DEFINITION 2: COMPLEMENT OF A SET

Let X be the Universal set and S be a subset of X. The complement of the set S is the set of all those elements of the Universal set X which do not belong to S. It is denoted by S^c.

Thus, if S is an arbitrary set contained in the Universal Set X, then the complement of S is the set

$$S^C = \{x : x \notin S\}.$$

Associated with each set S is the **Power set P (S)** of S consisting of all the subsets of **S. It is written as**

$$P(S) = \{A : A \subset S\}.$$

Now try the following exercise:

EXERCISE 3)

Let X be a universal set and let S be a subset of X.
Then justify the following by suitably choosing X and S.

(i) $P(\phi) = \{\phi\}$.
(ii) $(S^c)^c = S$.

Let us consider the sets S and T given as

$$S = \{1, 2, 3, 4, 5,\}, T = \{3, 4, 5, 6, 7\}.$$

Construct a new set {1, 2, 3, 4, 5, 6, 7}. Note that all the elements of this set have been taken from S or T such that no element of S and T is left out. This new set is called the union of the sets S and T and is denoted by S ∪ T.
Thus

$$S \cup T = \{1, 2, 3, 4, 5, 6, 7\}.$$

Again let us construct another set {3, 4, 5} . This set consists of the elements that are common to both S and T i.e. a set whose elements are in both S and T. This set is called the intersection of S and T. It is denoted by S ∩ T. Thus

$$S \cap T = \{3, 4, 5\}.$$

These notions of Union and Intersection of two sets can be generalized for any abstract sets in the following way: Note that all the sets under discussion will be treated as subsets of the Universal set X.

DEFINITION 3: UNION OF SETS

Let S and T be given sets. The collection of all elements which belong to S or to T is called the Union of S and T. It is expressed as

$$S \cup T = \{x : x \in S \text{ or } x \in T\}.$$

Note that when we say that $x \in S$ or $x \in T$, then it means that x belongs to S or x belongs to T or x belongs to both S and T.

DEFINITION 4: INTERSECTION OF SETS

The intersection S ∩ T of the sets S and T is defined to be the set of all those elements which belong to both S and T i.e.

$$S \cap T = \{x: x \in S \text{ and } x \in T\}.$$

Note that the sets are **disjoint** or **mutually exclusive** when $S \cap T = \phi$ i.e. **when their intersection is empty.**

You can now verify (or even prove) by means of examples the following laws of union and intersection of sets given in the next exercise.

EXERCISE 4)

Let A, B and C be any three sets. Then justify the following:

(i) $A \cup B = B \cup A$, $A \cap B = B \cap A$ **(Commutative laws).**

(ii) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ **(Associative laws).**

(iii) $\left.\begin{array}{l} A \cup (B \cap C) = (A \cup B)\,(A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{array}\right\}$ **(Distributive laws).**

(iv) $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ **(De Morgan laws).**

Also, you can easily see that

$$A \cup A = A, A \cap A = A, A \cup \phi = A, A \cap \phi = \phi.$$

Given any two sets S and T, we can construct a new set in such a way that it contains only those elements of one of the sets which do not belong to the other. Such a set is called the difference of the given sets. There will be two such sets denoted by **S—T** and **T—S**. For example, let

$$S = \{2, 4, 8, 10, 11\}, T = \{1, 2, 3, 4\}.$$

Then

$$S - T = \{8, 10, 11\}, T - S = \{1, 3\}.$$

Thus, we can define the difference of two sets in the following way:

DEFINITION 5: DIFFERENCE OF TWO SETS

Given two sets S and T, the difference S-T is a set consisting of precisely those members of S which are not in T.
Thus

$$S - T = \{x: x \in S \text{ and } x \in T\}.$$

Similarly, we can define **T—S.**

Consider a collection of sets S_i, where i varies over some index set J. This simply means that to each element $i \in J$, there is a corresponding set S_i. For example, the collection $\{S_1, S_2, S_3 ....\}$ could be expressed as $\{S_i\}_i \in N$, where N is the index set.

With the introduction of an index set, the notions of the Union and the Intersection of sets can be extended to an arbitrary collection of sets. For example

(i) $\bigcup_{i \in J} S_i = \{x : x \in S_i \text{ for at least one } i \in J\}$.

(ii) $\bigcap_{i \in J} S_i = \{x: x \in S_i \text{ for all } i \in J\}$.

(iii) $(\bigcup_{i \in J} S^i)^c = \bigcap_{i \in J} S_i^c$.

1.2.2 FUNCTIONS

Let S be the set of all books in IGNOU library and let N be the set of all natural numbers. Assign to each book the number of pages the book contains. Here each book corresponds to a unique natural number. In other words, there is a correspondence between the books and the natural numbers i.e. there is a rule or a mechanism by which we can associate to each book one and only one natural number. Such a rule or correspondence is named as a **function** or a **mapping** and is denoted by 'f'.

DEFINITION 6: FUNCTION

Let S and T be any two non-empty sets. A function f from S to T denoted as $f : S \rightarrow T$ is a rule which assigns to each element of the set S, a unique element in the set T.

The set S is called the domain of the function **f** and **T** is called its Co-domain. If an element x in S corresponds to an element y in T under the function **f**, then y is called the image of x under **f**. This is expressed by writing $y = f(x)$. The set $\{f(x): x \in S\}$ which is a subset of **T** is called the range of **f**. If Range of **f** = Co-domain of **f**, then f is called **onto or surjective function;** otherwise f is called an into function.

Thus, a function f: $S \rightarrow T$ is said to be onto if the range of S is equal to its codomain T.

Suppose $S = \{1, 2, 3, 4,\}$ and $T = \{1, 2, 3, 4, 5, 6\}$ and $f: S \rightarrow T$ is defined by $f(n) = n + 1 \ \forall \ n \in S$. Then the range of $f = \{2, 3, 4, 5\}$. This shows that f is an into function. On the other hand if $S = \{1, 2, 3, 4\}$, $T = \{1, 4, 9, 16\}$ and if $f: S \rightarrow T$ is defined by $f(n) = n^2$, then f is onto. You can verify that the range of f is, in fact, equal to T.

Refer back to the example on the books in IGNOU. It is just possible that two books may have the same number of pages. If it is so, then under this function, two different books shall have the same natural number as their image. However if for a function any two distinct elements in the domain thave distinct images in the co-domain, then the function is called **one-one** or **injective.**

Thus a function f is said to be one-one if distinct elements in the domain of f have distinct images or in other words, if $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ for any x_1, x_2 in the domain of f.

A function which is one-one and onto, is called a bijection or a 1-1 correspondence.

EXAMPLE 2: (i) Let $S = \{1, 2, 3\}$ and $T = \{a, b, c\}$ and a, let f: $S \rightarrow T$ be defined as $f(1) = f(2) = b$, $f(3) = c$. Then f is one-one and onto.

(ii) Let $N = \{1, 2, 3, 4 ....\}$ and f: $N \rightarrow N$ is defined as $f(n) = n + 1$. As 1 does not belong to the range of f, therefore f is not onto. However, f is one-one.

(iii) Let $S = \{1, -1, 2, 3, -3\}$ and let $T = \{1, 4, 9\}$. Define f: $S \rightarrow T$ by $f(n) = n^2$ $\forall n \in S$. Then f is not one-one as $f(1) = f(-1) = 1$, however f is onto.

DEFINITION 7: IDENTITY FUNCTION

Let S be any non-empty set. A function f: $S \rightarrow S$ defined by $f(x) = x$ for each x in S is called the identity function.

It is generally denoted by ^1S. It is easy to see that ^1S is one-one and onto.

DEFINITION 8: CONSTANT FUNCTION

Let S and T be any two non-empty sets. A function f: S→ T defined by f(x) = c for each x in S, where c is a fixed element of T, is called a constant function.

For example f: S→R defined as f(x) = 2, for every x in S, is a constant function. **Is this function one-one and onto?** Verify it.

DEFINITION 9 : EQUALITY OF FUNCTIONS

Any two functions with the same domain are said to be equal if for each point of their domain, they have the same image. Thus if f and g are any two functions defined on an non-empty set S, then

$$f = g \text{ if } f(x) = g(x), \forall x \in S.$$

In other words, **f = g** iff **f** and **g** are identical.

Let us now discuss another important concept in this section. This is about the composition or combination of any two functions. Consider the following situation:

Let S ={1, 2, 3, 4}, T = {1, 4, 9, 16}, N = {1, 2, 3, 4 ....} be any three sets. Let **f : S→ T** be defined by f(x) = x^2 ∀ x ∈ S and **g: T→** N be defined by **g(x) = x + 1, ∀ x ∈ T.** Then, by the function **f**, an element x ∈ S is mapped to **f(x)** = x^2. Further by the function **g** the element **f(x)** is mapped to **f(x)** + 1 = x^2 + 1. Denote this as **g(f(x))**. Define a function **h: S→ N** by **h**(x) = **g** (f(x)). This function **h maps each x in S to some unique element g(f(x))** = x^2+1 **of N. The function h is called the composition or the composite of the functions f and g.** Thus, we have the following definition:

DEFINITION 10: COMPOSITE OF FUNCTIONS

Let f: S→ **T and g: T→ V be any two functions. A function h: S→ V denoted as h = 'g ∘ f' and defined by**

$$h(x) = (g \circ f)(x) = g(f(x)), \forall x \in S$$

is called the composite of f and g.

Note that the domain of the composite function is the set S and its Co-domain is the set V.The set T which contains the range of f is equal to the domain of g.

But in general, the composition of the two functions is meaningful whenever the range of the first is contained in the domain of the second.

EXAMPLE 3: Let S = T = V - {1, 2, 3, 4 ....}. **Define**
f(x = 2x and g(x) = x + 5. Then
'g∘f' is defined as (g∘f) (x) = g(f(x)) = g(2x) = 2x + 5.

Note that we can also define **f∘g** the composite of **g** and **f.**
Here **(f∘g)** (x) = **f(g**(x)) = **f** (x+5) = 2 (x+5) = 2x + 10.
Also **(f∘g)** (1) = 12 and **(g∘f)** (1) = 7.
This shows that 'f∘g' need not be equal to 'g∘f'.

Let S = {1, 2, 3,} and T = {a, b, c}. Let **f: S→ T** be **f(**1) = a, **f** (2) = b, **f**(3) = c. Define a function **g: T→ S** as **g**(a) = 1, **g**(b) = 2 and **g** (c) = 3. Under the function **g,** the element f(x) in T is taken back to the element x in S. This mapping **g** is called the **inverse** of **f** and is given by **g(f(**x)) = x for each x in S. You may note that **f(g** (a)) = a, **f (g(**b)) = b and **f(g(**c)) = c. Thus, we have the following definition:

DEFINITION 11: INVERSE OF A FUNCTION

Let S and T be two non-empty sets. A function f: S → T is said to be invertible if there exists a function g: T → S such that

$$(g \circ f)(x) = x \text{ for each } x \text{ in } S$$

and

$$(f \circ g)(x) = x \text{ for each } x \text{ in } T.$$

In this case **g** is said to be the inverse of **f** and we write it as **g** = f^{-1}

You may ask: **Do all functions possess inverses?**

No, all functions do not possess inverses. For example, let S = {1, 2, 3} and T = {a, b}. If **f: S → T** is defined as **f**(1) = **f**(2)= **a** and **f**(3) = b, then **f** is not invertible. For if **g: S → T** is inverse of **f,** then

$$(g \circ f)(1) = g(f(1)) = g(a)$$

and

$$(g \circ f)(2)\ g(f(2)) = g(a).$$

Therefore

$$1 = 2 = g(a).$$

which is absurd.

This raises another question : **Under what conditions a function has an inverse? If a function f: S → T is one-one and onto, then it is invertible. Conversely, if f is invertible, then f is both one-one and onto. Thus if a function is one-one and onto, then it must have an inverse.**

1.3 SYSTEM OF REAL NUMBERS

You are quite familiar with some number systems and some of their properties. You will, perhaps recall the following properties:

(i) Any number multiplied by zero is equal to zero,

(ii) the product of a positive number with a negative number is negative,

(iii) the product of a negative number with a negative number is positive among others.

To illustrate these properties, you will most likely use the natural numbers or integers or even rational numbers. The questions, which begin to arise are: **What are these various types of numbers? What properties characterise the distinction between these various sets of numbers?**

In this section, we shall try to provide answers to these and many other related questions. Since we are dealing with the course on Real Analysis, therefore we confine our discussion to the system of real numbers. Nevertheless, we shall make you peep into the realm of a still larger class of numbers, the so called complex numbers.

The system of real numbers has been evolved in different ways by different mathematicians. In the late 19th Century, the two famous German mathematicians Richard Dedekind [1815-1897] and George Cantor [1845-1918] gave two independent approaches for the construction of real numbers. During the same time, an Italian mathematician, G. Peano [1858-1932] defined the natural numbers by the well-known Peano Axioms. However, we start with the set of natural numbers as the foundation and build up the integers. From integers, we construct the rational numbers and then through the set of rational numbers, we reach the stage of real numbers. This development of number system culminates into the set of complex numbers. A detailed study of the system of numbers leads us to a beautiful branch of Mathematics namely The Number Theory, which is beyond the scope of this course. However, we shall outline the general development of the system of the real numbers in this section. This is crucial to understand the characterization of the real numbers in terms of the algebraic structure to be discussed in Unit 2. Let us start our discussion with the natural numbers.

1.3.1 NATURAL NUMBERS

The notion of a number and its counting is so old that it is difficult to trace its origin. It developed much before the time of even the recorded history that its manner of development is based on conjectures and guesses. The mankind, even in the most primitive times had some number sense. The man, at least, had the sense of recognizing 'more' and 'less', when some objects were added to or taken out from a small collection. Studies have shown that even some animals possess such a sense. With the gradual evolution of society, simple counting became imperative. A tribe had to count how many members it had, how many enemies and how many friends. A shephard or a cowboy found it necessary to know if his flock of sheep or cows was decreasing or increasing in size. Various ways were evolved to keep such a count. Stones, pebbles, scratches on the ground, notches on a big piece of wood, small sticks, knots in a string or the fingers of hands were used for this purpose. As a result of several refinements of these counting methods, the numbers were expressed in the written symbols of various types called the digits. These digits were written differently according to the different languagues and cultures of the societies. In the ultimate development, the numbers denoted by the digits 1, 2, 3, ... became universally acceptable and were named as **natural numbers.**

Leopold Kronecker

Different theories have been advanced about the origin and evolution of natural numbers. An axiomatic approach, as evolved by G. Peano, is often used to define the natural numbers. Some mathematicians like L. Kronecker [1823-1891] have remarked that the natural numbers are a creation of God while all else is the work of man.

However, we shall not go into the origin of the natural numbers. In fact, we accept that the natural numbers are a gift of nature to the mankind.

We denote the set of all natural numbers as

$$N = \{1, 2, 3, ...\}.$$

One of the basic properties of these numbers is that there is a starting number 1. Then for each number there is a next number. This nextness property is an important idea that you may find fascinating with the natural numbers. You may think of any big natural number. Yet, you can always tell its next number. What's the next number after forty nine? After seventy seven? After one hundred twenty three? After three thousand and ninety nine? Thus you have an endless chain of natural numbers.

Some of the basic properties of the natural numbers are concerning the well-known fundamental operations of addition, multiplication, subtraction and division. You know that the symbol '+' is used for addition and the symbol '.' is used for multiplication. If we add or multiply any two natural numbers, we again get natural numbers. We express it by saying that the set of natural numbers is closed with respect to these operations.

However, if you subtract 2 from 2, then what you get is not a natural number. It is a number which we call zero denoted as '0'. The word zero in fact is a translation of the Sanskrit **'shunya'**. It is universally accepted that the concept of the number zero was given by the ancient Hindu mathematicians. You come across with certain concrete situations indicating the meaning of zero. For example the temperature of zero degrees is certainly not an absence of temperature.

After having fixed the idea of the number zero, it should not be difficult for you to understand the notion of negative natural numbers. You must have heard the weather experts saying that the temperature on the top of the hills is minus 5 degrees written as $-5°$. What does it mean? The simple and straight explanation is that -5 is the negative of 5 i.e. -5 is a number such that $5+(-5)=0$. Hence -5 is a negative natural number. Thus for each natural n, there is a unique number $-n$, called the negative of n such that

$$n + (-n) = 0.$$

1.3.2 INTEGERS

You have seen that in the set N of natural numbers, if we subtract 2 from 2 or 3 from 2, we do not get back natural numbers. Thus set of natural numbers is not closed with respect to the operation of subtraction. After the operation of subtraction is introduced, the need to include 0 and negative numbers becomes apparent. To make this operation valid, we must enlarge the system of natural numbers, by including in it the number 0 and all the negative natural numbers. This enlarged set consisting of all the natural numbers, zero and the negatives of natural numbers, is called the set of integers. It is denoted as

$$Z = \{.... -3, -2, -1, 0, 1, 2, 3, ...\}.$$

Now you can easily verify that the set of integers is closed with respect to the operations of addition, multiplication and subtraction.

The integers 1, 2, 3 .... are also called positive integers which are in fact natural numbers. The integers $-1, -2, -3, ...$ are called negative integers which are actually the negative natural numbers. The number 0, however, is neither a positive integer nor a negative one. The set consisting of all the positive integers and 0 is called the set of **non-negative integers.** Similarly we talk of the set of **non-positive integers.** Can you describe it?

1.3.3 RATIONAL NUMBERS

If you add or multiply the integers 2 and 3, then the result is, of course, an integer in each case. Also if you subtract 2 from 2 or 2 from 3, the result once again in each case, is an integer. What do you get, when you divide 2 by 3? Obviously, the result is not an integer. Thus if you divide an integer by a non-zero integer, you may not get an integer always. You may get the numbers of the form

$$\frac{1}{2}, \frac{1}{3}, \frac{-2}{3}, \frac{-4}{5}, \frac{5}{6} .... \text{so on.}$$

Such numbers are called rational numbers.

A rational number of form $\frac{-p}{q}$ or $\frac{p}{-q}$ is equivalently written as $\frac{-p}{q}$ where p and q are both positive integers.

Thus the set **Z** of integers is inadequate when the operation of division is introduced. Therefore, we enlarge the set **Z** to that of all rational numbers. Accordingly, we get a bigger set which includes all integers and in which division by non-zero integers is possible. Such a set is called the set of rational numbers. Thus a rational number is a number of the form $\frac{p}{q}$, $q \neq 0$, where p and q are integers. We shall denote the set of all rational numbers by **Q**. Thus,

$$\mathbf{Q} = \{ x = \frac{p}{q} : p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}.$$

EXERCISE 5)

Justify that

(i) N is a proper subset of Z.

(ii) Z is a proper subset of Q.

Now if you add or multiply any two rational numbers you again get a rational number. Also if you subtract one rational number from another or if you divide one rational number by a non-zero rational, you again get a rational numbers in each case. Thus the set **Q** of rational numbers looks to be a highly satisfactory system of numbers in the sense that the basic operations of addition, multiplication, subtraction and division are defined on it. However, **Q** is also inadequate in many ways. Let us now examine this aspect of **Q**.

Consider the equation $x^2 = 2$. We shall show that there is no rational number which satisfy this equation. In other words, we have to show that there is no rational number whose square is 2. We discuss this question in the form of the following example:

EXAMPLE 4 : Prove that there is no rational number whose square is 2.

SOLUTION : If possible, suppose that there is a rational number x such that $x^2 = 2$. Sinc x is a rational number, therefore x must be of the form

$$x = \frac{p}{q}, p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0.$$

Note that the integers p and q may or may not have a common factor. We assume that p and q have no common factor except 1.

Squaring both sides, we get

$$\frac{p^2}{q^2} = 2.$$

Then we have

$$p^2 = 2q^2.$$

This means that p^2 is even and hence p is even (verify it). Therefore, we can write $p = 2k$ for some integer k. Accordingly, we will have

$$p^2 = 4k^2 = 2q^2$$

or

$$q^2 = 2k^2.$$

Indeed, if p were odd, then p would be of the form $p = 2k+1$ for some integer k. Accordingly, then $p^2 = 4k^2 + 4k+1$ which is obviously odd. This contradicts the fact that p^2 is even. Hence p must be even.

Thus p and q are both even. In other words, p and q have 2 as a common factor. This contradicts our supposition that p and q have no common factor.

Hence there is no rational number whose square is 2.
Why don't you try the following similar exercises?

EXERCISE 6)
Show that there is no rational number whose square is 3.

EXERCISE 7)
There is no rational number x such that $x^2 = 5$

EXERCISE 8)
Use the argument of Example 6 to show that there is no rational number whose square is 4 and show where does the proof fail.

Thus you have seen that there are numbers which are not rationals. Such numbers are called irrational. In other words, a number is irrational if it cannot be expressed as p/q, $p \in Z$, $q \in Z$, $q \neq 0$. In this way, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ etc. are irrational numbers. In fact, such numbers are infinite. Rather, you will see in Unit 2 that such numbers are even uncountable. Also you should not conclude that all irrational numbers can be obtained in this way. For example, the irrational numbers e and π are not of this form. We denote by **I**, the set of all irrational numbers.

Thus we have seen that the set **Q** is inadequate in the sense that there are numbers which are not rationals.

A number which is either rational or irrational is called a real number. The set of real numbers is denoted by **R**. Thus the set **R** is the disjoint Union of the sets of rational and irrational numbers i.e. $\mathbf{R} = \mathbf{Q} \cup \mathbf{I}$, $\mathbf{Q} \cap \mathbf{I} = \phi$.

Now in order to visualise a clear picture of the relationship between the rationals and irrationals, their geometrical representation as points on a line is of great help. We discuss this in the next section.

1.4 THE REAL LINE

Draw a straight line L as shown in the Figure 1.

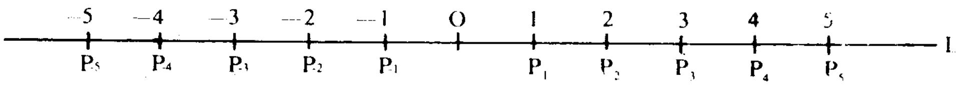


Fig. 1

Choose a point O on L and another point P_1 to the right of O. Associate the number 0 (zero) to the point O and the number 1 to the point P_1. We take the distance between the points O and P_1 as a unit length. We mark a succession of points P_2, P_3, ... to the right of P_1 each at a unit distance from the preceding one. Then associate the integers 2, 3, ... to the points P_2, P_3, respectively. Similarly, mark the points P_{-1}, P_{-2}, ... to the left of the point O. Associate the integers -1, -2, ... to the points P_{-1}, P_{-2} .... Thus corresponding to each integer, you have associated a unique point of the line L.

Now associate every rational number to a uniqe point of L. Suppose you want to associate the rational number $\frac{2}{7}$ to a point on the line L. Then $\frac{2}{7} = 2 \times \frac{1}{7}$ i.e. one unit is divided into seven parts, out of which 2 are to be taken. Let us see how you can do it geometrically

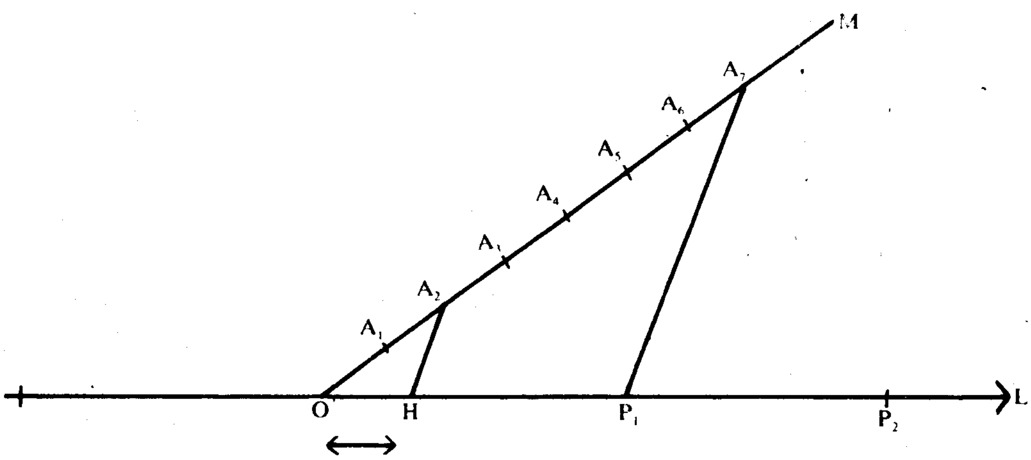


Fig. 2

Through O, draw a line O M inclined to the line L. Mark the points $A_1, A_2 \ldots A_7$ on the line OM at equal distances. Join $P_1 A_7$. Now if you draw a line through A_2 parallel to $P_1 A_7$ to meet the line L in H. Then H corresponds to the rational number $\frac{2}{7}$ i.e. OH = 2/7.

You can do likewise for a negative rational number. Such points, then, will be to the left of O.

EXERCISE 9)

Give the geometrical representation of $\frac{3}{7}$.

By having any line through O, you can see that the point P does not depend upon chosen line OM. Thus you have associated every rational number to a unique point on the line L.

Now arises the important question:

Have you used all the points of the line L while representing rational numbers on it?

The answer to this question is No. But how? Let us examine this.

At the point P_1 draw a line perpendicular to the line and mark A such that $P_1 A = 1$ unit. Cut off OB = OA on the line as shown in the Figure 3.

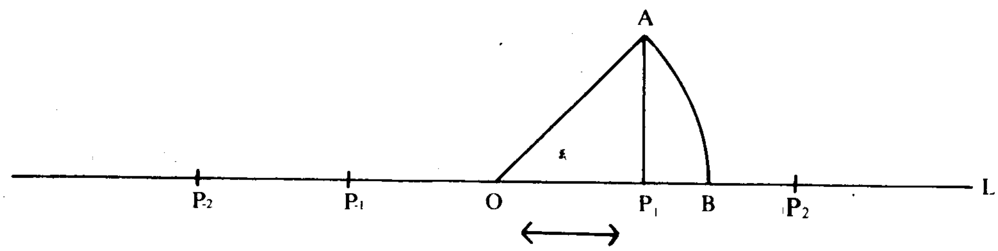


Fig. 3

Then B is a point which correspond to a number whose square is 2. You have already seen that there is no rational number whose square is 2. In fact, the length OA = $\sqrt{2}$ by Pythagorus Theorem. In other words, the irrational number $\sqrt{2}$ is associated with the point B on the line L. In this way, it can be shown that every irrational number can be associated to a unique point on the line L.

Thus, it can be shown that to every real number, there corresponds a unique point on the line L. In other words, all the real numbers are represented as points on a line. Is the converse true? That is to say, **does every point on the line correspond to a unique real number?** The very assumption that this happens i.e. every point on the line corresponds to a unique real number is known as the **Continuoum Hypothesis** or **Hypothesis of the Continuoum.** Therefore, hence onwards, we shall **say** that every real number corresponds to a unique point on the line and conversely every point on the line corresponds to a unique real number. In this sense, the line is called **the Real Line.**

Now let L be the real line.

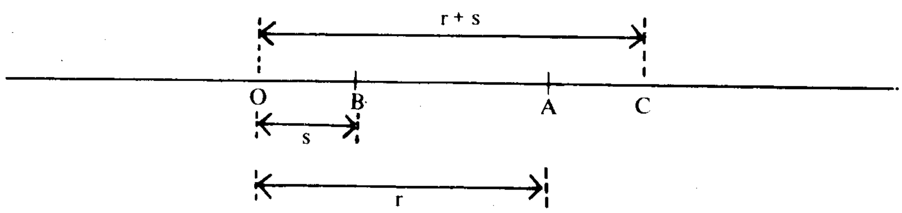


Fig. 4

We may define addition (+) and multiplication (.) of real numbers geometrically as follows:

Suppose A represents a real number r and B represents a real number s so that OA = r and OB = s. Shift OB so that O coincides with A. The point C which is the new position of B is defined to represent r+s. See the Figure 4.

The construction is valid for positive as well as negative values of r and s. A real number r is said to be positive if r corresponds to a point on the line L on the right of the point O. It is written as $r > 0$. Similarly, r is said to be negative if it corresponds to a point on the left of the point O and is written as $r < 0$. Thus if r is a real number, then either r is zero or r is positive or r is negative i.e. either $r = 0$ or $r > 0$ or $r < 0$. You should try the following exercise:

EXERCISE 10)
Construct r—s, r+ s and —r—s on the real line.

What about the product r s of two real numbers r and s? We shall consider the case when r and s are both positive real numbers.

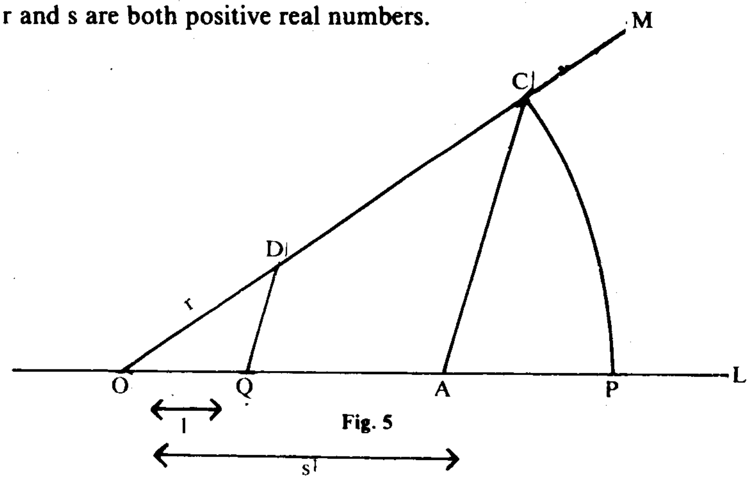


Fig. 5

Though O draw some other line OM. On L, let A represent the real number s. On OM take a point D so that OD = r. Let Q be a point on L so that OQ = 1 unit. Join Q D. Through A draw a straight line parallel to Q D to meet OM at C. Cut off OP on the line equal to OC. Then P represents the real number r.s.

Suppose s is a positive real number and r is a negative real number. Then, there exists a number r such that $r = -r'$ where r' is a positive real numbers. Therefore, the product rs can be defined on L as

$$rs = (-r')s = -(r's).$$

Similarly you can state that $rs = r(-s') = -(rs')$ where s is negative and $s = -s'$ for some positive s', while r is positive.

If both r and s are negative and $r = -r'$ and $s = -s'$ where r' and s' are positive real numbers, then we define

$$rs = r's' = (-r)(-s).$$

We can also similarly define $0. r = r. 0 = 0$ for each real number r.

1.5 COMPLEX NUMBERS

So far, we have discussed the system of real numbers. We have, yet, another system of numbers. For example, if you have to find the square root of a negative real number say −5, then you will write at as $\sqrt{-1}. \sqrt{5}$. You know that $\sqrt{5}$ is a real number but what about $\sqrt{-1}$? Again you can verify that a simple equation $x^2 + 1 = 0$ does not have a solution in the set of real numbers because the solution involves the square root of a negative real number. As a matter of fact, the problem is to investigate the nature of the number $\sqrt{-1}$ which we denote by such that $i^2 = -1$. Let us discuss the following example to know the nature of i.

EXAMPLE 5: Show that i is not a real number.

We claim that i is not a real number. If it is so, then either $i = 0$ or $i > 0$ or $i < 0$.

If $i = 0$, then $i^2 = 0$. This implies that $-1 = 0$ which is absurd. If $i > 0$, then $i^2 > 0$ which implies that $-1 > 0$. This is also absurd. Finally, if $i < 0$, then again $i^2 > 0$ which implies that $-1 > 0$. This again is certainly absurd. Thus i is not a real number. This number 'i' is called the imaginary number. The symbol 'i' is called **'iota'** in Greek language. This generates another class of numbers, the so called **complex numbers.**

The basic idea of extending the system of real numbers to the system of complex numbers arose due to the necessity of finding the solutions of the equations. like $x^2 + 1 = 0$ or $x^2 + 2 = 0$ and so on. The first contribution to the notion of such a number was made by **the most celebrated Indian Mathematician of the 9th century, Mahavira, who showed that a negative real number does not have a square root in the set of real numbers.** But it was an Italian mathematician, G. Cardon [1501-1576] who used imaginary numbers in his work without bothering about their existence. Due to notable contributions made by a large number of mathematicians, the system of complex numbers came into existence in the 18th century. Since we are dealing with real numbers, therefore, we shall not go into the detailed discussion of complex numbers. However, We shall give a brief introduction to the system of complex numbers. We denote the set of complex numbers as

$$C=\{z=a+i\,b, \text{ a and b real numbers}\}$$

In a complex number, $z = a + i\,b$, a is called its real part and b is called its imaginary part.

Any two complex numbers $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are equal if only if their corresponding real and imaginary parts are equal.

If $z_1 = a_1 + i\,b_1$ and $z_2 = a_2 + i\,b_2$ are any two complex numbers, then we define addition (+) and multiplication (.) as follows:

$$z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2)$$

$$z_1.z_2 = (a_1a_2 - b_1b_2) + i\,(a_1b_1 + a_2b_2).$$

The real numbers represent points on a line while complex numbers are identified as points on the plane.

EXERCISE 11)
Justify that R is a subset of C.

Before concluding this section, we would like to mention yet another classification of numbers as enunciated by some mathematicians. Consider the number $\sqrt{2}$. This is an example of what is called an **Algebraic Number** because it satisfies the equation

$$x^2 - 2 = 0.$$

A number is called an **Algebraic Number** if it satisfies a polynomial equation

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}\,x + a_n + a_n = 0$$

where the coefficients $a_0, a_1, a_2, \ldots\, a_n$ are **integers,** $a_n \neq 0$ and $n > 1$. The rational numbers are always algebraic numbers. The numbers defined in terms of the square root etc. are also treated as algebraic numbers. But there are some real numbers which are not algebraic. Such numbers are called the **Transcendental numbers.** The numbers π and e are transcendental numbers.

You may think that the operations of algebraic operations viz. addition, multiplication, etc. are the only aspects to be discussed about the set of real numbers. But certainly there are some more important aspects of the set of real numbers as points on the real line. We shall discuss these aspects in Unit 3 namely the point sets of the real line called also the topology of the real line. But prior to that, we shall discuss the structure of real numbers in Unit 2.

We conclude this unit by talking briefly about an important hypothesis closely linked with the system of natural numbers. This is called the **Principle of Induction.**

1.6 MATHEMATICAL INDUCTION

The Principle of Induction and the natural numbers are inseparable. In Mathematics, we often deal with the proofs of various theorems and formulas. Some of these are derived by the direct proofs, while some others can be proved by certain indirect methods. Consider, for example, the validity of the following two statements:

(i) The number 4 divides $5^n - 1$ for every natural number n.

(ii) The sum of the first n natural numbers is $\frac{n(n+1)}{2}$ i.e.

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

In fact, you can provide most of the verifications for both statements in the following way:

For (i), if $n = 1$, then $5^n - 1 = 5 - 1 = 4$ which is obviously divisible by 4;
if $n = 2$, then $5^2 - 1 = 24$, which is also divisible by 4;
if $n = 6$, then $5^6 - 1 = 15624$, which is indeed divisible by 4.

Similarly for (ii) if $n = 10$ then $1+2+ .... +10 = 55$, while the formula

$$\frac{n(n+1)}{2} = 55 \text{ when } n = 10.$$

Again, if $n = 100$, then also you can verify that in each way, the sum of the first hundred natural numbers is 5050 i.e.

$$\frac{n(n+1)}{2} = 5050 \text{ for } n = 100.$$

What do these statements have in common and what do they indicate? The answer is obvious that each statement is valid for every natural number.

Thus to a great extent, a large number of theorems, formulas, results etc. whose statement involves the phrase, **"for every natural number n"** are those for which an indirect proof is most appropriate. In such indirect proofs, clearly a criterion giving a general approach is applied. One such criterion is known as the criterion of Mathematical Induction. The principle of Mathematical Induction is stated (without proof) as follows:

Principle of Mathematical Induction

Suppose that, for each $n \in N$, P (n) is a statement about the natural number n. Also, suppose that

(i) P(1) **is true,**

(ii) if P(n) **is true,** then P(n+1) is also **true.**

Then P(n) **is true for every** $n \in N$.

Let us illustrate this principle by an example:

EXAMPLE 6: The sum of the first n natural numbers is $\frac{n(n+1)}{2}$

SOLUTION: In other words, we have to show that for each $n \in N$,

$$1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}$$

$$S_n = 1 + 2 + 3 + ... + n$$

$$= \sum_{k=1}^{n} k.$$

Let P(n) be the statement that

$$S_n = \frac{n(n+1)}{2}$$

We, then, have $S_1 = 1$ and $\frac{1(1+1)}{2} = 1$. Hence P(1) is true.

This proves part (i) of the Principle of Mathematical Induction. Now for (ii), we have to verify that if P(n) is true, then P(n+1) is also true. For this, let us assume that P(n) is true and establish that P(n+1) is also true. Indeed, if we assume that

$$S_n = \frac{n(n+1)}{2},$$

then we claim that

$$S_{n+1} = \frac{(n+1)\quad(N=2)}{2}$$

Indeed

$$\begin{aligned} S_{n+1} &= 1 + 2 + 3 + ... + n + (n+1) \\ &= S_n + (n+1) \\ &= \frac{1}{2}\, n(n+1) + (n+1) \\ &= \frac{(n+1)\,(n+2)}{2} \end{aligned}$$

Thus P(n + 1) is also true.

By using the Principle of Induction, you can prove that

(i) the sum of the squares of the first n natural numbers is

$$\frac{1}{6}\, n\,(n+1)\,(2n+1),$$

(ii) the sum of the cubes of the first n natural numbers is

$$\frac{1}{4}\, n^2\,(n+1)^2.$$

1.7 SUMMARY

We have recalled some of the basic concepts of Sets and Functions in section 1.2. A set is a well-defined collection of objects. Each object is an **element** or a **member** of the set. Sets are generally designated by capital letters and the members by small letters enclosed with **braces.** There are two ways to indicate the members of a set. The **tabular method** or **listing method** in which we list each element of a set individually and the **set-builder method** which gives a verbal description of the elements or a property that is common to all the elements of a set.

A set with a limited number of elements is a **finite set.** A set with an unlimited number of elements is an infinite set. A set with no elements is a **null-set.** A set **S** is a **subset** of a set **T** if every element of S is in **T.** The set **S** is said to be a **proper subset of T** if every element of S is in T and there is at-least one element of **T** which does not belong to S. The sets S and T are equal if S is a subset of T and T is a subset of **S.** The null set is a subset of every set and every set is a subset of itself.

The **Union** of two sets **S** and **T,** written as **S ∪ T,** includes all the elements of **S, T** or both S and **T.** The **intersection** of S and **T,** written as **S ∩ T,** includes all the elements that are common to both S and **T.** The **complement** of a set S in a Universal set **X** is the set denoted as S^c and it consists of all those elements of **X** which do not belong to S. The laws with respect to Union. Intersection and Complement have been asked in the form of exercises. Also, these notions have been extended to an arbitrary family of sets.

A **function f: S→ T** is a rule by which you can associate to each element of S, a unique element of T. The set S is the **domain** and the set **T** is the **co-domain** of **f.** The set {**f**(x): x∈ S} is the **Range** of **f** where **f**(x) is an image of x under **f.** The function **f** is **one-one** if $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ for any x_1, x_2 in the domain of **f.** It is said to be **onto** if the range of **f** is equal to the domain of **f.** A function **f** is said to be a one-one correspondence if it is both one-one and onto. A function **i: S → S** defined by $i(x) = x,\ \forall\, x \in S$ is called an **identity function,** while a function **f: S → T** is said to be **constant** if $f(x) = c\ \forall\, x \in S$, c being a fixed element of **T.**

Any two functions with the **same domain** are said to be **equal** if they have the same image for each point of the domain. The composite of the functions **f: S → T** and **g: T → V** is a function denoted as '**g o f**': **S → V** and defined by **(g o f) (x) = g (f(x)).** The function **f: S → T** is said to be invertible if there exists a function **g: T → S** such that both '**g o f**' and '**f o g**' are identity functions. Also, a function is invertible if it is both one-one and onto. The inverse of f exists if **f** is invertible and it is denoted as f^{-1}.

In Section 1.3, we have discussed the development of the system of numbers starting from the set of natural numbers. These are the following:

Natural Numbers (Positive Integers): $N = \{1, 2, 4 ....\}$

Integers: $\mathbf{Z} = \{.... 3, -2, -1, 0, 1, 2, 3 ....\}$

Rational Numbers: $\mathbf{Q} = \{\frac{p}{q} : p \in z, q \in \mathbf{Z}, q \neq 0\}$

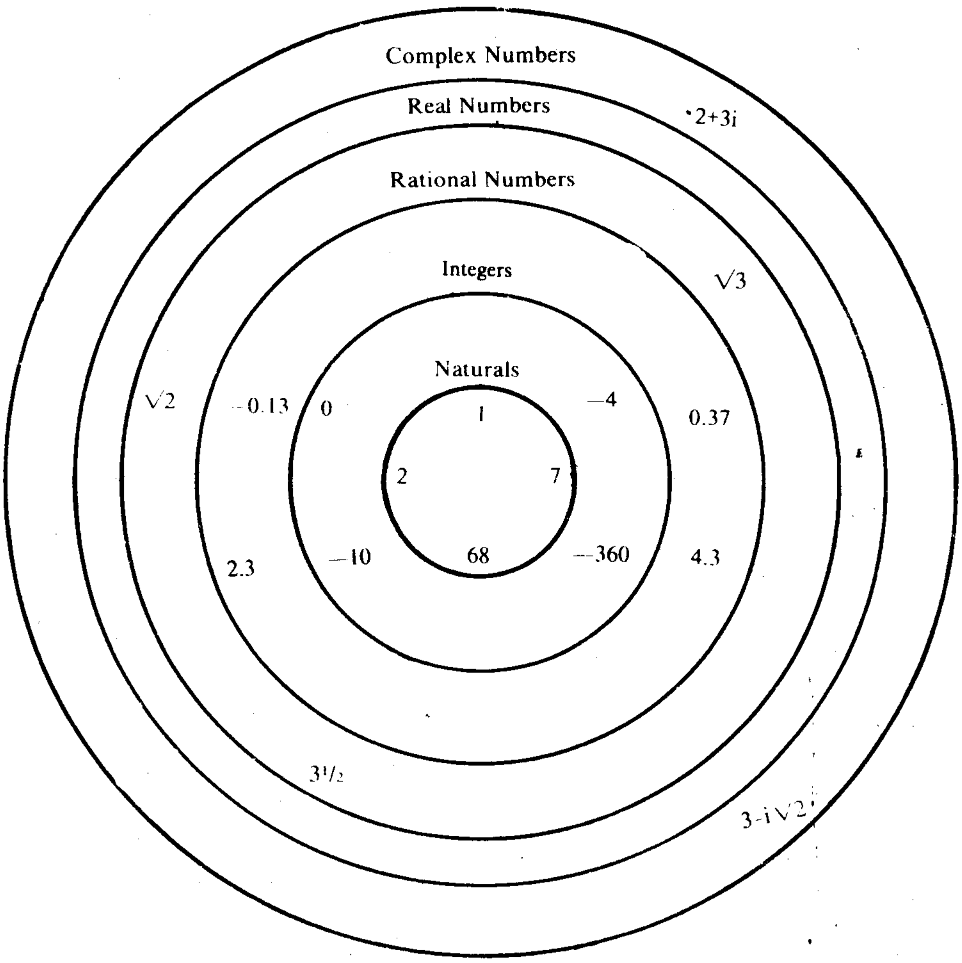
Irrational Numbers I

Real Numbers: **R** = Disjoint Union of Rational and Irrational Numbers

$$\mathbf{R} = \mathbf{Q} \cup \mathbf{I}, \quad Q \cap I = \phi$$

Complex Numbers $C = \{ z = x+iy : x \in \mathbf{R}, y \in \mathbf{R}\}, i=\sqrt{-1}$.

A mathematical development of the number systems is depicted in Figure 6:



A mathematical development of numbers systems

Fig. 6

In Section 1.4, we have discussed the geometrical representation of the real numbers and stated the continuum Hypothesis. According to this, every real number can be represented by a unique point on the line and every point on the line corresponds to a unique real number. In view of this, we call this line as **the Real Line.**

1.8 ANSWERS/HINTS/SOLUTIONS

E 1) (i) $A = \{n: n \in N, n \text{ is even}\}$.
$A = \{n: n \in N, n \text{ is odd}\}$.
$A = \{x: x \text{ is a president of India}\}$.

(ii) $A = \{1, 3, 5\}$.
$A = \{21, 22, 23, 24 \ldots\ldots\ldots 29,\}$.
$A = \{-1, -2, -3 \ldots\ldots\ldots\}$. .

E 2) (i) Each $n \in N \Longrightarrow n \in Z$ but not conversely.

(ii) There exist at least one $x \in R$ such that x does not belong to Q e.g. $x = \sqrt{2}$.

(iii) $A \subset B$ implies that $x \in A \Longrightarrow x \in B$.
$B \subset C$ implies that $x \in B \Longrightarrow x \in C$.
which shows that $x \in A \Longrightarrow x \in C$ and hence $A \subset C$.

E 3) (i) Since ϕ is a subset of everyset, therefore, ϕ is a subset of itself and hence $P(\phi) = \{\phi\}$.

(ii) Any choice of S and X will serve the purpose.

E 4) Choose any three sets (say finite sets) A, B, C. You can justify the required laws.

E 5) It is true because every natural is an integer, every integer is a rational but not conversely i.e. every rational may not be an integer and every integer may not be a positive integer. Give an example in each case.

E 6) (i) If possible, suppose that there is a rational x such that $x^2 = 3$.

Then $x^2 = \dfrac{p^2}{q^2}$ where $p \in \mathbf{Z}$, $q \in \mathbf{Z}$, $q \neq 0$. Suppose p and q have no common factor. Then

$$p^2 = 3q^2$$

which implies that p^2 is divisible by 3 and hence p is divisible by 3. Indeed if $p = 3k+1$, then $p^2 = 9k^2 + 6k+1$ which is not divisible by 3. Hence p must be divisible by 3 i.e. $p = 3k$. Then

$$p^2 = 9k^2$$

or

$$q^2 = 3k^2.$$

Thus both p and q are divisible by 3 which contradicts the assumption that p and q have no common factors. Hence the result.

E 7) Prove it in the same way.

E 8) In this case $\dfrac{p^2}{q^2} = 4 \Longrightarrow p^2 = 4q^2 \Longrightarrow p^2$ is even $\Longrightarrow p=2k$ (say) for some integer $k \Longrightarrow 4q^2 = 4k^2 \Longrightarrow q^2 = k^2$. Proof fails. Give suitable arguments.

E 9)

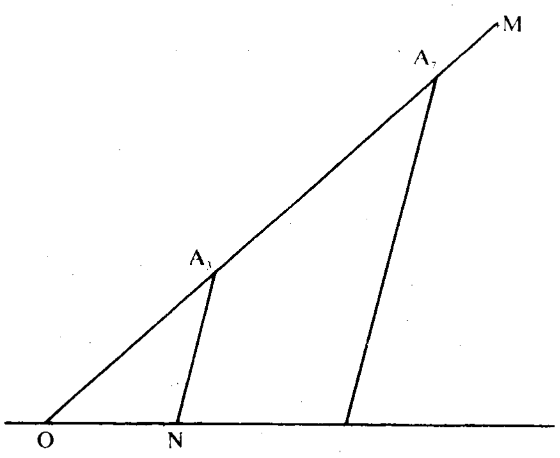


Fig. 7 (i)

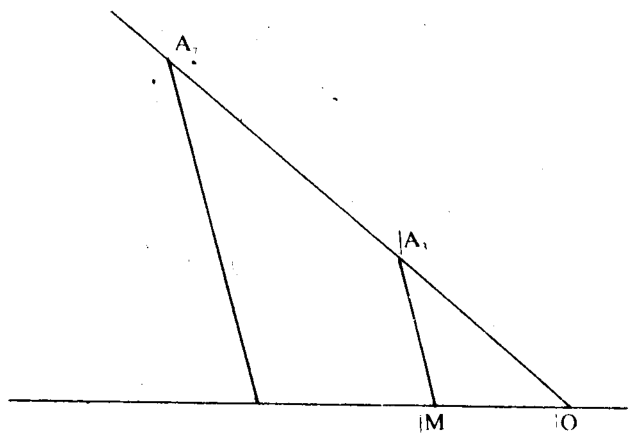


Fig. 7 (ii)

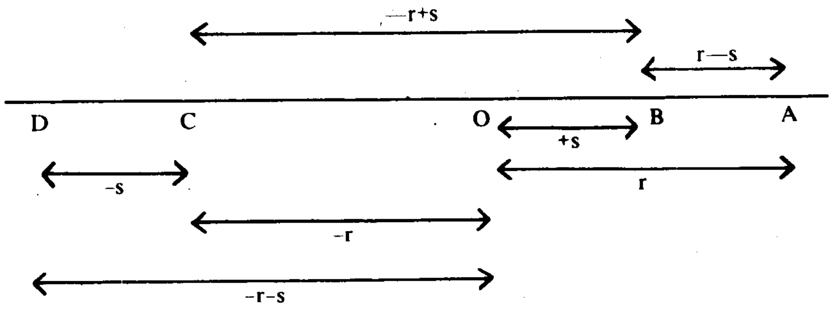


Fig. 8

E 10) BA = r−s

CB = −r+s

DO = −r−s

E 11) Every real number x can be expressed as a complex number e.g. z = x+iy. But every complex number z need not be a real number. Give an example. Think of $i = \sqrt{-1}$.

UNIT 2 STRUCTURE OF REAL NUMBERS

STRUCTURE

2.1 INTRODUCTION

In Unit 1 we have discussed the construction of real numbers from the rational numbers which, in turn, were constructed from integers. In this unit, we show that the set of real numbers has an additional property which the set of rational numbers does not have, namely it is a **complete ordered field.** The questions, therefore, that arise are: What is a **field?** What is an **ordered field?** What does it mean for an ordered field to be **complete?** In order to answers these questions we need a few concepts and definitions e.g. those of order **inequalities** and **intervals** in **R.** We shall discuss these concepts in Section 2.2. Also in this section, we shall explain the extended real number system.

You know that a given set is either finite or infinite. In fact a set is finite, if it contains just n elements where n is some natural number. A set which is not finite is called an infinite set. The elements of a finite set can be counted as one, two, three and so on, while those of an infinite set can not be counted in this way. Can you count the elements of the set of natural numbers? Try it. In Section 2.4, we shall show that this notion of counting can be extended in certain sense to even infinite sets.

OBJECTIVES

In this unit, you should, therefore, be able to

- identify the order relation in the set of real numbers and extended real number system,
- describe the field structure of the set of real numbers,
- discuss the order-completeness of the set of real numbers apply the notion of countability to various infinite sets.

2.2 ORDER RELATIONS IN REAL NUMBERS

In Section 1.3, we have demonstrated that every real number can be represented as a unique point on a line and every point on a line represents a unique real number. This helps us to introduce the notion of inequalities and intervals on the real line which we shall frequently use in our subsequent discussion through out the course.

You know that a real number x is said to be **positive** if it lies on the right side of O, the point which corresponds to the number 0 (zero) on the real line. We write it as $x > 0$. Similarly, a real number x is **negative**, if it lies on the left side of O. This is written as $x < 0$. If $x \geq 0$, then x is a **non-negative** real number. The real number x is said to be **non-positive** if $x \leq 0$.

Let x and y be any two real numbers. Then, we say that x is greater than y if $x-y>0$. We express this by writing $x>y$. Similarly x is less than y if $x-y<0$ and we write $x<y$. Also x is greater than or equal to y ($x \geq y$) if $x-y \geq 0$. Accordingly, x is less than or equal to y ($x \leq y$) if $x-y \leq 0$. Given any two real numbers x and y, exactly one of the following can hold:

either (i) $x>y$
or (ii) $x<y$
or (iii) $x=y$.

This is called the **law of trichotomy**. The order relation $\leq$ has the following properties:

PROPERTY 1: For any x, y, z in R,

(i) If $x \leq y$ and $y \leq x$, then $x=y$,
(ii) If $x \leq y$ an $y \leq z$, then $x \leq z$,
(iii) If $x \leq y$ then $x+z \leq y+z$,
(iv) If $x \leq y$ and $o \leq z$, then $x z \leq y z$.

The relation satisfying (i) is called **anti-symmetric**. It is called **transitive** if it satisfies (ii). The property (iii), shows that the inequality remains unchanged under addition of a real number. The property (iv) implies that the inequality also remains unchanged under multiplication by a non-negative real number. However, in this case the inequality gets reversed under multiplication by a non-positive real number. Thus, if $x \leq y$ and $z \leq 0$, then $x z \geq yz$. For instance, if $z=-1$, we see that

$$-2 \leq 4 \Longrightarrow 2(-1) \geq 4(-1) \Longrightarrow -2 \geq -4.$$

EXERCISE 1)

State the properties of order relation in the set R of real numbers with respect to the relation $\geq$ (is greater than or equal to) and illustrate the inequality under multiplication by a negative real number.

We state the following results without proof:

There lie an infinite number of rational numbers between any two distinct rational numbers.

As a matter of fact, something more is true.

Between any two real numbers, there lie infinitely many rational (irrational) numbers. Thus there lie an infinite number or real numbers between any two given real numbers.

2.2.1 INTERVALS

Now that the notion of an order has been introduced in **R**, we can talk of some special subsets of **R** defined in terms of the order relation. Before we formally define subset, we first introduce the notion of **'betweenness'**, which we have already used intuitively in the previous results. If 1, 2, 3 are three real numbers, then we say that 2 lies between 1 and 3. Thus, in general, if a, b and c are any three real numbers such that $a \leq b \leq c$ then we say that b lies 'between' a and c. Closely related to notion of betweeness is the concept of an interval.

DEFINITION 1: INTERVAL

An interval in R is an nonempty subset of R which has the property that, whenever two numbers a and b belong to it, all numbers between a and b also belong to it.

The set **N** of natural numbers is not an interval because while 1 and 2 belong to **N**, but 1.5 which lies between 1 and 2, does not belong to **N**.

We now discuss various forms of an interval.

Let a, $b \in \mathbf{R}$ with $a \leq b$.

(i) Consider the set $\{x \in \mathbf{R}: a \leq x \leq b\}$. This set is denoted by [a, b], and is called **a closed interval**. Note that the end points a and b are included in it.

(ii) Consider the set $\{x \in \mathbf{R} : a < x < b\}$. This set is denoted by]a, b[, and is called an **open interval**. In this case the end points a and b are not included in it.

(iii) The interval {x ∈ **R**: a ≤ x < b} is denoted by [a, b[.

(iv) The interval {x ∈ **R** : a < x ≤ b} is denoted by]a, b].

You can see the graph of all the four intervals in the Figure 1.

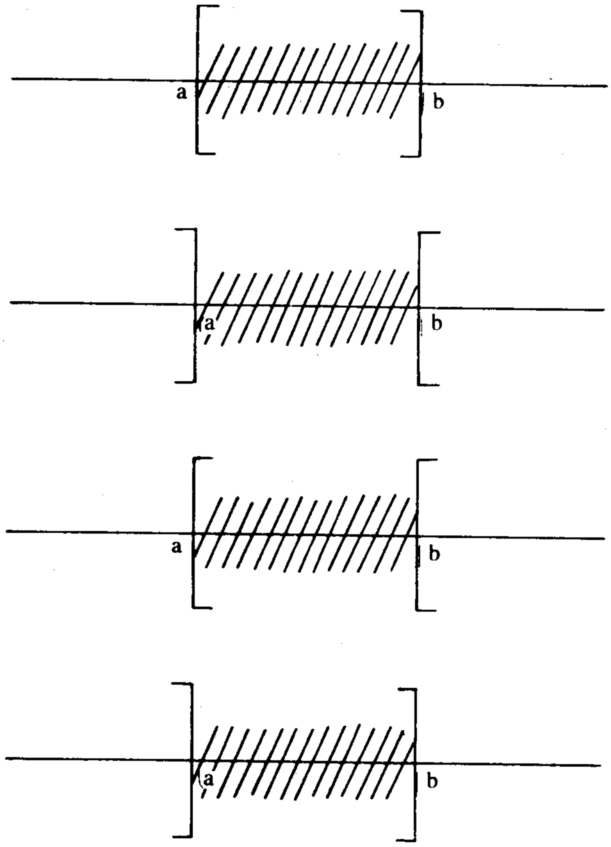


Fig. 1

Intervals of these types are called **bounded intervals.** Some authors also call them **finite intervals.** But remember that these are not finite sets. In fact these are infinite sets except for the case [a, a] = {a}.

You can easily verify that an open interval]a, b[as well as]a, b] and [a, b[are alway contained in the closed interval [a, b].

EXAMPLE 1: Test whether or not the union of any two intervals is an interval.

SOLUTION : Let [2, 5] and [7, 12] be two intervals. Then [2, 5] ∪ [7, 12] is not an interval as can be seen on the line in Figure 2.

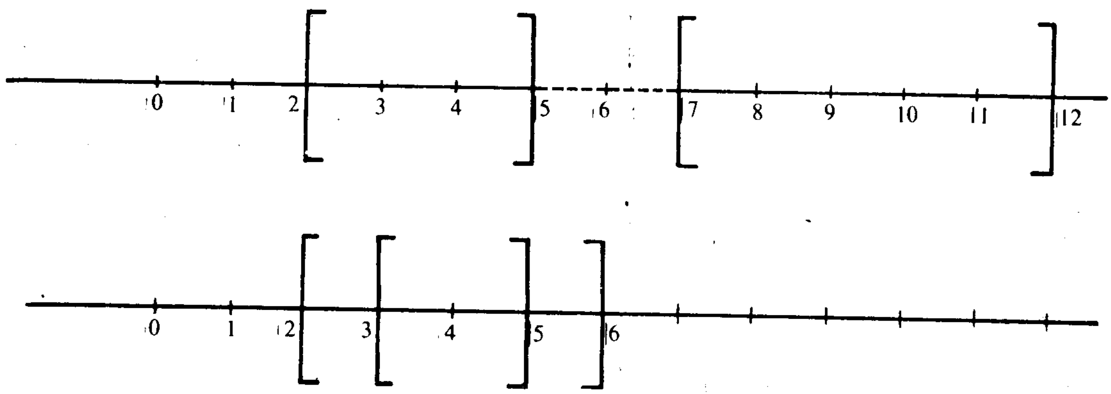


Fig. 2

However, if you take the intervals which are not disjoint, then the union is an interval For example, the union of [2, 5] and [3, 6] is [2, 6] which is an interval. **Thus the union of any two intervals is an interval provided the intervals are not disjoint.**

Now try the following exercise:

EXERCISE 2)
Give examples to show that the intersection of any two intervals may not be an interval. What happens, if the two intervals are not disjoint? Justify your answer by an example.

2.2.2 EXTENDED REAL NUMBERS

The notion of the **extended real number system** is important since we need it in this unit as well as in the subsequent units.

You are quite familiar with the symbols $+\infty$ and $-\infty$. You often call these symbols as **'plus infinity'** and **'minus infinity'** respectively. The symbols $+\infty$ and $-\infty$ are extremely useful. Note that these are **not real numbers.**

Let us construct a new set **R** by adjoining $-\infty$ and $+\infty$ to the set **R** and write it as

$$\mathbf{R} = \mathbf{R} \cup \{-\infty, +\infty\}.$$

Let us extend the order structure to **R** by a relation $<$ as $-\infty < x < +\infty$ for every $x \in R$. Since the symbols $-\infty$ and $+\infty$ do not represent any real numbers, you should, therefore, not apply any result stated for real numbers, to the symbols $+\infty$ and $-\infty$. The only purpose of using these symbols is that it becomes convenient to extend the notion of (bounded) intervals to **unbounded intervals** which are as follows:

Let a and b be any two real numbers. Then we adopt the following notations:

$$[a, \infty[= \{x \in \mathbf{R}: x \geq a\}$$
$$]a, \infty[= \{x \in \mathbf{R}: x > a\}$$
$$]-\infty, b] = \{x \in \mathbf{R}: x \leq b\}$$
$$]-\infty, b[= x \in \mathbf{R}: x < b\}$$
$$]-\infty, \infty[= x \in \mathbf{R}: -\infty < x < \infty\}.$$

You can see the geometric representation of these intervals in Figure 3.

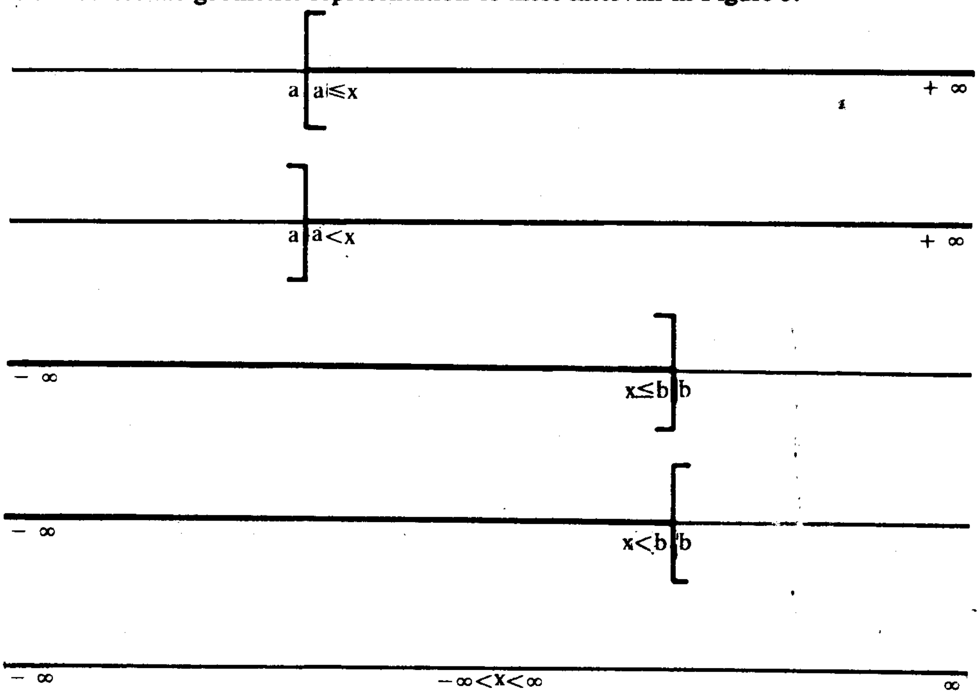


Fig. 3

All these unbounded intervals are also sometimes called **infinite intervals.**

You can perform the operations of addition and multiplication involving $-\infty$ and $+\infty$ in the following way: For any $x \in \mathbf{R}$, we have

$$x + (+\infty) = +\infty,$$
$$x + (-\infty) = -\infty,$$
$$x.(+\infty) = +\infty, \quad \text{if } x > 0$$
$$x.(+\infty) = -\infty \quad \text{if } x < 0$$
$$x.(-\infty) = -\infty, \quad \text{if } x > 0$$

$x.(-\infty) = +\infty$, if $x < 0$

$\infty + \infty = +\infty$, $-\infty - \infty = -\infty$

$\infty.(-\infty) = -\infty$, $(-\infty).(-\infty) = +\infty$.

Note that the operations $\infty - \infty$, $0.\infty$, $\frac{\infty}{\infty}$ are not defined.

2.3 ALGEBRAIC STRUCTURE

Richard Dedekind

During the 19th Century, a new trend emerged in Mathematics to use algebraic structures in order to provide a solid foundation for Calculus and Analysis. In this quest, several methods were used to characterise the real numbers. One of the methods was related to **the least upper bound principle** used by Richard Dedekind which we discuss in this section.

This leads us to the description of the real numbers as a complete ordered field. In order to define a complete ordered field, we need some definitions and concepts.

You are quite familiar with the operations of addition and multiplication on numbers, union and intersection on the subsets of a universal set. For example if you add or multiply any two natural numbers, the sum or the product is a natural number. These operations of addition or multiplications on the sets of numbers are examples of a binary operation on a set. In general, we can define a binary operation on a set in the following way:

DEFINITION 2: BINARY OPERATION

Given a non-empty set S, a binary operation on S is a rule which associates with each pair of elements of S, a unique element of S.

We denote this rule by symbols such as ., *, +. etc.

By an Algebraic Structure, we mean a non-empty set together with one or more binary operations defined on it. A field is an algebraic structure which we define as follows:

DEFINITION 3: FIELD STRUCTURE

A field consists of a non-empty set F together with two binary operations defined on it, denoted by the symbols '+' (addition) and '.' (multiplication) and satisfying the following axioms for any elements x, y, z of the set F.,

A_1: $x+y \in F$ **(Additive Closure)**

A_2: $x+(y+z) = (x+y)+z$ **(Addition is Associative)**

A_3: $x+y = y+x$ **(Addition is Commutative)**

A_4: There exists an element in F, denoted by '0' and called **the zero** or the zero element of F such that $x+0 = 0+x = x \ \forall \ x \in F$ **(Additive Identity)**

A_5: For each $x \in F$, there exists an element $-x \in F$ with the property $x+(-x) = (-x)+x = 0$ **(Additive Inverse)**

The element $-x$ is called **additive inverse** of x.

M_1 $x.y \in F$ **(Multiplicative Closure)**

M_2 $(x.y).z = x.(y.z)$ **(Multiplication is Associative)**

M_3 $x.y = y.x$ **(Multiplication is Commutative)**

M_4 There exists an element 1 different from 0 called the **unity** of F, such that $1.x = x.1 = x \ \forall \ x \in F$ **(Multiplicative Identity)**

M_5 For each $x \in F$, $x \neq 0$, there exists an element $x^{-1} \in F$ such that $x.x^{-1} = x^{-1}.x = 1$. **(Multiplicative Inverse)**

The element x^{-1} is called the **multiplicative inverse** of x.

D: $x.(y+z) = x.y + x.z$

$(x+y).z = x.z + y.z$. **(Multiplication is distributive over Addition).**

Since the unity is not equal to the zero i.e. $1 \neq 0$ in a field, therefore any field must contain at least two elements. Note that the axioms A_1 (closure under addition) and

M_1 (closure under multiplication) are unnecessary because the closures are implied in the definition of a binary operation. However, we include them, for the sake of emphasis. Now try the following exercises:

EXERCISE 3)

Show that the set {0, 1} forms a field under the operations '+' and '.' defined by the following tables:

+	0	1
0	0	1
1	1	0

.	0	1
0	0	0
1	0	1

EXERCISE 4)

Show that the zero and the unity are unique in a field.

Now, you can easily verify that all the eleven axioms are satisfied by the set of rational numbers with respect to the ordinary addition and multiplication. Thus, the set **Q** forms a field under the operations of addition and multiplication, and so does, the set **R** of all the real numbers.

EXERCISE 5)

Do the sets N (of natural numbers) and Z (set of integers) form fields? Justify your answers. Also verify that the set C of complex numbers is a field.

We state (without proof) some important properties satisfied by a field. They follow from the field axioms. Can you try?

PROPERTY 1

For any x, y, z in F,

1. $x+z = y+z \Longrightarrow x = y$,
2. $x.0 = 0 = 0.x$,
3. $(-x).y = -x.y = x.(-y)$,
4. $(-x).(-y) = x.y$,
5. $x.z = y.z,\ z \neq 0 \Longrightarrow x = y$,
6. $x.y = 0 \Longrightarrow$ either $x = 0$ or $y = 0$.

Thus by now you know that the sets **Q**, **R** and **C** form fields under the operations of addition and multiplication.

2.3.1 ORDERED FIELD

In Section 2.2, we defined the order relation $\leq$ in **R**. It is easy to see that this order relation satisfies the following properties:

PROPERTIES 2

Let x, y, z be any elements of **R**. Then

O_1: For any two elements x and y of **R**, one and only of the following holds:
(i) $x < y$, (ii) $y < x$, (iii) $x = y$,

O_2 $x \leq y,\ y \leq x \Longrightarrow x \leq z$,

O_3: $x \leq y \Longrightarrow x+z \leq y+z$,

O_4: $x \leq y,\ 0 < z \Longrightarrow x.z \leq y.z$

We express this observation by saying that the field **R** is an **ordered field** (i.e. it satisfies the properties $0_1 - 0_4$). It is easy to see that these properties are also satisfied by the field **Q** of rational numbers. Therefore, **Q** is also an **ordered field.** What about the field **C** of Complex numbers? Try it yourself as an exercise.

EXERCISE 6)

Show that the field C of Complex numbers is not an ordered field.

2.3.2 COMPLETE ORDERED FIELD

Although **R** and **Q** are both ordered fields, yet there is a property associated with the order relation which is satisfied by **R** but not by **Q.** This property is known as the **Order-Completeness,** introduced for the first time by Richard Dedekind. To explain this situation more precisely, we need a few more mathematical concepts which are discussed as follows:

Consider set $S = \{1, 3, 5, 7\}$. You can see that each element of S is less than or equal to 7. That is $x \leq 7$ for each $x \in S$. Take another set S, where $S = \{x \in \mathbf{R} : x \leq 16\}$. Once again, you see that each element of S is less than 18. That is, $x < 18$ for each $x \in S$. In both the examples, the sets have a special property namely that every element of the set is less than or equal to some number. This number is called an upper bound of the corresponding set and such a set is said to be bounded above. Thus, we have the following definition:

DEFINITION 4: UPPER BOUND OF A SET

Let $S \subset \mathbf{R}$. If there is a number $u \in \mathbf{R}$ such that $x \leq u$ for every $x \in S$, then S is said to be bounded above. The number u is called an upper bound of S.

EXAMPLE 2: Verify whether the following sets are bounded above: Find an upper bound of the set, if it exists.

(i) The set of negative integers $\{-1, -2, -3, ...\}$.

(ii) The set N of natural numbers.

(iii) The sets Z, Q and R.

SOLUTION : (i) The set is bounded above with -1 as an upper bound.
(ii) The set N is not bounded above
(iii) All the sets are not bounded above.

EXERCISE 7)

(i) Define a set which is bounded below. Also define a lower bound of a set.

(ii) Give at least two examples of a set (one of an infinite set) which is bounded below and mention a lower bound in each case.

(iii) Is the set of negative integers bounded below? Justify your answer.

Now consider a set $S = \{2, 3, 4, 5, 6, 7\}$. You can easily see that this set is bounded above because 7 is an upper bound of S. Again this set is also bounded below because 2 is a lower bound of S. Thus S is both bounded above as well as bounded below. Such a set is called a **bounded set.** Consider the following sets:

$S_1 = \{.... -3, -2, -1, 0, 1, 2, ...\}$,
$S_2 = \{0, 1, 2, \text{-------}\}$,
$S_3 = \{0, 1, 2, ...\}$.

You can easily see that S_1 is neither bounded above nor bounded below. The set S_2 is not bounded above while S_3 is not bounded below. Such sets are known as **Unbounded Sets.**

Thus, we can have the following definition:

DEFINITION 5: BOUNDED SETS

A set S is bounded if it is both bounded above and bounded below.

In other words, S has an upper bound as well as a lower bound. Thus if S is bounded, then, there exist numbers u (an upper bound) and v (a lower bound) such that $v \leq x \leq u$ for every $x \in S$.

If a set S is not bounded then S is called an unbounded set. Thus S is unbounded if either it is not bounded above or it is not bounded below.

EXAMPLE 3 : (i) Any finite set is bounded.

(ii) The set Q of rational numbers is unbounded

(iii) The set R of real numbers is unbounded

(iv) The set $P = \{\sin x, \sin 2x, \sin 3x, ..., \sin nx, ...\}$ is bounded because $-1 \leq \sin nx \leq 1$ for every n and x.

EXERCISE 8)

Test which of the following sets are bounded above, bounded below, bounded and unbounded.

(i) The intervals]a, b[, [a, b],]a, b] and [a, b[, where a and b are any two real numbers.

(ii) The intervals [2, ∞[,]−∞, 3[,]5, ∞ [and]−∞, 4].

(iii) The set {cos θ, cos 2 θ, cos 3 θ, ...}.

(iv) S = {x ∈ R: −a ≤ x ≤ a} for some a ∈ R.

You can easily verify that a subset of a bounded set is always bounded since the bounds of the given set will become the bounds of the subset.

Now consider any two bounded sets say S = {1, 2, 5, 7} and T = {2, 3, 4, 6, 7, 8}. Their union and intersection are given by

S ∪ T = {1, 2, 3, 4, 5, 6, 7, 8}

and

S ∩ T = {2, 7}.

Obviously S ∪ T and S ∩ T are both bounded sets. You can prove this assertion in general for any two bounded sets.

EXERCISE 9)

Prove that the union and the intersection of any two bounded sets are bounded.

Now consider the set of negative integers namely

S = {−1, −3, −2, −4, ...}.

You know that −1 is an upper bound of S. Is it the only upper bound of S? Can you think of some other upper bound of S? Yes, certainly, you can. What about 0? The number 0 is also an upper bound of S. Rather, any real number greater than −1 is an upper bound of S. You can find infinitely many upper bounds of S. However, you can not find an upper bound less than −1. Thus −1 is the least upper bound of S.

It is quite obvious that if a set S is bounded above, then it has an infinite number of upper bounds. Choose the least of these upper bounds. This is called the **least upper bound** of the set S and is known as the **Supremum** of the set S. (The word 'Supremum' is a Latin word). We formulate the definition of the Supremum of a set in the following way:

DEFINITION 6: THE SUPREMUM OF A SET

Let S be a set bounded above. The least of all the upper bounds of S is called the least upper bound or the Supremum of S. Thus, if a set S is bounded above, then a real number m is the supremum of S if the following two conditions are satisfied:

(i) m is an upper bound of S,

(ii) if k is another upper bound of S, then m ≤ k.

EXERCISE 10)

Give an example of an infinite set which is bounded below. Show that it has an infinite number of lower bounds and hence develop the concept of the greatest lower bound of the set.

The greatest lower bound, in Latin terminology, is called the **Infimum** of a set.

Let us now discuss a few examples of sets having the supremum and the infimum:

EXAMPLE 4: **Each of the intervals]a, b[, [a, b]]a, b], [a, b[has both the supremum and the infimum. The number a is the infimum and b is the supremum in each case. In case of [a, b] the supremum and the infimum both belong to the set whereas this is not the case for the set]a, b[. In case of the set]a, b], the infimum does not belong to it and the supremum belongs to it. Similarly, the infimum belongs to [a, b[but the supremum does not belong to it.**

Very often in our discussion, we have used the expressions **'the supremum', rather than a supremum.** What does it mean? It simply means that the supremum of a set, if it exists, is unique i.e. a set can not have more than one supremum. Let us prove it in the form of the following theorem:

THEOREM 1: Prove that the supremum of a set, if it exists, is unique.

PROOF : If possible, let there be two supremums (Suprema) say m and m′ of a set S.

Since m is the least upper bound of S, therefore by definition, we have

$$m \le m'.$$

Similarly, since m′ the least upper bound of S, therefore, we must have

$$m' \le m.$$

This shows that m = m′ which proves the theorem.

You can now similarly prove the following result:

EXERCISE 11)

Prove that the infimum of a set, if it exists, is unique.

In example 3, you have seen that supremum or the infimum of a set may or may not belong to the set. If the supremum of a set belongs to the set, then it is called the **greatest member** of the set. Similarly, if the infimum of a set belongs to it, then it is called the **least member** of the set.

EXAMPLE 5: (i) **Every finite set has the greatest as well as the least member.**

(ii) **The set N has the least member but not the greatest. Determine that number.**

(iii) **The set of negative integers has the greatest member but not the least member. What is that number?**

Try the following exercise:

EXERCISE 12)

Check which of the following sets have the greatest and the least member:

(i) $\{x: a \le x \le b\}$.

(ii) $\{x: a < x \le b\}$.

(iii) $\{x: a \le x < b\}$.

(iv) $\{x: a < x < b\}$.

(v) $[a, \infty[$, $]a, \infty[$.

(vi) $]-\infty, b]$, $]-\infty, b[$.

You have seen that whenever a set S is bounded above, then S has the supremum. In fact, this is true in general. Thus, we have the following property of **R** without proof:

PROPERTY 3: COMPLETENESS PROPERTY

Every non empty subset S of R which is bounded above, has the supremum.

Similarly, we have

Every non-empty subset S of R that is bounded below, has the infimum

In fact, it can be easily shown that the above two statements are equivalent.

Now, if you consider a non-empty subset S of **Q**, then S considered as a subset of **R** must have, by property 2, a supremum. However, this supremum may not be in **Q**. This fact is expressed by saying that **Q** considered as a field in its own right is not Order-Complete. We illustrate this observation as follows:

Construct a subset S of **Q** consisting of all those positive rational numbers whose squares are less than 2 i.e.

$$S = \{x \in \mathbf{Q}: x > 0, x^2 < 2\}.$$

Since the number $1 \in S$, therefore S is non empty. Also, 2 is an upper bound of S because every element of S is less than 2. Thus the set S is non-empty and bounded above. According to the Axiom of Completeness of **R**, the subset S must have the supremum in **R**. We claim that this supremum does not belong to **Q**.

Suppose m is the supremum of the set S. If possible, let m belong to **Q**. Obviously, then $m > 0$. Now either $m^2 < 2$ or $m^2 = 2$ or $m^2 > 2$.

Case (i) When $m^2 < 2$. Then a number y defined as

$$y = \frac{4+3m}{3+2m}$$

is a positive rational number and

$$y - m = \frac{2(2-m^2)}{3+2m}.$$

Since $m^2 < 2$, therefore $2 - m^2 > 0$. Hence

$$y - m = \frac{3(2-m^2)}{3+2m} > 0$$

which implies that $y > m$.
Again,

$$y^2 - 2 = \left(\frac{4+3m}{3+2m}\right)^2 - 2$$

$$= \frac{m^2-2}{(3+2m)^2}.$$

Since $m^2 < 2$, therefore

$$y^2 - 2 < 0 \text{ i.e. } y^2 < 2.$$

This shows that $y \in S$ and also it is greater than m (the supremum of S). This is absurd. Thus the case $m^2 < 2$ is not possible.

Case (ii) When $m^2 = 2$.

This means there exists a rational number whose square is equal to 2 which is again not possible, since you have already proved this in Section 1.3.

Case (iii) When $m^2 > 2$

In this case consider the positive rational number y defined in case (i). Accordingly, we have

$$y - m = \frac{2(2-m^2)}{3+2m} < 0 \text{ (check yourself)}$$

i.e. $y < m$.

Also $2 - y^2 = 2 - \left(\frac{4+3m}{3+2m}\right)^2 \; 2 = \frac{2-m^2}{(3+2m)^2}$

i.e. $2 - y^2 < 0$ or $y^2 > 2$,

which shows that y is an upper bound of S.

Thus y is an upper bound of S which does not belong to S. At the same time y is less than the supremum of S. This is again absurd. Thus $m^2 > 2$ is also not possible. Hence none of three possibilities is true. This means there is something wrong with our supposition. In other words, our supposition is false and therefore the set S does not possess the supremum in **Q**.

This justifies that the field **Q** of rational numbers is not order-complete.

Now you can also try a similar exercise.

EXERCISE 13)
Let S be a subset of all those positive rational numbers whose squares are less than 3 i.e.

$$S = \{x \in Q: x > 0 \text{ and } x^2 < 3\}.$$

Show that S is nonempty and bounded above but it does not have the least upper bound in Q.

2.4 COUNTABILITY

In Section 1.2, we recalled the notion of a set and certain related concepts. Subsequently, we discussed certain properties of the sets of numbers **N**, **Z**, **Q**, **R** and

C. A few more important properties and related aspects concerning these sets are yet to be examined. One such significant aspect is the countability of these sets. The concept of Countability of sets was introduced by George Cantor which, forms a corner stone of Modern Mathematics.

2.4.1 COUNTABLE SETS

George Cantor.

You can easily count the elements of a finite set. For example, you very frequently use the term **'one hundred rupees'** or **'fifty boxes', 'two dozen eggs'**, etc. These figures pertain to the **number** of elements of a set. Denote the number of elements in a finite set S by n (S). If S = {a, b, c, d}, then n **(S)** = 4. Similarly n (S) = 26, if **S** is the set of the letters of English alphabet. Obviously, then n (ϕ) = 0, where ϕ is the null set.

You can make another interesting observation when you count the number of elements of a finite set. While you are counting these elements, you are indirectly and perhaps unconsciously, using a very important concept of the one-one correspondence between two sets. Recall the concept of **one-one correspondence** from Section 1.2. Here one of the sets is a finite subset of the set of natural numbers and the other set is the set consisting of the articles/objects like rupees, boxes, eggs, etc. Suppose you have a basket of oranges. While counting the oranges, you are associating a natural number to each of the oranges. This, as you know, is a one-one correspondence between the set of oranges and a subset of natural members. Similarly, when you count the fingers of your hands, you are in fact showing a one-one correspondence between the set of the fingers with a subset, say $N_{10} = (1, 2, ... 10)$ of N.

Although, we have an intuitive idea of finite and infinite sets, yet we give a mathematical definition of these sets in the following way:

DEFINITION 7: FINITE AND INFINITE SETS

A set S is said to be finite if it is empty or if there is a positive integer k such that there is one-one correspondence between the elements of the set S and the set N_K = {1, 2, 3 ...., k}. A set is said to be infinite if it is not finite.

The advantage of using the concept of one-one correspondence is that it helps in studying the countability of infinite sets. Let E = {2, 4, 6, ...} be the set of even natural numbers. If we define a mapping **f**: **N** $\rightarrow$ **E** as

$$f(n) = 2n, \forall n \in N,$$

then we find that **f** is a one-one correspondence between N and E.

Consider another example. Suppose S = {1, 2, ... n} and T = $\{x_1, x_2, ... x_n\}$. Define a mapping **f: S** $\rightarrow$ **T** as

$$f(n) = x_n \ \forall n \in S.$$

Then again **f** is a one-one correspondence between S and T.

Such sets are known as equivalent sets. We define the equivalent sets in the following way:

DEFINITION 8: EQUIVALENT SETS

Any two sets are equivalent if there is one-one correspondence between them.

Thus if two sets S and **T** are equivalent, we write as S ~ T

You can easily show that **S, T** and P are any three sets such that **S ~ T** and **T ~ P,** then S ~ **P.**

The notion of the equivalent sets is very important because it forms the basis of the **'counting'** of the infinite sets.

Now, consider any two line segments AB and CD.

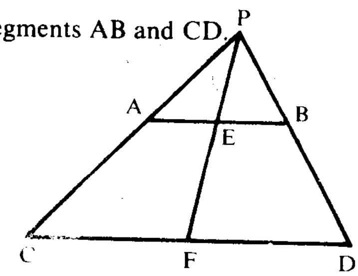


Fig. 4

Let M denote the set of points on AB and N the set of points on CD. Let us check whether M and N are equivalent.

Join CA and DB to meet in the point P. Let a line through P meet AB in E and CD in F. Define **f: M → N** as f(x) = y where x is any point on AB and y is any point on CD. In fact, as an example you can write as f (E) = F. The construction shows that f is a one-one correspondence. Thus M and N are equivalent sets.

The following are some examples of equivalent sets:

(i)]a. b] and] c, d[.

(ii)]0, 1] and] 0, 1[.

(iii) [0, 1], [0, 1[,]0, 1[and]0, 1]

(iv)]0, 1[and] 1, ∞[.

In fact, all the intervals are equivalent to one another.

Now, we introduce the following definition:

DEFINITION 9: DENUMERABLE AND COUNTABLE SETS

A set which is equivalent to the set of natural numbers is called a denumerable set. Any set which is either finite or denumerable, is called a Countable set.

Any set which is not countable is said to be an uncountable set.

EXAMPLE 6: (i) A mapping f: Z → N defined by

$$f(n) = \begin{cases} -2n & \text{if n is a negative integer} \\ 2n + 1 & \text{if n is non-negative integer,} \end{cases}$$

is a one-one correspondence. Hence Z ~ N. Thus **the set of integers is a denumerable set and hence a countable set.**

(ii) Let E denote the set of all even natural numbers. Then the mapping **f: N → E** defined as f(n) = 2n is a one-one correspondence. Hence the set E of **even natural numbers is a denumerable set and hence a countable set.**

(iii) Let D denote the set of all odd integers and E the set of even integers. Then the mapping f: E→ D, defined as f(n) = n + 1 is a one-one correspondence. Thus E ~ D. But, E ~ N, therefore D ~ N. Hence D is a denumerable set and hence a countable set.

We observe that a set S is denumerable if and only if it is of the form $\{ a_1 a_2, a_3 ....\}$ for distinct elements a_1, a_2, a_3 ..... For, in this case the mapping $f(a_n) = n$ is one-one mapping of S onto N i.e. the sets $\{a_1, a_2, a_3$ ----} and the set N are equivalent.

If we consider the set $S_2 = \{2, 3\ 4 ....\}$, we find that the mapping f: N → S_2 defined as f (n) = n+1 is one-one and onto. Thus S_2 is denumerable. Similarly if we consider $S_3 = \{3, 4 ....\}$ or $S_k = [k, k+1 ....]$, then we find that all these are denumerable sets and hence are countable sets.

We have seen that the set of integers is countable.

Now we discuss the countability of the rational and real numbers. Here is an interesting theorem:

THEOREM 2: Every infinite subset of a denumerable set is denumerable.

PROOF : Let S be a denumerable set. Then S can be written as

$$S = \{a_1, a_2, a_3 ....\}.$$

Let A be an infinite subset of S. We want to show that A is also denumerable.

You can see that the elements of S are designated by subscripts 1, 2, 3, ... Let n_1 be the smallest subscript for which $a_{n_1} \in A$. Then consider the set $A - \{a_{n_1}\}$ Again, in this new set, let n_2 be the smallest subscript such that $a_{n_2} \in A - \{a_{n_1}\}$.

Let n_k be the smallest subscript such that

$a_{n_k} \in A - \{a_{n_1}, a_{n_2}, ..., a_{n_{k-1}}\}$. Note that such an element a_{n_k}

always exists for each k ∈ N as A is infinite. For, then

$$A - \{a_{n_1}, a_{n_2}, ..., a_{n_k}\} \neq \phi$$

for each $k \in N$. Thus, we can write

$$A = \{a_{n_1}, a_{n_2}, a_{n_3}, ..., a_{n_k}, ...\}.$$

Define f: $N \rightarrow A$ by $f(k) = a_{n_k}$. Then it can be verified that f is a one-one correspondence. Hence A is denumerable. This completes the proof of the theorem.

EXERCISE 14)
Every subset of a countable set is countable.

Now consider the sets $S = [6, 8, 10, 12 ....]$and $T = \{3, 5, 7, 9, 11, ...\}$, which are both denumerable. Their union $S \cup T = \{3, 5, 6, 7, 8, 9 ....\}$ is an infinite subset of **N** and hence is denumerable. Again if $S = \{-1, 0, 1, 2\}$ and $T = \{20, 40, 60, 80, ...\}$, then we see that $S \cup T = \{-1, 0, 1, 2, 20, 40, 60, ...\}$ is a denumerable set. Note that in each case $S \cap T = \phi$. In fact, you can prove the following general results in the next exercise.

EXERCISE 14)

(i) If S and T are two denumerable sets, such that $S \cap T = \phi$, then $S \cup T$ is denumerable.

(ii) If S is denumerable and T is finite such that $S \cap T = \phi$, then also $S \cup T$ is denumerable.

(iii) The condition $S \cap T = \phi$ can be relaxed in (i) and (ii).

Thus, it follows, that **the union of any two countable sets is countable.**

Indeed, let S and T be any two countable sets. Then S and T are either fintie or denumerable.

If S and T are both finite, then $S \cup T$ is also a finite set and hence $S \cup T$ is countable.

If S is denumerable and T is finite, then also we know that $S \cup T$ is denumerable. Hence $S \cup T$ is countable. Again, if S is finite and T is denumerable, then again $S \cup T$ is denumerable and countable.

Finally, if both S and T are denumerable, then $S \cup T$ is also denumerable and hence countable.

In fact, this result can be extended to countably many countable sets and we prove the following theorem:

THEOREM 3 : The union of a countable number of countable sets is countable.

PROOF : Let the given sets be A_1, A_2, A_3 .... .

Denote the elements of these sets, using double subscripts as follows:

$$A_1 = \{a_{11},\ a_{12},\ a_{13}, ...\}$$

$$A_2 = \{a_{21},\ a_{22},\ a_{23}, ...\}$$

$$A_3 = \{a_{31},\ a_{32},\ a_{33}, ...\}$$

....

....

....

Note that the double subscripts have been used for the sake of convenience only. Thus a_{ij} is the jth element in the set A_i. Now let us try to form a single list of all elements of the union of these given sets.

One method for doing this is indicated in the following way:

$A_1 : a_{11}\ \ a_{12}\ \ a_{13}\ \ a_{14}$ ....

$A_2 : a_{21}\ \ a_{22}\ \ a_{23}\ \ a_{24}$....

$A_3 : a_{31}\ \ a_{32}\ \ a_{33}\ \ a_{34}$....

....

....

Relist the elements as indicated through the arrows. This is a scheme for making a single list of all the elements.

Following the arrows, you can easily arrive at the new single list:

$a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, a_{23} ....$

Note that while doing so, you must omit the duplicates, if any.

Now, if any of the sets $A_1, A_2, ...$ are **finite,** then this will merely shorten the final list. Thus, we have

$\bigcup_i A_i = \{a_{i1}, a_{i2}, ...\}\ i = 1, 2, 3, ...$

in which each element appears only once This set is countable which completes the proof of the theorem.

We are now in a position to discuss the countability of the sets of rational and real numbers.

2.4.2 COUNTABILITY OF REAL NUMBERS

We have already established that the sets **N** and **Z** are countable. Let us now consider the case of the set **Q** of rational numbers. For this we need the following theorems:

THEOREM 4 : The set of all rational numbers between [0, 1] is countable.

PROOF : Make a systematic scheme in an order for listing the rational numbers x where $0 \le x \le 1$, (without duplicates) of the following sets

$A_1 = \{0, 1\}$

$A_2 = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} ....\}$

$A_3 = \{\frac{2}{3}, \frac{2}{5}, \frac{2}{7} ....\}$

$A_4 = \{\frac{3}{4}, \frac{3}{5}, \frac{3}{7}, \frac{3}{8}, ...\}$

....

....

You can see that each of the above sets is countable. Their union is given by

$\bigcup_{i=1} A_i = \{0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, ..., = [0, 1],$

which is countable by Theorem 3.

THEOREM 5 : The set of all positive rational numbers is countable.

PROOF : Let Q_+ denote the set of all positive rational numbers. To prove that Q_+ is countable, consider the following sets :

$A_1 = \{1, 2, 3, ...\}$

$A_2 = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...\}$

$A_3 = \{\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, ...\}$

$A_4 = \{\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, ...\}$

....

....

Relist the elements of these sets in a manner as you have done in theorem 3 or as shown below:

$A_1 : 1, \quad 2, \quad 3, \quad 4, \quad ....$

$A_2 : \frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}, ...$

$A_3 : \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, ...$

$A_4 \ \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}, ...$

$A_5 \ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, ...$

....

....

You may follow the method of indicating by arrows for making a single list or you may follow another path as indicated here. Accordingly, write down the elements of Q_+ as they appear in the figure by the arrows, while omifting those numbers which are already listed to avoid the duplicates. We wil have the following list:

$$Q_+ = \{1, \frac{1}{2}, 2, \frac{1}{3}, \frac{2}{3}, 3, \frac{1}{4}, \frac{3}{4}, \frac{4}{3}, 4, ...\}$$
$$= \bigcup_i A_i \ (i = 1, 2, 3, ...)$$

which is countable by theorem 3. Thus Q_+ is countable.

Now let Q_- denote the set of all negative rational numbers. But Q_+ and Q_- are equivalent sets because there is one-one correspondence between Q_+ and Q_- given by $f: C \rightarrow Q_-$ as

$$f(x) = -x, \forall x \in Q_+$$

Therefore **Q** is also countable. Further {0} being a finite set is countable. Hence,

$$Q = Q_+ \cup \{o\} \cup Q_-$$

is a countable set. Thus, in fact, we have proved the following theorem:

THEOREM 6: The set Q of all rational numbers is countable.

You may start thinking that perhaps every infinite set is denumerable. This is not true. We have not yet discussed the countability of the set of real numbers or of the set of irrational numbers. To do so, we first discuss the countability of the set of real numbers in an interval (0, 1) which may be closed or open or semi-closed.

Consider the real numbers in the interval (0, 1).

Each real number in (0, 1) can be expressed in the decimal expansion. This expansion may be non-terminating or may be terminating e.g.

$$\frac{1}{3} = .333 ....$$

is an example of non-terminating decimal expansion, whereas

$$\frac{1}{4} = .25, \ \frac{1}{2} = .5, ...$$

are terminating decimal expansions. Even the terminating expansion can also be expressed as non-terminating expansion in the sense that you can write

$$\frac{1}{4} = .25 = .24999 ....$$

Thus, we agree to say that each real number (rational or irrational) in the (0, 1) can be expressed as a non-terminating decimal expansion in terms of the digits from 0 to 9.

Suppose $x \in (0, 1)$. Then it can be written as

$$x = .c_1 c_2 c_3 ....$$

where c_1, c_2.... take their values from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9} of ten digits. Similarly, let y be another real number in (0, 1). Then y can also be expressed as

$$y = .d_1 d_2 d_3 ....$$

We say that $x = y$ iff the digits in their corresponding positions in the expansions of x and y are identical. Thus, if there is even a single decimal place, say, 10th palce such that $d_{10} \neq c_{10}$, then $x \neq y$.

We now discuss the following result due to Georg Cantor.

THEOREM 7: The set of real numbers in the interval (0, 1) is not countable.

PROOF: Since the set of real numbers in (0, 1) is an infinite set, therefore, it is enough to show that the set of real numbers in (0, 1) is not denumerable

If possible, suppose that the set of real numbers in (0, 1) is denumerable. Then there is a one-one correspondence between N and the elements of (0, 1) i.e. there is a function $f: N \to (0, 1)$ which is one-one and onto. Thus, if

$$f(1) = x_1,\ f(2) = x_2,\ \ldots\ldots,\ f(k) = x_k,\ \ldots$$
$$\text{then } (0, 1) = \{x_1, x_2, \ldots, x_k \ldots\}.$$

We shall show that there is at least one real number in (0, 1) which is not an image of any element of N under f. In other words, there is an element of (0, 1) which is not in the list x_1, x_2 .....

Let x_1, x_2, ... be written as

$$x_1 = 0.\ a_{11}\ a_{12}\ a_{13}\ a_{14} \ldots$$
$$x_2 = 0.\ a_{21}\ a_{22}\ a_{23}\ a_{24} \ldots$$
$$x_3 = 0.\ a_{31}\ a_{32}\ a_{33}\ a_{34} \ldots$$
$$x_4 = 0.\ a_{41}\ a_{42}\ a_{43}\ a_{44} \ldots$$

....

....

....

From this we construct a real number

$$z = .\ b_1\ b_2\ b_3\ b_4 \ldots$$

where b_1, b_2 .... can take any digits from { 0, 1, 2, ... 9} in such a way that $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}$, ... Thus

$$z = .\ b_1\ b_2\ b_3 \cdots$$

is a real number in (0, 1) such that $z \neq x_1$ because $b_1 \neq a_{11}$, $z \neq x_2$ because $b_2 \neq a_{22}$. In general $z \neq x_n$ because $a_{nn} \neq b_n$. Therefore z is not in the list $\{x_1, x_2, x_3, \ldots\}$.

Hence (0, 1) is not countable.

We have already mentioned that the intervals [0, 1], [0, 1[,]0, 1] and]0, 1[are equivalent sets. Since the set of real numbers in (0, 1) is not countable, therefore none of the intervals is a countable set of real numbers.

Now you can easily conclude that the set of irrational numbers in (0, 1) is not countable. If possible, suppose that the set of irrational numbers in (0, 1) is countable. Also you know that the set of rational numbers in (0, 1) is countable and that the union of two countable sets is countable. Therefore, the union of the set of rational numbers and the set of irrational numbers in (0, 1) is countable i.e. the set of all real numbers in (0, 1) is countable which is not so. Hence the set of irrational numbers in (0, 1) is not countable.

In fact, every interval (a, b) or [a, b], (a,b], [a, b) is an uncountable set of real numbers.

Now what about the countability of the set **R** of real numbers?

Suppose that **R** is countable. Then an interval (0, 1), being an infinite subset of **R**, must be countable. But then, we have already proved that the set (0, 1) is not countable. Hence **R** can not be countable.

Thus even by the method of countability of sets, we have **established the much desired distinction between Q and R in the sense that Q is countable but R is not countable.**

Also, we observe that although R is not countable, yet it contains subsets which are countable. For example R has subsets as Q, Z and N which are countable. At the same time R is an infinite set. We sum up this observation in the form of the following theorem:

THEOREM 8 : Every infinite set contains a denumerable set.

PROOF: Let S be an infinite set. Consider some element of S. Denote it by a_1. Consider the set S − $\{a_1\}$. Now pick up an element from the new set and denote it by a_2.
Consider the set
S − $\{a_1, a_2\}$.

Proceeding in this way, having chosen a_{K-1}, you can have the set

S −$\{a_1, a_2 .... a_{k-1}\}$.

This set is always non-empty because S is an infinite set. Hence, we can choose an element in this set. Denote the element by a_k. This can be done for each k ∈ N. Thus the set

$\{a_1, a_2, ..., a_k ....\}$

is a denumerable subset of S and hence a countable subset of S. This proves the theorem.

The importance of this theorem is that it leads us to an interesting area of Cardinality of sets by which we can determine and compare the relative sizes of various infinite sets.

This, however, is beyond the scope of this course.

2.5 SUMMARY

In Section 2.2, we have discussed the order-relations (inequalities) in the set **R** of real numbers. Given any two real numbers x and y, either $x > y$ or $x = y$ or $x < y$.

This is known as the law of Trichotomy. Then we have stated a few properties with respect to the inequality '$\leq$'. The first property states that the inequality $\leq$ is antisymmetric. The second states the transitivity of $\leq$. The third allows us to add or subtract across the inequality, while preserving the inequality. The last property gives the situation in which the inequality is preserved if multiplied by a positive real number, while it is reversed if multiplied by a negative real number.

We have also defined the bounded and unbounded intervals. The bounded intervals are classified as open intervals, closed interval, semi-open or semi-closed intervals. The unbounded intervals are introduced with the help of the extended real number system $\mathbf{R} \cup \{-\infty, \infty)$ using the symbols $+\infty$ (called plus infinity and $-\infty$ (called minus infinity).

Section 2.3 deals with three important aspects of the real numbers: algebraic, order and the completeness. To describe these aspects, we have specified a number of axioms in each case. In the algebraic aspect, an algebraic structure called the field is used. A field is a non-empty set F having at least two distinct elements 0 and 1 together with two binary operations + (addition) and. (multiplication) defined on F such that both + and. are commulative, associative, 0 is the additive identity, 1 is the multiplicative identity, additive inverse exists for each element of F, multiplicative inverse exists for each element other than 0 and multiplication is distributive over addition. The second aspect is concerned with the Order Structure in which, we use the axioms of the law of trichotomy, the transitivity property, the property that preserve the inequality under addition and the property that preserve the inequality under multiplication by a positive real number.

In the completeness aspect, we introduce the notions of the supremum (or infimum) of a set and state the axiom of completeness. We find that both **Q** and **R** are ordered fields but the axiom of completeness distinguishes **Q** from **R** in the sense that **Q** does not satisfy the axiom of completeness. Thus, we conclude that **R** is a complete-ordered Field whereas Q is not a complete-ordered field.

Finally in Section 2.4, we introduce the notion of the countability of sets. A set is said to be denumerable if it is in one-one correspondence with the set of natural numbers. Any set which is either finite or denumerable is called a countable set. We have shown that the sets **N**, **Z** **Q** are countable sets but the sets I (set of irrational numbers) and **R** are not countable.

Thus in this unit, we have discussed the algebraic structure, the order structure and the countability of the real numbers.

2.6 ANSWERS/HINTS/SOLUTIONS

E1) Change $\leq$ into $\geq$ in the Property I.
Choose x, y and z to describe this property.
If $x \geq y$ and $z < 0$, then $x\,z \leq y\,z$.

E2) Take (2, 5) and (7, 12) as the intervals. Then $(2, 5) \cap (7, 12) = \phi$ which is not an interval. However, if you take (2,5) and (3, 6) as the intervals, then

$(2, 5) \cap (3, 6) = (3, 5)$

which is an interval. Note that the intervals (2, 5) and (7, 12) are disjoint but (2, 5) and (3, 6) are not disjoint. Thus you may conclude that the intersection of the intervals is an interval provided the intervals are not disjoint.

E3) Verify that all the axioms of a field are satisfied by the elements of the set {0, 1} with respect to binary operation + and. as defined in the given tables.

E4) Suppose there are two zeros of a Field F namely 0 and 0′. Then by definition we have

$0+x = x, \forall x \in F$

In particular, if $x = 0'$, then we have

$0 + 0' = 0'$

Again by definition we have

$0' + x = x, \forall x \in F$

Choose $x = 0$. Then we get

$0' + 0 = 0.$

It follows that $0 = 0'$.
Similarly you can prove the uniqueness of the unity.

E5) The set **N** does not form a field because its elements do not satisfy the axiom of additive inverse.

The set **Z** is not a field because the axiom of multiplicative inverse does not hold for Z.

The set $\mathbf{C} = \{z = x + iy: x \in \mathbf{R}, y \in \mathbf{R}\}$ forms a field under + and . defined as $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = [(x_1 + x_2) + i\,(y_1 + y_2)] \rightarrow z_1 + z_2 = \in \mathbf{C}$ for any $z_1, z_2 \in \mathbf{C}$.
Again $z_1 . z_2 = (x_1 + y_1) . (x_2 + iy_2)$
$= (x_1x_2 - y_1y_2) + i\,(x_1y_2 + y_2y_1)$

Now you can verify that all the axioms of a field are satisfied.

E6) The set C of complex numbers is a field but is not an ordered field, because order cannot be defined on C. Give an illustration. In Unit 1, we have already shown that the number $i = \sqrt{-1}$ is neither positive nor negative and also $i \neq 0$, because it is not a real number.

E7) (i) A set S ($S \subset R$) is said to be bounded below if it has a lower bound. A number $v \in S$ is said to be a lower bound of S if
$v \leq x \quad \forall \in S.$

(ii) Let $S = N$. Then N is bounded below. The number 1 is a lower bound of N. Any finite set is bounded below. You can name a lower bound of this set depending upon the choice of the set.

(iii) No. Because it has no lower bound.

E8) (i) All are bounded sets

(ii) $[2, \infty[$ and $]5, \infty[$ are **bounded below** with 2 and 5 as their respective lower bounds, whereas $]-\infty, 3[$ and $]-\infty, 4]$ are **bounded above,** with 3 and 4 as their respective upper bounds. Therefore, all the sets in this case are unbounded.

(iii) It is a bounded set with lower bound -1 and upper bound $+1$.

(iv) $S = \{a\}$, is a bounded set.

E9) Let S and T be any two bounded sets. Then by using definition of a bounded set, you can have the following: S is bounded means S has both a lower bound and an upper bound i.e. there exist v_1. (lower bound) and u_1. (upper bound) such that

$$v_1 \leq x \leq u_1, \forall x \in S.$$

Similarly since T is bounded, therefore, there exists v_2 and u_2 such that

$$v_2 \leq x \leq u_2 \forall x \in T$$

Now you know that

$$x \in S \cup T \Rightarrow x \in S \text{ or } x \in T$$

$$\Rightarrow v_1 \leq x \leq u_1 \text{ or } v_2 \leq x \leq u_2$$

Choose v = minimum of (v_1, v_2), u = maximum of (u_1, u_2).

Then

$$x \in S \cup T \Rightarrow v \leq x \leq u$$

$\Rightarrow S \cup T$ is a bounded set, because x is an arbitrary element of $S \cup T$.

As an illustration of this example Let $S = (1, 5)$ and $T = (2, 7)$. Obviously both S and T are bounded because both are open intervals i.e.

$$S = \{x: 1 \leq x \leq 5\}, T = \{x: 2 \leq x \leq 7\}$$

Obviously, then

$$S \cup T = \{x: 1 \leq x \leq 7\}$$

which is a bounded set.

Similarly, if you take the intersection of S and T, then you will have

$$S \cap T = \{x: 2 \leq x \leq 5\}$$

which is obviously a bounded set. Note that 2 is the maximum of the two lower bounds and 5 is the minimum of the upper bonds of S and T

You can similarly, prove that the intersection of any two bounded sets is a bounded set.

E10) The set N is bounded below only. The number 0 and negative integers are all lower bounds of N i.e. all the non-positive integers are lower bounds of N. Complete the solution.

E11) Proof is exactly similar to the proof for the uniqueness of the supremum. Do it yourself.

E12) (i) has both greatest and least

(ii) has the greatest

(iii) has the least

(iv) has none

(v) a as the least

(iv) b as the greatest.

E13) The set S is obviously non-empty and bounded above. We claim S has no least upper bound in Q.

If possible, suppose u is the least upper bound of S in Q.

Then either $u^2 < 3$ or $u^2 > 3$ or $u^2 = 3$.

(i) Suppose $u^2 < 3$. Define a rational number y as

$$y = u + \frac{1}{7}(3 - u^2)$$

Then it can be verified that $y > u$ and $y^2 < 3$. This shows that there exists a rational number y which belongs to S and is greater than the least upper bound of S. This is absurd. Hence $u^2 < 3$ is not possible.

(ii) Now suppose $u^2 > 3$. Define a rational number z as

$$z = \frac{u^2 + 3}{2u}$$

Then it can be verified that $z < u$ and z is an upper bound of S which is again a contradiction. Thus $u^2 < 3$ is also not possible.

(iii) Finally suppose $u^2 = 3$. This means there exists a rational number whose square is 3 which is not possible.

E14) Suppose S is a countable set. Then either S is finite or S is denumerable. Let A be a subset of S.
If S is finite, then A is also finite and hence A is countable.
If S is denumerable, then A is also denumerable as proved in theorem 2. Thus A is also countable. This completes the proof.

E15) (i) Let $S = \{a_1, a_2, a_3, ...\}$ and $T = \{b_1, b_2, ...\}$ be any two denumerable sets such that $S \cap T = \phi$.
Define a function $f: S \cup T \rightarrow N$ by
$f(a_n) = 2n$
$f(b_n) = 2n_{-1}$.

Then f is a one-one correspondence. Hence $S \cup T$ is denumerable.

Alternatively, you can actually list the elements of $S \cup T$ as $a_1, b_1, a_2, b_2, a_3, b_3....$
which is obviously a denumerable set.

In case $S \cap T\ \phi$ i.e. S and T have any elements in common, then the duplicates of any element already listed would simply be omitted when the same element is encountered again in the combined list.

(ii) Now let $S = \{a_1, a_2, ...\}$ and $T = \{b_1, b_2, ... b_k\}$ (a finite set) be any two sets. Define $f: S \cup T \rightarrow N$ by
$f(b_i) = i$, for $1 \leq i \leq k$
and
$f(a_n) = n + k$, for each n.

Then, you can verify that f is one-one and onto i.e. f is a one-one correspondence. Hence $S \cup T$ is denumerable.

(iii) You may note that since.
$S \cup T = (S - T) \cup (T - S) \cup (S \cap T)$, therefore, you can, in fact, relax the condition $S \cap T = \phi$ in both the cases (i) and (ii).

UNIT 3 TOPOLOGY OF THE REAL LINE

STRUCTURE

3.1 INTRODUCTION

You are quite familiar with an elastic string or a rubber tube or a spring. Suppose you have an elastic string. If you first stretch it and then release the pressure, then the string will come back to its original length. This is a physical phenomenon but in Mathematics, we interpret it differently. According to Geometry, the unstreched string and the stretched string are different since there is a change in the length. But you will be surprised to know that according to another branch of Mathematics, the two positions of the string are identical and there is no change. This branch is known as Topology, one of the most exciting areas of Mathematics.

The word 'topology' is a combination of the two Greek words 'topos' and 'logos'. The term 'topos' means the top or the surface of an object and 'logos' means the study. Thus 'topology' means the study of surfaces. Since the surfaces are directly related to geometrical objects, therefore there is a close link between Geometry and Topology. In Geometry, we deal with shapes like lines, circles, spheres, cubes, cuboids etc. and their geometrical properties like lengths, areas, volumes, congruences etc. In Topology, we study the surfaces of these geometrical objects and certain related properties which are called topological properties. What are these topological properties of the surfaces of a geometrical figure? We shall not answer this question at this stage. However, since our discussion is confined to the real line, therefore, we shall discuss this question pertaining to the topological properties of the real line. These properties are related to the points and subsets of the real line such as neighbourhood of a point, open sets, closed sets, limit points of a set of the real line etc. We shall, therefore, discuss these notions and concepts in this unit. However, prior to all these, we discuss the modulus of a real number and its relationship with the order relations or inequalities in Section 2.2.

OBJECTIVES

After reading this unit, you should, therefore, be able to

- define the modulus of a real number and its connection with the order relations in the real numbers
- describe the notion of a neighbourhood of a point on the line
- define an open set and give examples
- find the limit points of a set
- define a closed set and establish its relation with an open set
- explain the meaning of an open covering of a subset of real numbers and that of a compact set.

3.2 MODULUS OF A REAL NUMBER

You know that a real number x is said to be positive if x is greater than 0. Equivalently, if 0 represents a unique point O on the real line, then a positive real number x lies on the right side of O. Accordingly, we defined the inequality $x > y$ (in terms of this positivity of real numbers) if $x - y > 0$. You will recall from Section 2.2 that for the validity of the properties of order relations or the inequalities, such as the one concerning the multiplication of inequalities, it is essential to specify that some of the numbers involved should be positive. For example, it is necessary that $z > 0$ so that $x > y$ implies $xz > yz$. Again, the fractional power of a number will not be real if the number is negative, for instance $x^{1/2}$ when $x = -4$. Many of the fundamental inequalities, which you may come across in higher Mathematics, will involve such fractional powers of numbers. In this context, the concept of the absolute value or the modulus of a real member is important to which you are already familiar. Nevertheless, in this section, we recall the notion of the modulus of a real number and its related properties which we need for our subsequent discussion.

DEFINITION 1 : MODULUS OF A REAL NUMBER

Let x be any real number. The absolute value or the modulus of x denoted by $|x|$. is defined as follows :

$$|x| = x \text{ if } x > 0$$
$$= -x \text{ if } x < 0$$
$$= 0 \text{ if } x = 0.$$

You can easily see that

$$|-x| = |x|, \forall x \in R.$$

Note that $|-x|$ is different from $-|x|$.

3.2.1 PROPERTIES OF THE MODULUS OF A REAL NUMBER

Since the modulus of a real number is essentially a non-negative real number, therefore the operations of usual addition, subtraction, multiplication and division can be performed on these numbers. The properties of the modulus are mostly related to these operations.

PROPERTY 1 : For any real number x, $|x| = \text{Maximum of } \{x, -x\}$

PROOF: Since x is any real number, therefore either $x \geq 0$ or $x < 0$. If $x \geq 0$, then by definition, we have

$$|x| = x.$$

Also, $x \geq 0$ implies that $-x \leq 0$. Therefore,

$$\text{maximum of } \{x, -x\} = x = |x|$$

Again $x < 0$, implies that $-x > 0$. Therefore again maximum of $\{x, -x\} = -x = |x|$.

Thus

$$\text{Max. } \{x, -x\} = |x|$$

Now you can solve the following exercise :

EXERCISE 1)

Prove that $-|x| = \text{Min. } \{x, -x\}$ for any $x \in R$. Deduce that $-|x| \leq x$, for every $x \in R$. Illustrate it with an example.

Now consider the numbers $|5|^2$, $|-4.5|$, $|\frac{4}{5}|$. It is easy to see that

$|5|^2 = |5|\ |5| = 5.5 = 5^2 = |-5|^2$

$|-4.5| = |-20| = 20$ Also $|-4|.\ |5| = 4.5 = 20$

i.e. $|-4.5| = |-4|\ .\ |5|$

and

$$|\frac{4}{5}| = \frac{4}{5} \text{ and } \frac{|4|}{|5|} = \frac{4}{5} \text{ i.e.}$$

$$|\frac{4}{5}| = \frac{|4|}{|5|}.$$

All this leads us to the following properties:

PROPERTY 2: For any real number x

$$|x|^2 = x^2 = |-x|^2$$

PROOF: We know that $|x| = x$ for $x \geq 0$. Thus

$$|x|^2 = |x|\ |x| = x.x = x^2, \text{ for } x \geq 0$$

Again for $x < 0$, we know that $|x| = -x$. Therefore

$$|x^2| = |x|\ |x| = -x. - x = x^2.$$

Therefore, it follows that

$$|x|^2 = x^2 \text{ for any } x \in R.$$

Now vou should try the other part as an exercise.

EXERCISE 2)

Prove that $|-x|^2 = x^2$, for any $x \in R$.

PROPERTY 3 : For any two real numbers x and y, prove that

$$|x.y| = |x|\ .|y|.$$

PROOF: Since x and y are any two real numbers, therefore, either both are positive or one is positive and the other is negative or both are negative i.e. either $x \geq 0, y \geq 0$ or $x \geq 0, y \leq 0$ or $x \leq 0, y \geq 0$ or $x \leq 0, y \leq 0$. We discuss the proof for all the four possible cases separately.

Case (i): When $x \geq 0, y \geq 0$.
Since $x \geq 0$, therefore, we have, by definition,
$|x| = x, |y| = y$

Also $x \geq 0, y \geq 0$ imply that $xy \geq 0$ and hence

$$|xy| = xy = |x|\ |y|$$

which proves the property.

Case (ii): When $x \geq 0, y \leq 0$. Then obviously $x\,y \leq 0$. Consequently by definition, it follows that

$$|x| = x, |y| = -y, |xy| = -xy$$

Hence

$$|xy| = -xy = x(-y) = |x|\ |y|$$

which proves the property.
Case (iii): When $x \leq 0, y \geq 0$.
Interchange x and y in (ii).

Case (iv): When $x \leq 0, y \leq 0$, Then $x\,y \geq 0$. Accordingly, we have

$$|x| = -x, |y| = -y, |xy| = xy.$$

Hence

$$|xy| = xy = (-x)(-y) = |x|\ |y|$$

using the field properties stated in Section 3.3.
This concludes the proof of the property.

Alternatively, the proof can be given by using property 2 in following way:

$$|xy|^2 = (xy)^2 = x^2 y^2 = |x|^2.|y|^2$$
$$= (|x|.\ |y|)^2$$

Therefore

$$|xy| = \pm(|x|\ |y|)$$

Since $|xy|$, $|x|$ and $|y|$ are non-negative, therefore we take the positive sign only and we have

$$|xy| = |x|\ |y|$$

which proves the property.

You can use any of the two methods to try the following exercise.

EXERCISE 3)

For any two real numbers x and y ($y \neq 0$), prove that

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

The next property is related to the modulus of the sum of two real members. This is one of the most important properties and is known as **Triangular Inequality:**

PROPERTY 4: TRIANGULAR INEQUALITY

For any two real numbers x and y, prove that

$$|x + y| \leq |x| + |y|.$$

PROOF : For any two real numbers x and y, the number $x + y \geq 0$ or $x + y < 0$.
If $x + y \geq 0$, then by definition

$$|x + y| = x + y. \quad (1)$$

Also, we know that

$$|x| \geq x \qquad \forall\, x \in R$$
$$|y| \geq y \qquad \forall\, y \in R.$$

Therefore

$$|x| + |y| \geq x + y$$

or

$$x + y \leq |x| + |y|. \quad (2)$$

From (1) and (2), it follows that

$$|x + y| \leq |x| + |y|$$

Now, if $x + y < 0$, then again by definition, we have

$$|x + y| = -(x + y)$$

or

$$|x + y| = (-x) + (-y) \quad (3)$$

Also we know that (see property 1)

$$-x \leq |x| \text{ and } -y \leq |y|.$$

Consequently, we get

$$(-x) + (-y) \leq |x| + |y|$$

or

$$(-x) + (-y) \leq |x| + |y| \quad (4)$$

From (3) and (4), we get

$$|x + y| \leq |x| + |y|$$

This concludes the proof of the property.

You can try the following exercise similar to this property.

EXERCISE 4)

Prove that

$$|x - y| \geq \big|\, |x| - |y| \,\big|$$

for any real numbers x and y.

Now let us see another interesting relationship between the inequalities and the modulus of a real number.

By definition, $|x|$ is a non-negative real number for any $x \in \mathbf{R}$. Therefore, there always exists a non-negative real number u such that

either $|x| < u$ or $|x| > u$ or $|x| = u$.

Suppose $|x| < u$. Let us choose $u = 2$. Then

$$|x| < 2 \Rightarrow \text{Max. } \{-x, x\} < 2$$
$$\Rightarrow -x < 2,\ x < 2$$
$$\Rightarrow x > -2,\ x < 2$$
$$\Rightarrow -2 < x,\ x < 2$$
$$\Rightarrow -2 < x < 2.$$

i.e. $|x| < 2 \rightarrow -2 < x < 2$

Conversely, we have

$$-2 < x < 2 \Rightarrow -2 < x, x < 2$$
$$\Rightarrow 2 > -x, x < 2$$
$$\Rightarrow -x < 2, x < 2$$
$$\Rightarrow \text{Max.} \{-x, x\} < 2$$
$$\Rightarrow |x| < 2.$$

i.e.

$$-2 < x < 2 \Rightarrow |x| < 2$$

Thus, we have shown that

$$|x| < 2 \Longleftrightarrow -2 < x < 2.$$

This can be generalised as the following property.

PROPERTY 5: Let x and u be any two real numbers.

$|x| \leq u \Longleftrightarrow -u \leq x \leq u.$

PROOF : $|x| \leq u \Leftrightarrow \text{Max.} \{-x, x\} \leq u$
$\Leftrightarrow -x \leq u, x \leq u$
$\Leftrightarrow x \geq -u, x \leq u$
$\Leftrightarrow -u \leq x, x \leq u$
$\Leftrightarrow -u \leq x \leq u$

which proves the desired property.

The property 5 can be generalized in the form of the following exercise:

EXERCISE 5)
For any real numbers x, a and d,
$|x-a| \leq d \Longleftrightarrow a-d \leq x \leq a+d.$

EXAMPLE 1: Write the inequality $3 < x < 5$ in the modulus form.

SOLUTION : Suppose that there exists real numbers a and b such that
$a - b = 3, a + b = 5.$
Solving these equations for a and b, we get
$a = 4, b = 1.$
Accordingly,

$$3 < x < 5 \Leftrightarrow 4 - 1 < x < 4 + 1$$
$$\Leftrightarrow -1 < x - 4 < 1$$
$$\Leftrightarrow |x - 4| < 1$$

Now you can also try the following exercise.

EXERCISE 6)
(i) Write the inequality $2 < x < 7$ in the modulus form
(ii) Convert $|x - 2| < 3$ into the corresponding inequality.

3.3 NEIGHBOURHOODS

You are quite familiar with the word 'neighbourhood'. You use this word frequently in your daily life. Loosely speaking, a neighbourhood of a given point c on the real line is a set of all those points which are close to c. This is the notion which needs a precise meaning. The term 'close to' is subjectve and therefore must be quantified. We should clearly say how much 'close to'. To elaborate this, let us first discuss the notion of a neighbourhood of a point with respect to a (small) positive real number δ.

Let c be any point on the real line and let $\delta > 0$ be a real number. A set consisting of all those points on the real line which are at a distance of δ from c is called a neighbourhood of c. This set is given by

$$\{x \in R: |x - c| < \delta\}$$
$$= \{x \in R: c - \delta < x < c + \delta\}$$
$$=]c - \delta, c + \delta[$$

which is an open interval. Since this set depends upon the choice of the positive real number δ, we call it a δ-neighbourhood of the point c.

Thus, a δ-neighbourhood of a point c on the real line is an open interval $]c-\delta, c+\delta[$ $\delta > 0$ while c is the mid point of this neighbourhood. We can give the general definition of neighbourhood of a point in the following way:

DEFINITION 2: NEIGHBOURHOOD OF A POINT

A set P is said to be a Neighbourhood (NBD) of a point 'c' if there exists an open interval which contains c and is contained in p.

This is equivalent to saying that there exists an open interval of the form $]c-\delta, c+\delta[$ for some $\delta > 0$ such that

$$]c-\delta, c+\delta[\subset P.$$

EXAMPLE 2: (i) Every Open-interval $]a, b[$ is a NBD of each of its points.

(ii) A closed interval [a, b] is a NBD of each of its points except the end point i.e. [a, b] is not a NBD of the points a and b, because it is not possible to find an open interval containing a or b which is contained in [a, b]. For instance, consider the closed interval [0, 1]. It is a NBD of every point in [0, 1]. But, it is not a NBD of 0 because for every $\delta > 0$, $]-\delta, \delta[\subset [0, 1]$. Similarly [0, 1] is not a NBD of 1.

(iii) The null set ϕ is a NBD of each of its point in the sense there is no point in ϕ of which it is not a NBD.

(iv) The set R of real numbers is a NBD of each real number x because for every $\delta > 0$, the open interval $]x-\delta, x+\delta[$ is contained in R.

(v) The set Q of rational numbers is not a NBD of any of its points x because any open interval containing x will also contains an infinite number of irrational numbers and hence the open interval can not be a subset of Q.

Now try the following exercise corresponding to the example.

EXERCISE 7)

Examine the validity of the following statements. Justify your answer in each case.

(i) The interval $[a, b[$ is a neighbourhood of each of points.

(ii) The unit interval $]0, 1]$ is the neighbourhood of only its corresponding end points.

(iii) The set $\{x \in R: x \geq a\}$ is not a neighbourhood of any of points.

(iv) The set $\{x \in R: x < a\}$ is a neighbourhood of each of points.

(v) The singleton $\{x\}$ for an $x \in R$ is a neighbourhood of x.

(vi) A finite subset of R is not a neighbourhood of any of its points.

Now consider any two neighbourhoods of the point 0 say $]-\frac{1}{10}, \frac{1}{10}[$ and $]-\frac{1}{5}, \frac{1}{5}[$ as shown in the Figure 1.

$$\frac{-1}{5} \quad \frac{-1}{10} \quad 0 \quad \frac{1}{10} \quad \frac{1}{5}$$

The intersection of these two neighbourhoods is

$$]-\frac{1}{10}, \frac{1}{10}[\cap]-\frac{1}{5}, \frac{1}{5}[=]-\frac{1}{10}, \frac{1}{10}[$$

which is again a NBD of 0.

The union of these two neighbourhoods is $]-\frac{1}{5}, \frac{1}{5}[$.

which is also a NBD of 0

Let us now examine these results in general.

EXAMPLE 3: The intersection of any two neighbourhoods of a point is a neighbourhood of the point.

SOLUTION: Let A and B be any two NBDS of a point c in **R**. Then there exist open intervals $]c-\delta_1, c+\delta_1[$ and $]c-\delta_2, c+\delta_2)$ such that

$]c-\delta_1, c+\delta_1[\subset A$, for some $\delta_1 > 0$

$]c-\delta_2, c+\delta_2[\subset B$, for some $\delta_2 > 0$

Let $\delta = \text{Min.}\ \{\delta_1, \delta_2) =$ minimum of δ_1, δ_2.
This implies that $]c - \delta, c + \delta[\subset A \cap B$
which shows that $A \cap B$ is a NBD of c.

EXAMPLE 4: Show that the superset of a NBD of a point is also a NBD of the point.

SOLUTION: Let A be a NBD of a point c. Then there exists an open interval $]c - \delta, c + \delta[$, for some $\delta > 0$ such that

$$]c - \delta, c + \delta[\subset A.$$

Now let S be a super set which contains A. Then obviously

$$A \subset S \Longrightarrow [c - \delta, c + \delta[\subset S$$

which shows that S is also a NBD of c.

For instance, if $]\frac{1}{10}, \frac{1}{10}[$ is a NBD of the point 0.

Then $]-\frac{1}{5}\ \frac{1}{5}[$ is also a NBD of 0 as can be seen from the Figure 1.

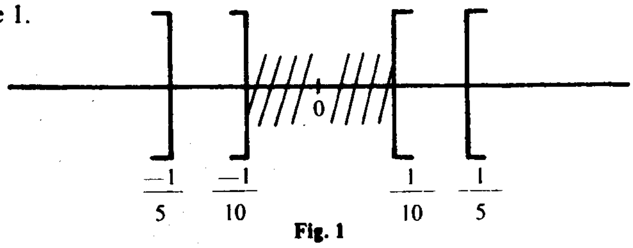


Fig. 1

Is a subset of a NBD of a point also a NBD of the point? Justify your answer.

Now you can try the following exercise.

EXERCISE 8)
Prove that the Union of any two NBDS of a point is a NBD of the point.

The conclusion of the Exercise 8, in fact, can be extended to a finite or an infinite or an arbitrary number of the NBDS of a point.

However, the situation is not the same in the case of intersection of the NBDS. It is true that the intersection of a finite number of NBDS of a point is a NBD of the point. But the intersection of an infinite collection of NBDS of a point may not be a NBD of the point. For example, consider the class of NBDS given by a family of open intervals of the form

$$I_1 =]-1, 1[, I_2 =]-\frac{1}{2}, \frac{1}{2}[\ I_3 =]-\frac{1}{3}, \frac{1}{3}[,$$

$$I_n =]-\frac{1}{n}, \frac{1}{n}[\ldots$$

which are NBDS of the point 0.
Then you can easily verify that

$$I_1 \cap I_2 \cap I_3 \cap I_4 \cap \ldots \cap I_n \cap \ldots$$

or
$$\bigcap_{n=1}^{\infty} I_n = \{0\}$$

which is not a NBD of 0.

3.4 OPEN SETS

You have seen from the previous examples and exercises that a given set may or may not be a NBD of a point. Also, a set may be a NBD of some of its points and not of its other points. A set may even be a NBD of each of its points as in the case of the interval $]a, b[$. Such a set is called an **open set.**

DEFINITION 3: A SET S IS SAID TO BE OPEN IF IT IS A NEIGHBOURHOOD OF EACH OF ITS POINTS.

Thus, a set S is open if for each x in S, there exists an open interval $]x - \delta, x + \delta[$, $\delta > 0$ such that

$$x \in]x - \delta, x + \delta[\subset S.$$

It follows at once that a set S is not open if it is not a NBD of even one of its points.

EXAMPLE 5: An open interval is an open set.

SOLUTION: Let]a, b[be an open interval. Then $a < b$. Let $c \in$]a, b[. Then $a < c < b$ and therefore

$c - a > 0$ and $b - c > 0$.

Choose

$\delta =$ Minimum of $\{b - c, c - a\}$
$= \text{Min}(b - c, c - a)$.
Note that $b - c > 0$, $c - a > 0$. Therefore $\delta > 0$.
Now $\delta \leq c - a \Longrightarrow a \leq c - \delta$
and $\delta \leq b - c \Longrightarrow c + \delta \leq b$.

i.e.
Therefore, $]c - \delta, c + \delta[\subset]a, b[$ and hence]a, b[is a NBD of c.

EXAMPLE 6: **(i) The set R of real numbers is an open set**
(ii) The null set ϕ is an open set
(iii) A finite set is not an open set
(iv) The interval]a, b] is not an open set.

You can solve the following exercise easily:

EXERCISE 9)
Test which of the following are open sets:

(i) An interval [a, b] for $a \in R$, $b \in R$, $a < b$

(ii) The intervals [0, 1[and]0, 1[

(iii) The set Q of rational numbers

(iv) The set N of natural numbers and the set Z of integers.

(v) The set $\{\frac{1}{n} : n \in N\}$

(vi) The intervals]a, ∞[and [a, ∞[for $a \in R$.

EXAMPLE 7: Prove that the intersection of any two open sets is an open set.

SOLUTION: Let A and B be any two open sets. Then we have to show that $A \cap B$ is also an open set. If $A \cap B = \phi$, then obviously $A \cap B$ is an open set. Suppose $A \cap B \neq \phi$.
Let x be an arbitrary element of $A \cap B$. Then $x \in A \cap B \Longrightarrow x \in A$ and $x \in B$.
Since A and B are open sets, therefore A and B are both NBDS of x. Hence $A \cap B$ is a NBD of x. But $x \in A \cap B$ is chosen arbitrarily. Therefore $A \cap B$ is a NBD of each of its points and hence $A \cap B$ is an open set. This proves the result.
In fact, you can prove that the intersection of a finite number of open sets is an open set. However, the intersection of an infinite number of open sets may not be an open set. Try the following exercises:

EXERCISE 10)
Give an example to show that intersection of an infinite number of open sets need not be an open set.
EXERCISE 11)
Prove that the union of any two open sets is an open set.

In fact, you can show that the union of an arbitrary family of open sets is an open set.

3.5 LIMIT POINT OF A SET

You have seen that the concept of an open set is linked with that of a neighbourhood of a point on the real line. Another closely related concept with the notion of neighbourhood is that of a limit point of a set. Before we explain the meaning of limit point of a set, let us study the following situations:

(i) Consider a set S = [1, 2[. Obviously the number 1 belongs to S. In any NBD of the point 1, we can always find points of S which are different from 1. For instance]0.5, 1.1[is a NBD of 1. In this NBD, we can find the point 1.05 which is in S but at the same time we note that $1.05 \neq 1$.

(ii) Consider another set $S = \{\frac{1}{n} : n \in N.\}$. The number 0 does not belong to this set.

Take any NBD of 0 say,]−0.1, 0.1[. The number $\frac{1}{20} = 0.05$ of S is in this NBD of 0. Note that $0.05 \neq 0$.

(iii) Again consider the same set S of (ii) in which the number 1 obviously belongs to S. We can find a NBD of 1, say]0.9, 1.1[in which we can not find a point of S different from 1.

In the light of the three situations, we are in a position to define the following:

DEFINITION 4: LIMIT POINT OF A SET

A number p is said to be a limit point of a set S of real numbers if every neighbourhood of p contains at least one point of the set S different from p.

EXAMPLE 8: (i) In the set S = [1, 2[, the number 1 is a limit point of S. This limit point belongs to S. The set $S = \{\frac{1}{n} : n \in N\}$ has only one limit point 0. You may note that 0 does not belong to S.

(ii) Every point in Q, (the set of rational numbers), is a limit point of Q, because for every rational number r and $\delta > 0$, i.e.] $r - \delta$, $r + \delta$ [has at least one rational number different from r. This is because of the reason that there are infinite rationals between any two real numbers. Now, you can easily see that every irrational number is also a limit point of the set Q for the same reason.

(iii) The set N of natural numbers has no limit point because for every real number a, you can always find $\delta > 0$ such that]$a - \delta$, $a + \delta$[does not contain a point of the set N other than a.

(iv) Every point of the interval]a, b] is its limit point. The end points a and b are also the limit points of]a, b]. But the limit point a does not belong to it whereas the limit point b belongs to it.

(v) Every point of the set [a, ∞[is a limit point of the sets. This is also true for]−∞, a[.

Now try the following exercise and justify your answer.

EXERCISE 12)

(i) Does the set Z possess a limit point?

(ii) Every point of R, the set of real numbers is a limit point of R. Is it true?

(iii) Is every point of an open interval]a, b[a limit point of]a, b[? What about the end points a and b?

(iv) Is every point of a closed interval [a, b] is its limit point? what about the end points a and b?

(v) Is every point of the sets]a, ∞[and]− ∞, a] a limit point of the set?

EXERCISE 13)

Show that a given point p is a limit point of a set S if and only if every neighbourhood of p contains an infinite number of members of S.

From the foregoing examples and exercises, you can easily observe that

(i) **A limit point of set may or may not belong to the set,**
(ii) **A set may have no limit point,**
(iii) **A set may have only one limit point.**
(iv) **A set may have more than one limit point.**

The question, therefore, arises: **"How to know whether or not a set has a limit point?"** One obvious fact is that a **finite set can not have a limit point.** Can you give a reason for it? Try it. But then there are examples where even an infinite set may not have a limit point e.g. the sets N and Z do not have a limit point even though they are infinite sets. However, it is certainly clear that a set which has a limit point, must necessarily be an infinite set. Thus our question takes the following form:

"What are the conditions for a set to have a limit point?"

This question was first studied by a Czechoslovakian Mathematician, Bernhard Bulzano [1781-1848] in 1817 and he gave some ideas.

Unfortunately, his ideas were so far ahead of their time that the world could not appreciate the full significance of his work. It was only much later that Bulzano's work was extended by Karl Weierstrass [1815-1897], a great German Mathematician, who is known as the "father of analysis". It was in the year 1860 that Weierstrass proved a fundamental result, now known as Buzano-Weierstrass Theorem for the existence of the limit points of a set. We state and prove this theorem as follows:

Karl Weierstrass

3.5.1. BOLZANO—WEIERSTRASS THEOREM

THEOREM 1: Every infinite bounded subset of R set has a limit point in R.

PROOF: Let S be an infinite and bounded subset of **R**. Since A is bounded, therefore A has both a lower bound as well as an upper bound. (Recall the definition of a bounded set from Section 2.3.)

Let m be a lower bound and M be an upper bound of A. Then obviously

$$m \leq x \leq M, \forall x \in A.$$

Construct a set S in the following way:
S = {x ∈ R: x exceeds at most finite number of the elements of A}. Now, let us examine the following two questions:
(i) Is S a non-empty set?
(ii) Is S also a bounded set?

Indeed S is non-empty because m ∈ S since m ≤ x, ∀ x ∈ A. Also M is an upper bound of S because no number greater than or equal to M can belong to S. Note that M can not belong to S because it exceeds an infinite number of elements of A.

Since the set S is non-empty and bounded above, therefore by the axiom of completeness (see Section 2.3), S has the supremum in R. Let p be the supremum of S. We claim that p is a limit point of the set A.

In order to show that p is a limit point of A, we must establish that every NBD of p has at least one point of the set A other than p. In other words, we have to show that every NBD of p has an infinite number of elements of A. For this, it is enough to show that any open interval] p − δ, p + δ [, for an δ > 0 contains an infinite number of members of set A. For this, we proceed as follows:

Since p is the supremum of S, therefore, by the definition of the Supremum of a set (see Section 2.3), there is at least one element y in S such that y > p −δ for some δ>0. Also y is a member of S, therefore y exceeds at the most a finite number of the elements of A. In other words, if you visualise it on the line as shown in the figure 2, the number of elements of A lying on the left of p − δ is finite at the most. But

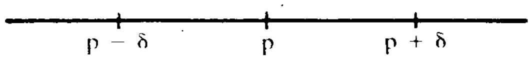


Fig. 2

certainly, the number of elements of A lying on the right side of the point p − δ is infinite.

Again since p is the supremum of S, therefore by definition p + δ can not belong to S. In other words, p + δ exceeds an infinite number of elements of A. This means that there lie an infinite number of elements of A on the left side of the point p + δ .

Thus we have shown that there lies an infinite number of elements of A on the right side of p − δ and also there is an infinite number of elements of A on the left side of p + δ. What do you conclude from this? In other words, what is the number of elements of A in between (within) the interval] p − δ, p + δ[? Indeed, this number is infinite i.e. there is an infinite number of elements of A in the open interval

$]p - \delta, p + \delta[$. Hence the interval $]p - \delta, p + \delta[$ contains an infinite number of elements of A for some $\delta > 0$. Since $\delta > 0$ is chosen arbitrarily, therefore every interval $]p - \delta, p + \delta[$ has an infinite number of elements of A. Thus every NBD of p contains an infinite number of elements of A. Hence p is a limit point of the set A.

This completes the proof of the theorem.

EXAMPLE 9 (i) The intervals [0, 1],]0, 1[,] 0, 1], [0, 1 [are all infinite and bounded sets. Therefore each of these intervals has a limit point. In fact, each of these intervals has an infinite number of limit points because every point in each interval is a limit point of the interval.

(ii) The set [a, ∞[is infinite and unbounded set but has every point as a limit point. This shows that the condition of boundedness of an infinite set is only sufficient in the theorem.

EXERCISE 14)
Give examples of the following:

(i) At least four infinite bounded sets indicating their corresponding limit points.

(ii) At least three unbounded (and infinite) sets each having a limit point.

(iii) An infinite and unbounded set having no limit point.

From the previous examples and exercises, it is clear that it is not necessary for an infinite set to be bounded to possess a limit point. In other words, a set may be unbounded and still may have a limit point. However for a set to have a limit point, it is necessary that it is infinite.

Another obvious fact is that a limit point of a set may or may not belong to the set and a set may have more than one limit point. In the next section, we shall further study how sets can be characterized in terms of their limit points.

3.6 CLOSED SETS

In Section 3.5, you have seen that a limit point of a set may or may not belong to the set. For example, consider the set $S = \{x \in R : 0 \leq x < 1\}$. In this set, 1 is a limit point of S but it does not belong to S. But if you take $S = \{x: 0 \leq x \leq 1\}$, then all the limit points of S belong to S. Such a set is called a **closed set**. We define a closed set as follows:

DEFINITION 5: CLOSED SET

A set is said to be closed if it contains all its limit points.

EXAMPLE 10:

(i) Every closed and bounded interval such as [a, b] and [0, 1] is a closed set.

(ii) An open interval is not a closed set. Check Why?

(iii) The set R is a closed set because every real number is a limit point of R and it belongs to R.

(iv) The null set ϕ is a closed set.

(v) The set $S = \{\frac{1}{n} : n \in N\}$ is not a closed set. Why?

(vi) he set]a, ∞[is not a closed set, but]−∞, a] is a closed set.

You can try the following exercise:

EXERCISE 15)
Check whether or not the following sets are closed sets:

(i) The set Q of rational numbers

(ii) The set N of natural numbers

(iii) The set Z of integers

(iv) A finite set of real numbers

(v) The set $S = \{x \in R: a \leq x \leq b\}$

(vi) The sets $[a, \infty[$ and $]-\infty, a[$

You may be thinking that the word open and closed should be having some link. If you are guessing some relation between the two terms, then you are hundred per cent correct. Indeed, there is a fundamental connection between open and closed sets. What exactly is the relation between the two? Can you try to find out? Consider, the following subsets or **R**:

(i) $]0, 4[$

(ii) $[-2, 5]$

(iii) $]0, \infty[$

(iv) $]-\infty, 6]$.

The sets (i) and (iii) are open while (ii) and (iv) are closed. If you consider their complements, then the complements of the open sets are closed while those of the closed sets are open. In fact, we have the following concrete situation in the form of theorem 2:

THEOREM 2: A set is closed if and only if its comlement is open.

PROOF: We assume that S is a closed set. Then we prove that its complement S^c is open.

To show that S^c is open, we have to prove that S^c is a NBD of each of its points. Let $x \in S^c$. Then $x \in S^c \Rightarrow x \notin S$. This means x is not a limit point of S because S is given to be a closed set. Therefore there exists à $\delta > 0$ such that $]x - \delta, x + \delta[$ contains no points of S. This means that $]x - \delta, x + \delta[$ is contained in S^c. This further implies that S^c is a NBD of x. In other words, S^c is an open set, which proves the assertion.

Conversely, let a set S be such that its complement S^c is open. Then we prove that S is closed.

To show that S is closed, we have to prove that every limit point x of S belongs to S. Suppose $x \in S$. Then $x \in S^c$

This implies that S^c is a NBD of x because S^c is open. This means that there exists an open interval $]x - \delta, x + \delta[$, for some $\delta > 0$ such that

$$]x - \delta, x + \delta[\subset S^c$$

In other words, $]x - \delta, x + \delta[$ contains no point of S. Thus x is not a limit point of S which is a contradiction. Thus our supposition is wrong and hence $x \in S$ is not possible. In other words, the limit point x belongs to S and thus S is a closed set.

Note that the notions of open and closed sets are not mutually exclusive. In other words, if a set is open, then it is not necessary that it can not be closed. Similarly, if a set is closed, then it does not exclude the possibility of its being open. In fact there are sets which are both open and closed and there are sets which are neither open nor closed as you must have noticed in the various examples we have given in our discussion. For example the set R of all the real numbers is both an open set as well as a closed set. Can you give another example? What about the null set. Again Q, the set of rational numbers is neither open nor closed.

EXERCISE 16)
Give examples of two sets which are neither closed nor open.

In Section 3.4, we have discussed the behaviour of the union and intersection of open sets. Since closed sets are closely connected with open sets, therefore, it is quite natural that we should say something about the union and intersection of closed sets. In fact, we have the following results.

EXAMPLE 11 : Prove that the union of two closed sets is a closed set.

SOLUTION : Let A and B be any two closed sets. Let $S = A \cup B$. We have to show that S is a closed set. For this, it is enough to prove that the complement S^c is open. Now

$$S^c = (A \cup B)^c = B^c \cap A^c = A^c \cap B^c$$

Since A and B are closed sets, therefore A^c and B^c are open sets. Also, we have proved in Section 3.4 that the intersection of any two open sets is open. Therefore $A^c \cap B^c$ is an open set and hence S^c is open.

This result can be extended to a finite number of closed sets. You can easily verify that **the union of a finite number of closed sets is a closed set. But note that the union of an arbitrary family of closed sets may not be closed.**

For example, consider the family of closed sets given as

$$S_1 = [1, 2],\ S_2 = \left[\frac{1}{2}, 2\right],\ S_3 = \left[\frac{1}{3}, 2\right], \ldots$$

and in general

$$S_n = \left[\frac{1}{n}, 2\right] \ldots \text{ for } n = 1, 2, 3, \ldots$$

Then,

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \ldots \cup S_n \cup \ldots$$

$$=]0, 2]$$

which is not a closed set.

Now try the following exercise:

EXERCISE 17)
Prove that the intersection of an arbitrary family of closed sets is closed.

DEFINITION 6 : DERIVED SET

The set of all limit points of a given set S is called the derived set and is denoted by S′.

EXAMPLE 12 : (i) Let S be a finite set. Then $S' = \phi$

(ii) $S = \{\frac{1}{n} : n \in N\}$, the derived set $S' = \{0\}$

(iii) The derived set of R is given by $R' = R$

(iv) The derived set of Q is given by $Q' = R$

We define another set connected with the notion of the limit point of a set. This is called the **closure of a set.**

DEFINITION 7: CLOSURE OF A SET

Let S be any set of real numbers ($S \subset R$). The closure of S is defined as the union of the set S and its derived set S. It is denoted by $\bar{S}$. Thus

$$\bar{S} = S \cup S'$$

In other words, the closure of a set is obtained by the combination of the elements of a given set S and its derived set $\bar{S}'$.

For example, $\bar{S}$ of $S = \{\frac{1}{n} : n \in N\}$ is given by

$$\bar{S} = \{\frac{1}{n},\ n \in N\} \cup \{0\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$$

Similarly, you can verify that

$$\bar{Q} = Q \cup Q' = Q \cup R = R$$

$$\bar{R} = R \cup R' = R \cup R = R$$

3.7 COMPACT SETS

We discuss yet another concept of the so called compactness of a set. The concept of compactness is formulated in terms of the notion of an open cover of a set.

DEFINITION 8: OPEN COVER OF A SET

Let S be a set and $\{G_\alpha\}$ be a collection of some open subsets of R such that $S \subset \cup G_\alpha$. Then $\{G_\alpha\}$ is called an open cover of S.

EXAMPLE 13: Verify that the collection $G_n = \{G\}^{\infty}$ where $G_n =]-n, n[$ is an open cover of the set R.

SOLUTION:

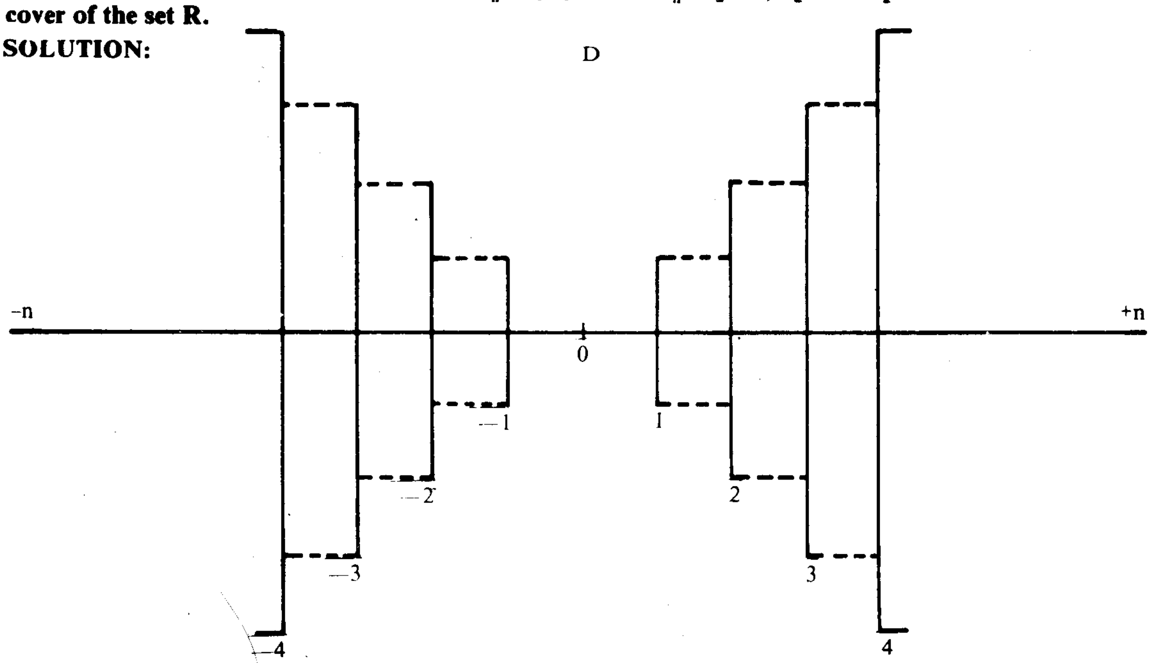


Fig. 3

As shown in the Figure 3, we see that every real number belongs to some G_n. Hence

$$R = \bigcup_{n=1}^{\infty} G_n$$

EXAMPLE 14: Examine whether or not the following collections are open covers of the interval [1, 2].

(i) $G_1 = \{]\frac{1}{4}, \frac{5}{4}[,]\frac{3}{4}, \frac{7}{4}[,]\frac{5}{3}, \frac{9}{4}[\}$

(ii) $G_2 = \{]\frac{1}{2}, \frac{5}{4}[,]\frac{3}{2}, \frac{9}{4}[\}$

SOLUTION: (i) Plot the subsets of G_1 on the real line as shown in the Figure 4.

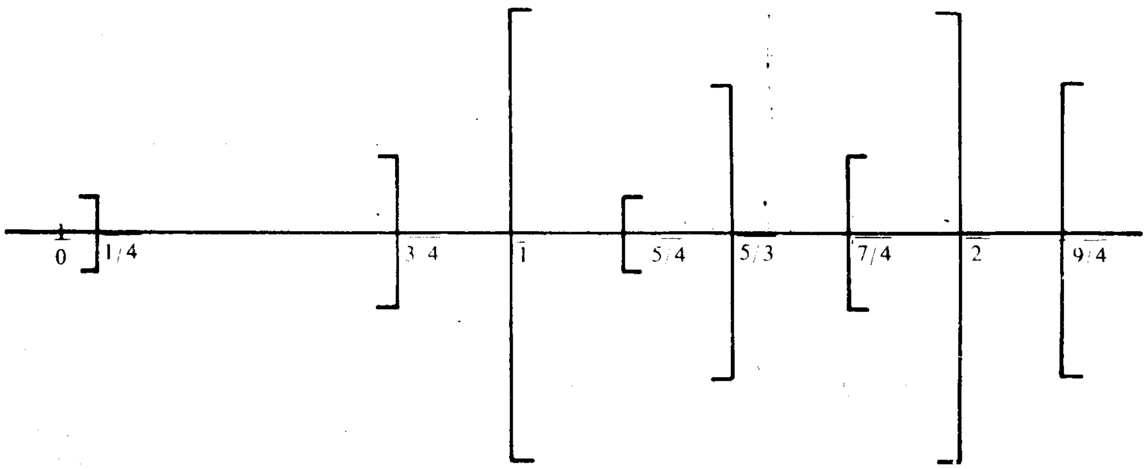


Fig. 4

From the figure, it follows that every element of the set $S = [1, 2] = \{u: 1 \leq x \leq 2]$ belongs to at least one of the subsets of G_1. Since each of the subsets in G_1 is an open set, therefore G_1 is an open cover of S.

(ii) Again plot the subsets of G_2 on the real line as done in the case of (i).

You will find that none of the points in the interval $[\frac{5}{4}, \frac{3}{2},]$ belongs to any of the subsets of G_2. Therefore G_2 is not an open cover of S.

Now try the following exercise:

EXERCISE 18)
Verify whether the following collections are open covers of the corresponding sets mentioned in each case :

(i) $G_1 = \{]n, n+2[: n \in Z\}$ **of R**

(ii) $G_2 = \{]n, n+1[: n \in Z\}$ **of R**

(iii) $G_3 = \{] \frac{1}{n+2}, \frac{1}{n}, [: n \in N\}$ **of] 0, 1[.**

Now consider the set [0, 1] and two classes of open covers of this set namely G_1 and G_2 given as

$$G_1 = \{] -1 - \frac{1}{n}, 1 + \frac{1}{n}, [\}_{n=1}^{\infty} \quad G_2 = \{] -1 - \frac{1}{2n}, 1 + \frac{1}{2n} [\}_{n=1}^{\infty}$$

You can see that $G_1 \subset G_2$. In this case, we say that G_2 is a subcover of G_1. In general, we have the following definition:

DEFINITION 9: SUBCOVER AND FINITE SUBCOVER OF A SET

Let G be an open cover of a set S. A subcollection E of G is called a subcover of S if E too is a cover of S. Further, if there are only a finite number of sets in E, then we say that E is called a finite subcover of the open cover G of S. Thus if G is an open cover of a set S, then a collection E is a finite subcover of the open cover G of S provided the following conditions hold:

(i) **E is contained in D**

(ii) **E is a finite collection**

(iii) **E is itself also a cover of S.**

EXERCISE 19)
Give an example of an infinite set S such that there is an open cover G of S which admits of a finite subcover of G

From the forgoing example and exercise, it follows that an open cover of a set may or may not admit of a finite subcover. Also, there may be a set whose every open cover contains a finite subcover. Such a set is called a compact set. We define a compact set in the following way:

DEFINITION 10: COMPACT SET

A set is said to be compact if every open cover of the set admits of a finite subcover of the set.

For example, consider the finite set $S = \{1, 2, 3\}$ and an open cover $\{G_\alpha\}$ of S. Let G^1, G^2, G^3, be the sets in G which contain 1, 2, 3 respectively. Then $\{G^1, G^2, G^3\}$ is a finite subcover of $\{G_\alpha\}$. Thus S is a compact set. In fact, you can show that every finite set in R is a compact set.

The collection $G = \{]-n, n[: n \in N\}$ is an open cover of R but does not admit of a finite subcover of R. Therefore the set R is not a compact set.

Thus you have seen that every finite set is always compact. But an infinite set may not may not be a compact set. The question, therefore, arises, "**What is the criteria to determine whether a given set is compact?**" This question has been settled by a beautiful theorem known as **Heine-Borel Theorem** named in the honour of the

German mathematician H.E. Heine [1821-1881] and the French mathematician F. E.E. Borel [1871-1956], both of whom were pioneers in the development of Mathematical Analysis.

We state this theorem without proof.

THEOREM 3: Heine-Borel Theorem

Every closed and bounded subset of R is compact.
The immediate consequence of this theorem is that every bounded and closed interval is compact.

3.8 SUMMARY

1. In Section 3.2, we have defined the absolute value or the modulus of a real number and discussed certain related properties. The modulus of real number x is defined as

$$|x| = x \quad \text{if } x \geq 0$$
$$= -x \quad \text{if } x < 0.$$

Also, we have shown that

$$|x - a| < d \Leftrightarrow a - d < x < a + d$$

2. In Section 3.3, we have discussed the fundamental notion of NBD of a point on the real line i.e. first we have defined it as a δ – neighbourhood and then, in general, as a set containing an open interval with the point in it.
3. With the help of NBD of a point we have defined, in Section 3.4, an open set in the sense that a set is open if it is a NBD of each of its points.
4. We have introduced the notion of the limit point of a set in Section 3.4. A point p is said to be a limit point of a set S if **every** NBD of p contains a point of S different from p. This is equivalent to saying that a point p is a limit point of S if every NBD of p contains an infinite number of the members of S. Also, we have discussed Bulzano-Weiresstrass theorem which gives a sufficient condition for a set to possess a limit point. It states that an infinite and bounded set must have a limit point. This condition is not necessary in the sense that an unbounded set may have a limit point.
5. The limit points of a set may or may not belong to the set. However, if a set is such that every limit point of the set belongs to it, then the set is said to be a closed set. The concept of a closed set has been discussed in Section 3.6. Here, we have also shown a relationship between a closed set and an open set in the sense that a set is closed if and only if its complement is open. Further, we have also defined the Derived set of a set S as the set which consists of all the limit points of the set S. The Union of a given set and its Derived set is called the **closure** of the set. Note the distinction between a closed set and the closure of a set S.
6. Finally, we have introduced another topological notion in Section 3.7. It is about the open cover of a given set. Given a set S, a collection of open sets such that their Union contains the set S is said to an open cover of S. A set S is said to be compact if every open cover of S admits of a finite subcover. The criteria to determine whether a given set is compact or not, is given by a theorem named **Heine-Borel Theorem** which **states that every closed and bounded subset of R is compact.** An immediate consequence of this theorem is that every bounded and closed interval is compact.

3.9 ANSWERS/HINTS/SOLUTIONS

E 1) If $x \geq 0$, then $-x \leq 0$
Min $(x, -x) = -x$
Also $|x| = x$. Hence
$-|x| = -x =$ Min $(x, -x)$.
If $x < 0$, $-x > 0$. Hence
Min $(x, -x) = x$. Also
$|x| = -x$. Therefore
$-|x| = x =$ Min $(x, -x)$.

E 2) Follow the method as explained in property 2)

E 3) $\left|\frac{x}{y}\right| = \left(\frac{x}{y}\right)$ (By property 2).

$= \frac{x}{y} = \frac{|x|}{|y|}$ (Again by property 2)

$= \left|\frac{x}{y}\right|$

$\Leftrightarrow \left|\frac{x}{y}\right| = \pm \frac{|x|}{|y|} = \frac{|x|}{|y|}$. Why?

E 4) $|x - y|^2 = (x - y)^2 = x^2 + y^2 - 2xy$
$\geq |x|^2 + |y|^2 + 2(-|xy|)$
$= |x|^2 + |y|^2 - 2|x||y| = (|x| - |y|)^2 = ||x| - |y||^2$
Therefore
$|x - y| \geq \pm |x| - |y|$
Hence $|x - y| \geq |x| - |y|$. Why?

E 5) Follow the procedure explained in the property 5.
$|x - a| \leq d \Leftrightarrow -d \leq (x - a) \leq d$
$\Leftrightarrow a - d \leq x \leq a + d$

E 6) (i) Choose a and b such that $a + b = 7$, $a - b = 2$.

Then $2a = 9$ or $a = \frac{9}{2}$ and $2b = 5$ or $b = \frac{5}{2}$.

Hence

$2 < x < 7 \Leftrightarrow \frac{9}{2} - \frac{5}{2} < x < \frac{9}{2} + \frac{5}{2}$

$\Leftrightarrow \frac{5}{2} < x - \frac{9}{2} < \frac{5}{2}$

$\Leftrightarrow -5 < 2x - 9 < 5 \Leftrightarrow |2x - 9| < 5$

(ii) $|x - 2| < 3 \Leftrightarrow -3 < x - 2 < 3$
$\Leftrightarrow 2 - 3 < x < 3 + 2$
$\Leftrightarrow -1 < x < 5$.

E 7) (i) [a, b [is a NBD of each of its points except the point b.

(ii) Easy to solve.

(iii) Each of these is a NBD of each of its points except the point a.

(iv) Same as (iii)

(v) It is not a NBD. Explain why?

(vi) It is also not a NBD of any of its points. Why?

E 8) Let S and T be any two NBDs of a point c.
We have to show that $S \cup T$ is a NBD of c. Now $c \in S \cup T \Longrightarrow c \in T$. If $c \in S$, then since S is a NBD of c and $S \subset S \cup T$, therefore $S \cup T$ is also a NBD of c.

E 9) (i) [a, b] is not an open set. Why?

(ii)]0, 1 [is open while [0, 1 [is not open. Give reasons.

(iii) It is not an open set because it is not a NBD of any of its points.

(iv) Neither N nor Z is an open set.

(v) It is not open

(vi)]a, ∞[is open but [a, ∞[is not open.

E 10) Example is given after E 8). Look for it.

E 11) Use the method of E 8).

E 12) (i) No. Give reasons.

(ii) Every NBD of an arbitrary real number contains an infinite number of real numbers.

(iii) Yes the end points are also the limit points but they do not belong to it.

(iv) Yes. The end points, in this case, are also the limit points but they belong to the interval.

(v) Yes. Elaborate it.

E 13) (i) Any four bounded intervals

(ii) The sets Q, R, and the set

$$\{\frac{1}{n} : n \in N\}$$

(iii) Z

E 14) Let p be a limit point of a set S. We have to prove that

(i) if every NBD of p contains at least one member of S different from p, then every NBD of p contains an infinite number of the members of S.

(ii) If every NBD of p contains an infinite number of members of S, then it must have at least one member of S other than p.

The result (ii) is obvious. Therefore it is enough to prove (i).

Since every NBD of p contains atleast one member of S different from p, therefore let $]p - \delta_1, p + \delta_1, [$ be a NBD such that it has a member x_1 of S and $x_1 \neq p$. Suppose $|x_1 - p| = \delta_2$ where $\delta_2 < \delta_1$. Consider the NBD $]p - \delta_2, p + \delta_2[$ of p. Then by definition, $]p - \delta_2, p + \delta_2[$ must have an element say x_2 of S such that $x_2 \neq p$. But since $\delta_2 < \delta_1$, therefore $]p - \delta_1, p + \delta_1[$ contains two elements x_1, x_2 of S which are different from p. Continuing like this, you can show that the NBD $]p - \delta_1, p + \delta_1[$ contains an infinite number of the members of S.

E 15) (i) **Not closed.**

(ii) **Closed**

(iii) **Closed**

(iv) **Closed**

(v) **Both are closed sets**

(vi) **First is closed and the second is not closed.**

E 16) (i) $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

(ii) $[0, 1[$.

E 17) Consider an arbitrary family of closed sets such that their intersection is nonempty. Let x be a limit point of this intersection. Then every NBD $]x-\delta, n + \delta[$ for some $\delta > 0$, of x contains an infinite numbers of this intersection and hence of each member of the given family. Therefore x is a limit point of each member of this family of closed sets. Hence x belongs to each member of the family and therefore x belongs to the intersection. Hence the intersection is also a closed set.

E 18) (i) $D = \{ \ldots,]-3, -1[,]-2, 0[,]-1, 1[,]0, 2[\ldots\ldots \}$.
Since every $x \in R$ belongs to at least of the subsets of D, therefore D is an open cover of R.

Similarly verify (ii) and (iii).

E 19) Any suitable example will be acceptable.

UNIT 4 REAL FUNCTIONS

STRUCTURE

4.1 INTRODUCTION

Leonard Euler

Real Analysis is often referred to as the **Theory of Real Functions.** The word **'function'** was first introduced in 1694 by L.G. Leibniz [1646-1716], a famous German mathematician, who is also credited along with Isacc Newton for the invention of **Calculus.** Leibniz used the term function to denote a quantity connected with a curve. A Swiss mathematician, L. Euler [1707-1783] treated function as an expression made up of a variable and some constants. Euler's idea of a function was later generalized by an eminent French mathematician J. Fourier [1768-1830]. Another German mathematician, L. Dirichlet (1805-1859) defined function as a relationship between a variable (called an **independent variable**) and another variable (called the **dependent variable**). This is the definition which, you know, is now used in Calculus.

The concept of a function has undergone many refinements. With the advent of Set Theory in 1895, this concept was modified as a correspondence between any two non-empty sets. Given any two non-empty sets S and T, a function f from S into T, denoted as $f: S \to T$, defines a rule which assigns to each $x \in S$, a unique element $y \in T$. This is expressed by writing as $y = f(x)$. This definition, as you will recall, was given in Section 1.2. A function $f: S \to T$ is said to be a

(i) **Complex-valued function of a complex variable** if both S and T are sets of complex numbers;

(ii) **Complex-valued function of a real variable** if S is a set of real numbers and T is a set of complex numbers;

(iii) **Real-valued function of a complex variable** if S is a set of complex numbers and T is a set of real numbers;

(iv) **Real-valued function of a real variable** if both S and T are some sets of real numbers.

Since we are dealing with the course on Real Analysis, we shall confine our discussion to those functions whose domains as well as co-domains are some subsets of the set of real numbers. We shall call such functions as **Real Functions.**

In this unit, we shall deal with the algebraic and transcendental functions. Among the transcendental functions, we shall define the trigonometric functions, the exponential and logarithmic functions. Also, we shall talk about some special real functions including the bounded and monotonic functions. We shall frequently use these functions to illustrate various concepts in Blocks 3 and 4.

OBJECTIVES

After going through this unit, you should be able to

→identify various types of algebraic functions

→define the trigonometric and the inverse trigonometric functions

→describe the exponential and logarithmic functions

→discuss some special functions including the bounded and monotonic functions.

4.2 ALGEBRAIC FUNCTIONS

In Unit 1, we identified the set of natural numbers and built up various sets of numbers with the help of the algebraic operations of addition, subtraction, multiplication, division etc. In the same way, let us construct new functions from the real functions which we have chosen for our discussion. Before we do so, let us review the algebraic combinations of the functions under the operations of addition, subtraction, multiplication and division on the real-functions.

4.2.1 ALGEBRAIC COMBINATIONS OF FUNCTIONS

Let **f** and **g** be any two real functions with the same domain $S \subset$ **R** and their co-domain as the set R of real numbers. Then we have the following definitions:

DEFINITION 1: SUM AND DIFFERENCE OF TWO FUNCTIONS

(i) **The Sum of f and g, denoted as f + g, is a function defined from S into R such that**

$(f + g)(x) = f(x) + g(x), \forall x \in S.$

(ii) **The Difference of f and g, denoted as f − g, is a function defined from S to R such that**

$(f - g)(x) = f(x) - g(x), \forall x \in S.$

Note that both f(x) and **g (x)** are elements of **R**. Hence each of their sum and difference is again a unique member of **R.**

DEFINITION 2: PRODUCT OF TWO FUNCTIONS

Let f: S → R and g: S → R be any two functions. The product of f and g, denoted as f.g, is defined as a function f. g: S → R by

$(f \,.\, g)(x) = f(x) \,.\, g(x), \forall x \in S.$

DEFINITION 3: SCALAR MULTIPLE OF A FUNCTION

Let f: S → R be a function and
k be some fixed real number. Then the scalar multiple of 'f' is a function
k f S → R defined by

$(kf(x) = k.\, f(x), \forall x \in S.$

This is also called the scalar multiplication.

DEFINITION 4: QUOTIENT OF TWO FUNCTIONS

Let f: S → R and g: S → R be any two functions such that g(x) ≠ 0 for each x in S. Then a function $\frac{f}{g}$: S → R defined by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \forall x \in S$$

is called the quotient of the two functions.

EXERCISE 1)

Let f, g, h be any three functions, defined on S and taking values in R, as f (x) = ax^2, g(x) = bx for every x in S, where a, b, are fixed real numbers. Find f + g, f − g, f . g, f/g and kf, when k is a constant.

4.2.2 NOTION OF AN ALGEBRAIC FUNCTION

You are quite familiar with the equations $ax + b = 0$ and $ax^2 + bx + c = 0$, where a, b, c $\in$ **R**, **a ≠ 0**. These equations, as you know are, called **linear (or first degree) and**

quadratic (or **second degree**) equations, respectively. The expressions **$ax + b$** and $ax^2 + bx + c$ are, respectively, called the first and second **degree polynomials in x. In** the same way an expression of the form $ax^3 + bx^2 + cx + d$ ($a \neq 0$, a, b, c, $d \in \mathbf{R}$) is called a third degree polynomial (cubic polynomial) in x. In **general, an expression of** the form $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$ **where** $a_0 \neq 0$, $a_i \in \mathbf{R}$, $i = 0, 1, 2, ...,\ n$, **is called an nth degree polynomial in x.**

A function which is expressed in the form of such a polynomial is called a polynomial function. Thus, we have the following definition:

DEFINITION 5: POLYNOMIAL FUNCTION

Let a_i ($i = 0, 1, ..., n$) be fixed real numbers where n is some fixed non-negative integer. Let S be a subset of R. A function f: S → R defined by

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n,\ \forall\ x \in S,\ a_0 \neq 0$$

is called a polynomial function of degree n.

Let us consider some particular cases of a polynomial function on **R**:

Suppose **f: S → R** is such that

(i) $f(x) = k$, $\forall x \in S$ (k is a fixed real number). This is a polynomial function. This is generally called a **constant function** on S.

For example,

$f(x) = 2$, $f(x) = -3$, $f(x) = \pi$, $\forall\ x \in \mathbf{R}$, are all constant functions.

(ii) One special case of a constant function is, obtained by taking $k = 0$ i.e. when $f(x) = 0$, $\forall\ x \in S$.

This is called the **zero function** on S

EXERCISE 2)
Draw the graph of a constant function. Draw the graph of the zero function.

Let f: S → R be such that

(iii) $f(x) = a_0 x + a_1$, $\forall\ x \in S$, $a_0 \neq 0$.

This is a polynomial function and is called a **linear function** on S. For example, $f(x) = 2x + 3$, $f(x) = -2x+3$, $f(x) = 2x - 3$, $f(x) = -2x - 3$, $f(x) = 2x$ for every $x \in S$ are all linear functions

(iv) The function **f: S → R** defined by

$f(x) = x$, $\forall\ x \in S$

s called the **identity function** on S.

(v) **f: S → R** given as

$f(x) = a_0 x^2 + a_1 x + a_2$, $\forall\ x \in \mathbf{R}$, $a_0 \neq 0$.

is a polynomial function of degree two and is called a **quadratic function** on S.

For example, $f(x) = 2x^2 + 3x - 4$, $f(x) = x^2 + 3$, $f(x) = x^2 + 2x$, $f(x) = -3x^2$, for every $x \in S$ are all quadratic functions.

DEFINITION 6: RATIONAL FUNCTION

A function which can be expressed as the quotient of two polynomial functions is called a rational function.

Thus a function f: S → **R** defined by

$$f(x) = \frac{a_0 x^n + a_1 x^{n-1} + ... + a_n}{b_0 x^m + b_1 x^{m-1} + ... + b_m},\ \forall\ x \in S.$$

is called a **rational function.**

Here $a_0 \neq 0$ $b_0 \neq 0$, $a_i, b_j \in R$ where i, j are some fixed real numbers and the polynomial function in the denominator is never zero.

EXAMPLE 1: The following are all rational functions on R.

$$\frac{2x+3}{x^2+1}, \quad \frac{4x^2+3x+1}{3x-4} \quad (x \neq \frac{4}{3}) \quad \text{and} \quad \frac{3x+5}{x-4} \quad (x \neq 4).$$

The functions which are not rational are known as **irrational functions.** A typical example of an irrational function is the **square root function** which we define as follows:

DEFINITION 7: SQUARE ROOT FUNCTION

Let S be the set of non-negative real numbers. A function f: S → R defined by

$$f(x) = \sqrt{x}, \ \forall \ x \in S$$

is called the square root function.

You may recall that $\sqrt{x}$ is the non-negative real number whose square is x. Also it is defined for all $x \geq 0$.

EXERCISE 3)

Draw the graph of the function $f(x) = \sqrt{x}$ for $x \geq 0$.

Polynomial functions, rational functions and the square root function are some of the examples of what are known as **algebraic functions.** An algebraic function, in general, is defined as follows

DEFINITION 8: ALGEBRAIC FUNCTION

An algebraic function f: S → R is a function defined by y = f(x) if it satisfies identically an equation of the form

$$p_0(x)\,y^n + p_1(x)\,y^{n-1} + \dots + p_{n-1}(x)\,y + p_n(x) = 0$$

where $p_0(x)$, $p_1(x)$, ... $p_{n-1}(x)$, $p_n(x)$ are Polynomials in x for all x in S and n is a positive integer.

EXAMPLE 2: Show that f: R → R defined by

$$f(x) = \sqrt{\frac{x^2-3x+2}{4x-1}}$$

is an algebraic function.

Solution

Let $y = f(x) = \dfrac{\sqrt{x^2-3x+2}}{\sqrt{4x-1}}$

Then $(4x-1)\,y^2 - (x^2-3x+2) = 0$

Hence f(x) is an algebraic function.

In fact, any function constructed by a finite number of algebraic operations (addition, subtraction, multiplication, division and root extraction) on the identity function and the constant function, is an **algebraic function.**

EXAMPLE 3: The functions f : R → R defined by

(i) $f(x) = \dfrac{(x^2+2)\sqrt{x-1}}{x^2+4}$

or $f(x) = \dfrac{x^3-2x}{\sqrt{x.(3x^2+5)}}$

are algebraic functions.

EXAMPLE 4: Prove that every rational function is an algebraic function.

SOLUTION : Let f: R → R be given as

$$f(x) = \frac{p(x)}{q(x)}, \quad \forall \ x \in R,$$

where p(x) and q(x) are some polynomial functions such that $q(x) \neq 0$ for any $x \in R$. Then we have

$$y = f(x) = \frac{p(x)}{q(x)}$$

$$q(x)\,y - p(x) = 0$$

which shows that $y = f(x)$ can be obtained by solving the equation

$$q(x)\,y - p(x) = 0.$$

Hence f(x) is an algebraic function.

EXERCISE 4)
Verify that a function f: R → R defined by
$f(x) = \sqrt{x + \sqrt{x}}$
is an algebraic function.

A function which is not algebraic is called a **Transcendental Function.** Examples of elementary transcendental functions are the trigonometric functions, the exponential functions and the logarithmic functions, which we discuss in the next section.

4.3 TRANSCENDENTAL FUNCTIONS

In Unit 1, we gave a brief introduction to the algebraic and transcendental numbers. Recall that a number is said to be an algebraic if it is a root of an equation of the form

$$a_0 x^n + a_1 x^{n-1} + \dots x + a_{n-1} x + a_n = 0$$

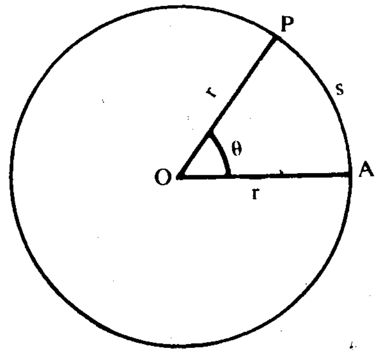
with integral coefficients and $a_0 \neq 0$, where n is a positive integer. A number which is not algebraic is called a transcendental number. For example the numbers e and π are transcendental numbers. In fact, the set of transcendental numbers is uncountable.

Based on the same analogy, we have the transcendental functions. In Section 4.2, we have discussed algebraic functions. The functions that are non-algebraic are called transcendental functions. In this section, we discuss some of these functions.

4.3.1 TRIGONOMETRIC FUNCTIONS

You are quite familiar with the trigonometric functions from the study of Geometry and Trigonometry. The study of Trigonometry is concerned with the measurement of the angles and the ratio of the measures of the sides of a triangle. In Calculus, the trigonometric functions have an importance much greater than simply their use in relating sides and angles of a triangle. Let us review the definitions of the trigonometric functions sin x, cos x and some of their properties. These functions form an important class of real functions.

Consider a circle $x^2 + y^2 = r^2$ with radius r and centre at O. Let P be a point on the circumference of this circle. If θ is the radian measure of a central angle at the centre of the circle as shown in the Figure 1.



then you know that the length of the arc AP = s is given by

$$s = \theta r.$$

You also know that the trigonometric ratios sin θ, cos θ were defined for an angle θ measured in degrees or radions. We now, define, sin θ, cos θ for a real number θ.

If we put r = 1, then we got $\theta = s$. Also the equation of circle becomes $x^2 + y^2 = 1$. This, as you know, is known as the Unit Circle. Let C represent this circle with centre O and radius 1. Suppose the circle meets the X-Axis at a point A as shown in the Figure 2.

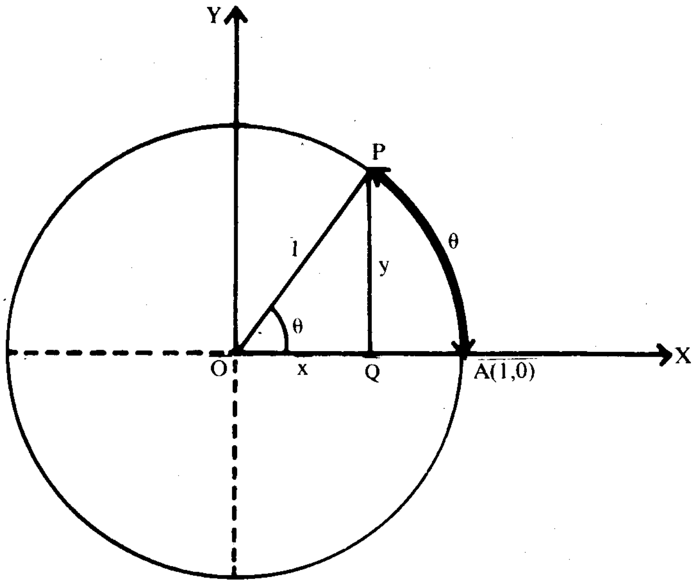


Fig. 2

Take a point **P** on the circumference of this Circle. Then the length AP = s = θ where $0 \leq \theta \leq 2\pi$. Also corresponding to any angle $\theta \in [0, 2\pi]$, there lies a point **P** on the unit circle such that the length of AP is equal to θ.

Now let x be any real number, then corresponding to each x, there exists an integer k such that

$$x = 2k\pi + \theta, 0 \leq \theta \leq 2\pi.$$

Corresponding to θ there exists a unique point **P** on the unit circle. We, define

$\sin x = \mathbf{PQ}$

$\cos x = \mathbf{OQ}.$

Then, it follows that

$\sin(2\pi + x) = \sin x$
$\cos(2\pi + x) = \cos x$

for each $x \in \mathbf{R}$. It also follows that

$$\sin 0 = 0, \sin \frac{\pi}{2} = 1, \sin \pi = 0, \sin \frac{3\pi}{2} = -1$$

and

$$\cos 0 = 1, \cos \frac{\pi}{2} = 0, \cos \pi = -1, \cos \frac{3\pi}{2} = 0.$$

In general,

$$\text{Sin } n\pi = 0, \text{ and } \cos\left((2n+1)\frac{\pi}{2}\right) = 0$$

for every integer n.

You can easily see that as θ increases from o to π 2, P Q increases from 0 to 1 and O Q decreases from 1 to 0. Further as θ

increases from $\frac{\pi}{2}$ to π, P Q decreases from 1 to 0 and O Q decreases from 0 to -1. Again as θ increases from π to $\frac{3\pi}{2}$, PQ decreases from 0 to -1 and O Q increases from -1 to 0. As θ increases from $\frac{3\pi}{2}$ to 2π, O Q increases from 0 to 1 and P Q increases from -1 to 0. Since Sin $(2\pi + x)$ = Sin x and Cos $(2\pi + x)$ = Cos x for each real x, the graphs of these functions take the shapes as shown in figure 3.

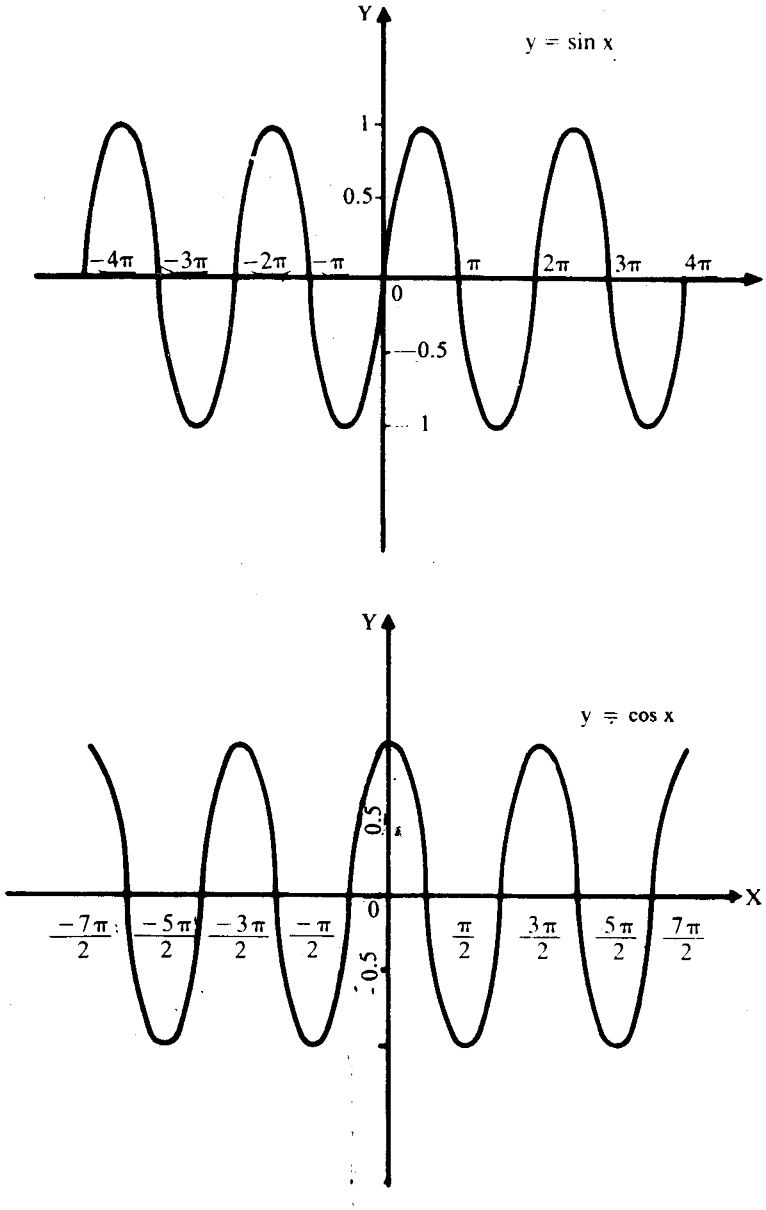


Fig. 3.

Thus, we define sin x and cos x as follows:

DEFINITION 9: SINE FUNCTION

A function f: R ⟶ R defined by

$$f(x) = \sin x, \ \forall x \in R$$

is called the sine of x. We often write y = sin x.

DEFINITION 10: COSINE FUNCTION

A function f: R ⟶ R defined by

$$f(x) = \cos x, \ \forall x \in R$$

is called the cosine of x and we write

$$y = \cos x.$$

Note that the range of each of the sine and cosine, is $[-1, 1]$. In terms of the real functions sine and cosine, the other four trigonometric functions can be defined as follows:

(i) A function $f: S \longrightarrow R$ defined by

$f(x) = \tan x = \dfrac{\sin x}{\cos x}$, $\cos x \neq 0$, $\forall x \in S = R - \{(2n+1)\frac{\pi}{2}\}$
is called the

Tangent Function. The range of the tangent function is $]-\infty, +\infty[= R$ and the
domain is $S = R - \{(2n+1)\frac{\pi}{2}\}$, where n is a non-negative integer.

(ii) A function $f: S \longrightarrow R$ defined by

$f(x) = \cot x = \dfrac{\cos x}{\sin x}$, $\sin x \neq 0$, $\forall x \in S = R - \{n\pi\}$,

is said to be the **Cotangent Function.** Its range is also same as its co-domain i.e. range $=]-\infty, \infty[= R$ and the domain is $S = R - \{n\pi\}$, where n is a non-negative integer.

(iii) A function $f: S \longrightarrow R$ defined by

$f(x) = \sec x = \dfrac{1}{\cos x}$, $\cos x \neq 0$, $\forall x \in S = S - \{2n+1)\frac{\pi}{2}\}$.

is called the **Secant Function.** Its range is the set

$S =]-\infty, -1] \cup [1, \infty[$ and domain is $S = R - \{2n+1)\frac{\pi}{2}\}$,

(iv) A function $f: S \longrightarrow R$ defined by

$f(x) = \text{cosec } x = \dfrac{1}{\sin x}$, $\sin x \neq 0$, $x \in S = R - \{n\pi\}$,

is called the Cosecant function. Its range is also the set $S =]-\infty, -1] \cup [1, \infty[$ and domain is $S = R - \{n\pi\}$.

The graphs of these functions are shown in the Figure 4 on pages 76-77.

EXAMPLE 5: Let $S = [-\frac{\pi}{2}, \frac{\pi}{2}]$. Show that the function $f: S \longrightarrow R$ defined by

$f(x) = \sin x, \forall x \in S$

is one-one. When is f only onto? Under what conditions f is both one-one and onto?

SOLUTION: Recall from Unit 1 that a function f is one-one if
$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

for every x_1, x_2 in the domain of f.

Therefore, here we have for any $x_1, x_2 \in S$,

Recall the trigonometric identities which you have learnt in your previous study of Trigonometry.

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$

$$\Rightarrow \sin x_1 - \sin x_2 = 0$$

$$\Rightarrow 2\sin\frac{x_1 - x_2}{2}\cos\frac{x_1 + x_2}{2} = 0$$

$$\Rightarrow \text{Either } \sin\frac{x_1 - x_2}{2} = 0, \text{ or } \cos\frac{x_1 + x_2}{2} = 0.$$

If $\sin\frac{x_1 - x_2}{2} = 0$, then $\frac{x_1 - x_2}{2} = 0, \pm\pi, +2\pi, \ldots$

If $\cos\frac{x_1 + x_2}{2} = 0$, then $\frac{x_1 + x_2}{2} = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

Since $x_1, x_2 \in [-\frac{\pi}{2}\ \frac{\pi}{2}]$. therefore we can only have

$$\frac{-\pi}{2} \leq \frac{x_1 - x_2}{2} \leq \frac{\pi}{2}$$

and

$$\frac{-\pi}{2} \leq \frac{x_1 + x_2}{2} \leq \frac{\pi}{2}$$

Thus $\frac{x_1 - x_2}{2}$ 0 i.e. $x_1 = x_2$. Also, if $\frac{x_1 + x_2}{2} = \pm \frac{\pi}{2}$

i.e. then $x_1 + x_2 = \pm \pi$.

Since $x_1, x_2 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

therefore $x_1 = x_2 = \frac{\pi}{2}$ or $x_1 = x_2 = -\frac{\pi}{2}$

Hence $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$, which proves that f is one-one. Then function $f(x) = \sin x$ defined as such, is not onto because you know that the range of sin x is $[-1, 1] \neq R$.

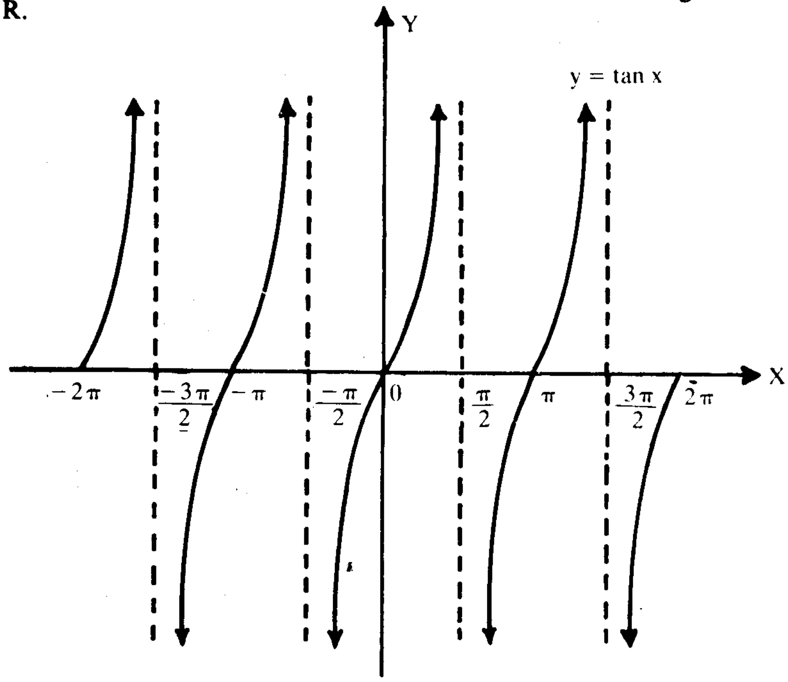


Fig. 4 (i)

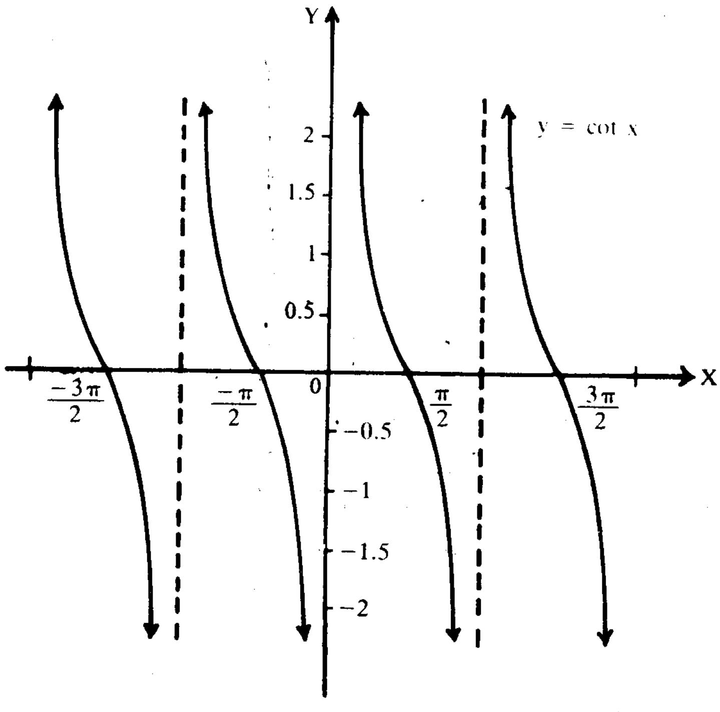


Fig. 4 (ii)

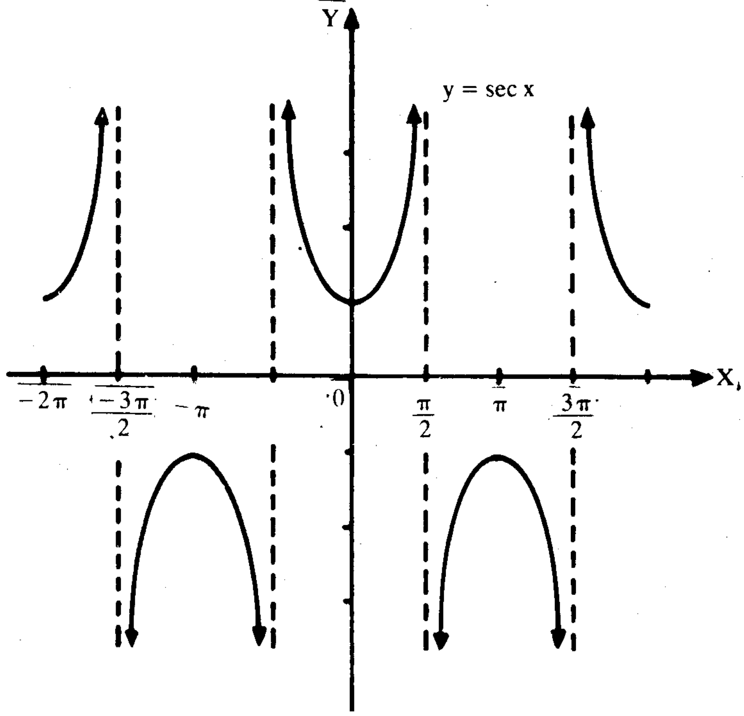


Fig. 4 (iii)

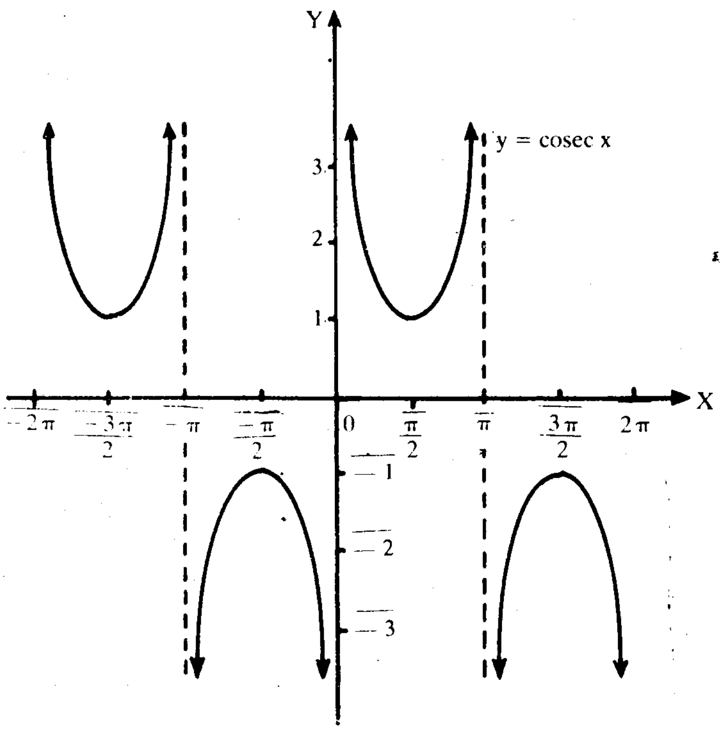


Fig. 4 (iv)

If you define f: R ⟶ [−1, 1] as

$$f(x) = \sin x, \ \forall\, x \in R.$$

Then f is certainly onto. But then it is not one-one. However the function.

f: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$ defined by

$$f(x) = \sin x, \ \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

is both one-one and onto.

EXERCISE 5)

Two functions g and h are defined as follows :

(i) g: S—> R defined by
$g(x) = \cos x, x \in S = [0, \pi]$

(ii) h: S—> defined by
$h(x) = \tan x, x \in S =]-\frac{\pi}{2}, \frac{\pi}{2}[$

Show that the functions are one-one. Under what conditions the functions are one-one and onto?

4.3.2 INVERSE TRIGONOMETRIC FUNCTIONS

In Section 1.2 we discussed inverse functions. You know that if a function is one-one and onto, then it will have an inverse. If a function is not one-one and onto, then sometimes it is possible to restrict its domain in some suitable manner such that the restricted function is one-one and onto. Let us use these ideas to define the inverse trigonometric functions. We begin with the inverse of the sine function.

Refer to the graph of f(x)= sin x in figure 3. The X-Axis cuts the curve y = sin x at the points = 0 $x = \pi$, $x = 2\pi$, ... This shows that function f(x) = x is not one—one. However, we have already shown in example 5 that if we restrict the domain of

$f(x) = \sin x$ to the interval $[-\pi/2, \pi/2]$, then the function

$f: [-\frac{\pi}{2} \frac{\pi}{2}] \to [-1, 1]$ defined by

$f(x) = \sin x, -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

is one-to-one as well as onto. Hence it will have the inverse. The inverse function is called the inverse sine of x and is denoted as $\sin^{-1} x$. In other words,

$y = \sin^{-1} x$ means $x = \sin y$

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $-1 \leq x \leq 1$

Thus, we have the following definition:

Be careful about the notation used. The superscript − 1 that appears in $y = \sin^{-1}$ is not an exponent, but is the symbol f^{-1} used to denote the inverse of a function f. To avoid this, notation y = arcsin, x instead of $y = \sin^{-1} x$ is used sometimes.

DEFINITION 11: INVERSE SINE FUNCTION

A function $g : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by

$g(x) = \sin^{-1} x, \forall x \in [-1, 1]$

is called the inverse sine function.

Again refer back to the graph of f(x) = cos x in figure 3. You can easily see that cosine function is also not one-one. However, if you restrict the domain of f(x) = cos x to the interval $[0, \pi]$, then the function f: $[0, \pi] \longrightarrow [-1, 1]$ defined by

$f(x) = \cos x.\ 0 \leq x \leq \pi$

is one-one and onto. Hence it will have the inverse. The inverse function is called the **inverse cosine** of x and is denoted by $\cos^{-1}x$ (or by arc cosx). In other words,

$y = \cos^{-1}x$ means $x = \cos y$
where $0 \leq y \leq \pi$ and $-1 \leq x \leq 1$

Thus, we have the following definition:

DEFINITION 12: A function g: $[0, \pi] \longrightarrow [-1, 1]$ defined by
$g(x) = \cos^{-1} x, \forall x \in [0. \pi]$

is called the inverse cosine function.

You can easily see from Figure 4 that the tangent function, in general, is not one-one. However, again if we restrict the domain of f (x) = tan x to the interval $]-\pi/2, \pi/2[$, then the function

f: $(-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow R$ defined by

$f(x) = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

is one-one and onto. Hence it has an inverse. The inverse function is called the **inverse tanget** of x and is denoted by $\tan^{-1} x$ (or by arctan x). In other words,

$y = \tan^{-1} x$ means $x = \tan y$

where $-\frac{\pi}{2} < y < \frac{\pi}{2}$ and $-\infty < x < +\infty$.

Thus, we have the following definition:

DEFINITION 13: Inverse Tangent Function

A function g: $(-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow R$ defined by

$$g(x) = \tan^{-1} x, \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

is called the inverse tangent function.

Now you can try the following exercise to define the remaining three inverse trigonometric functions:

EXERCISE 6)
Define the inverse contangent, inverse secant and inverse cosecant function. State their domains and ranges.

Now, before we proceed to define the logarithmic and exponential functions, we need the concept of the monotonic functions. We discuss these functions as follows:

4.3.3 MONOTONIC FUNCTIONS

Consider the following functions:

(i) $f(x) = x, \forall x \in R$.

(ii) $f(x) = \sin x, \forall x \in [-\pi/2, \pi/2]$.

(iii) $f(x) = -x^2 \forall x \in [0, \infty[$.

(iv) $f(x) = \cos x \forall x \in [0, \pi]$.

Out of these functions, (i) and (ii) are such that for any x_1, x_2 in their domains,

$$x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$$

where as (iii) and (iv) are such that for any x_1, x_2 in their domains,

$$x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2).$$

The functions given in (i) and (ii) are called monotonically increasing while those of (iii) and (iv) are called monotonically decreasing. We define these functions as follows:

Let f: $S \rightarrow R$ ($S \subset R$) be a function.

(i) **It is said to be a monotonically increasing function on S if**
$x_1 < x_2 \Longrightarrow f(x_1) \leq f(x_2)$ for any $x_1, x_2 \in S$

(ii) **It is said to be a monotonically decreasing function on S if**
$x_1 < x_2 \Longrightarrow f(x_1) \geq f(x_2)$ for any $x_1, x_2 \in S$.

(iii) **The function f is said to be a monotonic function on S if it is either monotonically increasing or monotonically decreasing.**

(iv) the function f is said to be **strictly increasing** on S if
$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$, for $x_1, x_2 \in S$.

(v) **It is said to be strictly decreasing on** S if
$x_1 < x_2 \Longrightarrow f(x_1) > f(x_2)$, for $x_1, x_2 \in S$.

You can notice immediately that if f is monotonically increasing then $-f$ i.e. $-f: \mathbf{R} \to \mathbf{R}$ defined by $(-f)(x) = -f(x), \forall x \in \mathbf{R}$ is monotonically decreasing and vice-versa.

EXAMPLE 6: Test the monotonic character of the function f: R → R defined as

$$f(x) = \begin{cases} x^2, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$$

SOLUTION: For any $x_1, x_2 \in \mathbf{R}, x_1 \leq 0, x_2 \leq 0$

$$x_1 < x_2 \Longrightarrow x_1^2 > x_2^2 \Longrightarrow f(x_1) > f(x_2)$$

which shows that f is strictly decreasing.

Again if $x_1 > 0, x_2 > 0$, then
$x_1 < x_2 \Longrightarrow x_1^2 < x_2^2 \Longrightarrow -x_1^2 > -x_2^2 \Longrightarrow f(x_1) > f(x_2)$
which shows that f is strictly decreasing for $x > 0$. Thus f is strictly decreasing for every $x \in R$.

Now, we discuss an interesting property of a strictly increasing function in the form of the following theorem:

THEOREM 1: Prove that a strictly increasing function is always one-one.

PROOF: Let f: S → T be a strictly increasing function. Since f is strictly increasing, therefore,
$x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$ for any $x_1, x_2 \in S$.

Now to show that f: S → T is one-one, it is enough to show that

$$f(x_1) = f(x_2) \Longrightarrow x_1 = x_2.$$

Equivalently, it is enough to show that distinct elements in S have distinct images in T i.e. $x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$, for $x_1, x_2 \in S$.

Indeed,

$$\begin{aligned} x_1 \neq x_2 &\Longrightarrow x_1 < x_2 \text{ or } x_1 > x_2 \\ &\Longrightarrow f(x_1) < f(x_2) \text{ or } f(x_1) > f(x_2) \\ &\Longrightarrow f(x_1) \neq f(x_2) \end{aligned}$$

which proves the theorem.

EXAMPLE 7: Let f: S → T be a strictly increasing function such that f(S) = T. Then prove that f is invertible and f^{-1}: T → S is also strictly increasing.

SOLUTION: Since f: S → T is strictly increasing, therefore, f is one-one. Further, since f (S) = T, **therefore f is onto. Thus f is one-one and onto. Hence f is invertible. In other words, f^{-1}: T → S exists.**

Now, for any $y_1, y_2 \in T$, we have $y_1 = f(x_1)$, $y_2 = f(x_2)$. If $y_1 < y_2$, then we claim $x_1 < x_2$.
Indeed if $x_1 \geq x_2$, then $f(x_1) \geq f(x_2)$ (why?).
This implies that $y_1 \geq y_2$ which contradicts that $y_1 < y_2$.
Hence $y_1 < y_2 \Longrightarrow x_1 < x_2 \Longrightarrow f^{-1}(y_1) < f^{-1}(y_2)$
which shows that f^{-1} is strictly increasing.

You can similarly solve the following exercise for a strictly decreasing function:

EXERCISE 7)
Let f: S → T be a strictly decreasing function such that f(S) = T. Show that f is invertible and f^{-1}: T → S is also strictly decreasing.

4.3.4 LOGARITHMIC FUNCTION

You know that a definite integral of a function represents the area enclosed between the curve of the function, X-Axis and the two Ordinates. You will now see that this can be used to define logarithmic function and then the exponential function.

We consider the function $f(x) = \frac{1}{x}$ for $x > 0$. We find that the graph of the function is as shown in the Figure 5.

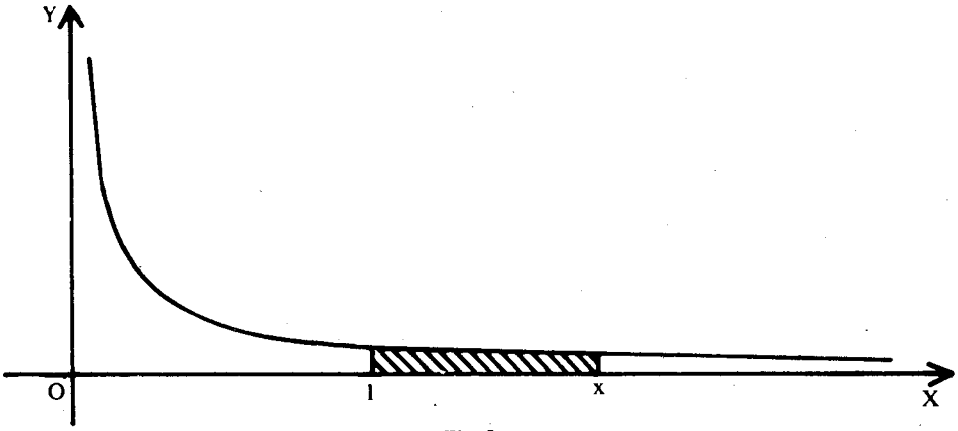


Fig. 5

DEFINITION 14: LOGARITHMIC FUNCTION

For $x \geq 1$, we define the natural logarithmic function log x as

$$\log x = \int_1^x \frac{1}{t}\, dt$$

In the Figure 5, log x represents the area between the curve $f(t) = \frac{1}{t}$, X – Axis and the two ordinates at 1 and at x. For $0 < x < 1$, we define

$$\log x = -\int_x^1 \frac{1}{t}\, dt$$

This means that for $0 < x < 1$, log x is the negative of the area under the graph of $f(t) = \frac{1}{t}$, X - Axis and the two ordinates at x and at 1

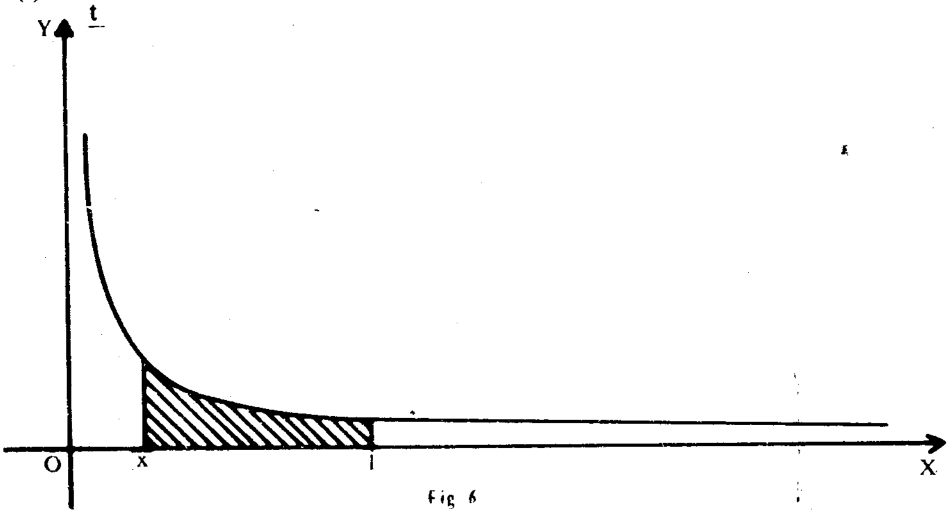


Fig 6

We also see by this definition that

$\log x < 0$ if $0 < x < 1$

$\log 1 = 0$

and

$\log x > 0$ if $x > 1$.

It also follows by definition that if $x_1 > x_2 > 0$, then $\log x_1 > \log x_2$. This shows that log x is strictly increasing. The reason for this is quite clear if we realise by $\log x_1$ as the area under the graph as shown in the Figure 7.

The logarithmic function defined here is called the Natural logarithmic function. For any $x > 0$, and for any positive real number $a \neq 1$, we can define

$$\log_a x = \frac{\log x}{\log a}$$

This function is called the logarithmic function with respect to the base a.

If $a = 10$, then this function is called **the common logarithmic function.**

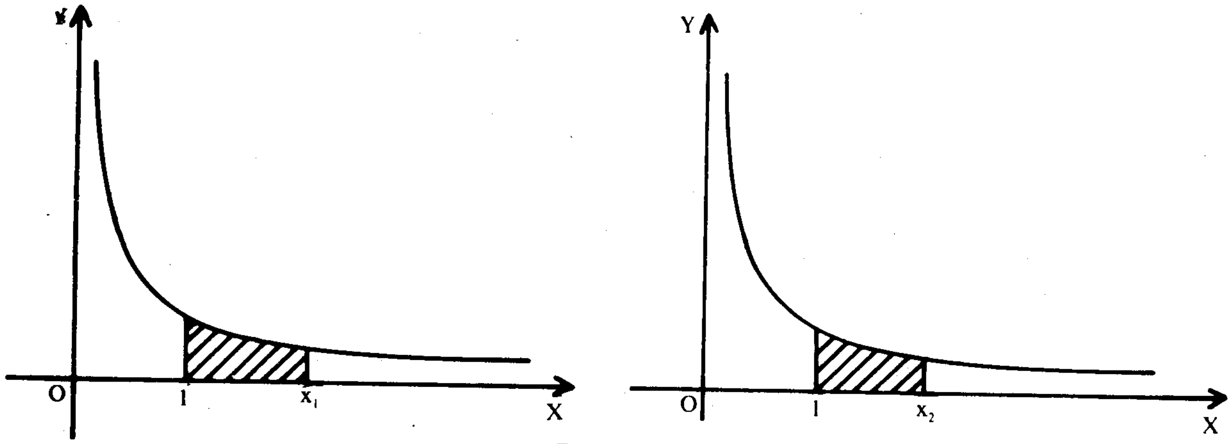


Fig. 7

Logarithmic function to the base a has the following properties

(i) $\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2$

(ii) $\log_a \left[\frac{x_1}{x_2}\right] = \log_a x_1 - \log_a x_2$

(iii) $\log_a x^m = m \log_a x$ for every integer m.

(iv) $\log_a^a = 1$.

(v) $\log_a^1 = 0$

By the definition of log x, we see that log 1 = 0 and as x becomes larger and larger, the area covered by the curve $f(t) = \frac{1}{t}$, X — axis and the ordinates at 1 and x, becomes larger and larger. Its graph is as shown in the Figure 8.

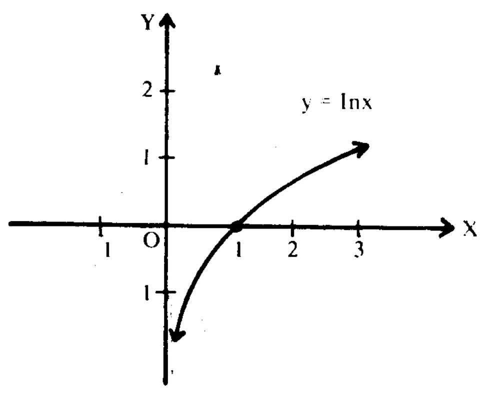


Fig. 8

You already know what is meant by inverse of a function. You had also seen in Unit 1 that if f is 1 — 1 and onto, then f is invertible. Let us apply that study to logarithmic function.

4.3.5 EXPONENTIAL FUNCTION

We now come to define exponential function. We have seen that

$\log x:]0, \infty[\rightarrow R$

is strictly increasing function. The graph of the logarithmic function also shows that

$\log x:]0, \infty[\rightarrow R$

is also onto. Therefore this function admits of inverse function. Its inverse function, **called the Exponential function, Exp (x) has domain as the set R of all real numbers and range as]0, ∞[.** If

$\log x = y$, then $\text{Exp}(y) = x$.

The graph of this function is the mirror image of logarithmic function as shown in the Figure 9.

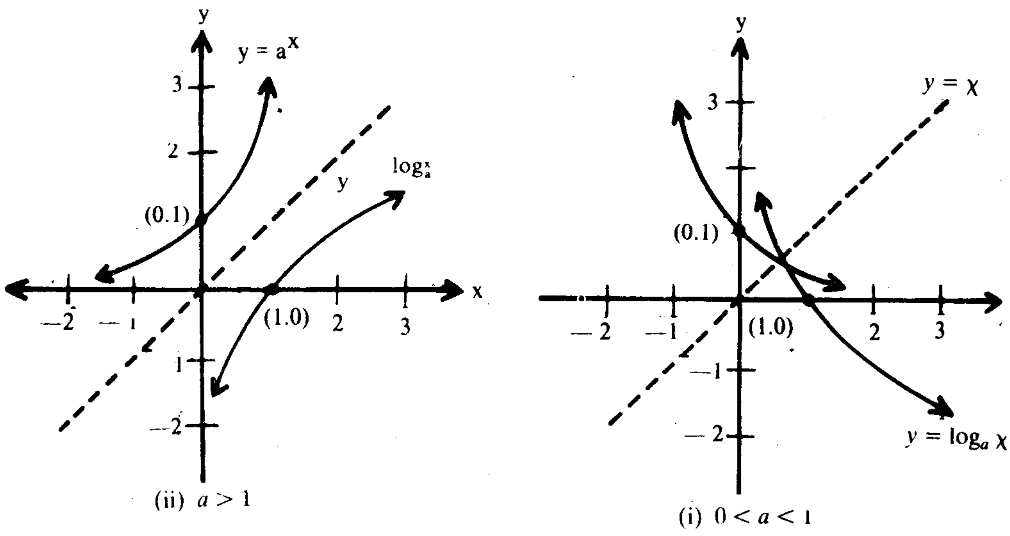


Fig. 9

The Exp (x) satisfies the following properties

(i) $\text{Exp}(x + y) = \text{Exp } x \text{ Exp } y$

(ii) $\text{Exp}(x - y) = \text{Exp } x/\text{Exp } y$

(iii) $(\text{Exp } x)^n = \text{Exp}(nx)$

(iv) $\text{Exp}(0) = 1$

We now come to define a^x for $a > 0$ and x any real number. We write

$$a^x = \text{Exp}(x \log a)$$

If x is any rational number, then we know that $\log a^x = x \log. a$. Hence $\text{Exp}(x \log a) = \text{Exp}(\log a^x) = a^x$. Thus our definition agrees with the already known definition of a in case x is a rational number. The function a^x satisfies the following properties

(i) $a^x a^y = a^{x+y}$

(ii) $\dfrac{a^x}{a^y} = a^{x-y}$

(iii) $(a^x)^y = a^{xy}$

(iv) $a^x b^x = (ab)^x$, $a > 0$, $b > 0$.

Denote $E(1) = e$, so that $\log e = 1$. The number e is an irrational number and its approximation say upto five places of decimals is 2.71828. Thus

$$e^x = \text{Exp}(x \log e) = \text{Exp}(x).$$

Thus Exp (x) is also denoted as e^x and we write for each $a > 0$, $a^x = e^x \log a$

EXAMPLE 8: Plot the graph of the function f: R → R defined by $f(x) = 2^x$.

SOLUTION:

x	−2	−1	0	1	2
2^x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4

The required graph takes the shape as shown in the Figure 10.

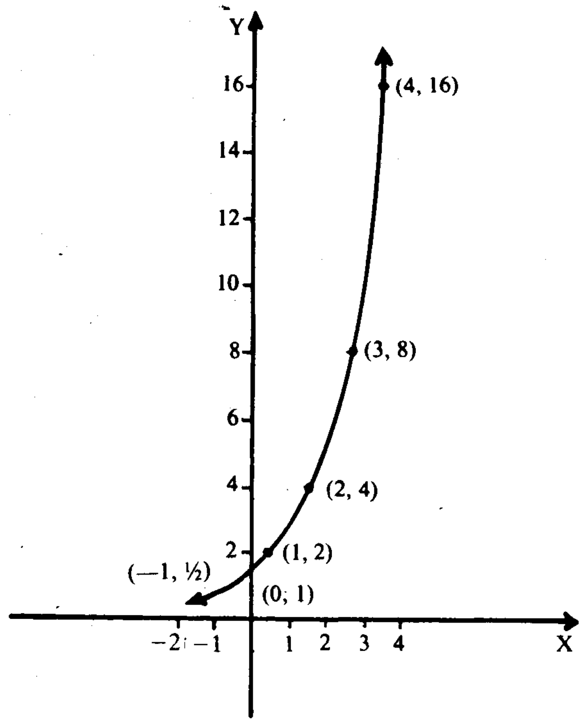


Fig. 10

EXERCISE 8)
Show the graph of f: R → R defined by $F(x) = (\frac{1}{2})^x$

4.4 SOME SPECIAL FUNCTIONS

So far, we have discussed two main classes of real functions—Algebraic and Transcendental. Some functions have been designated as special functions because of their special nature and behaviour. Some of these special functions are of great interest to us. We shall frequently use these functions in our discussion in the subsequent units and blocks.

1. Identity Function

We have already discussed some of the special functions in Section 4.2. For example, the **Identity function i : R → R,** defined as i (x) = x, ∀ x ∈**R** has already been discussed as an algebraic function. However, this function is sometimes, referred to as a special function because of its special characteristics, which are as follows:

(i) domain of i = Range of i = Codomain of i

(ii) The function i is one-one and onto. Hence it has an inverse i^{-1} which is also one-one and onto.

(iii) The function i is invertible and its inverse i^{-1} =

(iv) The graph of the identity function is a straight line through the origin which forms an angle of 45° along the positive direction of X-Axis as shown in the Figure 11.

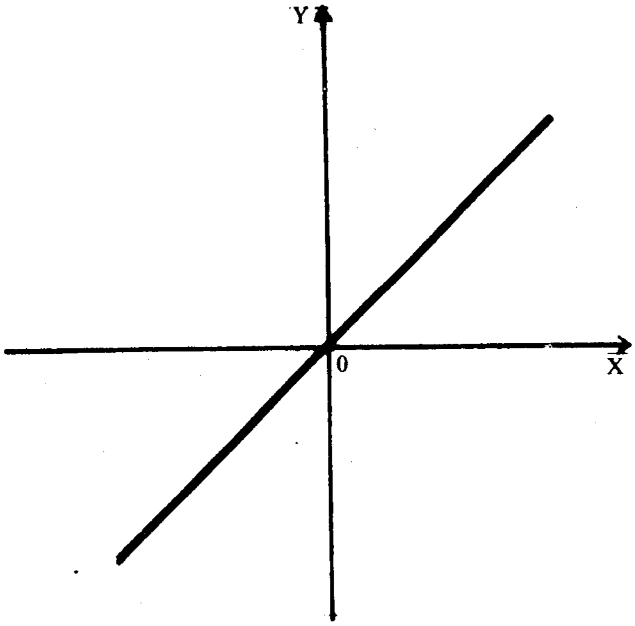


Fig. 11

2. Periodic Function

You know that
$\sin(2\pi + x) = \sin(4\pi + x) = \sin x$,
$\tan(\pi + x) = \tan(2\pi + x) = \tan x$.

This leads us to define a special class of functions, known as **Periodic** functions. All trigonometric functions belong to this class.

A function f: S → R is said to be periodic if there exists a positive real number k such that

$$f(x + k) = f(x),\ \forall\ x \in S$$

where S ⊂ R.

The smallest such positive number k is called the period of the function.

You can verify that the functions sine, cosine, secant and cosecant are periodic each with a period 2π while tangent and cotangent are periodic functions each with a period π.

EXERCISE 9)
Find the period of the function f where $f(x) = |\sin^3 x|$

---

3. Modulus Function

The modulus or the absolute (numerical) value of a real number has already been defined in Unit 1. Here we define the **modulus (absolute value) function** as follows:

Let S be a subset of R. A function f: S → R defined by

$$f(x) = |x|,\ \forall\ x \in S$$

is called the modulus function.

In short, it is written as Mod function.

You can easily see the following properties of this function:

(i) The domain of the Modulus function may be a subset of **R** or the set **R** itself.

(ii) The range of this function is a subset of the set of non-negative real numbers.

(iii) The Modulus function **f: R → R** is not an onto function. (Check why?).

(iv) The Modulus function **f: R → R** is not one-one. For instance, both 2 and − 2 in the domain have the same image 2 in the range.

(v) **The modulus function f: R → R** does not have an inverse function (why)?

(vi) **The graph** of the Modulus function is **R → R** given in the Figure 12.

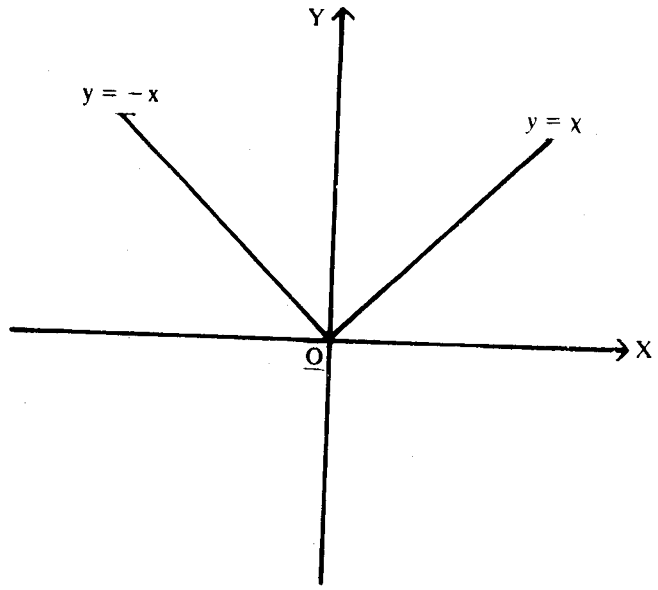


Fig. 12

It consists of two straight lines:

(i) $y = x \ (y \geq 0)$

and (ii) $y = -x \ (y \geq 0)$

through 0, the origin, making an angle of $\pi/4$ and $3\pi/4$ with the positive direction of X-axis.

4. Signum Function

A function f: R → R defined by

$$f(x) = \begin{cases} \dfrac{|x|}{x} & \text{when } x \neq 0 \\ x & \text{when } x = 0; \end{cases}$$

or equivalently by:

$$f(x) = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0. \end{cases}$$

is called the signum function. It is generally written as sgn (x).

Its range set is $\{-1, 0, 1\}$. Obviously sgn x is neither one-one nor onto. The graph of sgn x is shown in the Figure 13.

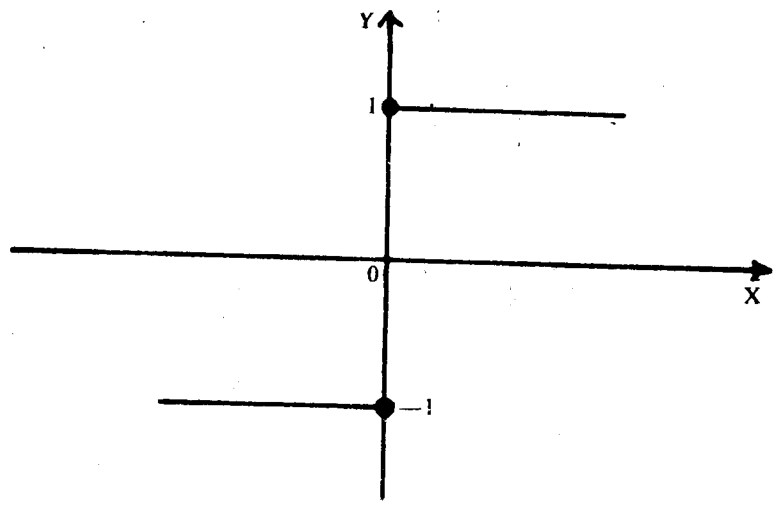


Fig. 13

5. Greatest Integer Function

Consider the number 4.01. Can you find the **greatest integer** which is less than or equal to this number? Obviously, the required integer is 4 and we write it as $[4.01] = 4$.

Similarly, if the symbol $[x]$ denotes the greatest integer contained in x then we have

$$[3/4] = 0, [5.01] = 5,$$

$$[-.005] = -1 \text{ and } [-3.96] = -4.$$

Based on these, the Greatest integer function is defined as follows:

A function f: R → R defined by

$f(x) = [x], \forall x \in R$

where $[x]$ is the largest integer less than or equal to x is called the greatest integer function:

The following properties of this function are quite obvious:

(i) The domain is R and the range is the set Z of all integers.

(ii) The function is neither one-one nor onto

(iii) If n is any integer and x is any real number such that x is greater than or equal to n but less than n + 1 i.e. if $n \leq x < n + 1$ (for some integer n) then $[x] = n$ i.e.

The graph of the greatest integer function is shown in the Figure 14.

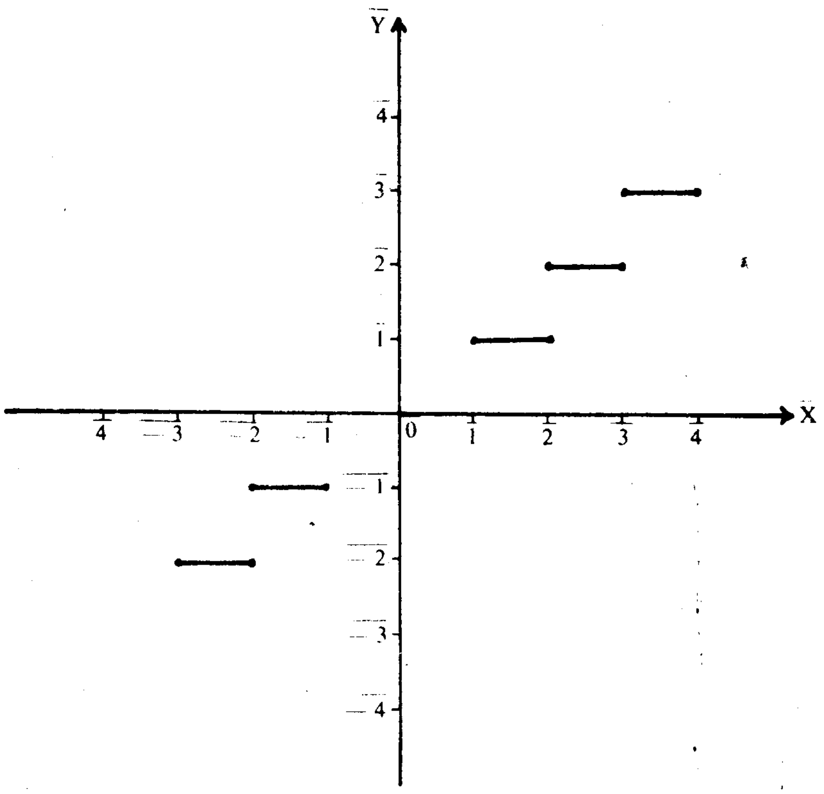


Fig. 14

EXAMPLE 6: Prove that

$$[x + m] = [x] + m, \forall x \in R, m \in Z.$$

SOLUTION: You know that for every $x \in$ **R**, there exists an integer n such that

$$n \leq x < n + 1.$$

Therefore

$$n + m \leq x + m < n + 1 + m$$

and hence

$$[x + m] = n + m = [x] + m$$

which proves the result.

EXERCISE 10)

Test whether or not the function f: R → R defined by $f(x) = x-[x]\ \forall\ x \in R$, is periodic. If it is so, find its period.

---

6. Even and Odd Functions

Consider a function f: **R** → **R** defined as

$f(x) = 2x, \forall x \in \mathbf{R}.$

If you change x to −x, then you have

$f(-x) = 2(-x) = -2x - f(x).$

Such a function is called an **odd function.**

Now, consider a function f:**R** → **R** defined as

$f(x) = x^2\ \forall x \in \mathbf{R}$

Then changing x to −x we get

$f(-x) = (-x)^2 = x^2 = f(x)$

Such a function is called an **even function.**

The definitions of even and odd functions are as follows:

A function f: R → R is called even if $f(-x) = f(x), \forall x \in R$.

It is called odd if $f(-x) = -f(x), \forall x \in R$

EXAMPLE 7: Verify whether the function f: R → R defined by

(i) $f(x) = \sin^2 x + \cos^3 2x$

$f(x) = \sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}$

are even or odd.

SOLUTION: (i) $f(x) = \sin^2 x + \cos^3 2x.\ \forall x \in \mathbf{R}$

$\Longrightarrow (-x) = \sin^2(-x) + \cos^3 2(-x)$

$= \sin^2 x + \cos^3 2x = f(x), \forall x \in R$

$\Longrightarrow$ f is an even function.

(ii) $f(x) = \sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}, \quad \forall x \in \mathbf{R}$

$\Longrightarrow f(-x) = \sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}$

$= -f(x), \forall x \in \mathbf{R}$

$\Longrightarrow$ f is an odd function.

EXERCISE 11)

Determine which of the following functions are even or odd or neither:

(i) $f(x) = x$ (ii) a constant function

(iii) sin x, cos x, tan x,

(iv) $f(x) = \dfrac{x-4}{x^2-9}, \forall x \in \mathbf{R}, x \subset \{-3, 3\}$

---

7. Bounded Functions

In Unit 2, you were introduced to the notion of a bounded set, upper and lower bounds of a set. Let us now extend these notions to a function.

You know that if f: S → **R** is a function, (S ⊂ **R**), then

$\{f(x) : x \in S\},$

is called the range set or simply the range of the function f.

A function is said to be bounded if its range is bounded.

Let f: S → **R** be a function. It is said to be **bounded above** if there exists a real number K such that

$f(x) \le K.\ \forall x \in S$

The number K is called an **upper bound of f.** The function f is said to be bounded below if there exists a number k such that

$f(x) \ge k\ \forall x \in S$

The number k is called a lower bound of f.

A function f: S → R, which is bounded above as well as bounded below, is said to be bounded. This implies that there exist two real numbers k and K such that $k \leq f(x) \leq K \; \forall \; x \in S$.

This is equivalent to say that a function f: S → R is bounded if there exists a real number M such that

$$|f(x)| \leq M, \; \forall \; x \in S.$$

A function may be bounded above only or may be bounded below only or neither bounded above nor bounded below.

Recall that sin x and cos x are both bounded functions. Can you say why? It is because of the reason that the range of each of these functions is [−1, 1].

EXAMPLE 8: A function f: R → R defined by

(i) $f(x) = -x^2, \forall \; x \in R$ is bounded above with 0 as an upper bound

(ii) $f(x) = x, \forall \; x \geq 0$ is bounded below with 0 as a lower bound

(iii) $f(x) = \sqrt{1-x^2}$ for $|x| \leq 1$ is bounded because $|f(x)| \leq 1$ for $|x| \leq 1$.

Try the following exercise.

EXERCISE 12)
Test which of the following functions with domain and co-domain as R are bounded and unbounded:

(i) $f(x) = \tan x$

(ii) $f(x) = [x]$

(iii) $f(x) = e^x$

(iv) $f(x) = \log x$

EXERCISE 13)
Suppose f: S → R and g: S → R are any bounded functions on S. Prove that f + g and f. g are also bounded functions on S.

4.5 SUMMARY

In this unit, we have discussed various types of real functions. We shall frequently use these functions in the concepts and examples to be discussed in the subsequent units throughout the course.

In Section 4.2, we have introduced the notion of an algebraic function and its various types. A function f: S → R (S ⊂ R) defined as $y = f(x), \forall \; x \in S$ is said to be algebraic if it satisfies identically an equation of the form
$p_0(x) y^n + p_1(x) y^{n-1} + p_2(x) y^{n-2} + ... + p(x) y + p_n(x) = 0$,
where $p_0(x), p_1(x), ..., p_n(x)$ are polynomials in x for all $x \in S$ and n is a positive integer. In fact, any function constructed by a finite number of algebraic operations—addition, subtraction, multiplication, division and root extraction—is an algebraic function. Some of the examples of algebraic functions are the polynomial functions, rational functions and irrational functions.

But not all functions are algebraic. The functions which are not algebraic, are called transcendental functions. These have been discussed in Section 4.3. Some important examples of the transcendental functions are trigonometric functions, logarithmic functions and exponential functions which have been defined in this section. We have defined the monotonic functions also in this section.

In Section 4.4, we have discussed some special functions. These are the identity function, the periodic functions, the modulus function, the signum function, the greatest integer function, even and odd functions. Lastly, we have introduced the bounded functions and discussed a few examples.

4.6 ANSWERS/HINTS/SOLUTIONS

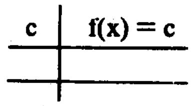
E1) $(f+g)(x) = ax^2 + bx$

$(f-g)(x) = ax^2 - bx$

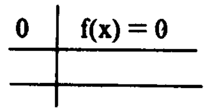
$(g{:}g)(x) = ax^2 . bx = abx^3$

$(f/g)(x) = \dfrac{ax^2}{bx} = \dfrac{ax}{b}$ provided $b \neq 0$, $x \neq 0$.

$kf = k\,ax^2$

E2) 

constant function



Zero function (X—Axis)

E3) $y = f(x) = \sqrt{x} \Rightarrow y^2 = x$. Now draw the graph.

E4) $y = f(x) \Rightarrow y = \sqrt{x + \sqrt{x}}$

$\Rightarrow y^2 = x + \sqrt{x}$

$\Rightarrow y^2 - x = \sqrt{x}$

$\Rightarrow (y^2 - x)^2 = x$

$y^2 - 2y^2 x + x^2 - x = 0$

which shows that $y = f(x)$ is an algebraic function.

E5) (i) Let $x_1, x_2 \in]0, \pi[$. Then

$f(x_1) = f(x_2) \Rightarrow \cos x_1 = \cos x_2$

$\Rightarrow \cos x_1 - \cos x_2 = 0$

$(2 \sin \dfrac{x_1 + x_2}{2}) \sin \dfrac{(x_2 - x_1)}{2} = 0$

$\Rightarrow$ either $\sin\left(\dfrac{x_1 + x_2}{2}\right) = 0$ or $\sin \dfrac{x_2 - x_1}{2} = 0$

$\sin \dfrac{x_2 - x_1}{2} = 0 \Rightarrow \dfrac{x_2 - x_1}{2} = 0, \pm\pi, \pm 2\pi \ldots$

$\cos \dfrac{x_1 + x_2}{2} = 0 \Rightarrow \dfrac{x_1 + x_2}{2} = \pm \dfrac{\pi}{2}, \pm \dfrac{3\pi}{2}, \ldots$

$\dfrac{x_2 - x_1}{2} = 0 \Rightarrow x_1 = x_2$

$\dfrac{x_2 - x_1}{2} = \pm\pi \Rightarrow x_2 = \pm 2\pi + x_1$

which is not possible.

Hence $\dfrac{x_2 - x_1}{2} = \pm\pi$ is not possible.

Thus, the only possibility is

$\dfrac{x_2 - x_1}{2} = 0$ which means $x_2 = x_1$

Again the only possibility is that

$\dfrac{x_1 + x_2}{2} = \pm \dfrac{\pi}{2} \Rightarrow x_1 + x_2 = \pm\pi$

$\Rightarrow 2x_1 = \pm\pi \Rightarrow x_1 = \pm \dfrac{\pi}{2}$

$\Rightarrow x_2 = \pm \dfrac{\pi}{2} \Rightarrow x_1 = x_2$

Hence $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

In other words $f(x) = \cos x$ is one-one in $[0, \pi]$.

Now, the range of cos x is $[-1, 1] \neq \mathbf{R}$. Therefore cos x, defined from $[0, \pi]$ to $\mathbf{R}$ is not onto. But, if cos x is defined from $[0, \pi]$ to $[-1, 1]$, then it is certainly one-one and onto.

(ii) Do it yourself.

E6) (i) **Cotangent Inverse.**
$y = \cot^{-1}$ means $x = \cot y$
where $0 < y < \pi$ and $-\infty$ and $-\infty < x < +\infty$.

(ii) Secant Inverse
$y = \sec^{-1} x$ means $x = \sec y$

where $0 \leq y \leq \pi$, $y \neq \frac{\pi}{2}$ and $|x| \geq |$.

(iii) Cosecant inverse
$y = \text{cosec}^{-1} x$ means $x = \text{cosec } y$

where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$ and $|x| \geq |$.

E7) (i) Let f: S ——> T be a strictly decreasing function.
Let x_1, x_2 be any two distinct elements of S. Then

$x_1 \neq x_2 \implies x_1 < x_2, x_1 > x_2$
$\implies f(x_1) > f(x_2), f(x_1) < f(x_2)$
$\implies f(x_1) \neq f(x_2)$

which shows that f is one-one.
Since f(S) = T, therefore f is onto. Thus f is one-one and onto and hence f is invertible. In other words f^{-1} exists i.e. f^{-1}: T ——> S is defined.

For $y_1, y_2 \in T$, we have $y_1 = f(x_1)$, $y_2 = f(x_2)$ for some x_1, x_2 in S. If $y_1 < y_2$, we claim that $x_1 > x_2$. If not, then $x_1 \leq x_2$ which implies that $f(x_1) > f(x_2)$ i.e., $y_1 > y_2$. This is a contradiction. Hence $y_1 < y_2 \implies x_1 > x_2 \implies f^{-1}.(y_1) > f^{-1}(y_2)$
Thus, f^{-1} is also strictly increasing.

E8) Follow the method of example. 8.

E9) Since $f(\pi + x) = |\sin^3(\pi + x)| = |-\sin^3 x| = \sin^3 x$,
therefore π is the period of f. You may note that π is the least such number satisfying the above relation.

E10) The function $f(x) = x - [x]$ is periodic with period 1 because 1 is the only least number such that
$f(x + 1) = (x + 1) - [x+1] = (x + 1) - [x] - 1 = x - [x]$
$= f(x)$.

E11) (i) odd (ii) even

(iii) sin x is odd, cos x is even, tan x is odd

(iv) $f(-x) = \dfrac{(-x)-4}{(-x)^2-9} = \dfrac{-x-4}{x^2-9}$ which shows that f is neither even nor odd

E12) (i) It is unbounded because its range is $]-\infty, +\infty[$

(ii) $|x|$ is bounded below with 0 as a lower bound.

(iii) ex is bounded below because its range is $]0, \infty[$

(iv) log x is unbounded.

E13) Since f and g are given to be bounded functions, therefore there exist numbers k_1, K_1 and k_2, K_2 such that
$k_1 \leq f(x) \leq K_1 \; \forall \; x \in S$
$k_2 \leq g(x) \leq K_2 \; \forall \; x \in S$.

(i) **Since $(f+g)(x) = f(x) + g(x), \forall \; x \in S$**
therefore $k_1 + k_2 \leq f(x) + g(x) \leq K_1 + K_2, \forall \; x \in S$

$\Rightarrow k \leq (f+g)(x) \leq K \; \forall \; x \in S$
where $k = k_1 + k_2$, $K = K_1 + K_2$ are some real numbers
Thus $f(x) + g(x)$ is a bounded function.

(ii) We know $(f.g)(x) = f(x).\, g(x) \; \forall \; x \in S$. Since f and g are bounded, therefore, we can find m_1, m_2 such that
$|f(x)| \leq m_1 |$, $|g(x)| \leq m_2$, $\forall \; x \in S$.
Then
$$|(f.g)(x)| = |f(x).\, g(x)|$$
$$= |f(x)|\,|g(x)|$$
$$\leq m_1.\, m_2 \; \forall \; x \in S$$
which shows that f.g is bounded.

REVIEW

Attempt the following self-assessment questions and verify your answers given at the end:

1. Test whether the following are rational numbers:
   (i) $\sqrt{17}$ (ii) $\sqrt{8}$ (iii) $3 + \sqrt{2}$
2. The inequality $x^2 - 5x + 6 < 0$ holds for
   (i) $x < 2, x < 3$ (ii) $x > 2, x < 3$
   (ii) $x < 2, x > 3$ (iii) $x > 2, x > 3$
3. If a, b, c, d are real numbers such that
   $a^2 + b^2 = 1, c^2 + d^2 = 1,$
   then show that $ac+bd \leq 1$.
4. Prove that $|a + b + c| \leq |a| + |b| + |c|$
   for all $a, b, c, \in R$.
5. Show that
   $|a_1 + a_2 + ... + a_n| \leq |a_1| + |a_2| + ... + |a_n|$
   for $a_1, a_2, ... a_n \in R$.
6. Which of the following sets are bounded above? Write the supremum of the set if it exists.
   (i) $\{\pi, e\}$ (ii) $\bigcup_{n=1}^{\infty} [2n, 2n + 1]$

   (ii) $\left\{n + \frac{(-1)^n}{n} : n \in N\right\}$ (iv) $\{x \in Q : x^2 < 2\}$

   (v) $\{x \in R : x < 0\}$ (vi) $\{\frac{1}{n} : n \in N$ and n is prime$\}$

   (vii) $\{x^2 : x \in R\}$ (viii) $\left\{\text{Cos}\left(\frac{n\pi}{3}\right) : n \in N\right\}$

   (ix) $\{2n : n \in Z\}$ (x) $\{x \in R : x \leq 2\} \cup \{x \in R: x>2\}$
7. Find which of the sets in question 6 are bounded below. Write the inferqum if it exists.
8. Which of the sets in question 6 are bounded and unbounded.
9. Test whether the following statements are true or false :
   (i) The set Z of integers is not a NBD of any of its points.
   (ii) The interval]0, 1] is a NBD of each of its points
   (iii) The set]1, 3[$\cup$] 4, 5 [is open.
   (iv) The set [a, ∞[$\cup$]$-\infty$, a] is not open.
   (v) N is a closed set.
   (vi) The derived set of Z is non-empty.
   (vii) Every real number is a limit point of the set Q of rational numbers.
   (viii) A finite bounded set has a limit point.
   (ix) [4, 5] $\cup$ [7, 8] is a closed set.
   (x) Every infinite set is closed.
10. Justify the following statements:
    (i) The identity function is an odd function.
    (ii) The absolute value function is an even function.
    (iii) The greatest integer function is not onto.
    (iv) The tangent function is periodic with period π.
    (v) The function $f(x) = |x|$ for $-2 \leq x \leq 3$ is bounded.
    (vi) The function $f(x) = e^x$ is not bounded
    (vii) The function $f(x) = \sin x$, for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is monotonically increasing.
    (viii) The function $f(x) = \cos x$ for $0 \leq x \leq \pi$ is monotonically decreasing.
    (ix) The function $f(x) = \tan x$ is strictly increasing for $x \in [0, \frac{\pi}{2}]$.
    (x) $f(x) = \frac{\sqrt{2x^2-3x+2}}{3x-2}$ is an algebraic function

ANSWERS

1. None is a rational number
2. For (ii) only since $2 < x < 3$.
3. $(b-d)^2 \geq 0 \implies b^2+d^2 \geq 2bd \implies bd \leq \frac{1}{2}$ and $(a-c)^2 \geq 0 \implies a^2+c^2 \geq 2ac \implies ac \leq \frac{1}{2}$ $\implies ac+bd \leq 1$.
4. Use the triangle inequality.
5. Use the principle of Induction.
6. (i) π; (iv) 2

   (v) 0 (vi) $\frac{1}{2}$ (vii) 1

   (ix) and (x) are unbounded.
7. (i) e
   (ii) 2
   (iii) 0
   (vi) 0
   (viii) -1
8. All the sets are unbounded except the (i)
9. (i) True (ii) False (iii) True (iv) False (v) True (vi) False (vii) True (viii) False (ix) True (x) False.

NOTES

NOTES

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2

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NOTATIONS AND SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	is a member of (belongs to)
$\notin$	is not a member of (does not belong to)
$\subset$	is a subset of (is contained in)
$\not\subset$	is not a subset of (is not contained in)
$\supset$	is a superset
$\cup$	Union
$\cap$	intersection
ϕ	empty set
$\Rightarrow$	implies
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\sim$	equivalence relation
$\forall$	for all
$\exists$	there exists
$\cdot$	multiplication
$+$	addition
$-$	subtraction
sup	supremum
inf	infimum
min	minimum
max	maximum
$\circ$	composition
f'	derivative of f
f^{-1}	inverse of a function f
exp	exponential
log	logarithm
In	natural logarithm
sgn	signum
[x]	greatest integer not exceeding x
$\|x\|$	absolute value of x or Modulus of x
$\mathbf{R}^{+}$	set of positive real numbers
R	set of real numbers
I	Set of irrational numbers
Q	set of rational numbers
Z	set of integers
N	set of natural numbers
F	field
C	set of complex numbers
[a, b]	closed interval
]a, b[	open interval
]a, b]	semi-open interval (open at left)—semi-closed interval
[a, b[	semi-open interval (open at right)—semi-closed interval
$+\infty$	infinity
$-\infty$	minus infinity
Σ	sum
$\sum_{n=1}^{\infty} u_n$	infinite series
(s_n)	sequence
S^c	complement of S
S'	derived set of S
$\overline{S}$	closure of S

Greek Alphabets

α	Alpha
β	Beta
γ	Gama
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
i	Iota
λ	Lambda
μ	Mu
ν	Nu
ξ	exi
π	Pi
Π	(capital Pi)
ρ	Rho
σ (Σ)	Sigma (capital Sigma)
τ	Tou
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

REFERENCES

We have discussed the course material in the course on Real Analysis in a complete form. We believe that the discussion is quite exhaustive in each unit. Nevertheless, you may like to refer to some books for some more understanding of the concepts or may be you need some additional readings. For this, we give below a list of books which may be available at your Study Centre or in a nearby institution.

1. *Real Analysis* by S.C. Malik, Wiley Eastern Limited.
2. *Elements of Real Analysis* by Shanti Narayan, S. Chand & Company Ltd.
3. *Foundations of Analysis the Theory of Limits* by Herbert S. GasPill P.P. Narayanaswami.
4. *Introduction to Mathematical Analysis* by C.R.J. Claphan, Routldge & Kegan Paul (London).
5. *Mathematical Analysis* by M.D. Hattan, Hodder and Stonghton (London).

BLOCK 2 SEQUENCES AND SERIES

PREVIEW

In Block 1, you were introduced to the system of real numbers and real functions. The main purpose at that stage was to build the foundation for a careful study of Real Analysis. At this level, we shall begin this study with the process of limits. For this, we shall follow the paths of a few 18th and 19th century mathematicians who were mainly concerned with the notion of a limit — the basic tool of Analysis. We shall, therefore, in the first instance, deal with the notion of a limit in the context of the convergence of sequences and series. Having absorbed the basic idea in this relatively simple situation, you will, then, be in a position to appreciate more rigorous forms of limit.

Although sequences and series are usually introduced in elementary Calculus and Algebra Courses, it is not possible at that stage to pay much attention to the rigorous discussion of the definitions and the proofs of the statements/theorems. In this block, therefore, we shall discuss these topics with emphasis on the mathematical rigor. The block has three units namely Units 5, 6 and 7.

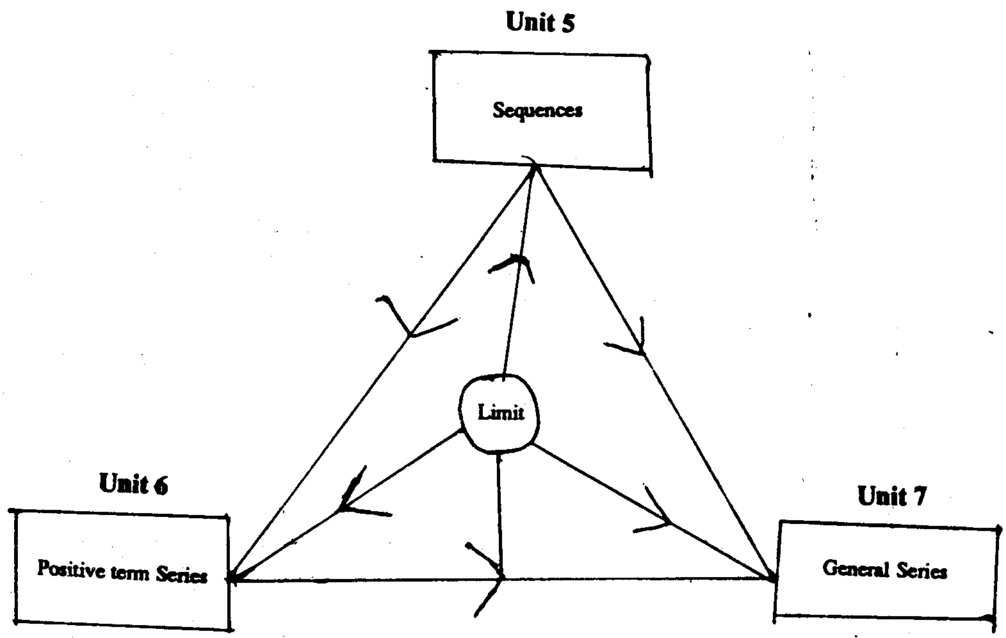
In Unit 5, (the first unit of this block) you will be introduced to the notion of a sequence, a sub-sequence, bounded and monotonic sequences. Thereafter, the concept of the limit of a sequence and hence a convergent sequence is discussed.

After having understood the concept of the convergence of a sequence, you will learn the criteria to test the convergence of a given sequence. This criteria is popularly known as Cauchy's Criteria for the convergence of sequences. Then, we discuss the algebra of the convergent sequences.

Unit 6, (the second unit of this block) begins with the introduction of an infinite series and its convergence. In this unit, we confine our discussion to the convergence of the infinite series of positive terms and a few tests of their convergence.

In Unit 7, (the last unit of the block), we deal with the convergence of the general series -- the series with both positive and negative terms. You will learn a few methods of testing the convergence of such series. Finally, we talk of the absolute convergence and the conditional convergence of the general series.

The following figure depicts how the notion of limit is a common link among the three units of this block.



UNIT 5 SEQUENCES

Structure

5.1 INTRODUCTION

In Unit 2, you were introduced to the structure of the real numbers. In Unit 3, some interesting properties of the system of real numbers were discussed. In addition to these properties, there are several other fascinating features of the real numbers. In this unit, we discuss one such feature. This is related to the problem of obtaining the sum of an infinite number of real numbers.

You know that it is easy to find the sum of a finite number of real numbers. The addition of an infinite number of real numbers, however, poses some problem. Apparently, you may conclude that it is not possible to add an infinite number of real numbers. But an infinite sum i.e. the sum of an infinite number of real numbers is not artificial. Under certain limiting processes, it is possible to give a meaning to an infinite sum of the form

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\dots \infty.$$

Recall that this is the infinite sum of a Geometrical Progression with first term 1 and common ratio $\frac{1}{2}$.

To obtain the infinite sums, we need the notion of a sequence of real numbers, and its convergence to a limit. What is, then, a sequence? What is the meaning of the convergence of a sequence? What is the criteria to determine the convergence of a sequence? We shall try to find answers for these questions. Also we shall discuss a few related concepts such as boundedness and monotoncity of sequences. We shall frequently use these concepts in the Units 6 and 7 as well as later on in Blocks 3 and 5.

Objectives

After studying this unit, you should, therefore, be able to

- define a sequence and its subsequence
- discuss a bounded and a monotonic sequence
- find the limit of a sequence, if it exists
- verify whether a given sequence is convergent or not
- use the criteria for the convergence of a sequence and define the Cauchy sequence.

5.2 REAL SEQUENCES

Very often you use the word 'sequence' in your daily life in several ways. You talk of 'a sequence of events' or 'arranging the library books in a sequence' and so on. Intuitively the idea of a sequence is that of a progression or succession of numbers, e.g. the first, the second, the third and so on. For example, if you want to evaluate $\sqrt{2}$ up to many decimal places, you can arrange its approximate values as 1.4, 1.4 1, 1.414, 1.4142, ... and so on. Thus, intuitively, a sequence of real numbers would mean a succession of real numbers x_1, x_2 ....; where x_1 is the first element, x_2 being the second element, ... and so on. Hence, you may say that a sequence is an ordered collection of numbers. But in Mathematics, we define a sequence as a special type of function in the following way:

Recall from unit 1 that f is a function from a nonempty set A to a nonempty set B, if to each element $x \in A$, there is assigned a unique element $f(x) \in B$.

If a function s has its domain as the set N of natural numbers and range in B, then s is called a sequence. Let us study the following two examples:

EXAMPLE 1: Let a function s: $N \to R$ be defined as

$$s(n) = 2n-1.$$

Then $s(1) = 1$, $s(2)=3$, $s(3) = 5$,..........., and so on.

This function s: $N \to R$ is called a sequence.

Let us write $s(n) = s_n \ \forall \ n \in N$. Then obviously,

$s_1 = 1, s_2 = 3, s_3 = 5$ ..... and so on.

The set $\{s_1, s_2, \ldots\ldots, s_n \ldots\ldots\}$ forms the range of the sequence s: $N \to R$. The values $s_1, s_2, \ldots\ldots$ are called the terms of the sequence. The sequence is generally written as $(s_n)_{n \in N}$ or $(s_n)_{n=1}^{\infty}$ or simply (s_n).

EXAMPLE 2: Let a function s: $N \to R$ be defined as

$$s(n) = s_n = \frac{1}{3n-1}.$$

Then $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{5}$, $s_3 = \frac{1}{8}$, $s_4 = \frac{1}{11}$, ... and so on.

The sequence s_n is given by

$$(s_1, s_2, \ldots\ldots) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots\ldots\right).$$

EXAMPLE 3: Let s: $N \to R$ be defined by

$$s(n) = 1.$$

In this case, $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, ...
where 1 is the first element of N, 2 is the second element of N, 3 is the third element of N, and so on. Thus in this case the sequence $(s_n)_{n \in N}$ is such that $s_1 = s(1) = 1$, is the first term of the sequence, $s_2 = s(2) = 1$ is the second term of the sequence, $s_3 = s(3) = 1$ is the third term of the sequence, and so on.

In all these examples, we have taken the range as a subset of real numbers. Such sequences are called **Real Sequences**. A formal definition of a Real Sequence is as follows:

DEFINITION 1: REAL SEQUENCE

A real sequence is a function s from the set N of natural numbers to the set R of real numbers whose values are denoted by $(s_1, s_2, \ldots\ldots)$ or by $(s_n)_{n \in N}$, or by s(n), where $s(n) = s_n$ for n = 1, 2, 3, ... The number s_n is called the nth term of the sequence.

We shall use the notation (s_n) throughout our discussion. Thus, the sequence in Example 1 is $(2n-1)$, the sequence in Example 2 is $\left(\frac{1}{3n-1}\right)$ and the sequence in Example 3 is (1^n) or (1, 1, 1, ...). It is important to distinguish between a sequence and its set of values since the validity of many results depends on whether we are working with a sequence or a set. We shall always use parentheses ( ) to denote a sequence and the braces { } to signify a set. The sequence $(s_1, s_2, s_3, \ldots\ldots)$ should not be confused with the set $\{s_1, s_2, s_3, \ldots\ldots\}$. For instance (1, 1, 1 ......) is a sequence whose first term is 1, second term is 1, third term is 1, and so on, whereas the set {1, 1, 1, ... } is just the singleton {1}. Hence to make the distinction clear, we sometimes write this sequence as (1^n) or $(1)_{n\in N}$.

Let us look at a few more examples of Real Sequences.

EXAMPLE 4: (i) The expression (1, 2, 3, 4, ...) is a sequence. This is the sequence $(n)_{n\in N}$ or (n).

(ii) A sequence such as (c, c, c, c, ...) where every term is the same number c, is called a constant sequence. This is the sequence $(c)_{n\in N}$.

(iii) The sequence $(-1)^n$ has terms (−1, 1, −1, 1, −1, 1, ...).

EXAMPLE 5: Let a sequence (s_n) be given as

$s_1 = 1, s_2 = 1, s_{n+1} = s_n + s_{n-1}$ for n=2, 3, 4, 5, ...

In this case (s_n) becomes (1, 1, 2, 3, 4, 5, 8 .... m, n, m+n ....).

Leonardo Fibonacci

This sequence is called the **Fibonacci Sequence** given by an eminent **Italian mathematician L. Fibonacci [1175–1250].**

It has many fascinating and interesting properties. Also, it has lot of applications particularly in puzzles and riddles in Mathematics. In fact, Fibonacci, was inspired by Hindu-Arabic methods of calculation. He found this sequence when he was trying to solve the following problem:

"How many pairs of rabbits can be produced from a single pair in a year if every month each pair begets a new pair which from the second month on becomes productive?"

We shall, however, not go into the detailed discussion on Fibonacci Sequences at this stage.

Note that not merely the numbers which occur but the order in which they occur is vital in defining a sequence. For example, the sequences (1, 2, 3, 4, ...) and (2, 1, 3, 4, ...) are two different sequences although their sets of values are same.

Now consider the sequence (s_n) given by

$$s_n = (-1)^n n^2.$$

Form a sequence (t_k) whose terms are the positive terms of the sequence (s_n).

Then the terms of (s_n) are (−1, 4, −9, 16, −25, 36, −49, 64, ...)

and the terms of (t_k) are (4, 16, 36, 64. ....).

Obviously (t_k) is obtained by selecting, in order, an infinite number of the terms of (s_n). In such a case, we say that (t_k) is a subsequence of the sequence (s_n).

Now, we are ready to formulate the concept of a subsequence in the form of the following definition:

DEFINITION 2: SUBSEQUENCE

Let (s_n) be a sequence. Let $n_1, n_2, n_3, ...$ be natural numbers such that $n_1 < n_2 < n_3 < ....$ Then a sequence

$(s_{n_1}, s_{n_2}, s_{n_3}, ...) = \left(s_{n_k}\right) = (t_k)_{k\in N}$ or (t_k)

is called a subsequence of (s_n).

In other words, a subsequence of a sequence is a sequence obtained by omitting some terms of the original sequence and not disturbing the relative positions of the remaining terms. For instance, $(s_1, s_3, s_5, ...)$ and $(s_2, s_4, s_6, ...)$ are subsequences of the sequence $(s_1, s_2, s_3, s_4, ...)$. But $(s_2, s_1, s_4, s_3 ....)$ is not a subsequence of $(s_1, s_2, s_3, s_4 ....)$. The sequence $(-1)^n$ has two subsequence namely (1, 1, 1, ...) and (−1, −1, −1, ...).

Note that

$n_1 < n_2 < n_3 <........ < n_{k-1} < n_k < n_{k+1} <........$ defines an infinite subset of **N**, namely, $\{n_1, n_2, n_3 ....\}$. Conversely, you can say that every infinite subset of **N** can be described by

$$n_1 < n_2 < n_3 < ....$$

Thus, a subsequence of (s_n) is a sequence obtained by selecting, in order, an infinite subset of the terms of the sequence. Now try the following exercise:

EXERCISE 1

Which of the following sequences are subsequences of the sequence (1, 2, 3, 4, ...,)?

i) (1, 0, 1, 0, 1, 0, ...)
ii) (1, 3, 6, 10, 15, ...)
iii) (1, 1, 1, 2, 1, 3, 1, 4, 1, 5, ...).

Since a sequence is a function from N to R and N has the natural order, it makes sense to talk about a sequence being bounded and a sequence being monotonic. Recall from Unit 4, the definitions of bounded and monotonic functions. You can easily deduce the definition of a bounded sequence because a sequence, as you know, is a special type of function.

DEFINITION 3: BOUNDED SEQUENCE

A sequence s: N → R is said to be bounded if its range is bounded in R. In other words, a sequence (s_n) is bounded if there exists a number K > 0 such that

$$|s_n| \le K \text{ for } n = 1, 2, 3, \ldots\ldots$$

For instance, the sequence in example 1 is not bounded whereas the sequence in example 3 is bounded. What about the sequence in example 2 ?. Is it bounded or not? Verify it yourself. Again, the sequence in Exercise 1 (i) is bounded, while the sequence in Exercise 1 (ii) is not bounded. What about the sequence in Exercise 1 (iii) ?

Just as, in Unit 4, we defined a function which is bounded below only or bounded above only, you can similarly define a sequence which is bounded below only or bounded above only or both bounded below and bounded above i.e. bounded.

EXERCISE 2

Define a sequence which is bounded below only or bounded above only. Give an example for each. Verify whether an Arithmetical Sequence (a, a+d, a+2d, ...), d ≠ 0 is bounded or bounded below only or bounded above only.

You know that a function is said to be unbounded if it is bounded below only or bounded above only or none of the two. Similarly, you can define an unbounded sequence.

EXERCISE 3

Examine which of the sequence in exercise 1 are bounded and unbounded.

Again, we discussed the monotonic functions in Unit 4. Since a sequence is a special type of function, therefore, we can say something about a monotonic sequence. First, let us study the following examples:

Consider the sequence ($\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$, ...).

Here $s_1 = s(1) = \frac{1}{2}$, $s_2 = s(2) = \frac{2}{3}$, $s_3 = s(3) = \frac{3}{4}$, and so on. You can see that $s_1 < s_2 < s_3 < s_4 < \ldots\ldots$, that is, $s(1) < s(2) < s(3) < \ldots\ldots$ In other words, the sequence preserves the order in N. Such a sequence is called a **Monotonically Increasing Sequence.**

Again consider the sequence ($1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, ...). Here, the inequalities are reversed

$$1 > \frac{1}{2} > \frac{1}{3} > \frac{1}{4} > \ldots\ldots\ldots\ ;$$

that is,

$$s(1) > s(2) > s(3) > s(4) > \ldots\ldots$$

In this case the sequence **reverses the order in** N. Such a sequence is called a **Monotonically Decreasing Sequence.**

Also look at the sequence ($1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}$ ....).

You can see that this sequence neither preserves nor reverses the order in N. Such a sequence is neither monotonically increasing nor monotonically decreasing.

A sequence which is either monotonically increasing or monotonically decreasing is called a **Monotonic Sequence.** We have the formal definition as follows:

DEFINITION 4: MONOTONIC SEQUENCE

A sequence ($s_1, s_2,$ ....) is called a monotonic sequence if either $s_1 \leq s_2 \leq s_3$, ... , or $s_1 \geq s_2 \geq s_3 \geq$ .... In the first case, the sequence is called a monotonically increasing sequence, while in the second case it is called a monotonically decreasing sequence.

Try the following exercises:

EXERCISE 4

Which of the following sequences are monotonic?

i) (sin n)

ii) (tan n)

iii) $\left(\frac{1}{1 + n^2}\right)$

iv) $(2n + (-1)^n)$.

EXERCISE 5

i) Show that a subsequence of a monotonic sequence is also monotonic.

ii) Do there exist sequences which are both monotonically increasing and monotonically decreasing?

We state a theorem (without proof) which we shall use in the next section:

THEOREM 1: Every Sequence has a Monotonic Subsequence.

For example, the sequence $(\frac{1}{n})$ has a subsequence

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\ldots\ldots\right)$$

which is monotonic. Verify it yourself. Is it monotonically increasing or decreasing?

5.3 CONVERGENT SEQUENCES

The basic tool of Analysis is the notion of a limit and the simplest form of a limit is that of the limit of a sequence. A real number s is called the limit of a sequence (s_n) if a large number of values of (s_n) are close to s.

For example, consider the sequence $(\frac{1}{n})$. It is intuitively clear that the terms of the sequence "approach" the number 0 as n becomes larger and larger. In other words, we can say that the n^{th} term of the sequence is 'as close to 0' as you prescribe for "sufficiently large n". See figure 1 for the limit of the sequence $(\frac{1}{n})$.

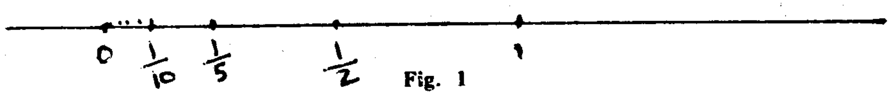


Fig. 1

Alternatively, we can say that the sequence $(\frac{1}{n})$ approaches the limit 0 if $\left|\frac{1}{n} - 0\right|$ can be made as small as possible for larger and larger values of n. Note that such a behaviour is not true of the sequence (n). Check why?

Thus, we have the following definition of the limit of a sequence:

DEFINITION 5: LIMIT OF A SEQUENCE

A real number s is said to be a limit of a real sequence (s_n) if for every $\varepsilon > 0$, there exists a number $m \in N$ such that for every integer $n \in N$,

$$|s_n - s| < \varepsilon \qquad \text{for } n > m$$

or

$$n > m \Rightarrow |s_n - s| < \varepsilon.$$

Since the number m depends upon the choice of $\varepsilon > 0$. Therefore, sometimes the number m is denoted as m_ε

The condition that a sequence (s_n) has a limit s is often expressed by saying that the sequence (s_n) tends to a limit.

We say also that a sequence (s_n) is said to be convergent if there exists a number s (called the limit of the sequence) such that $|s_n - s|$ can be made "as small as possible" for "all sufficiently large values of n". Note, however, that it is not always possible to guess the behaviour for a given sequence. For instance, our intuition is not sharp enough to tell if the sequence $(n \sin \frac{1}{n})$ converges. We therefore, need a precise definition of the convergence of a sequence, and also some criteria to determine whether a given sequence converges or not. We shall first define convergence of sequences in this section, and later on take up the criteria of convergence of sequences in the next section.

DEFINITION 6: CONVERGENCE OF A SEQUENCE

A real sequence (s_n) is said to converge to a real number s (called the limit of the sequence) if for a given $\varepsilon > 0$, there exists a positive integer m such that

$$|s_n - s| < \varepsilon \text{ for } n > m$$

or

$$n > m \Rightarrow |s_n - s| < \varepsilon.$$

We express the above fact in several ways. We say that

i) the sequence (s_n) is convergent to a real number s.

or

ii) $\lim\limits_{n\to\infty} s_n = s$,

or

iii) $s_n \to s$ as $n \to \infty$.

The number s is called the limit of the sequence (s_n).

Here, we emphasize the existence a real numbers.

The convergence of (s_n) to s can be reviewed in the language of neighbourhood (Unit 3) as follows:

We say that $\lim\limits_{n\to\infty} s_n = s$ if and only if the sequence is in every neighbourhood of s.

To say it differently, (s_n) converges to s if, given any $\varepsilon > 0$, all the elements of the sequence, possibly omitting a finite number of elements of the sequence, must be at a distance less than ε from s. Geometrically, $(s_n) \to s$ if, given $\varepsilon > 0$, you should be able to cut off an initial segment of the sequence,

$$[s_1, s_2, \ldots s_m]$$

such that, every member of the 'tail' s_{m+1}, s_{m+2} .... is in the interval $]s-\varepsilon, s+\varepsilon[$. The initial segment that must be cut off depends on ε i.e. how close to s are the tail elements. See the figure 2.

Figure 2 illustrate the limit concept graphically. Consider the horizontal strip of width 2ε generated by the lines, $y = s - \varepsilon$ and $y = s+\varepsilon$. A given term, s_n, of the sequence, (s_n), lies inside this strip exactly if the inequality $|s_n - s| < \varepsilon$. Thus, for the number s, to be the limit of the sequence, (s_n), we must be able to specify a point, m, on the X-axis, such that for all n lying to the right of m, the corresponding term, s_n, gets trapped within the horizontal strip.

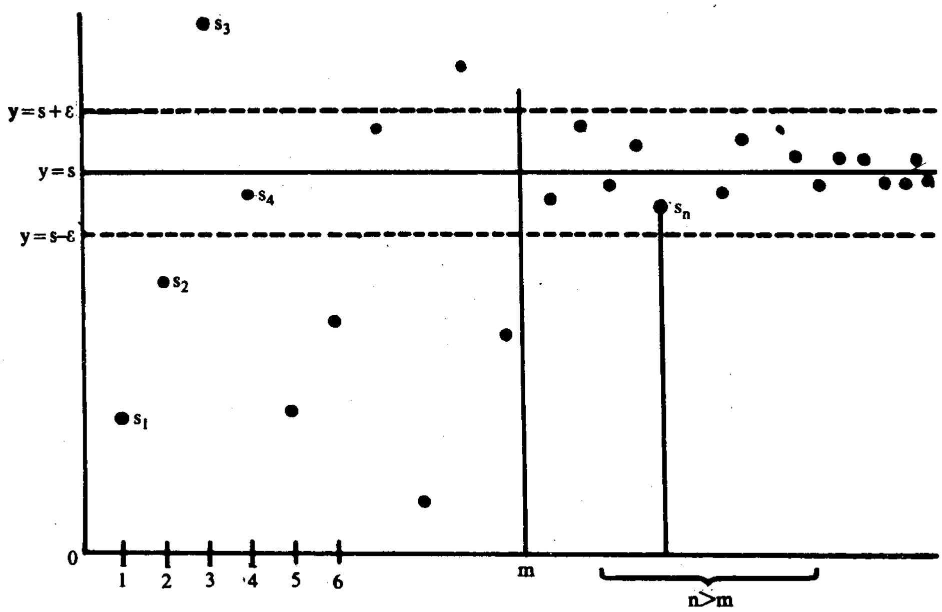


Fig. 2

Thus, if (s_n) is a sequence having the number s as a limit, we write

$$\lim_{n\to\infty} s_n = s$$

(or simply $\lim s_n = s$) and say the limit as n tends to infinity of s_n is s or the sequence (s_n) converges to the limit s.

Let us look at some examples.

EXAMPLE 6: Discuss the convergence of the sequence $(\frac{1}{n^2})$.

SOLUTION : Intuitively, you can see that

$$\lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Therefore the sequence $(\frac{1}{n^2})$ converges to the limit 0.

Let us demonstrate the application of the definition 6.

For this, our task is to consider an arbitrary $\varepsilon > 0$ and show that there exists a number m (depending upon ε) such that

$$n > m \text{ implies that } \left|\frac{1}{n^2} - 0\right| < \varepsilon.$$

Therefore, we expect our proof to start with "Let $\varepsilon > 0$" and to end with something like

"Hence $n > m$ implies $\left|\frac{1}{n^2} - 0\right| < \varepsilon$.

In between the proof should specify an m and then specify that m has the desired property, namely $n > m$ does indeed imply $\left|\frac{1}{n^2} - 0\right| < \varepsilon$

As you know that quite often (for example in the proof of the trigonometric identities), we use **the if method** i.e. we initially work backward from our desired conclusion, but in the formal proof we must make sure that our steps are reversible. In the present example, we want $\left|\frac{1}{n^2} - 0\right| < \varepsilon$ and we want to know how big n must be. So, we will operate on this inequality algebraically and try to solve it for n. Thus we want $\frac{1}{n^2} < \varepsilon$.

By multiplying both sides by n^2 and dividing both sides by ε, we find that we want

$$\frac{1}{\varepsilon} < n^2$$

or $\frac{1}{\sqrt{\varepsilon}} < n$.

If our steps are reversible, then we can see that

$$n > \frac{1}{\sqrt{\varepsilon}} \text{ implies that } \left|\frac{1}{n^2} - 0\right| < \varepsilon.$$

This suggests that we should put

$$m = \frac{1}{\sqrt{\varepsilon}}.$$

Thus, we proceed as follows

Let $\varepsilon > 0$: Let $m = \frac{1}{\sqrt{\varepsilon}}$. Then

$$n > m \Rightarrow n > \frac{1}{\sqrt{\varepsilon}}$$

$$\Rightarrow n^2 > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{n^2} - 0\right| < \varepsilon$$

which proves that the sequence $(\frac{1}{n^2})$ converges to the limit 0.

EXERCISE 6

Use definition 6 to prove that the sequence $(\frac{1}{n})$ converges to the limit 0.

EXAMPLE 7: Test the convergence of the following sequences

i) (s_n) where $s_n = \frac{1}{n}$

ii) (1, –1, 1, –1, ...)

SOLUTION: Look at the sequence $s_n = (n)$. This sequence cannot converge. For, suppose the sequence converges to a limit s. Taking $\varepsilon = \frac{1}{2}$, all but finitely many elements of s_n should lie in the interval $]\, s - \frac{1}{2}, s + \frac{1}{2}\, [$. But this is clearly not so. Thus the sequence (n) does not converge.

ii) Suppose the sequence $[(-1)^n]$ converges to some number s. Taking $\varepsilon = \frac{1}{2}$, all but finitely many elements of this sequence must lie in the interval $]\, s - \frac{1}{2}, s + \frac{1}{2}\, [$. That means that both 1 and –1 are in $]\, s - \frac{1}{2}, s + \frac{1}{2}\, [$. But that is impossible because the distance between 1 and –1 is 2, while the length of the interval $]\, s - \frac{1}{2}, s + \frac{1}{2}\, [$ is 1. Thus the sequence (1, –1, 1, –1, ...) does not converge.

In example 7, we have come across two sequences which are not convergent. Such sequences are called divergent sequences.

Thus a sequence is said to be divergent if it is not convergent. Can a sequence converge to two different limits? We answer this question in the following theorem:

THEOREM 2: If a sequence is convergent, then its limit is unique.

PROOF: Suppose that a sequence (s_n) has two distinct limits s and s′. Then $s \neq s'$. Let us assume, without loss of generality that $s > s'$. Take $\varepsilon = \frac{s - s'}{3}$.

Since (s_n) → s, all but finitely many elements of the sequence are in the interval] s−ε, s+ε [. For the same reason, all but finitely many elements of the sequence are in the interval

] s−ε, s+ε [. Look at the figure 3.

Fig 3

The two intervals] s' − ε, s'+ε [and] s−ε, s+ε [are disjoint, and a tail of the sequence cannot be contained in both the intervals. Hence, it is not possible for the sequence (s_n) to converge to two different limits. This proves the theorem.

Having defined convergence of sequence, the question that remains to be settled is: **How to test the convergence of a given sequence?** Let us, first, obtain necessary conditions for the convergence of a sequence. We state and prove the following theorem:

THEOREM 3: Every convergent sequence is bounded.

PROOF: Let (s_n) be a sequence which converges to s. Then, there is a number m ∈ N such that

$$|s_n - s| < \varepsilon \text{ for } n > m.$$

With ε = 1, we obtain m in N so that

$$n > m \Rightarrow |s_n - s| < 1.$$

Using triangle inequality (Unit 3), we see that

$$|s_n - s| \geq \big| |s_n| - |s| \big|$$

or

$$\big| |s_n| - |s| \big| \leq |s_n - s| < 1$$

$$\Rightarrow \big| |s_n| - |s| \big| < 1$$

$$\Rightarrow |s_n| < |s| + 1$$

Thus

$$n > m \Rightarrow |s_n| < |s| + 1$$

Choose K = Max. { $|s| + 1$, $|s_1|$, $|s_2|$, ... $|s_m|$ }.

Then, we have

$$|s_n| \leq K.$$

for all n ε N. Hence (s_n) is bounded.

Our choice of ε = 1 was just arbitrary. You can, in fact, try the proof for some other value of ε.

From this theorm, it follows that if a sequence is convergent, then it is always bounded. Is the converse true? That is to say that if a sequence is bounded, then, is it convergent? The answer is No. For example the sequence (−1, 1, −1, 1, −1, 1 ....) is bounded but not convergent. Thus boundedness is a necessary condition for the convergence of a sequence. It is not a sufficient condition. However, for a monotonic sequence, boundedness is both necessary and sufficient. We prove this in the form of the following theorem:

THEOREM 4: A monotonic sequence converges if and only if it is bounded

PROOF: We already know that every convergent sequence is bounded. Thus, it is enough to prove that a monotonic and bounded sequence is convergent

We shall prove this assertion for a monotonically increasing sequence. Let (s_n) be a monotonically increasing sequence which is bounded above.

Let S denote the set $\{ s_n : n \in N \} = \{ s_1, s_2, \ldots\}$.

This means that $s_1 \le s_2 \le s_3 \le \ldots\ldots\ldots$ Since (s_n) is bounded above, therefore, there exists an upper bound for the sequence $(s_1, s_2, s_3 \ldots\ldots\ldots)$. Thus (s_n) has the least upper bound, u (say). We claim that the sequence (s_n) converges to u.

Indeed, let $\varepsilon > 0$ be some real number. Since $u - \varepsilon$ is not an upper bound for S, therefore

$$u - \varepsilon < u.$$

This means that there exists some integer m such that

$$s_m > u - \varepsilon$$

Since (s_n) is an increasing sequence, therefore

$$s_m < s_n \ \forall\, n > m.$$

But $s_n \le u$ for all n. Therefore

$$n > m \Rightarrow s_n > s_m$$
$$\Rightarrow s_n > u - \varepsilon$$
$$\Rightarrow |s_n - u| < \varepsilon$$

which shows that (s_n) converges to the limit u.

Now you can similarly prove the theorem for a monotonically decreasing sequence as the following exercise:

EXERCISE 7

Let (s_n) be a monotonic decreasing sequence which is bounded below. Show that the sequence converges to its greatest lower bound.

EXERCISE 8

i) Suppose that the sequence (s_n) converges to s and $s_n \le A$ for every n. Show that $s \le A$.

ii) Suppose the sequence (s_n) converges to s and $s_n \ge a$ for every n. Show that $s \ge a$.

EXERCISE 9

i) Suppose that the sequence (s_n) converges to s. Show that every subsequence of (s_n) also converges to s.

ii) 'If a sequence does not converge, then no subsequence of it can converge'. Is the statement true? Justify your answer.

In fact, it can be shown that a sequence (s_n) converges to s **if and only if each of its subsequence converges to the same limit s.** However, if a sequence is such that its subsequences converge to different limits, then the sequence will not be convergent. For example, let

(s_n) be a sequence with $s_n = (-1)^n$.

Then it has two subsequences namely $(-1, -1, -1 \ldots\ldots\ldots)$ and $(1, 1, 1 \ldots\ldots\ldots)$ which converge to -1 and 1 respectively. But the sequence itself is not convergent.

EXERCISE 10

Suppose the sequence (s_n) converges to s. Show that the sequence $(|s_n|)$ converges to $|s|$. Give an example to show that the converse is not true.

We now state the following theorem (without proof) which is also sometimes referred to as the **Bolzano-Weirstrass Theorem:**

THEOREM 5: Every bounded sequence of real numbers contains a convergent subsequence.

We have already talked about the divergent sequences.

You have encountered two examples of divergent sequences namely (1, 2, 3, 4, ... } and (1, – 1, 1, –1, ...). The sequence (1, 2, 3, 4, ...) is said to be divergent sequence because its terms become "too big", whereas (1, –1, 1, –1, ...) is divergent because it has two limits namely 1 and –1.

We now give a formal definition of a divergent sequence.

DEFINITION 7: DIVERGENT SEQUENCE

A sequence is said to be divergent if it is not convergent. A sequence (s_n) diverges to + ∞ if given any real number ε > 0, there is a positive integer m such that $s_n > \varepsilon$ for all n > m.

In this case, we also write lim $s_n = \infty$ or $s_n \to \infty$ as $n \to \infty$. Such a sequence is said to be divergent sequence. You can similarly define the divergence of a sequence (s_n) to –∞.

We can write this as $\lim\limits_{n \to \infty} s_n = -\infty$.

EXAMPLE 8: Show that the sequence ($\sqrt{n}$) diverges to + ∞.

SOLUTION: Let ε > 0 be given. Then $\sqrt{n} > \varepsilon$ for all $n > \varepsilon^2$.

This shows that the sequence $\sqrt{n} \to +\infty$ as $n \to \infty$.

Hence the sequence ($\sqrt{n}$) diverges to +∞.

You can easily see that if a sequence (s_n) diverges to +∞ or –∞, then the sequence is unbounded.

Some divergent later diverge to +∞; while some other diverge to – ∞.

EXAMPLE 9: Show that the sequence $(1 + (-1)^n)$ is divergent.

SOLUTION: This sequence has two limits namely 0 and 2. Therefore it is divergent.

To end this section, let us look at a very important sequence, which will be frequently used in the later sections:

EXAMPLE 10: i) If $0 \le x < 1$, then (x^n) converges to 0
ii) If x = 1, then (x^n) converges to 1
iii) If x > 1, then (x^n) diverges to +∞.

SOLUTION: i) If x = 0, then the sequence (x^n) is the constant sequence (0, 0, 0 ....). Hence it converges to zero. Let 0 < x < 1. Then

$$x^{n+1} = x.\, x^n < x^n.$$

Hence the sequence (x^n) is a monotonic decreasing sequence which is bounded below by 0. Hence the sequence converges to its greatest lower bound. Therefore, it is enough to show that 0 is the greatest lower bound of (x^1, x^2, x^3,........).

We know that 0 is a lower bound of the sequence. Therefore it is enough to show that if r > 0, then r is not a lower bound of the sequence. Let r > 0. We wish to show that, for some n ∈ **N**. $x^n < r$. That is, for some n ∈ **N**, n log x < log r. (Recall from Unit 4 that log x is an increasing function). This is equivalent to finding n ∈ **N** such that $n > \frac{\log r}{\log x}$.

Surely there are infinitely many such n's. That shows that, when 0 < x < 1, lim $x^n = 0$. Thus, when $0 \le x < 1$, the sequence (x^n) converges too.

ii) If x = 1, the sequence (x^n) is the constant sequence (1, 1, 1, ...) and thus converges to 1.

iii) Let x > 1. Then, since $x^{n+1} > x^n$ the sequence (x^n) is a monotonic increasing sequence. To show that the sequence diverges to +∞, it is sufficient to show that the sequence is unbounded.

Let M > 0 be any number. Then

$x^n > M$

$\Rightarrow \log x^n > \log M$ (why ?)

$\Rightarrow n \log x > \log M$

$\Rightarrow n > \frac{\log M}{\log x}$

Which shows that the sequence (x^n) is unbounded and hence diverges to $+\infty$.

EXERCISE 11

i) Show that if $-1 < x < 0$, then (x^n) converges to 0.

ii) Discuss the nature of the sequence (x^n) when $x = -1$, and $x < -1$.

EXERCISE 12

Suppose (s_n) and (t_n) are monotonic increasing bounded sequences such that $s_n \leq t_n$ for each n. Prove that

$$\lim_{n\to\infty} s_n \leq \lim_{n\to\infty} t_n$$

We conclude this section by discussing a very important example of a convergent sequence.

EXAMPLE 13: Let $s_n = n^{\frac{1}{n}}$. Show that $\lim_{n\to\infty} s_n = 1$,

that is, $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

SOLUTION: Write $s_n = n^{\frac{1}{n}} = 1 + x_n$. Then $x_n \geq 0$.

By using Binomial theorem, we have

$$n = (1 + x_n)^n = 1 + n x_n + \frac{n(n-1)}{2} x_n^2 + \ldots\ldots + x_n^n$$

$$> \frac{n(n-1)}{2} x_n^2 \ \forall\, n \geq 2$$

This implies $x_n^2 < \frac{2}{n-1}$ for $n \geq 2$.

Hence $x_n < \sqrt{\frac{2}{n-1}}$ for $n \geq 2$.

Let $\varepsilon > 0$ be given. Then

$$|x_n| = x_n < \sqrt{\frac{2}{n-1}} < \varepsilon^2, \text{ for, } n > \frac{2}{\varepsilon^2} + 1.$$

Let $m \in N$ such that $m > \frac{2}{\varepsilon^2} + 1$. Then, $x_n < \varepsilon$ for all $n > m$

hence $|x_n - 1| < \varepsilon$ for all $n > m$. That is, (x_n) converges to 1.

5.4 CRITERIA FOR THE CONVERGENCE OF SEQUENCES

In section 5.3, we have defined a sequence (s_n) to be convergent if we are able to find a number s such that $s_n \to s$ as $n \to \infty$. In some cases, it may be easy enough to guess the existence of such a number s. But, quite often, it is not easy to do so. For example, consider the sequence $\left(\frac{1}{n} \sin n\right)$. Does this sequence converge? We cannot say anything unless we know the limit of this sequence. What we really need is a criterion for the convergence of a sequence (s_n) which uses only the terms of the given sequence — and not to search for a possible limit of the sequence from among the infinite set **R**. That is precisely what we intend to do in this section. In fact, we shall obtain a necessary and sufficient condition for the convergence of a sequence in this section.

We state, first, a necessary condition for a sequence to converge. You will see later that the condition is also sufficient for a sequence to converge but under certain additional restrictions.

THEOREM 6: A sequence (s_n) is said to be convergent, if for a given $\varepsilon > 0$, there exists a (natural) number $m \in N$ such that

$$|s_n - s_k| < \varepsilon$$

whenever $n > m$, $k > m$ ($n > k$).

PROOF: Let the sequence (s_n) converge to **a number s (say).** Let $\varepsilon > 0$ be given.

Then $\frac{\varepsilon}{2} > 0$. Since $s_n \to s$ as $n \to \infty$, therefore for $\frac{\varepsilon}{2}$, there exists a natural number m such that

$$|s_n - s| < \frac{\varepsilon}{2} \text{ whenever } n > m.$$

Now, suppose n, k > m. Then

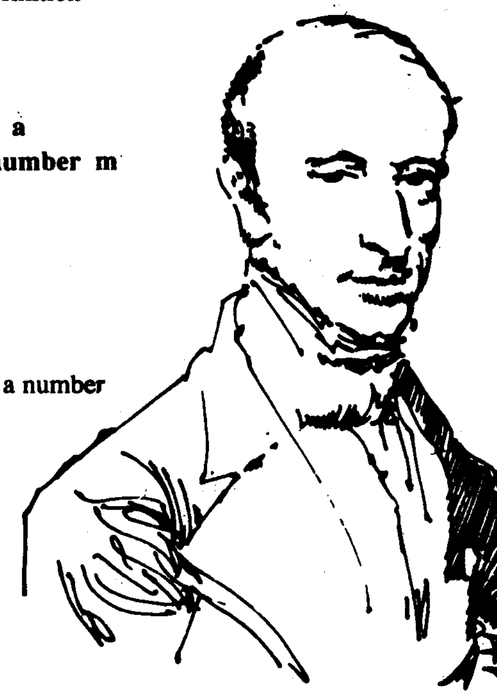
$$|s_n - s_k| = |(s_n - s) - (s_k - s)|.$$

$$\leq |s_n - s| + |s_k - s| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

Thus, $|s_n - s_k| < \varepsilon$ for all n, k > m.

Use Triangle Inequality as discussed in Unit 3

The theorem says that if the terms of the sequence (s_n) get close to some number, then they get close to each other. Motivated by the above theorem, we have the following definition due to A.L. Cauchy, an eminent French mathematician.



A.L. Cauchy

DEFINITION 9: CAUCHY SEQUENCE:

A sequence (s_n) of real numbers is called a Cauchy sequence or a fundamental sequence, if, for each $\varepsilon > 0$, there exists a natural number m such that

$$|s_n - s_k| < \varepsilon \textbf{ for all } n > m, k > m \; (n > k).$$

We state and prove the following theorem:

THEOREM 7: Every Cauchy Sequence is bounded

PROOF: Let (s_n) be a Cauchy Sequence. Then by definition, it follows that fpr a number ε say $\varepsilon = 1$, there is a positive integer m such that

$$|s_n - s_k| < 1 \text{ whenever } n > m, k > m.$$

In particular

$$|s_n - s_{m+1}| < 1 \text{ for all } n > m,$$

In other words (Unit 3)

$$|s_n| \leq |s_{m+1}| + 1 \text{ for all } n > m.$$

Thus if

$$A = \text{Max}\left\{|s_1|, |s_2|, \ldots\ldots\ldots, |s_m|, |s_{m+1}| + 1\right\},$$

then

$$|s_n| \leq A \text{ for all } n \in N.$$

This concludes that the Sequence is bounded.

THEOREM 8: Every Cauchy Sequence is convergent

PROOF: Let (s_n) be a Cauchy Sequence. Then by theorem 7, it follows that (s_n) is bounded. Using Bolzano-Weirstrass theorem (theorem 5), we can conclude that the sequence (s_n) must have a convergent subsequence say (s_{n_r}). Suppose that

$$s_{n_r} \to p \text{ as } r \to \infty.$$

We claim that

$$s_n \to p \text{ as } n \to \infty.$$

Indeed, let $\varepsilon > 0$ be any real number. Then $\frac{1}{2}\varepsilon > 0$. Hence, there exists a natural number q such that for any r > q we have

$$\left|s_{n_r} - p\right| < \frac{1}{2}\varepsilon.$$

Again since (s_n) is a Cauchy Sequence, therefore there exists a natural number k such that for $k > m$, and any $n > m$,

we have

$$\left|s_n - s_k\right| < \frac{1}{2}\varepsilon.$$

Now for an $n > m$, choose r so large that $n_r > m$ and $r > q$. Then

$$\left|s_{n_r} - p\right| < \frac{1}{2}\varepsilon.$$

is satisfied. Also

$$\left|s_n - p\right| < \frac{1}{2}\varepsilon.$$

is satisfied with $k = n_r$. Thus, for $n > m$,

$$\begin{aligned}\left|s_n - p\right| &= \left|s_n - s_{n_r} + s_{n_r} - p\right| \\ &\le \left|s_n - s_{n_r}\right| + \left|s_{n_r} - p\right| \text{ (Triangle Inequality)} \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon.\end{aligned}$$

Thus, given an $\varepsilon > 0$, we have found an m such that for any $n > m$,

$$\left|s_n - p\right| < \varepsilon.$$

This shows that $s_n \to p$ as $n \to \infty$ which concludes the proof.

Theorems 6, 7 and 8 are sometimes combined into one theorem which is popularly known as **Cauchy's General Principle of Convergence or Cauchy's Criteria for the Convergence of sequences:**

This is stated in the following way:

THEOREM 9: Cauchy's General Principle of Convergence

A necessary and sufficient condition for the convergence of a sequence (s_n) is that for every $\varepsilon > 0$, there exists a positive integer m such that

$$\left|s_n - s_k\right| < \varepsilon \text{ whenever } n, k > m \ (n > k)$$

or

$$n > m,\ k > m,\ (n > k) \text{ imply that } \left|s_n - s_k\right| < \varepsilon.$$

Thus a sequence of real numbers converges if and only if it is a Cauchy Sequence.

Let us illustrate this by the following example:

EXAMPLE 10: Show that the sequence ($\frac{1}{n}$) is a Cauchy Sequence. What about (n^2)?

SOLUTION: For any two integers n and k such that $n > m$, $k > m$, we have

$$\left|s_n - s_k\right| = \left|\frac{1}{n} - \frac{1}{k}\right| = \left|\frac{k-n}{nk}\right| < \frac{1}{k}$$

If $\varepsilon > 0$ be given, then by taking $k > \frac{1}{\varepsilon}$, we see that

$$\frac{1}{k} < \varepsilon$$

and hence

$$\left|\frac{1}{n} - \frac{1}{k}\right| < \varepsilon.$$

In other words

$$\left|\frac{1}{n} - \frac{1}{k}\right| < \varepsilon \text{ whenever } n > m, k > m$$

which shows that the sequence $(\frac{1}{n})$ is convergent.

The sequence (n^2), however, is not a Cauchy Sequence.

For, if n and k be any two integers, then

$|n^2 - k^2| = |(n-k)(n+k)| > |2k| > 1,$

whatever m may be. Choose $\varepsilon = 1$. Then you can easily see that there does not exist a positive integer m such that

$$n^2 - k^2 < \varepsilon.$$

whenever $n > m$, $k > m$. Thus the sequence (n^2) is not convergent.

The Cauchy's Criteria is sometimes described in the following way:

A sequence (s_n) is said to be a Cauchy Sequence if for any $\varepsilon > 0$, there is a positive integer m such that

$$|s_{n+k} - s_n| < \varepsilon, \text{ whenever } k > 0.$$

The advantage of the Cauchy Criteria is that we are able to test the convergence of a sequence without necessarily knowing the value of its limit. Example 10 has justified the utility of Cauchy Criteria. To further elaborate this assertion, consider a sequence (r_n) of rational numbers which converges to $\sqrt{2}$. If (r_n) is treated as a sequence of real numbers, then we have a real number $\sqrt{2}$, as the limit of the sequence (r_n). Thus the sequence (r_n) satisfies the definition of convergence. However, if we treat (r_n) as a sequence of rational numbers and if our definition of convergence requires us to find a rational number which is the limit of (r_n), then (r_n) is no longer convergent since $\sqrt{2}$ is not a rational number. This lack of convergence has arisen not due to the change of the sequence in any way. In fact we have merely modified the context in which we are considering the sequence by changing the underlying field in which the sequence is being discussed. By changing the context in which the sequence is considered has no effect on whether the sequence is Cauchy or not. Thus, it also implies that there is no difference between convergent sequences of real numbers and Cauchy Sequences of real numbers. This is true because of the axiom of completeness of R, the set of real numbers. But then this is not the case if we confine our sequences to the field of rational numbers. In view of this, the completeness of R is also described by saying that **the system of real numbers is complete if every Cauchy Sequence in R has a limit in R.**

5.5 ALGEBRA OF CONVERGENT SEQUENCES

In this section, we shall discuss the behaviour of convergent sequences with respect to algebraic operations like addition, multiplication, and so on. Recall that a sequence of real numbers is a function $s : N \to R$. Since, the sum, product, etc. of real-valued functions are defined, you can easily define the sum, product etc. of sequences.

DEFINITION 10: COMPOSITION OF CONVERGENT SEQUENCES

Let (s_n) and (t_n) be two convergent sequences of real numbers. Let $\alpha \in R$. Then,

i) $(s_n) + (t_n) = (s_n + t_n)$

ii) $(s_n)(t_n) = (s_n t_n)$

iii) $\alpha(s_n) = (\alpha s_n)$

we shall show that convergence and limits are preserved under these operations.

THEOREM 10: If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then

$$\lim_{n\to\infty} (s_n + t_n) = s + t.$$

PROOF: Let $\varepsilon > 0$ be given. Then $\varepsilon/2 > 0$.

Since $\lim_{n\to\infty} s_n = s$, there is $m_1 \in N$ such that

$\left| s_n - s \right| < \frac{\varepsilon}{2}$ whenever $n \geq m_1$.

Since $\lim_{n\to\infty} t_n = t$, there is $m_2 \in N$ such that

$\left| t_n - t \right| < \frac{\varepsilon}{2}$ whenever $n \geq m_2$.

Let $m = \max(m_1, m_2)$,

Then, for all $n \geq m$, we have

$$\left|(s_n + t_n) - (s + t)\right| \leq \left| s_n - s \right| + \left| t_n - t \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $(s_n + t_n)$ converges to $s + t$.

The next theorem is easier to prove.

THEOREM 11: If $\lim_{n\to\infty} s_n = s$ and $\alpha \in R$, then $\lim_{n\to\infty} (\alpha s_n) = \alpha s$

PROOF: The theorem is obvious if $\alpha = 0$.

So, let us consider the case when $\alpha \neq 0$.

Let $\varepsilon > 0$ be given. Then $\frac{\varepsilon}{|\alpha|} > 0$, because $|\alpha| > 0$.

Since (s_n) converges to s, there is $m \in N$ such that

$\left| s_n - s \right| < \frac{\varepsilon}{|\alpha|}$ whenever $n \geq m$.

Thus, if $n \geq m$, then we have $\left|\alpha s_n - \alpha s\right| = |\alpha| \left|s_n - s\right| < |\alpha| \frac{\varepsilon}{|\alpha|} = \varepsilon$.

That is, $\lim_{n\to\infty} (\alpha s_n) = \alpha . s$.

Now you can easily solve following exercise:

EXERCISE 13

Let $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Let $\alpha, \beta \in R$. Show that $\lim_{n\to\infty} (\alpha s_n + \beta t_n) = \alpha s + \beta t$.

Deduce that $\lim_{n\to\infty} (s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n$.

Now to prove that the limit of the product of two convergent sequences is the product of their limits, we need the following theorem:

THEOREM 12: If $\lim_{n\to\infty} (s_n) = s$, then $\lim_{n\to\infty} s_n^2 = s^2$.

PROOF: Let $\varepsilon > 0$ be given. We have to find $m \in N$ such that

$\left| s_n^2 - s^2 \right| < \varepsilon$ for all $n > m$, that is,

$\left| s_n - s \right| \left| s_n + s \right| < \varepsilon$ for all $n > m$.

Since (s_n) converges, therefore it is bounded. Hence there exists a real number K such that $\left| s_n \right| \leq K$ for all n.

Since $\lim_{n\to\infty} s_n = s$, we have $|s| \leq K$. Hence

$\left| s_n + s \right| \leq \left| s_n \right| + \left| s \right| \leq 2K$ for all n.

Since $\lim_{n\to\infty} s_n = s$, there is an $m \in N$

such that $\left| s_n - s \right| < \frac{\varepsilon}{2K}$ for all $n \geq m$.

Hence, whenever n > m, we have

$$\left|s_n^2 - s^2\right| = \left|s_n - s\right| \left|s_n + s\right| < \frac{\varepsilon}{2K} . 2K = \varepsilon.$$

This proves that $\lim_{n\to\infty} s_n^2 = s^2$.

Now we are ready to prove the following theorem:

THEOREM 13: Let $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then

$$\lim_{n\to\infty} (s_n . t_n) = s.t.$$

PROOF: we use the well-known algebraic identity $ab = \frac{1}{4}[(a + b)^2 - (a - b)^2]$

Now, as $n\to\infty$, $s_n + t_n \to s + t$, therefore using theorem 12, we get

$$(s_n + t_n)^2 \to (s + t)^2$$

Also $s_n - t_n \to s - t$. Therefore again using theorem 12, we have

$$(s_n - t_n)^2 \to (s - t)^2$$

Hence, $(s_n + t_n)^2 - (s_n - t_n)^2 \to (s + t)^2 - (s - t)^2$.

Finally, using the algebraic identity, we get

$$s_n t_n = \frac{1}{4}\left[(s_n + t_n)^2 - (s_n - t_n)^2\right] \to \frac{1}{4}\left[(s + t)^2 - (s - t)^2\right] = st.$$

Verify that all the steps are justified. Notice that this proof uses no ε. The technique of using the algebraic identity to deal with the product is called **polarization**.

Finally, we turn our attention to the quotient of convergent sequences. For this, we again need the following theorem:

THEOREM 14: If $\lim_{n\to\infty} s_n = s$, and $s \neq 0$, then

$$\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}.$$

PROOF: To prove the theorem, we need $s \neq 0$ so that $\frac{1}{s}$ can be defined. But what about $\frac{1}{s_n}$? It some $s_n = 0$, then $\frac{1}{s_n}$ is not defined. To overcome this difficulty we may assume, without loss of generality, that all the s_n's are non-zero.

We shall discuss the proof for the case s > 0. The case s < 0 can be discussed by applying the case for s > 0 to the sequence $(-s_n)$.

Let ε> 0 be given. We must find m ∈ **N** such that

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| < \varepsilon \text{ whenever } n > m.$$

Since $\lim_{n\to\infty} s_n = s$, therefore there exists $m_1 \in$ **N** such that

$$\left|s_n - s\right| < \frac{\varepsilon}{2}$$

whenever $n > m_1$.

This implies that $s_n > \frac{s}{2}$ for all $n > m_1$.

Similarly, there exists $m_2 \in$ N such that

$$\left|s_n - s\right| < \frac{s^2 \varepsilon}{2} \text{ whenever } n > m_2.$$

Let $m = \max(m_1, m_2)$.

If n > m, then we have

$$\frac{\left|s_n - s\right|}{\left|s_n\ s\right|} = \frac{1}{\left|s_n\right|\left|s\right|} \left|s_n - s\right| < \frac{s^2 \varepsilon}{2} . \frac{1}{\frac{s}{2}.s} = \varepsilon$$

This proves that $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$

Now, you can prove the following theorem:

THEOREM 15: If $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$,

then $$\lim_{n\to\infty} \frac{s_n}{t_n} = \frac{s}{t}$$

EXAMPLE 11: Prove that $\lim_{n\to\infty} \left(\frac{2n^3 + 5n}{4n^3 + n^2}\right) = \frac{1}{2}$

SOLUTION: You have seen earlier that $\lim_{n\to\infty} \frac{1}{n} = 0$. Consequently,

$$\lim_{n\to\infty} \frac{2n^3 + 5n}{4n^3 + n^2} = \frac{\lim_{n\to\infty} (2 + \frac{5}{n^2})}{\lim_{n\to\infty} (4 + \frac{1}{n})} = \frac{2}{4} = \frac{1}{2}$$

(Divide the numerator and the denominator by n^3.)

What we have proved is that, if $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t = t$, then it is true that $\lim_{n\to\infty} (s_n + t_n) = s + t$. In other words, convergence of the sequences (s_n) and (t_n) is sufficient for the convergence of $(s_n + t_n)$. It is possible for $(s_n + t_n)$ to be convergent even if (s_n) and (t_n) do not converge.

Similar remarks are true for sequence (αs_n) and $(s_n t_n)$. Now try the following exercises:

EXERCISE 14

i) Given an example of divergent sequences (s_n) and (t_n) such that $(s_n + t_n)$ converges.

ii) Given an example of a divergent sequence (s_n) and a convergent sequence (t_n) such that $(s_n t_n)$ converges.

EXERCISE 15

Show that if (s_n) is a bounded sequence and if (t_n) converges to 0, then $(s_n t_n)$ converges to 0.

We have discussed the algebra of convergent sequences. Is there an algebra of divergent sequences? The following results do justify that there is algebra of divergent sequences also.

If $\lim_{n\to\infty} s_n = +\infty$ and $\lim_{n\to\infty} t_n = +\infty$, then

I $\lim_{n\to\infty} (s_n + t_n) = +\infty$,

II $\lim_{n\to\infty} (s_n t_n) = +\infty$,

III If (s_n) diverges to $+\infty$ and if (t_n) converges, then $(s_n + t_n)$ diverges to $+\infty$.

You can similarly try to formulate some similar results for the sequences diverging to minus infinity.

5.6 SUMMARY

In this unit, we have initiated the study of the limiting process by introducing the notion of a sequence and other related concepts. In section 5.2, we have defined a sequence, a subsequence and a few types of sequences such as bounded and monotonic sequences etc. We have confined our discussion to the real sequences. A real sequence is a special type of real function whose domain is the set N of natural numbers and the range is a subset of the set **R** of real numbers. If $s_n : N \to R$ is a sequence, then its values are denoted by $s_1, s_2, \ldots\ldots\ldots$ The sequence is generally denoted by (s_n) where the values $s_1, s_2, \ldots\ldots\ldots\ldots$ are known as its terms. A sequence (t_n) is called a subsequence of the sequence (s_n) if all terms of t_n are taken in order from those of (s_n). A sequence (s_n) is said to be bounded if there exists a real number K such that $|s_n| \leq K$ for every $n \in N$. A sequence (s_n) is said to be monotonically increasing if $s_n \leq s_{n+1}$ for every $n \in N$ and it is said to be monotonically decreasing if $s_n \geq s_{n+1}$ for every $n \in N$. The sequence (s_n) is said to be strictly increasing if $s_n < s_{n+1}$ (strict inequality) for every $n \in N$ and strictly decreasing if $s_n > s_{n+1}$ for every $n \in N$. A sequence which is either increasing or decreasing is said to be a monotonic sequence.

Section 5.3 deals with the convergence of a sequence. When a sequence (s_n) possesses a limit as $n \to \infty$, then it is said to be convergent. In other words, we say that a sequence (s_n) converges to a limit s if for a given $\varepsilon > 0$, there exists a positive integer m such that

$$|s_n - s| < \varepsilon \;\forall\; n > m.$$

A sequence which is not convergent is said to be a divergent sequence. This is due to the reason that (s_n) is unbounded or because (s_n) does not have a unique limiting value.

We have proved that a convergent sequence is always bounded. According to Bolzano-Weirstrass Theorem every bounded sequence has a convergent subsequence. Similarly a bounded and monotonic sequence is convergent.

In Section 5.4, we have discussed Cauchy's Criteria to test the convergence of a sequence without taking the botheration of finding the limit of the sequence. This criteria states that a sequence (s_n) is convergent if and only if for an $\varepsilon > 0$, there exists a positive integer m such that

$$|s_n - s_k| < \varepsilon, \text{ for all } n, k > m \;(n > k).$$

Finally in section 5.5, we have discussed the algebra of convergent sequences i.e. the sum, difference, product and the quotient of two convergent sequences is a sequence which is convergent under certain necessary restrictions.

5.7 ANSWERS/HINTS/SOLUTIONS

E 1) ii) is a subsequence of (n) while (i) and (iii) are not subsequence of (n).

E 2) Since $d \neq 0$, either $d > 0$ or $d < 0$. Let us consider the case $d > 0$. The case $d < 0$ is similar and can be proved in an analogous way.

You have to show that (a, a+d, a+2d, ...) is unbounded.

Clearly it is bounded below by a. We show that it cannot have an upper bound.

Suppose $m > 0$ is any number. Then, there is a number a+nd such that $a+nd > m$ for some $n \in N$. Thus there are infinitely such positive integers. Hence (a, a+d, a+2d ....) is unbounded.

E 3) i) (1, 0, 1, 0 ....) is bounded, 1 is an upper bound and 0 is a lower bound.

ii) (1, 3, 6, 10, 15, ...) is not bounded above.

iii) (1, 1, 2, 1, 3, 1, 4, 1, 5, ...) is not bounded above.

E 4) i) is not monotonic.

ii) is also not monotonic.

iii) is monotonically decreasing since $\frac{1}{1+1^2} > \frac{1}{1+2^2} > \frac{1}{1+3^2} > \ldots\ldots$

iv) The sequence is monotonically increasing.

The sequence is {1, 5, 5, 7, 7, 9, 9, ... }

E 5) i) Suppose (s_n) is a monotonic increasing sequence. That is,

$s_1 \leq s_2 \leq s_3 \leq$ ....

Let (s_{n_k}) be a subsequence. This means that

$n_1 < n_2 < n_3$ ....

Hence $s_{n_1} \leq s_{n_2} \leq s_{n_3} \leq$ ....

In other words (s_n) is monotonically increasing.

ii) Yes. Suppose (s_n) is both an increasing and a decreasing sequence. This means that $s_1 \geq s_2 \geq s_3 \geq$ .... and $s_1 \leq s_2 \leq s_3 \leq$ .... By the law of trichotomy (do you recall it), this can happen if and only if $s_1 = s_2 = s_3 = ...$ In other words constant sequences are the only sequences which are both increasing and decreasing.

E 6) Proof is similar to that of example 6.

E 7) We have $s_1 \geq s_2 \geq s_3 \geq$ .... and we are given that the sequence is bounded. Hence, by the completeness axiom, the set { s_1 , s_2 .... } has the greatest lower bound. Let a be the greatest lower bound.

Hence $s_n \geq a$ for every $n \in N$.

Let $\varepsilon > 0$ be given. Then $a + \varepsilon$ is not a lower bound of the sequence. Hence $s_m < a + \varepsilon$ for some $m \in N$. But then $a + \varepsilon > s_m \geq s_{m+1} >$ ...... But then $a - \varepsilon > s_m \geq s_{m+1} \geq s_{m+2}$ ...... $\geq a$. That is, s_n lies in the interval $]\, a - \varepsilon,\ a + \varepsilon\, [$ for all $n > m$. In other words $\lim\limits_{n \to \infty} s_n = a$.

E 8) i) We prove this by contradiction.

If possible, let $s > A$.

Take $\varepsilon = s - A$.

Since (s_n) converges to s, there exists $m \in N$ such that $|s - s_n| < \varepsilon$ for all $n > m$.

Hence $s_n > s - \varepsilon = M$, for all $n > m$ which is a contradiction of the hypothesis that $s_n \leq A$ for all n.

Hence $s \leq A$.

ii) This is entirely analogous to Part (i). You have merely to reverse the inequalities.

E 9) i) Given that (s_n) converges to s.

Let (s_{n_r}) = ($s_{n_1}, s_{n_2}, s_{n_3}$, ...) be a subsequence of (s_n).

This means, by definition, that $n_1 < n_2 < n_3 <$..........

To show that (s_{n_r}) converges to s, let $\varepsilon > 0$ be given.

Since (s_n) converges to s, there exists $m \in N$ such that $|s - s_n| < \varepsilon$ whenever $n \geq m$.

Since $n_1 < n_2 < n_3 <$........ is an increasing sequence of natural numbers, there is an integer $i \in N$ such that n_i, n_{i+1}, n_{i+2} .... are all $\geq m$.

Hence $\left|s - s_{n_r}\right| < \varepsilon$ whenever $r \geq i$.

That is, (s_{n_r}) converges to s.

ii) The statement is false. For instance, consider the sequence (1, 0, 1, 0 ....). This sequence does not converge. However (1, 1, 1, ...) is a subsequence which converges to 1.

E 10) Given that (s_n) converges to s.

To show that ($|s_n|$) converges to $|s|$.

Let $\varepsilon > 0$ be given

Since $|s_n - s| < \varepsilon$ whenever $n > m$,

then, $\big||s_n| - |s|\big| \leq |s_n - s|$. [Refer to Unit 3].

Hence, $\big||s_n| - |s|\big| < \varepsilon$ whenever $n > m$.

Hence ($|s_n|$) converges to $|s|$.

The converse is not true, as you can see by considering the non-convergent sequence (1, −1, 1, −1 ....).

E 11) A sequence (s_n) is said to diverge to $-\infty$ if, given any real number $k < 0$, there is $m \in N$ such that $s_n \leq k$ for all $n \geq m$.

E 12) Let $-1 < x < 0$. Then, $0 < |x| < 1$.

Let $\varepsilon > 0$ be given. Then

$|0 - x^n| = |x|^n$.

$|x|^n < \varepsilon$ if $n \log |x| < \log \varepsilon$.

Remembering that $\log |x| < 0$ since $0 < |x| < 1$.

$$n > \frac{\log \varepsilon}{\log |x|}.$$

Hence $|0 - x^n| < \varepsilon$ for all $n > \dfrac{\log \varepsilon}{\log |x|}$.

Hence (x^n) converges to 0.

ii) If $x = -1$, then the sequence becomes

(−1, 1, −1, +1 , ...), which we know, oscillates finitely.

If $x < -1$, then (x^n) oscillates infinitely.

E 12) Let $\lim\limits_{n\to\infty} s_n = s$ and $\lim\limits_{n\to\infty} t_n = t$.

Since (t_n) is an increasing sequence, therefore let t be the l.u.b. of the set { t_1, t_2, ... }. That is $t \geq t_n$ for every n. Hence $t \geq t_n \geq s_n$.

for every n. Thus t is an upper bound for { s_1, s_2, ... }.

Since { s_n } is an increasing sequence, therefore s is the l.u.b. of the set { s_1, s_2, ... }

Hence $s \leq t$.

E 13) $\lim\limits_{n\to\infty} (\alpha s_n) = \alpha s$ by Theorem 11.

Similarly $\lim\limits_{n\to\infty} (\beta t_n) = \beta t$ by Theorem 11.

$\lim\limits_{n\to\infty} (\alpha s_n + \beta t_n) = \lim\limits_{n\to\infty} (\alpha s_n) + \lim\limits_{n\to\infty} (\beta t_n)$ by Theorem 10

$= \alpha s + \beta t$.

$\lim\limits_{n\to\infty} (-t_n) = -\lim\limits_{n\to\infty} s_n = -t$ by Theorem 11.

Hence $\lim\limits_{n\to\infty} (s_n \; t_n) = \lim\limits_{n\to\infty} s_n + \lim\limits_{n\to\infty} (-t_n) = s-t$.

E 14) i) Let $s_n = n$ and $t_n = -n$.

Then (s_n) diverges, (t_n) diverges, but

($s_n + t_n$) is the constant sequence (0, 0, 0, . . .) which converges to 0.

ii) Let $s_n = n$ and $t_n = \frac{1}{n}$.

Then (s_n) diverges, while ($s_n \; t_n$) is the constant sequence (1, 1, 1, ...) which converges to 1.

E 15) Let $|s_n| \leq k$, for all n.

Let $\varepsilon < 0$ be given.

Since (t_n) converges to 0, there is $m \in N$ such that $|t_n| = |t_n - 0| < \frac{\varepsilon}{K}$ for all $n > m$.

Hence, for all $n > m$,

$|s_n t_n - 0| = |s_n t_n| = |s_n||t_n| \leq K.\frac{\varepsilon}{K} = \varepsilon.$

Hence $(s_n t_n)$ converges to 0.

UNIT 6 POSITIVE TERM SERIES

Structure

6.1 INTRODUCTION

In the Unit 5, you were introduced to the notion of a real sequence and its convergence to a limit. It was also stated that one of the main aims of discussing the real sequences and its convergence was to find a method of obtaining the sum of an infinite number of real numbers. In other words, we have to give a meaning to the infinite sums of the forms

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\ldots\ldots\ldots$$

$$1 + 2 + 3 + 4 + 5 + \ldots\ldots\ldots\ldots$$

where the ' .... ' is interpreted to indicate the remaining infinite number of additions which have to be performed. The clear explanation of this concept will, then, lead us to conclude that it is possible to achieve the addition of an **infinite number of real numbers,** by using the limiting process of the real sequences.

To give a satisfactory meaning to the summation of the infinite number of the terms of a sequence, we have to define a summation which is popularly known as an **infinite series.** The infinite series have been classified mainly into two categories — **the positive term series and the general series. What are, then, the positive term series and the general series? We shall try to find answers for these questions.** The summation of an infinite series of real numbers is directly connected with the convergence of the associated real sequences. We shall, therefore, **give a meaning to the term associated sequence for an infinite series and hence its convergence which will lead us ultimately to find the sum of an infinite series.**

Although the famous Greek philosopher and mathematician, Archimedes had summed up the well-known Geometric Series, yet other results on infinite series did not appear in Europe until the 14th century when Nicole Oresme [1330-1382] showed that the Harmonic series diverges. Since then, a lot of work has been going on in this direction. There is evidence that this type of work was known in India also as early as in 1550. Indeed, even modern work has shown evidence of the discovery of a number of mathematical ideas pertaining to the infinite series in China, India and Persia much before they came to be known in Europe. In the 17th century, there seemed to be little concern for the convergence of the infinite series. But during the 18th century, two French mathematicians D' Alembert and Cauchy devised remarkable tests for the convergence of infinite series under certain conditions which we shall discuss in this unit. Also, we shall discuss, in this unit, a few more tests for the convergence of the infinite series when these two basic tests fail to help us in knowing the convergence of the Infinite Series.

Objectives

Therefore, after studying this unit, you should be able to

- define an infinite series as well as a positive term series
- define the associated sequence of partial sums associated with an infinite series and hence its convergence
- use the **Ratio** and **Root Tests** to determine the convergence of infinite series
- apply Integral Test and few more tests to discuss convergence of positive term series.

6.2 INFINITE SERIES

Consider the sequence $(\frac{1}{n})$. Its terms as you know, are

$$1, \frac{1}{2}, \frac{1}{3} \ldots\ldots\ldots$$

With the help of the terms of this sequence, form an expression

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots \text{ up to } \infty,$$

which is nothing but a summation of the infinite number of terms of $(\frac{1}{n})$. Such an expression is known as an infinite series.

In general, we define an infinite series as follows:

DEFINITION 1: INFINITE SERIES

If (u_n) be a sequence of real numbers, then the expression

$$u_1 + u_2 + u_3 + \ldots\ldots\ldots\ldots\ldots\ldots \infty$$

is called an infinite series of real numbers.

The series is generally written in the form $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$, where $u_1, u_2, \ldots\ldots, u_n, \ldots\ldots$ are respectively called the first term, the second term, ... the nth term ...., of the series. We shall write just "infinite series" or "series" in place of "infinite series of real numbers".

EXAMPLE 1: i) The series a + (a+d) + (a+2d) + ... is an infinite series.

You are familiar with it. It is an Arithmetic Series with 'a' as the first term and 'd' as the common difference.

ii) The series $a+ar+ar^2 + \ldots\ldots\ldots$ is an infinite series.

You know that it is a Geometric Series with 'a' as the first term and 'r' as the common ratio.

iii) The expression 1 −1 + 1 −1 .... is an infinite series.

It has been formed by using the terms of sequence (1, −1, 1, −1 ....)

Consider the series

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots\ldots\ldots$$

We find the sum of first term, first two terms, first three terms, ..., first n terms of the series and denote them by $s_1, s_2, s_3, \ldots\ldots\ldots s_n$ respectively. Then, we have

$s_1 = 1$ (sum of the first term)

$s_2 = 1 + \frac{1}{2}$ (sum of the first two terms)

$s_3 = 1 + \frac{1}{2} + \frac{1}{4}$ (sum of the first three terms)

....

....

$s_n = 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots$ (sum of the first n terms)

$$= \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}}$$

$$= 2 [1 - (\frac{1}{2})^n]$$

From the series, we get a sequence $(s_1 , s_2, s_3 \ldots\ldots\ldots) = (s_n)$.

The sequence (s_n) is called sequence of partial sums of the series or the sequence associated with the series. In general, we define the associated sequence of any series as follows:

DEFINITION 2: SEQUENCE OF PARTIAL SUMS (ASSOCIATED SEQUENCE)

Consider the series $\Sigma\ u_n$. We form the sums $s_1 , s_2, s_3 ,\ldots\ldots\ldots$ as follows:

$s_1 = u_1$

$s_2 = u_1 + u_2$

$s_3 = u_1 + u_2 + u_3$

....

....

$s_n = u_1 + u_2 + \ldots\ldots\ldots\ldots + u_n$

Form the sequence $(s_n) = (s_1, s_2 \ldots\ldots\ldots\ldots s_n)$ where

$s_1, s_2 , s_3 , \ldots\ldots\ldots\ldots s_n$ are respectively called the first partial sum, the 2nd partial sum, the 3rd partial sum, ..., the nth partial sum. The sequence (s_n) is called sequence of partial sums of the series $\Sigma\ u_n$ or the associated sequence of the series $\Sigma\ u_n$.

EXERCISE 1

Find the sequence of partial sums of the following series:

i) $1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots\ldots$

ii) $1 + 2 + 3 + \ldots\ldots\ldots\ldots$

iii) $1 - 1 + 1 - 1 + \ldots\ldots\ldots\ldots$

Having associated a sequence to a series, we are in a position to define convergence of a series and then, give a meaning to an infinite sum.

DEFINITION 3: CONVERGENCE OF A SERIES

Let $\sum_{n=1}^{\infty} u_n$ be an infinite series with associated sequence (s_n) of partial sums where $s_n = u_1 + u_2 + \ldots\ldots\ldots\ldots u_n$. If the sequence (s_n) converges to s. we say that the series $\Sigma\ u_n$ converges to s and we write $\sum_{n=1}^{\infty} u_n = s$

and call 's' the sum of the series $\sum_{n=1}^{\infty} u_n$.

If the sequence (s_n) diverges, we say that the series $\sum_{n=1}^{\infty} u_n$ diverges. If the sequence (s_n) diverges to $+\infty$ or $-\infty$, we write $\sum_{n=1}^{\infty} u_n = +\infty$ or $\sum_{n=1}^{\infty} u_n = -\infty$ respectively.

Note that when $\sum_{n=1}^{\infty} u_n$ converges, the symbol $\sum_{n=1}^{\infty} u_n$ is used to denote not only the infinite series, but also its sum.

EXAMPLE 2: Examine the convergence of the following infinite series:

i) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

ii) $\sum_{n=1}^{\infty} n$

iii) $\sum_{n=1}^{\infty} [1+(-1)^{n+1}]$

SOLUTION: Consider the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Here

$$s_n = \frac{1}{2} + \frac{1}{2^2} + \dots\dots\dots + \frac{1}{2^n}.$$

From your knowledge of geometric series, you should be able to see that

$$s_n = 1 - \frac{1}{2^n}.$$

Since (s_n) converges to 1, the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1 and $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$

ii) Consider the series 1 + 2 + 3 + ... In this case,

$$s_n = 1 + 2 + 3 + \dots\dots\dots + n = \frac{n(n+1)}{2}.$$

The sequence (s_n) is divergent, thus the series is divergent.

iii) Consider the series $\sum_{n=1}^{\infty} [1+(-1)^{n+1}]$

Here (s_n) is 2 or 0 according as n is odd or even. In this case (s_n) = (2, 0, 2, 0, ...) is divergent. Therefore, the series is divergent.

Note that the following results follow immediately from the definition of convergence of the series.

I. The addition, omission or change of a finite number of terms of a series does not affect its behaviour regarding its convergence or divergence.

II. Multiplying the terms of a series by a non-zero number does not affect its behaviour as regards its convergence or divergence.

EXERCISE 2

i) Let a+ (a+d) + (a+2d) + ... = $\sum_{n=1}^{\infty}$ (a+ (n–1)d) be an arithmetic series. Prove that the series diverges to $+\infty$ or $-\infty$ according as $d > 0$ or $d < 0$. What can you say if $d = 0$?

ii) If a series $u_1 + u_2 + u_3 + \dots\dots\dots$ converges to s, then prove that $u_2 + u_3 + u_4 + \dots\dots\dots$ converges to $s-u_1$.

iii) Prove that the series $\sum_{n=1}^{\infty} \frac{1}{n\,(n+1)}$ converges to the sum 1.

The following theorem are immediate consequence of the theorems on sequences which you have studied in Unit 5.

THEOREM 1: If $\sum_{n=1}^{\infty} u_n$ converges to s and $\sum_{n=1}^{\infty} v_n$ converges to t, then

i) $\sum_{n=1}^{\infty} (u_n + v_n)$ converges to s+t.

and

ii) $\sum_{n=1}^{\infty} cu_n$ converges to cs for $c \in \mathbf{R}$.

PROOF: Let (s_n) and (t_n) be the sequences of partial sums of

$\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ respectively. Then the n^{th} partial sum of $\sum_{n=1}^{\infty} (u_n + v_n)$

$(u_1 + v_1) + (u_2 + v_2) + \dots\dots\dots + (u_n + v_n)$

$= (u_1 + u_2 + \dots\dots u_n) + (v_1 + v_2 + \dots\dots\dots + v_n) = s_n + t_n.$

We know that $\lim_{n\to\infty} (s_n + t_n) = \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s+t.$

Hence $\sum_{n=1}^{\infty} (u_n + v_n)$ converges to s+t. This proves (i).

For (ii), the n^{th} partial sum of $\sum_{n=1}^{\infty} cu_n$ is

$cu_1 + cu_2 + \dots\dots cu_n = c(u_1 + u_2 + \dots\dots\dots u_n) = cs_n.$

Since $\lim_{n\to\infty} (cs_n) = c \lim_{n\to\infty} s_n = cs$, therefore the series $\sum_{n=1}^{\infty} cu_n$ converges to cs.

With the same method you can easily prove that

$\sum_{n=1}^{\infty} (u_n - v_n) = s - t.$

In the following theorem, we have shown that if the series is convergent, then all the terms after some stage must become arbitrary small. What about the converse? We shall answer this question also at the end of the theorem:

THEOREM 2: If $\sum_{n=1}^{\infty} u_n$ is a convergent series, then $\lim_{n\to\infty} u_n = 0.$

PROOF: Suppose $\sum_{n=1}^{\infty} u_n = s$. Then $\lim_{n\to\infty} s_n = s$,

where (s_n) is the sequence of partial sums of $\sum_{n=1}^{\infty} u_n$.

Since $\lim_{n\to\infty} s_n = s$, therefore $\lim_{n\to\infty} s_{n-1} = s$.

Since $u_n = s_n - s_{n-1}$ therefore

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$

Thus, $\lim_{n\to\infty} u_n = 0$ is a necessary condition for $\sum_{n=1}^{\infty} u_n$ to converge.

The contrapositive form of the theorem states that if $\lim_{n\to\infty} u_n \neq 0$, then $\sum_{n=1}^{\infty} u_n$ cannot converge. This is used as a simple test for convergence of a series. The converse of the above theorem is not true i.e. the condition $\lim_{n\to\infty} u_n = 0$, is not a sufficient condition for the convergence of $\sum_{n=1}^{\infty} u_n$. In other words, there are divergent series $\sum_{n=1}^{\infty} u_n$ with $\lim_{n\to\infty} u_n = 0$.

For example, the series $\sum \frac{1}{n}$ is not convergent, although $\lim_{n\to\infty} \frac{1}{n} = 0$.

In Unit 5, you have learnt the Cauchy criterion for the convergence of a sequence. Closely connected with this, is a theorem which is known as Cauchy's Principle of convergence of infinite series. We state and prove this theorem as follows:

THEOREM 3: CAUCHY PRINCIPLE OF CONVERGENCE.

The series $\sum_{n=1}^{\infty} u_n$ converges if, and only if, given $\varepsilon > 0$, there exists $m \in N$ such that $\left| \sum_{r=k+1}^{n} u_r \right| < \varepsilon$ whenever $n, k > m$ $(n > k)$

PROOF: Let (s_n) be the sequence of partial sums of the series $\sum u_n$. By Cauchy's Principle of convergence of sequences, (s_n) is convergent iff given $\varepsilon > 0$, there exists $m \in N$ such that $|s_n - s_k| < \varepsilon$ for $n, k > m$, $(n > k)$

But $s_n - s_k = \sum_{r=k+1}^{n} u_r$ and since convergence of the sequence (s_n) implies the convergence of $\sum u_n$, it follows that $\sum u_n$ is convergent iff given $\varepsilon > 0$, there exists $m \in N$ such that

$$\left| \sum_{r=k+1}^{n} u_r \right| < \varepsilon \text{ for } n, k > m. \ (n > k)$$

You can use Cauchy's Principle of convergence as a test for convergence of a series. See the following example:

EXAMPLE 3: Test the convergence of the series $1 + \frac{1}{2} + \frac{1}{3} + \dots\dots$

SOLUTION : You know that this series is called an Harmonic Series. Suppose the series converges. By Cauchy's Principle of convergence, given $\varepsilon > 0$, $\exists\, m_1 \in N$ such that

$$\left| \sum_{r=k+1}^{n} u_r \right| < \varepsilon \text{ for } n, k > m_1 \ (n > k)$$

Take $k = m$ and $n = 2m$ where $m > m_1$. Then

$$|u_{m+1} + u_{m+2} + \dots\dots\dots + u_{2m}| < \varepsilon$$

i.e $\frac{1}{m+1} + \frac{1}{m+2} + \dots\dots + \frac{1}{2m} < \varepsilon$

But L.H.S. $> \frac{1}{2m} + \frac{1}{2m} + \dots\dots\dots\dots + \frac{1}{2m}$

$$= m\frac{1}{2m} = \frac{1}{2}.$$

This is a contradiction, if $\varepsilon < \frac{1}{2}$ say $\frac{1}{3}, \frac{1}{4}$ etc. Hence, the series does not converge.

EXERCISE 3

i) **Find the sum of the series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots\dots\dots\dots$**

ii) **Does $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)$ converge ?**

iii) **Prove that if $\sum_{n=1}^{\infty} u_n$ converges and $\sum_{n=1}^{\infty} v_n$ diverges, then $\sum_{n=1}^{\infty} (u_n + v_n)$ diverges.**

iv) **Give an example of a series $\sum_{n=1}^{\infty} u_n$ such that $(u_1 + u_2) + (u_3 + u_4) + \dots\dots\dots\dots$ converges, but $u_1 + u_2 + u_3 + u_4 + \dots\dots\dots\dots\dots$ does not converge.**

The infinite series have been divided into two major classes: The positive term series and the series with arbitrary terms both positive and negative terms, called the general series.

The easiest series to deal with are those with positive terms and most of the tests for convergence of a series are for series of positive terms. We shall study the positive term series in this unit while the series with arbitrary terms (general series) will be discussed in Unit 7.

6.3 SERIES OF POSITIVE TERMS

A series Σu_n where $u_n > 0$ for all n is called a series of positive terms or a positive term series.

Recall that the behaviour of a series is defined in terms of its associated sequence of partial sums. The sequence (s_n) associated with a series Σu_n of positive terms is a monotonic increasing sequence, since $s_{n+1} - s_n = u_{n+1} > 0$. We know that a monotonic sequence converges if it is bounded, and diverges to $+\infty$ if it is unbounded. Thus, we have the following theorem:

THEOREM 4: Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms with associated sequence (s_n). Then $\sum_{n=1}^{\infty} u_n$ converges if (s_n) is bounded, and $\sum_{n=1}^{\infty} u_n$ diverges to $+\infty$ if (s_n) is unbounded.

For example, consider the Harmonic Series $\Sigma \frac{1}{n}$. Let (s_n) be the sequence of partial sums of the series. We claim that the sequence (s_n) is unbounded. Indeed

$s_1 = 1$

$s_2 = 1 + \frac{1}{2} = \frac{3}{2}$

$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > \frac{3}{2} + \frac{1}{4} + \frac{1}{4} = 2.$

$s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 2 + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{2}.$

In general, it can be shown that $s_{2^n} > \frac{n+2}{2}$. This shows that (s_n) is itself unbounded. Hence

$\sum_{n=1}^{\infty} \frac{1}{n}$ divergers to ∞.

Let us consider another important example of the well-known Geometric Series:

EXAMPLE 4: (Geometric Series)

The series $\sum_{n=1}^{\infty} x^n$ converges to $\frac{1}{1-x}$ if $0 < x < 1$

and $\sum_{n=1}^{\infty} x^n$ diverges to ∞ if $x \geq 1$.

(The series $\sum_{n=1}^{\infty} x^n$ is called the Geometric Series)

SOLUTION: If $x \geq 1$, then the sequence (s_n) associated with the series

$\sum_{n=1}^{\infty} x^n$ of positive terms is unbounded. Indeed

$s_n = 1+x+x^2 + \ldots\ldots + x^{n-1} \geq 1+1+ \ldots +1 = n,$

But (n) is unbounded and hence divergent. Therefore (s_n) is divergent and hence the given series is divergent for $x \geq 1$.

For the case $0 < x < 1$, we have $s_n = 1+x+x^2 \ldots\ldots\ldots + x^{n-1}$.

$= \frac{1-x^n}{1-x}$

$< \frac{1}{1-x}$ for $n = 1, 2, 3, \ldots\ldots\ldots$

Therefore (s_n) is bounded and hence $\sum_{n=1}^{\infty} x^n$ is convergent.

EXERCISE 4

i) Show that if $u_1 + u_2 + \ldots\ldots$ converges to s, then so does $u_1 + 0 + u_2 + 0 + u_3 + 0 + \ldots\ldots.$ More generally, show that any number of zero terms may be inserted anywhere (or removed from anywhere) in a convergent series without affecting its convergence or its sum.

ii) Determine the convergence or divergence of the series $\sum_{n=1}^{\infty} \log\left(1 + \frac{1}{n}\right)$.

From Exercise 4 (i), it follows that the behaviour of a series of non-negative terms is determined by that of a series of positive terms. In other words, the convergence or divergence of the positive term series and the non-negative term series is same.

Now, let us study some tests of convergence of the positive term series. In Section 6.4, we discuss some general tests and in Section 6.5, we shall study some special tests of convergence.

6.4 GENERAL TESTS OF CONVERGENCE

So far, you have seen the convergence of a series. It is defined in terms of convergence of its associated sequence of partial sums. However, it is not always easy to find the sequence (s_n) and its convergence. Then, how to test the convergence of such series? For this, we state and prove some general tests for the convergence of series of positive terms.

COMPARISON TEST

The most common tests of convergence of the positive term series are the comparison tests. In these tests, we compare the series $\Sigma\, u_n$ with a series $\Sigma\, v_n$ with known behaviour. Accordingly, we decide whether the series $\Sigma\, u_n$ is convergent or divergent. This is sometimes, reworded by saying that the behaviour of the series $\Sigma\, u_n$ in terms of its convergence or divergence is dominated by the behaviour of the series $\Sigma\, v_n$. In other words, we say that if a positive term series $\Sigma\, u_n$ is dominated by a positive term series $\Sigma\, v_n$ which is convergent, then $\Sigma\, u_n$ is also convergent. Similarly, if a positive term series $\Sigma\, u_n$ dominates another positive term series $\Sigma\, v_n$ and $\Sigma\, u_n$ is divergent, then $\Sigma\, v_n$ is also divergent.

We discuss the comparison tests in the form of the following theorem:

THEOREM 5: Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be any two series of positive terms.

I) Suppose there exists a positive real number k such that $u_n < k\, v_n\ \forall\, n$. Then if Σv_n is convergent, Σu_n is convergent and if Σu_n is divergent, Σv_n is divergent.

II) Suppose $\lim_{n\to\infty} \frac{u_n}{v_n} = A$, where A is a finite non-zero real number. Then Σu_n and Σv_n converge or diverge together.

III) Suppose there exists a positive integer m such that $\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}}$ for $n \geq m$. Then if Σv_n is convergent, Σu_n is convergent and if Σu_n is divergent, Σv_n is divergent.

PROOF: I. Since $u_n < kv_n$ for all n, therefore

$u_1 + u_2 + \ldots + u_n < k\,(v_1 + v_2 \ldots + v_n)\ \forall n.$

Suppose $\Sigma\, v_n$ is convergent.

Since $\Sigma\, v_n$ converges, there must exist a positive number A such that

$v_1 + v_2 + \ldots\ldots + v_n < A,\ \forall\, n$

Consequently, $u_1 + u_2 + \ldots\ldots + u_n < kA,\ \forall\, n.$

This means the sequence of partial sums of the series Σu_n is bounded above by kA and bounded below by 0, i.e. the sequence of partial sums of Σu_n is bounded and also monotonic. Hence

Σu_n is convergent.

Similarly you can show that Σu_n is divergent implies that Σv_n is divergent.

II) Since Σu_n and Σv_n are two series of positive terms, therefore

$\frac{u_n}{v_n} > 0, \forall n,$

which implies that $\lim_{n\to\infty} \frac{u_n}{v_n} \geq 0.$

In other words, $A \geq 0$.

But by our assumption $A \neq 0$, therefore $A > 0$.

Now let us choose an $\varepsilon > 0$ (however small) such that $A - \varepsilon > 0$.

Since $\lim_{n\to\infty} \frac{u_n}{v_n} = A$, therefore there exists a positive integer m

such that

$\left|\frac{u_n}{v_n} - A\right| < \varepsilon \qquad \forall n > m$

or $A - \varepsilon < \frac{u_n}{v_n} < A + \varepsilon \qquad \forall n > m$

or $(A - \varepsilon) v_n < u_n < (A + \varepsilon) v_n \qquad \forall n > m$

Consider

$u_n < (A + \varepsilon) v_n \qquad \forall n > m$

Using (I), it follows that if Σv_n converges, Σu_n also converges. Further if Σu_n diverges, then Σv_n also diverges.

Now consider the inequality

$(A - \varepsilon)v_n < u_n \qquad \forall n > m$

Then

$v_n < \frac{1}{(A-\varepsilon)} u_n \qquad \forall n > m$

Thus, again it follows that if Σu_n converges, then Σv_n also converges and if Σv_n diverges, then Σu_n diverges. Hence Σu_n and Σv_n converge or diverge together.

You may note that in the case when convergence of Σu_n follows from the convergence of Σv_n, then A may or may not be zero. But conversely when the convergence of Σv_n follows from the convergence of Σu_n, then A must not be zero ($A \neq 0$).

III) Putting n=m, m+1, m+2, m+3, ... n–2, n–1 in the given inequality, we get

$$\frac{u_m}{u_{m+1}} > \frac{v_m}{v_{m+1}}$$
$$\frac{u_{m+1}}{u_{m+2}} > \frac{v_{m+1}}{v_{m+2}}$$
$$\cdots\cdots$$
$$\cdots\cdots$$
$$\frac{u_{n-1}}{u_n} > \frac{v_{n-1}}{v_n}$$

Multiplying the corresponding sides of the above inequalities, we get

$$\frac{u_m}{u_n} > \frac{v_m}{v_n} \qquad \forall\, n > m$$

i.e. $$\frac{u_n}{u_m} < \frac{v_n}{v_m} \qquad \forall\, n > m$$

$$u_n < \frac{u_m}{v_m} v_n \qquad \forall\, n > m.$$

Since m is a fixed integer, u_m and v_m both are positive, therefore

$\frac{u_m}{v_m}$ is a positive fixed number. Let $\frac{u_m}{v_m} = k$, where k is a fixed positive number. Then obviously, by using I, it follows that if $\Sigma\, v_n$ converges, then $\Sigma\, u_n$ converges and if $\Sigma\, u_n$ diverges, then $\Sigma\, v_n$ diverges.

This completes the proof of the theorem.

EXAMPLE 5: Test the convergence of the following series:

i) $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2^{n-1}+1} + \dots\dots$

ii) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots\dots\dots + \frac{1}{\sqrt{n}} + \dots\dots$

iii) $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots + \frac{1}{n!} + \dots\dots$

SOLUTION: (i) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \dots\dots\dots + \frac{1}{2^{n-1}+1} + \dots\dots$$

Compare the series with the convergent geometric series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \dots\dots\dots + \frac{1}{2^{n-1}} + \dots\dots$$

Clearly $\frac{1}{2^{n-1}+1} < \frac{1}{2^{n-1}}$ for each n. That is, each term of the first series is less than the corresponding term of the second series. Hence, by the Comparison Test (I) the given series converges.

ii) Consider the series $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots\dots\dots + \frac{1}{\sqrt{n}} + \dots$

Let us compare this series with the Harmonic Series

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} + \dots\dots$$

You have seen earlier that the Harmonic Series diverges.

Now, for each n, $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$. In other words, each term of the given Series is greater than the corresponding term of the harmonic series. Hence, by the Comparison Test (I) the series diverges.

iii) In the series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots + \frac{1}{n!} + \dots\dots$$

you know that n! is 'n factorial'. That is $n! = n(n-1) \dots 3.2.1$.

Does this converge? With which series shall we compare it? Let us examine it.

We know that deletion of a finite number of terms does not alter the convergence or divergence of a series. So, let us consider the terms of the above series from the third term onwards.

Now, $\frac{1}{2!} = \frac{1}{2}$

$$\frac{1}{3!} = \frac{1}{3 \times 2} < \frac{1}{2 \times 2} = \frac{1}{2^2}$$

$$\frac{1}{4!}=\frac{1}{4\times 3\times 2}<\frac{1}{2\times 2\times 2}=\frac{1}{2^3}$$

$$\frac{1}{n!}=\frac{1}{n(n-1)\ldots 3.2}<\frac{1}{2\times 2\times \ldots \times 2}=\frac{1}{2^{n-1}}$$, and so on.

We also know that $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots$ converges. Hence, by Comparison Test (I), the given series converges. This series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\ldots+\frac{1}{n!}+\ldots\ldots\ldots$ is a very important series. The number to which it converges, is denoted by e, which, as you know, is called the **exponential number or transcendental number.**

You would have noted that, in order to use the Comparison Test I, you must have a large number of known convergent and divergent series.

Let us now discuss important series which is frequently used for the Comparison Tests. This is known as the p-series namely $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots\ldots+\frac{1}{n^p}+\ldots\ldots\ldots,$ where p is a positive real number. Let us investigate the behaviour of the p-series for different values of p.

The p-series is one of most important series. Its behaviour changes from divergence to convergence as we go from p=1 to p > 1.
We state and prove the following theorem known as p-test for its convergence which depends upon the values of p.

THEOREM 5 : (p-Test)

A positive term series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ $p>0$ is convergent if $p>1$ and is divergent if $p\leq 1$.

PROOF: There are three cases namely $p = 1$, $p < 1$ and $p > 1$. We discuss these cases as follows:

Case 1: Let $p = 1$. The series is just the Harmonic Series, which has already been shown to be divergent.

Case 2: Let $p < 1$.
Since $p < 1$, $n^p \leq n$ and hence $\frac{1}{n^p}\geq\frac{1}{n}$ for each n. In other words, each term of the series is greater than the corresponding term of the divergent series $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}+\ldots.$
Hence, in this case also, p-series diverges.

Case 3 : Let $p > 1$.

To consider this case, we use the following series for comparison:

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{2^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{8^p}+\ldots\ldots\ldots$$

(The pattern should be clear).

It is clear that each term of the p-series is less than or equal to the corresponding term of this series. We claim that this series converges. Indeed, it is clear that

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)+\ldots\ldots\ldots$$

$$=\frac{1}{1^p}+\frac{2}{2^p}+\frac{4}{4^p}+\ldots\ldots\ldots$$

which is a geometric series with common ratio $\frac{2}{2^p}<1$.

Thus this series converges. Hence by Comparison Test I, the p-series also converges.

This completes the proof of the theorem.

EXERCISE 5

In the proof of theorem 5, we have grouped some terms of the series of positive terms. Prove that such grouping does not affect the nature of the series.

EXAMPLE 6: Show that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

SOLUTION: For large values of n, $\frac{n}{n^2+1}$ behaves like $\frac{n}{n^2}$ i.e. $\frac{1}{n}$.

For comparison, take the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which, you know, diverges.

Now, $u_n = \frac{n}{n^2+1}$, $v_n = \frac{1}{n}$.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ which is non-zero and finite.

Hence by the Comparison Test (II), it follows that the given series diverges.

Try the following exercises.

EXERCISE 6

Determine whether the following series are convergent?

i) $\frac{1}{4} + \frac{1}{7} + \frac{1}{10} + \dots\dots\dots + \frac{1}{3n+1} + \dots\dots\dots\dots$

ii) $\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n^2 - 4}$

EXERCISE 7

Show that if the series $\sum_{n=1}^{\infty} u_n$ of positive terms converges, then $\sum_{n=1}^{\infty} u_n^2$ also converges.

6.5 SOME SPECIAL TESTS OF CONVERGENCE

In Section 6.4, we discussed some general tests to know the convergence or divergence of infinite series. These tests enable us to deal with a fairly large number of positive term series. However, the scope is really quite limited and we are forced to look for other tests to handle a few more series. In this section, we shall develop some special tests which can be used to test the convergence of a still a larger number of infinite series. We begin our discussion with the two basic tests which are more useful and frequently employed.

The first test, called the Ratio Test, is due to J. D'Alembert [1717-1783] and the other called the Root Test is due to Cauchy, both eminent French Mathematicians.

In the Ratio Test, we discuss the convergence of a given series by studying the sequence of the ratios of the consecutive terms. Comparison Test needs another series with known behaviour for the purpose of comparison but in ratio test we use only the terms of the given series. We now state and prove D'Alembert's Ratio Test.

THEOREM 6: D'ALEMBERT'S RATIO TEST

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$$

i) If $L < 1$, then $\sum_{n=1}^{\infty} u_n$ converges.

ii) If $L > 1$, then $\sum_{n=1}^{\infty} u_n$ diverges.

iii) If $L = 1$, the test fails to give any define information about the convergence of the series.

PROOF: Case (i) Let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L < 1$.

Let r be a real number such that $L < r < 1$. Choose a number $\varepsilon > 0$ such that $L+\varepsilon = r$.

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$, there exists a positive integer m such that

$$\left|\frac{u_{n+1}}{u_n} - L\right| < \varepsilon \text{ for } n \geq m$$

i.e. $L-\varepsilon < \frac{u_{n+1}}{u_n} < L + \varepsilon$ for $n \geq m$

But $L + \varepsilon < r$, therefore

$$\frac{u_{n+1}}{u_n} < r \text{ for } n \geq m$$

i.e. $u_{n+1} < r\, u_n$ for $n \geq m$.

Jean-Le-Rond D'Alembert

Thus,

$u_{m+1} < r\, u_m$ for $n = m$

$u_{m+2} < r\, u_{m+1} < r^2 U_m$, for $n = m+1$

$u_{m+3} < r\, u_{m+2} < r^3 U_m$, for $n = m+2$.

In general, $u_{m+k} < r^k U_m$ for $k = 1, 2, 3, \ldots\ldots.$

Hence, by Comparison Test, $u_{m+1} + u_{m+2} \ldots\ldots$ converges, since it is dominated by the convergent geometric series

$u_m r + u_m r^2 \ldots\ldots + u_m r^k \ldots$

Therefore, the series $\sum_{n=1}^{\infty} u_n$ also converges.

Case (ii) Let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L > 1$.

Choose a number $\varepsilon > 0$ such that $L-\varepsilon > 1$.

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$, there exists a positive integer m such that

$$\left|\frac{u_{n+1}}{u_n} - L\right| < \varepsilon \text{ for } n \geq m$$

i.e. $L-\varepsilon < \frac{u_{n+1}}{u_n} < L + \varepsilon$ for $n \geq m$

Thus

$$\frac{u_{n+1}}{u_n} > L - \varepsilon > 1 \text{ for } n \geq m.$$

That is, $u_{n+1} > u_n$ for all $n \geq m$. This means that $\lim_{n\to\infty} u_n \neq 0$.

Hence $\sum_{n=1}^{\infty} u_n$ cannot converge. Thus $\sum_{n=1}^{\infty} u_n$ is divergent to ∞.

Case (iii) $L = 1$. The test fails because the series may converge or may diverge. The reason is that there are convergent series of positive terms with

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L.$$

and there are divergent series of positive terms with $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$.

For instance, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Here $u_n = \frac{1}{n}$, so that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

On the other hand, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Here $u_n = \frac{1}{n^2}$, so that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1.$$

Note that if, in the statement of the D'Alembert's Test, had we taken $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = L$ then $L > 1$ would imply convergence, and $L < 1$ would imply divergence of $\sum_{n=1}^{\infty} u_n$.

You may also note that if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \infty$ then the series $\sum u_n$ is divergent. You may prove it by applying the procedure of the case (ii).

EXAMPLE 7: Test the convergence of the series

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \ldots\ldots\ldots\ldots + \frac{2n-1}{2^n} + \ldots\ldots\ldots\ldots$$

SOLUTION: Here $u_n = \frac{2n-1}{2^n}$. So that $u_{n+1} = \frac{2n+1}{2^{n+1}}$.

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(2n+1)2^n}{2^{n+1}(2n-1)} = \frac{1}{2} \lim_{n\to\infty} \frac{1+\frac{1}{2n}}{1-\frac{1}{2n}} = \frac{1}{2}.$$

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{2} < 1$, the series converges.

EXERCISE 8

i) Show, using the Ratio Test, that the series $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots\ldots\ldots\ldots$ converges.

ii) For what positive values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converge?

iii) Find all positive values of x for which the series $1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$ converges.

iv) Test for convergence the series $\sum_{n=1}^{\infty} \frac{5^n}{(2n+1)!}$.

You have seen that D'Alembert's Ratio Test fails to give any definite information about the convergence or divergence or the series in some situations. In such cases, sometimes Cauchy's Root Test is helpful. But mostly Cauchy's Root Test is more suitable for those series whose nth term contains n, n^2 etc. in the exponent. In the Root Test, the convergence of a given series is based on the behaviour of the sequence formed by taking the nth root of the terms of the given series. Let us state and prove this test as the following theorem:

THEOREM 7: CAUCHY'S ROOT TEST

Let $\sum^{\infty} u_n$ be a series of positive terms such that $\lim_{n\to\infty} u_n^{1/n} = L$

i) If $L < 1$, then $\sum_{n=1}^{\infty} u_n$ converges.

ii) If $L > 1$, then $\sum_{n=1}^{\infty} u_n$ diverges.

iii) If $L = 1$, the test fails and the series may converge or diverge.

PROOF: Case (i) Let $L < 1$.

Choose a real number r such that $L < r < 1$.

Let $\varepsilon > 0$ be a number such that $L+\varepsilon = r$.

Since $\lim_{n\to\infty} u_n^{1/n} = L$, there exists $m \in N$ such that

$$\left|u_n^{1/n} - L\right| < \varepsilon \qquad \text{for } n \geq m$$

i.e. $L-\varepsilon < u_n^{1/n} < L+\varepsilon$ for $n \geq m$

$u_n^{1/n} < r$ for $n \geq m$

i.e. $u_n < r^n$ for $n \geq m$.

Since $\sum r^n$ is a geometric series with common ratio r which is less than 1, therefore it is convergent. Thus by Comparison Test, it follows that $\sum u_n$ is Convergent.

Case (ii) Let $L > 1$. Choose a real number s such that $L > s > 1$:

Let $\varepsilon > 0$ be a number such that $L-\varepsilon = s$.

Since $\lim u_n^{\frac{1}{n}} = L$, there exists $m \in N$ such that

$\left| u_n^{1/n} - L\right| < \varepsilon$ for $n \geq m$

i.e. $L-\varepsilon < u_n^{1/n} < L + \varepsilon$ for $n \geq m$.

$s < u_n^{1/n}$ for $n \geq m$

i.e. $s^n < u_n$ for $n \geq m$.

Since $\sum s^n$ is a geometric series with common ratio s which is greater than 1, therefore it is divergent. Hence by Comparison Test, $\sum u_n$ is divergent.

Case (iii) Let $L = 1$. In this case, the test fails to furnish any definite information about the convergence or divergence of the series. For example, consider the convergent series

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ and the divergent series $\sum_{n=1}^{\infty} \frac{1}{n}$. In both the cases $\lim_{n\to\infty} u_n^{1/n} = 1$.

EXAMPLE 8: Test for Convergence the Series

$$\frac{2}{1} + \left(\frac{3}{3}\right)^2 + \left(\frac{4}{5}\right)^3 + \ldots\ldots + \left(\frac{n+1}{2n-1}\right)^n + \ldots\ldots\ldots\ldots$$

SOLUTION: Here $u_n = \left(\frac{n+1}{2n-1}\right)^n$.

Since n occurs in the exponent of u_n, so we apply Cauchy's Root Test. Here

$$u_n^{1/n} = \frac{n+1}{2n-1}.$$

Therefore

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} < 1.$$

Hence the series converges.

In the above example, if you wish to apply the Ratio Test, you will have to evaluate

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} \text{ i.e.}$$

$$\lim_{n\to\infty} \left[\left(\frac{n+2}{2n+1}\right)^{n+1} \left(\frac{2n-1}{n+1}\right)^{n}\right]$$

which is certainly not easy. Now we take an example where the Ratio Test fails, but the Root Test gives a definite answer.

EXAMPLE 9: Consider the series $\sum_{n=1}^{\infty} u_n$ where

$$u_n = \begin{cases} 2^{-n-\sqrt{n}} & \text{if } n \text{ is odd} \\ 2^{-n+\sqrt{n}} & \text{if } n \text{ is even} \end{cases}$$

SOLUTION: Let us try the Ratio Test.

$$\frac{u_{2n+1}}{u_{2n}} = \frac{2^{-2n-1-\sqrt{(2n+1)}}}{2^{-2n+\sqrt{(2n)}}} = 2^{-1-\sqrt{(2n)}-\sqrt{(2n+1)}}$$

Hence $\lim_{n\to\infty} \frac{u_{2n+1}}{u_{2n}} = 0.$

On the other hand. $\frac{u_{2n}}{u_{2n-1}} = \frac{2^{-2n+\sqrt{(2n)}}}{2^{-2n+1-\sqrt{(2n-1)}}} = \frac{1}{2} \cdot 2^{\sqrt{2n}+\sqrt{2n-1}}$

Hence $\lim_{n\to\infty} \frac{u_{2n}}{u_{2n-1}} = \infty.$

In other words, the sequence $\frac{u_{n+1}}{u_n}$ does not have a limit.

Thus, the ratio test is not applicable in this case. However, the root test is applicable as is evident from the following:

For $(u_{2n})^{\frac{1}{2n}} = 2^{-1+\frac{1}{\sqrt{2n}}}$ so that $\lim_{n\to\infty} (u_{2n})^{\frac{1}{2n}} = \frac{1}{2}$

and $(u_{2n+1})^{\frac{1}{2n+1}} = 2^{-1+\frac{1}{\sqrt{2n+1}}}$ so that $\lim_{n\to\infty} (u_{2n+1})^{\frac{1}{2n+1}} = \frac{1}{2}$

Thus, $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \frac{1}{2} < 1.$

Hence the series converges.

You may note that whatever the ratio test determines the nature of a series, so does the root test. In other words, if $\lim_{n\to\infty} \frac{u_{2n+1}}{u_n} = L$ then it is true that $\lim_{n\to\infty} u_n^{\frac{1}{n}} = L$. But converse may not be true as is clear from the above example. Thus the root test is more powerful than the ratio test.

EXERCISE 9

Test the following series for convergence:

i) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

ii) $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}}$

Sometimes to discuss the nature of a series, we associate an integral to the series and discuss its convergence which is easier. This method is given by Cauchy's Integral Test which we now discuss.

Before introducing the integral test, you may recall some preliminaries regarding the Integral Calculus which you have already done in your previous studies.

Let f be a real valued function with domain $[a, \infty[$.

Suppose that f(x) is such that $\int_a^t f(x)\,dx$ has a meaning for every $t \geq a$.

Then we write

$$\phi(t) = \int_a^t f(x)dx.$$

If $\lim_{t\to\infty} \phi(t)$ exists, then we say that the integral $\int_a^\infty f(x)\,dx$ is convergent or that it exists. In that case, we write

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

If $\lim_{t\to\infty} \phi(t)$ does not exist, then it follows that $\int_a^\infty f(x)\,dx$ does not exist.

If $\lim_{t\to\infty} \phi(t) = \infty$, then the integral $\int_a^\infty f(x)\,dx$ is said to be divergent.

For example, let $f(x) = \frac{1}{x^{1/3}}$ be a function defined on the interval $[1, \infty[$. Then, we have

$$\phi(t) = \int_1^t \frac{dx}{x^{1/3}} = \left| \frac{x^{2/3}}{\frac{2}{3}} \right|_1^t = \frac{3}{2}\left[t^{2/3}-1\right].$$

Since $\lim_{t\to\infty} \phi(t) = \infty$, therefore the integral $\int_1^\infty f(x)\,dx$ is not convergent.

Let $f(x) = \frac{1}{x^4}$ be another function defined on the interval $[1, \infty[$.

Then, we have

$$\phi(t) = \int_1^t \frac{dx}{x^4} = \left| \frac{x^{-3}}{-3} \right|_1^t$$

$$= \frac{1}{3}\left[1 - \frac{1}{t^3}\right].$$

Hence $\lim_{t\to\infty} \phi(t) = \frac{1}{3}$.

In this case, we say that $\int_a^\infty f(x)\,dx$ is convergent and that its value is $\frac{1}{3}$.

THEOREM 8: CAUCHY'S INTEGRAL TEST

Let f be a real valued function with domain $[1, \infty[$ such that

i) $f(x) \geq 0, \forall\ x \geq 1$ (f is non-negative)

ii) $x < y \Rightarrow f(x) > f(y)$, (f is a monotonically decreasing function)

iii) f(x) be integrable for $x > 1$ such that $f(n) = u_n$ i.e. f (n) is associated with the series $\Sigma\, u_n$.

Then $\sum f(n)$ is convergent if and only if $\int_a^\infty f(x)dx$ is convergent and $\sum f(n)$ is divergent if and only if

$\int_a^\infty f(x)\,dx$ is divergent.

i.e. $\int_a^\infty f(x)\,dx$ and $\sum_{n=1}^{\infty} u_n$ converge or diverge together.

PROOF: Since f is a decreasing function on [1, ∞[, we have
$f(n) \le f(x) \le f(n-1)$ for $n-1 \le x \le n$, n=2, 3 ....

Consequently,

$$\int_{n-1}^{n} f(n)\,dx \le \int_{n-1}^{n} f(x)\,dx \le \int_{n-1}^{n} f(n-1)\,dx$$

i.e. $f(n) \le \int_{n-1}^{n} f(x)\,dx \le f(n-1)$ for n = 2, 3 ....

Thus,

$$\sum_{k=2}^{n} f(k) \le \sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx \le \sum_{k=2}^{n} f(k-1),$$

But,

$$\sum_{k=2}^{n} \int_{k-1}^{k} f(x)\,dx = \int_{1}^{n} f(x)\,dx \text{ and } \sum_{k=2}^{n} f(k-1) = \sum_{k=1}^{n-1} f(k).$$

Therefore, for $n \ge 2$,

$$\sum_{k=2}^{n} u_k \le \int_{1}^{n} f(x)\,dx \le \sum_{k=1}^{n-1} u_k \text{ i.e. } s_n - u_1 \le \int_{1}^{n} f(x)\,dx \le s_n - u_n$$

where (s_n) denotes the sequence of partial sums of the series $\sum u_n$. Therefore,

$$u_n \le s_n - \int_{1}^{n} f(x)\,dx \le u_1$$

If we write $A_n = s_n - \int_1^n f(x)\,dx$, we have

$$A_{n+1} - A_n = (s_{n+1} - s_n) - \left(\int_1^{n+1} f(x)\,dx, - \int_1^{n} f(x)\,dx\right)$$

$$= u_{n+1} - \int_1^{n+1} f(x)\,dx \le 0$$

Therefore, $A_{n+1} \le A_n\ \forall\, n$. Thus the sequence (A_n) is monotonically decreasing sequence. Also $A_n \ge u_n \ge 0\ \forall\, n$, therefore the sequence (A_n) is bounded below. Consequently (A_n) is convergent.

Now

$$s_n = A_n + \int_1^n f(x)\,dx$$

The convergence of (A_n) implies that (s_n) and $\left(\int_1^n f(x)\,dx\right)$ converge or diverge together.

Hence $\sum u_n$ and $\int_1^\infty f(x)\,dx$ converge or diverge together.

The idea underlying the Cauchy's Integral Test and its proof is self-explanatory from the Figure 1.

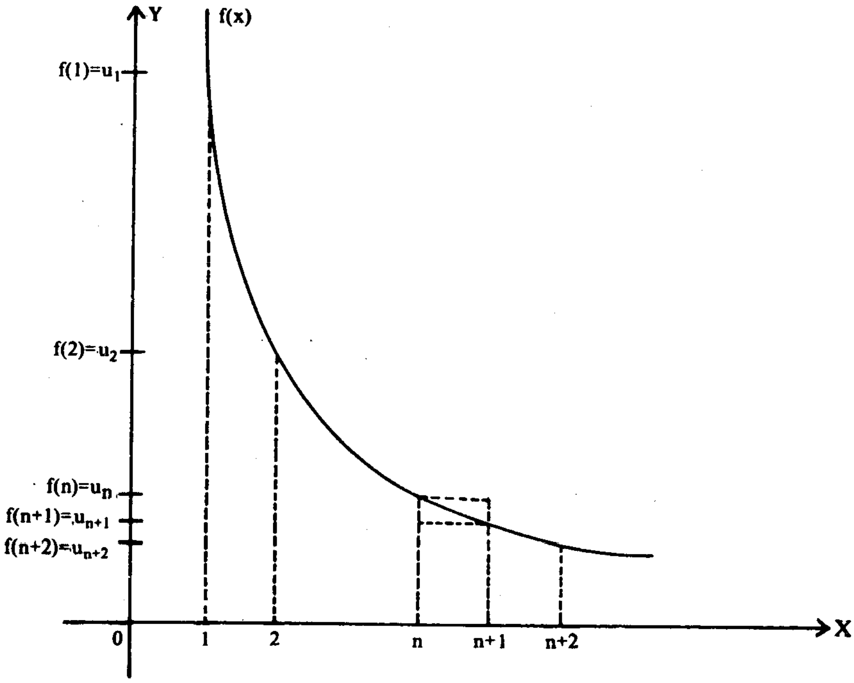


Fig. 1

You may note that if the conditions of Cauchy's Integral Test are satisfied for $x \geq k$ (a positive integer), then $\sum_{n=k}^{\infty} u_n$ and $\int_k^{\infty} f(x)\,dx$ converge or diverge together. This can be seen from the following example:

EXAMPLE 10: Discuss the Convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, p > 0 by using the Integrai Test.

SOLUTION : Here $u_n = \frac{1}{n^p}$

Let $f(x) = \frac{1}{x^p}$

For $p > 0$, f is a decreasing, positive integrable function. So by Cauchy's Integral Test,

$\sum_{n=1}^{\infty} \frac{1}{n^p}$, and $\int_1^{\infty} f(x)\,dx$ converge or diverge together.

$$\int_1^{x} f(x)\,dx = \int_1^{x} \frac{dx}{x^p}$$

$$= \begin{cases} \log x \text{ if } p = 1 \\ \frac{x^{1-p} - 1}{1-p} \text{ if } p \neq 1 \end{cases}$$

$$\to \begin{cases} \infty \text{ if } 0 < p \leq 1 \\ \frac{1}{p-1} \text{ if } p > 1 \end{cases}$$

as $x \to \infty$

Therefore

$\int_1^{\infty} f(x)\,dx$ converges for $p > 1$ and diverges for $0 < p \leq 1$ and hence the series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $0 < p < 1$.

EXAMPLE 11: Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, **where p > 0.**

SOLUTION: Let $f(x) = \frac{1}{x(\log x)^p}$ for $x > 2$.

If $p > 0$, then f is a positive, decreasing, integrable function on $[2, \infty[$. Hence by Cauchy's Integral Test, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ and $\int_2^{\infty} \frac{1}{x(\log x)^p}$ converge or diverge together.

We have, for $p > 0$

$$\int_2^x \frac{dx}{x(\log x)^p} = \begin{cases} \log(\log x) - \log(\log 2) & \text{if } p = 1 \\ \frac{(\log x)^{1-p} - (\log 2)^{1-p}}{1-p} & \text{if } p \neq 1 \end{cases}$$

$$\rightarrow \begin{cases} \infty \text{ if } p \leq 1 \\ \frac{(\log 2)^{1-p}}{p-1} \text{ if } p > 1 \end{cases}$$

as $x \rightarrow \infty$

This shows that $\int_2^{\infty} f(x)\,dx$ converges if $p > 1$ and diverges if $0 < p \leq 1$. Therefore the given series $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

EXERCISE 10

Discuss the convergence of the series

$$\sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^p} \quad (p > 0).$$

In general, it is difficult to determine whether an arbitrary positive term series is covergent or divergent. There is no single universal test or method that will deal with all possible cases. We have discussed several useful tests including the popular ones like the Ratio Test and the Root Test. Most of these tests have been derived in some way from one of the forms of the comparison test. We now discuss some more tests which may be applied when all the earliest tests fail. In particular, some of these new tests will be helpful when the Ratio Test and the Root Test fail. We have selected only two tests to be discussed in our course. These are Raabe's Test and Gauss's Test.

Raabe (1801-1859) was Professor at Zurich. He made lot of important contributions to Geometry and Analysis. He gave a test for the convergence of a series of positive terms, which is often decisive when the D'Alembert's Test fails. We state the test, without proof and discuss examples to illustrate its use.

THEOREM 9: RAABE'S TEST

Let $\sum_{n=1}^{\infty} u_n$ **be a series of positive numbers such that**

$$\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = L$$

Then i) $\sum_{n=1}^{\infty} u_n$ **converges if** $L > 1$

ii) $\sum_{n=1}^{\infty} u_n$ **diverges, if** $L < 1$

iii) the test is inconclusive, if L = 1.

Let us look at an example.

EXAMPLE 12 : Test the convergence of the series

$$\frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \ldots\ldots\ldots\ldots$$

SOLUTION : Here $u_n = \frac{2.4.6 \ldots\ldots (2n+2)}{3.5.7 \ldots\ldots (2n+3)}$

$$\text{Hence } \frac{u_n}{u_{n+1}} = \frac{2n+5}{2n+4} \text{ and}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2n+5}{2n+4} = 1.$$

Note that the cases $L = +\infty$ and $L = -\infty$ are also included in (i) and (ii) in the test.

Thus, the ratio test fails. But

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{n}{2n+4} = \frac{1}{2} < 1.$$

Hence, by Raabe's Test $\sum_{n=1}^{\infty} u_n$ diverges.

EXERCISE 11

Test the convergence of the series

$$\frac{1}{2}x + \frac{1.3}{2.4}x + \frac{1.3.5}{2.4.6}x^2 + \ldots\ldots\ldots\ldots\ldots (x > 0)$$

We end this section by discussing Gauss's Test.

Gauss (1777-1855) an eminent German mathematician, gave a very powerfũl test for convergence which is applicable if Raabe's Test fails. It is not essential that first we apply Ratio Test, then Raabe's Test (if Ratio Test fails) and finally Gauss's Test (if Raabe's Test also fails). We can straightaway apply Gauss Test. Both D'Alembert's Ratio Test and Raabe's Test are included in this test. We only state this test and then illustrate it by an example.

THEOREM 10: GAUSS'S TEST

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms. Suppose

$$\frac{u_n}{u_{n+1}} = a + \frac{b}{n} + \frac{r_n}{n^p}$$

where a, b, p ε R, a > 0, p > 1 and (r_n) is a bounded sequence.

Then:

i) $\sum_{n=1}^{\infty} u_n$ converges, if $a > 1$

ii) $\sum_{n=1}^{\infty} u_n$ diverges, if $a < 1$

iii) $\sum_{n=1}^{\infty} u_n$ converges, if $a = 1, b > 1$

iv) $\sum_{n=1}^{\infty} u_n$ diverges, if $a = 1, b \le 1$

EXAMPLE 13: Test the convergence of the series

$$\frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2 7^2} + \ldots.$$

SOLUTION: In this case,

$$u_n = \frac{2^2.4^2.6^2 \cdots\cdots (2n)^2}{3^2.5^2.7^2\cdots\cdots (2n+1)^2}$$

Thus

$$\frac{u_n}{u_{n+1}} = \frac{(2n+3)^2}{(2n+2)^2} = \left(1 + \frac{3}{2n}\right)^2 \Big/ \left(1 + \frac{1}{n}\right)^2$$

$$= \left(1 + \frac{9}{4n^2} + \frac{3}{n}\right) \Big/ \left(1 - \frac{2}{n} + \frac{3}{n^2} - \frac{4}{n^3} \cdots\cdots\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2}\left(-3/4 + \text{powers of } \frac{1}{n}\right)$$

Here $r_n = -\frac{3}{4} +$ powers of $\frac{1}{n}$. Therefore, (r_n) is a bounded sequence. Since the coefficient of $\frac{1}{n}$ is 1, by Gauss's Test, the given series is divergent.

EXERCISE 12

Discuss the convergence of the series

$$1 + \frac{\alpha.\beta}{1!\gamma} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{2!\,\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)\,(\alpha+2)\,\beta(\beta+1)\,(\beta+2)}{3!\,\gamma(\gamma+1)\,(\gamma+2)}x^3 + \ldots\ldots$$

where x, α, β, γ are positive numbers.

6.6 SUMMARY

In this unit, you have been introduced to the notion of an infinite series and the concept of convergence of an infinite series. Series of positive terms were taken up for consideration, and various tests of convergence of series of positive terms were discussed.

In Section 6.2 we gave the definition of an infinite series and gave a meaning of the infinite sum as its convergence. Although an infinite summation seems to be artificial, yet by using the powerful tools of the limit concept, we are able to give a very concrete meaning to an infinite sum. The convergence of an infinite series is shown in terms of the convergence of the associated (corresponding) sequence of partial sums of its terms. The basic technique is to find an explicit formula for the nth term of the sequence of partial sums. The convergence of this sequence implies the convergence of the corresponding series.

Infinite series have been divided into two major classes — the one with positive terms and the other consisting of the arbitary terms. In Section 6.3, we deal with the positive term series. The notion of convergence of these series has been introduced. General tests of convergence such as the comparison tests under different conditions and the p-test have been discussed in Section 6.4.

In 6.5, we have discussed some special tests. Notable among these are the two basic tests – D'Alembert's Ratio Test and Cauchy's Root Test. We also studied another important test– Cauchy's Integral Test. Finally, we have discussed two more useful tests namely Raabe's Test and Gauss's Test for the convergence of the positive term series.

6.7 ANSWERS/HINTS/SOLUTIONS

E 1) i) $\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots + \frac{1}{n}\right)$

ii) $\left(\frac{n(n+1)}{2}\right)$

iii) (1, 0, 1, 0, ...)

E 2) i) For the series a+(a+d) + (a+2d) + ..., the n^{th} partial sum is

$$s_n = \frac{n}{2}\,[2a + (n-1)d],\ n = 1, 2, 3, \ldots\ldots$$

If d = 0, then s_n = na.

If a ≠ 0, then (s_n) = (na) is divergent.

If a = 0, then (s_n) = (0) and it converges to 0.

If d = 0. the series diverges if a ≠ 0

and converges if a = 0.

Now suppose d > 0. Let k > 0 be given.

$s_n > k$ if $\frac{n}{2}[2a+(n-1)d] > k$

Hence $s_n > k$ if 2a + (n–1) d > k.

This happens whenever (n–1) d > k –2a,

that is, whenever n > k+d–2a.

Let m ∈ N such that m > k+d–2a.

Then $s_n > k$ whenever n ≥ m.

Hence (s_n) diverges to +∞. Therefore, by definition, the series

$$\sum_{n=1}^{\infty} [a+(n-1)d] \text{ diverges to } +\infty.$$

You should be able to take care of the case d < 0.

ii) Let (s_n) be the sequence associated with u_1+u_2+ ....

and (t_n) the sequence associated with u_2+u_3+ ....

Clearly $t_{n-1} = s_n - u_1$. (n ≥ 2)

Hence $\lim\limits_{n\to\infty} t_{n-1} = \lim\limits_{n\to\infty} s_n - u_1 = s - u_1$,

Therefore, $u_2 + u_3 + \ldots\ldots$ converges to $s - u_1$.

iii) Hence $s_n = \frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)}$.

$$= (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \ldots\ldots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}$$

Thus, $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (1 - \frac{1}{n+1}) = 1$.

Hence, the series $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.

E3) i) For the geometric series $1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots\ldots$

$$s_n = 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \ldots\ldots + (-1)^{n-1}\frac{1}{4^{n-1}}$$

$$= \frac{1-(-\frac{1}{4})^n}{1-(-\frac{1}{4})} = \frac{1-(-\frac{1}{4})^n}{5/4} = \frac{4}{5}\,[1-(-\frac{1}{4})^n].$$

$$\lim_{n\to\infty} s_n = \frac{4}{5}\left[1-\lim_{n\to\infty}(-\frac{1}{4})^n\right] = \frac{4}{5}$$

Hence the series $1-\frac{1}{4}+\frac{1}{16}-\frac{1}{64}+$ .... converges to $\frac{4}{5}$.

[Recall from Unit 5 that $\lim_{n\to\infty} x^n = 0$ if $-1 < x < 1$].

ii) Here $u_n = \frac{n+1}{n+2}$ and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n+1}{n+2} = 1 \neq 0$.

Hence, $\sum_{n=1}^{\infty} u_n$ does not converge.

iii) If (s_n) and (t_n) are sequences of partial sums of Σu_n and Σv_n respectively, then sequence of partial sums of $\Sigma(u_n + v_n)$ is (s_n+t_n) which is divergent, since (s_n) is convergent and (t_n) is divergent.

Hence $\sum_{n=1}^{\infty}(u_n + v_n)$ is divergent.

iv) Consider the series $1-1+1-1+1-1+$ ....

Here $u_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even.} \end{cases}$

You can see that the n^{th} partial sum s_n is given by

$s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

Hence (s_n) is not convergent. Thus the series $1-1+1-1+$ .... is not convergent

However the series $(1-1)+(1-1)+(1-1)+$ .... is the series $0+0+0+$ .... which, clearly, coverges to 0.

E4) i) If (s_n) and (t_n) are the sequences associated with the series $u_1 + u_2 +$.........., and $u_1 + 0 + u_2 + 0 +$.............,

you can see that $t_n = \begin{cases} s_m & \text{if } n = 2m \\ s_m & \text{if } n = 2m-1 \end{cases}$

That is, the sequence $(t_1, t_2, t_3, \ldots)$ is

$(s_1, s_1, s_2, s_2, s_3, s_3 \ldots)$. It is easy to see that

$(s_1, s_1, s_2, s_2, s_3, s_3, \ldots)$ also converges to s.

ii) Here, $s_n = \log(1+1) + \log(1+\frac{1}{2}) + \ldots + \log(1+\frac{1}{n})$

$= \log(2) + \log(\frac{3}{2}) + \log(\frac{4}{3}) + \ldots \log\left(\frac{n+1}{n}\right)$

$= \log[2.\frac{3}{2}.\frac{4}{3} \ldots \frac{n+1}{n}] = \log(n+1)$.

$\lim_{n\to\infty} s_n = \lim_{n\to\infty}\log(n+1) = \infty$.

Hence the series $\sum_{n=1}^{\infty} \log(1+\frac{1}{n})$ diverges to ∞.

E5) In E 3 (iv) you saw that grouping terms of a series attered its behaviour.

However, such a thing will not happen in case of a series of positive terms. That is the purpose of this exercise.

Suppose $\sum_{n=1}^{\infty} u_n$ is a convergent series of positive terms. Hence (s_n) is an increasing sequence of positive tenns bounded above, where

$s_n = u_1 + u_2 + \dots\dots\dots u_n$.

If the terms of $\sum_{n=1}^{\infty} u_n$ are grouped, and if (t_n) is the sequence associated with the new series, then, it is easy to see that (t_n) is a subsequence of (s_n) and hence converges to the same limit as that of (s_n).

To show that a divergent series of positive terms remains divergent under grouping is easier to prove. Try it.

E 6) i) Here $u_n = \frac{1}{3n+1}$. Take $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{3n+1} = 3.$

Also $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Hence, $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ diverges.

ii) Here $u_n = \frac{\sqrt{n}}{n^2-4}$. Take $v_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2-4} = 1.$

Also $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is the p-series will $p = 3/2 > 1$, and hence converges.

Thus $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2-4}$ converges.

E 7) Since $\sum_{n=1}^{\infty} u_n$ converges, $\lim_{n\to\infty} u_n = 0$.

Hence, there is $M \in N$ such that $u_n < 1$ for $n \geq M$.
So, for $n \geq M$, $u_n^2 < u_n$.

Hence, by Comparison Test I, $\sum_{n=1}^{\infty} u_n^2$ converges.

E 8) i) Here $u_n = \frac{1}{n!}$ $u_{n+1} = \frac{1}{(n+1)!}$

Hence, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

ii) Let us first use the ratio test. Here

$u_n = \frac{x^n}{n}, u_{n+1} = \frac{x^{n+1}}{n+1}.$

Hence $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(\frac{x^{n+1}}{n+1} \frac{n}{x^n}\right) = x \lim_{n\to\infty} \frac{n}{n+1} = x.$

Thus, when $x < 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges, and when $x > 1$, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ diverges.

It remains to consider the case $x = 1$ because then ratio test fails. When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, which diverges.

Thus, finally, $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges if $0 < x < 1$

and diverges if $x \geq 1$.

iii) Here $u_n = n\,x^{n-1}$, $u_{n+1} = (n+1)\,x^n$.

Hence $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x \lim_{n\to\infty}\left(\frac{n+1}{n}\right) = x$.

So, $\sum_{n=1}^{\infty} nx^{n-1}$ converges when $x < 1$ and diverges when $x > 1$ by D'Alembert Ratio Test and test fails for $x = 1$.

However, when $x = 1$, the series becomes

$1+2+3+4+ \ldots\ldots\ldots$, which obviously diverges.

iv) Here $u_n = \frac{5^n}{(2n+1)!}$ $u_{n+1} = \frac{5^{n+1}}{(2^{n+3})}$.

Hence $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{5^n}$

$= 5 \lim_{n\to\infty} \frac{1}{(2n+3)\ (2n+2)} = 0 < 1.$

Hence the series converges.

E 9) i) $u_n = \frac{1}{(\log n)^n}$. So $(u_n)^{\frac{1}{n}} = \frac{1}{\log n}$

$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1.$

Hence $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$ converges.

ii) $u_n = \frac{1}{n^n} \cdot u_n^{\frac{1}{n}} = \frac{1}{n}$.

$\lim_{n\to\infty} u_n^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges.

E 10) Let $f(x) = \frac{1}{x \log x\ [\log \log x)^p}$

$f(x)$ is a decreasing, positive, integrable function of x and

$f(n) = \frac{1}{n \log n\ [\log (\log n)]^p}$ for $n \geq 3$

Hence $\sum_{n=3}^{\infty} \frac{1}{n \log n\ [\log (\log n)]^p}$ and

$$\int_3^{\infty} \frac{dx}{x \log x\ [\log (\log x)]^p}$$

converge or diverge together.

$$\int_3^x \frac{dx}{x \log x\,[\log(\log x)]^p} = \begin{cases} \log(\log(\log x)) - \log(\log(\log 3)) & \text{if } p = 1 \\ \dfrac{[\log(\log x)]^{1-p} - [\log(\log 3)]^{1-p}}{1-p} & \text{if } p \neq 1. \end{cases}$$

Hence the integral converges when $p > 1$ and diverges when $p \le 1$. Hence, also, the given series converges when $p > 1$ and diverges when $p \le 1$.

E 11) Here $u_n = \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots\ldots.(2n)} x^n$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\,\frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$$

By Ratio Test, the series converges if $x < 1$ and diverges if $x > 1$.

If $x = 1$, Ratio Test fails.

When $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$ so that

$\lim_{n\to\infty} \left(\frac{u_n}{u_{n+1}} - 1\right) = \frac{1}{2}$ and so the series diverges by Raabe's Test.

Hence the series is convergent for $x > 1$ and divergent for $x \ge 1$.

E 12) Here $u_n = \frac{\alpha(\alpha+1)\ldots.(\alpha+n-1)\,\beta(\beta+1)\ldots.(\beta+n-1)}{1.2\ldots\ldots.n\,r(r+1)\ldots..(r+n-1)} x^n$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)(r+n)}{(\alpha+n)(\beta+n)}\,\frac{1}{x} = \frac{n^2+(r+1)n+r}{n^2+(\alpha+\beta)n+\alpha\beta}\,\frac{1}{x}$$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = x$ and so by Ratio Test, series converges for $x < 1$ and diverges for $x > 1$:

For $x = 1$, $\frac{u_n}{u_{n+1}} = 1 + \frac{(\gamma+1-\alpha-\beta)\,n + (\gamma-\alpha\beta)}{n^2 + (\alpha+\beta)n + \alpha\beta}$

$= 1 + \frac{(\gamma+1-\alpha-\beta)}{n} + \frac{s_n}{n^2}$ for some bounded sequence s_n.

By Gauss' Test, the given series converges if

$\gamma+1-\alpha-\beta > 1$ i.e. $\gamma > \alpha+\beta$ and diverges when $\gamma+1-\alpha-\beta \le 1$.

i.e. $\gamma \le \alpha+\beta$.

Hence (i) if $x < 1$, series converges for all positive value of α, β, γ

ii) if $x > 1$, series diverges for all positive values of α, β, γ

iii) if $x = 1$, series converges for $r > \alpha+\beta$ and diverges for $\gamma \le \alpha+\beta$.

UNIT 7 GENERAL SERIES

Structure

7.1 INTRODUCTION

In Unit 6, we dealt with the positive term series. Accordingly, we developed the convergence tests which are applicable only to the positive term series. But, as you are aware that, an infinite series need not be always a positive term series. In fact, an infinite series, in general, can have both an infinite number of negative terms as well as an infinite number of positive terms. The series which have both negative and positive terms may be classified into two major categories. The first category consists of those infinite series whose terms are alternately positive and negative. Such series are called **Alternating Series.** The other category is one in which terms need not necessarily be alternately positive and negative that is to say, the infinite series whose terms are mixed and do not follow any pattern of being positive or negative.
For example, the infinite series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

has alternately positive and negative terms, whereas the infinite series

$$1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\ldots\ldots\ldots\ldots\ldots\ldots$$

does not follow any specific pattern.

The question, therefore, arises: **How to test the convergence of such infinite series?** The convergence tests discussed in Unit 6 are not suitable enough for the purpose because these tests in their present form cannot be applied to these series. Hence, we have to either modify these tests or devise new tests of convergence for such series.

We shall discuss an important test applicable to the series with terms alternately positive and negative. This test is known as **Leibnitz test.**

To determine the convergence of other infinite series having no pattern of positive and negative terms, we shall introduce the notions of **Absolute** and **Conditional Convergence** of the infinite series. Finally, we shall have a brief discussion on the method of **Rearrangement** of series. There are a few more categories of general series viz. Power-Series which we intend to discuss in some other advanced level course in Analysis at a later stage.

Objectives

Therefore, after studying this unit, you should be able to

- recognize an Alternating Series
- apoly the Leibnitz Test to know the convergence of an Alternating Series
- identify an absolutely convergent series and a conditionally Convergent Series
- have an idea about the method of rearrangement of the terms of an infinite series to know its convergence or divergence.

7.2 ALTERNATING SERIES

In this section, we shall consider series whose terms are alternately positive and negative. Such series are called **'Alternating Series'.**
For example, the infinite series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \quad ....$$

and

$$-1 + 2 - 3 + 4 - 5 + \; ....$$

are alternating series. Formally, we define an alternating series in the following way :

DEFINITION 1 : An infinite series $\sum_{n=1}^{\infty} u_n$ is called an Alternating Series if any two consecutive terms of the series are of opposite sign.

An Alternating Series may, thus, be written as $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \; ....$

where, each $u_n > 0$.

If the first term is negative, then it can be written as

$$\sum_{n=1}^{\infty} (-1)^n u_n = -u_1 + u_2 - u_3 + u_4 + ...$$

Gottfried Wilhelm Leibnitz

The second series can be obtained from the first if you multiply each term of the first series by -1. Therefore, it is enough to discuss the convergence of the first series.
There is a very simple test for the convergence of an Alternating Series provided there is a sequence (s_n) of partial sums of u_1, u_2,............... which is monotonically decreasing and convergent to 0. This test is known as Leibnitz Test after the name of Leibnitz, the eminent German mathematician.

THEOREM 1: (Leibnitz Test).

Let $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ be an Alternating Series such that

(i) $u_i > 0 \; \forall \; i = 1, 2, 3, ...$

(ii) $u_1 \geq u_2 \geq u_3 \geq$ .... i.e.(u_n) is a monotonically decreasing sequence

(iii) $\lim_{n \to \infty} u_n = 0.$

Then the series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ is convergent.

PROOF : Note that we are assuming that the odd terms of thc alternating series are positive and the even terms are negative.
Let (s_n) be the sequence associated with the series

$u_1 - u_2 + u_3 - u_4 + ...$

Let us first consider the partial sums with odd index namely those ending in positive terms i.e.

$s_1, s_3, s_5,$ ....

Then, we have

$s_1 = u_1 > 0$

$s_3 = u_1 - u_2 + u_3 = s_1 - (u_2 - u_3) \leq s_1$ since $u_2 \geq u_3$.

Similarly, we have

$s_5 = u_1 - u_2 + u_3 - u_4 + u_5$

$\;\; = s_3 + u_5 - u_4$

$s_5 - s_3 \leq u_5 - u_4 \leq 0$

i.e. $s_5 \leq s_3 \leq s_1$.

In general,

$s_{2n+1} = s_{2n-1} - (u_{2n} - u_{2n+1}) \leq s_{2n-1}$.

Thus $s_1 \geq s_3 \geq \ldots\ldots\ldots\ldots \geq s_{2n-1} \geq s_{2n+1} \geq \ldots\ldots\ldots\ldots$

This shows that (s_{2n-1}) is a monotonic decreasing sequence. Also

$s_{2n-1} = (u_1 - u_2) + (u_3 - u_4) + \ldots\ldots + (u_{2n-3} - u_{2n-2}) + u_{2n-1} \geq 0.$

This shows that (s_{2n-1}) is bounded below with 0 as the lower bound.

Therefore (s_{2n-1}) is a monotonic decreasing sequence which is bounded below. Hence it is convergent.

Suppose (s_{2n-1}) converges to a limit s.

Then $\lim_{n\to\infty} s_{2n-1} = s.$

In the same way, you can compute for the sequence of even partial sums and show that

$$s_{2n+2} \geq s_{2n}$$

i.e. the sequence (s_{2n}) or even partial sums is monotonically increasing.

Also

$$s_{2n} = u_1 - \left[(u_2 - u_3) + (u_4 - u_5) + \ldots\ldots\ldots\ldots + \left(u_{2n-2} - u_{2n-1}\right) + u_{2n}\right]$$

But $u_{i+1} \leq u_i \ \forall\ i = 2, 3, \ldots\ldots\ldots\ldots$

Therefore $u_2 - u_3 \geq 0$

$u_4 - u_5 \geq 0$

....

....

Hence

$s_{2n} = u_1$ – some positive quantity (number) which, shows that

$s_{2n} < u_1 \ \forall\ n.$

Thus (s_{2n}) is bounded above.

Hence (s_{2n}) is convergent. Suppose it converges to a limit t. Then

$\lim_{n\to\infty} s_{2n} = t.$

Since $u_{2n} = s_{2n} - s_{2n-1}$, we have, therefore, by the condition (ii)

$0 = \lim_{n\to\infty} u_{2n} = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n-1} = s - t.$

Thus $s = t$.

Thus both (s_{2n}) and (s_{2n-1}) converge to the same limit s.

Hence

$\lim_{n\to\infty} s_{2n} = s.$

Finally, we shall show that the sequence (s_n) converges to s. Let $\varepsilon > 0$ be given. Since the sequence (s_{2n}) converges to s, therefore there exists a positive integer m_1 such that

$|s_{2n} - s| < \varepsilon \ \forall\ (2n) > m_1$

Similarly, given $\varepsilon > 0$, there exists a positive integer m_2 such that

$|s_{2n-1} - s| < \varepsilon \ \forall\ (2n-1) > m_2$

Thus it follows that

$|s_n - s| < \varepsilon \ \forall\ n > \max.(m_1, m_2)$

This implies that (s_n) converges to s or

$\lim_{n\to\infty} s_n = s.$

Therefore it follows that the alternating series

$u_1 - u_2 + u_3 - u_4 + \ldots\ldots\ldots$

converges to the limit s, which, in fact, is the sum of the series.

EXERCISE 1

Prove Theorem 1 for the Alternating Series of the form

$\sum_{n=1}^{\infty} (-1)^n u_n$ [that is, where the odd terms are negative and even terms are positive].

From Unit 6, you know that the condition $\lim_{n\to\infty} u_n = 0$ is necessary for the convergence of every infinite series $\sum_{n=1}^{\infty} u_n$. But according to Leibnitz Test, if the given infinite series is an alternating series decreasing in absolute values, then the condition $\lim_{n\to\infty} u_n = 0$ is also sufficient for the convergence. Let us now study some examples and exercises:

EXAMPLE 1: Test the convergence of the Alternating Series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots\ldots$$

SOLUTION: This series is known as the Alternating Harmonic Series.

In this series the conditions of the Leibnitz Test are satisfied.

Here (i) each $u_i > 0$ $i = 1, 2, \ldots\ldots$

(ii) $u_1 > u_2 > u_3 > u_4 \ldots\ldots\ldots$

(iii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0.$

Hence, the series is convergent. The limit or sum of this series is well-known and is equal to log 2.

(See Example 6 in this section).

EXERCISE 2

Show that $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{3n+2}$ diverges.

EXAMPLE 2: Test the convergence of the Alternating Series

$3 - 3^{\frac{1}{2}} + 3^{\frac{1}{3}} - 3^{\frac{1}{4}} + 3^{\frac{1}{5}} \ldots\ldots\ldots$

SOLUTION: Here since $3 > 3^{\frac{1}{2}} > 3^{\frac{1}{3}} > \ldots\ldots$, therefore first and second conditions of the Leibnitz test are satisfied.

However, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} 3^{\frac{1}{n}} = 1 \neq 0$. (Recall from Unit 5).

Since the third condition of the Leibnitz test is not satisified, therefore the given series is divergent.

EXERCISE 3

For what values of p does the series

$\frac{1}{1^p} - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots\ldots\ldots$ converge?

Now consider once again the **Harmonic Series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots\ldots\ldots$$

You know that this series diverges. But the Alternating Harmonic Series (as discussed in Example 1) namely

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots$$

converges. Thus, we have a series that converges only because some of its terms are negative. If all the negative terms are replaced by the corresponding positive terms, then the convergence is demolished. To study this phenomenon in a more general way, we introduce the notions of absolute convergence and conditional convergence in the next section.

7.3 ABSOLUTE AND CONDITIONAL CONVERGENCE

Consider the following two series:

i) $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \ldots\ldots\ldots$

and

ii) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots\ldots$

By the Leibnitz Test, both series converge.

Again consider the two series

iii) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots\ldots\ldots$ and

iv) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \ldots\ldots\ldots$

obtained from (i) and (ii) by replacing each term by its positive value.

The series (iii) converges, while the series (iv) diverges.

This leads us to divide the convergent series into two classes, namely, the absolutely convergent series and the conditionally convergent series, which we define as follows:

DEFINITION 2: Let $\sum_{n=1}^{\infty} u_n$ be an infinite series of real numbers.

i) If $\sum_{n=1}^{\infty} |u_n|$ converges, then we say that the series $\sum_{n=1}^{\infty} u_n$ converges absolutely.

ii) If $\sum_{n=1}^{\infty} u_n$ converges but $\sum_{n=1}^{\infty} |u_n|$ diverges, we say that $\sum_{n=1}^{\infty} u_n$ converges conditionally.

Thus the series (i) converges absolutely, while series (ii) converges conditionally.

Note that in (ii) we have defined $\sum_{n=1}^{\infty} u_n$ to be conditionally convergent if $\sum_{n=1}^{\infty} u_n$ is convergent but $\sum_{n=1}^{\infty} |u_n|$ is divergent. In (i) we have defined $\sum_{n=1}^{\infty} u_n$ to be absolutely convergent if $\sum_{n=1}^{\infty} |u_n|$ is convergent. but we have not said anything about the behaviour of $\sum_{n=1}^{\infty} u_n$ itself.

EXAMPLE 3: (i) the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is absolutely convergent.

(ii) The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{(Alternating Harmonic Series)}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$$

are conditionally convergent.

The following theorem provides that we can produce examples of absolutely convergent series by changing algebraic signs of some or all of the terms of a convergent series of positive terms.

THEOREM 2: If an infinite series is absolutely convergent, then it is convergent.

PROOF: Let Σu_n be an absolutely convergent series i.e. $\Sigma |u_n|$ is convergent. Then we have to prove that Σu_n is convergent.

Let (s_n) be the sequence of partial sums of Σu_n. Then

$s_n = u_1 + u_2 + \ldots\ldots\ldots\ldots + u_n$.

It is enough to show that (s_n) is a Cauchy sequence.

Let (t_n) be the sequence associated with the series $\Sigma |u_n|$. Since $\Sigma |u_n|$ is convergent, therefore (t_n) is also convergent. Thus (t_n) is a Cauchy sequence. In other words, for an $\varepsilon > 0$, there exists a positive integer m such that

$|t_n - t_k| < \varepsilon$ for $n > m$, $k > m$.

Suppose $n > k$. Then

$$|s_n - s_k| = |u_{k+1} + u_{k+2} + \ldots\ldots\ldots\ldots + u_n|$$

$$\leq |u_{k+1}| + |u_{k+2}| + \ldots\ldots\ldots\ldots + |u_n| \text{ (Recall from Unit 3).}$$

$$= |t_n - t_k| < \varepsilon.$$

Which shows that (s_n) is a Cauchy sequence. This completes the proof of the theorem.

Thus every absolutely convergent series is convergent. The converse, however, is not true. That is to say that if a series is convergent, then it may not be absolutely convergent. Can you give an example ? Try it.

EXAMPLE 4: Test the absolute and conditional convergence of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$$

SOLUTION: Here $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \frac{1}{2n+1}$

Let $v_n = \frac{1}{n}$ for $n = 1, 2, 3, \ldots\ldots\ldots\ldots$

Then the series $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Also,

$$\lim_{n\to\infty} \frac{|u_n|}{v_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2}.$$

Hence by comparison test, it follows that

$\sum_{n=1}^{\infty} |u_n|$ is divergent. Thus

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^{n+1}}$ is not absolutely convergent.

However, since $\frac{1}{2n+1} > \frac{1}{2n-1} \forall n$.

and $\lim_{n\to\infty} \frac{1}{2n+1} = 0$, therefore by Leibnitz test, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is convergent. In other words, it is a case of conditional convergence.

Now try the following exercises.

EXERCISE 4

i) **Test the convergence and absolute convergence of the series**

$$\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{1}{n^3} + \frac{1}{n^5}\right)$$

ii) **Determine the values of p for which the series**

$1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \frac{1}{5^p}$ .... **converges conditionally.**

All the tests of convergence of infinite series discussed in Unit 6 for series of positive terms can be used to decide absolute convergence of general series.

EXAMPLE 5: Test the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$, $x \in R$, **for convergence.**

SOLUTION: Here $u_n = \frac{\cos nx}{n^2}$

Since $|\cos nx| \le 1$, we obtain $|u_n| \le \frac{1}{n^2}$.

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, (Recall from Unit 6, why it so?)

This means that the series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent.

Hence, the series $\sum_{n=1}^{\infty} u_n$ i.e. $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is convergent.

Therefore $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ converges absolutely for all $x \in R$.

EXERCISE 5

i) **Let $\sum_{n=1}^{\infty} u_n$ be absolutely convergent and (s_n) be a bounded sequence.**

Prove that $\sum_{n=1}^{\infty} s_n u_n$ is also absolutely convergent.

ii) **Can a series of positive terms be conditionally convergent?**

7.4 REARRANGEMENT OF SERIES

You know that to find the sum of a finite number of real numbers, the order in which they are added does not matter. But this is not the case when you have to find the sum of infinite series of numbers. The order in which the terms occur in an infinite series may affect its nature and the sum i.e. the convergence of the series.

In this section, we shall discuss this aspect of the infinite series. This is, also, sometimes referred to as the **Rearrangement Convergence,** where every rearrangement of the series converges. We first have the following definition:

DEFINITION 3: Let $\sum_{n=1}^{\infty} u_n$ be an infinite series. Let π be a one-to-one function from N onto N. Then $\sum_{n=1}^{\infty} u_{\pi(n)}$ is said to be a rearrangement of $\sum_{n=1}^{\infty} u_n$.

For example $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots\ldots\ldots\ldots$ is a rearrangement of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots\ldots\ldots\ldots\ldots$$

We state two results (without proof) which will indicate the effect on the convergence of a series due to the order in which the terms occur in the series.

I If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series converging to s, then every rearrangement of $\sum_{n=1}^{\infty} u_n$ also converges to s.

Thus, the order in which the terms occur is immaterial in absolutely convergent series.

What of conditionally convergent series? To answer this question, we state the following result:

II Let $\sum_{n=1}^{\infty} u_n$ be a conditionally convergent series. Given any $\alpha \in R$, there is a rearrangement of the series $\sum_{n=1}^{\infty} u_n$ which converges to α.

Let us give an illustration of a Rearrangement and show how the sum or the convergence is altered:

EXAMPLE 6: Show that

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots = \log 2.$$

Evaluate the sum of the rearranged series

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots\ldots\ldots\ldots\ldots$$

SOLUTION: Set a number r_n as

$$r_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ldots\ldots\ldots + \frac{1}{n} - \log n$$

It is a well-known result that the sequence (r_n) converges to a limit r. Let us, therefore, assume that

$$\lim_{n\to\infty} r_n = r.$$

Let (s_n) denote the sequence of the partial sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\ldots\ldots\ldots$$

and (t_n) denote the sequence of the partial sum of the rearrangement of the series namely

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots\ldots\ldots\ldots$$

Then, we have

$$s_{2n} = 1 - \frac{1}{2} + \frac{1}{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{2n-1} - \frac{1}{2n}$$

$$= \left(1 + \frac{1}{2} + \ldots\ldots\ldots + \frac{1}{2n}\right) - 2\left[\frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots\ldots + \frac{1}{2n}\right]$$

$$= \left(1 + \frac{1}{2} + \dots\dots\dots + \frac{1}{2n}\right) - \left[1 + \frac{1}{2} + \dots\dots\dots\dots + \frac{1}{n}\right]$$

$$= [r_{2n} + \log 2n] - [r_n + \log n]$$

$$= [r_{2n} - r_n] + \log 2n - \log n$$

$$= [r_{2n} - r_n] + \log \frac{2n}{n}$$

$$= [r_{2n} - r_n] + \log 2$$

Since (r_n) is convergent, therefore (r_n) is a Cauchy Sequence. Consequently, there exists $m \in N$ such that $|r_{2n} - r_n| < \epsilon$ for $n \geq m$ where $\varepsilon > 0$ is any number.

This implies that

$$\lim_{n \to \infty} s_{2n} = \log 2.$$

Now, it is easy to show that

$$\lim_{n \to \infty} s_n = \log 2.$$

For the sequence t_n, we have

$$t_{3n} = \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \dots\dots + \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right)$$

$$= \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \dots\dots + \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right)$$

$$= \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots\dots\dots\dots\dots + \frac{1}{4n-1} + \frac{1}{4n}\right]$$

$$- \frac{1}{2}\left[1 + \frac{1}{2} + \dots\dots\dots\dots + \frac{1}{2n}\right] - \frac{1}{2}\left[1 + \frac{1}{2} + \dots\dots\dots\dots + \frac{1}{n}\right]$$

Thus,

$$t_{3n} = (r_{4n} + \log 4n) - \frac{1}{2}(r_{2n} + \log 2n) - \frac{1}{2}(r_n + \log n)$$

$$= \left(r_{4n} - \frac{1}{2} r_{2n} - \frac{1}{2} r_n\right) + \frac{3}{2} \log 2$$

Again since (r_n) is a Cauchy Sequence, therefore,

$$\lim_{n \to \infty} t_{3n} = \frac{3}{2} \log 2.$$

Since $t_{2n+1} = t_{3n} + \frac{1}{4n+1}$ **and** $t_{3n+2} = t_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3}$.

Therefore

$$\lim_{n \to \infty} t_n = \frac{3}{2} \log 2.$$

This shows that the arrangement of a conditional convergent series may change its sum.

EXERCISE 6

Suppose $\sum_{n=1}^{\infty} u_n$ **is a series of positive terms diverging to** $+\infty$**. Show that every rearrangement of** $\sum_{n=1}^{\infty} u_n$ **also diverges to** $+\infty$**.**

7.5 SUMMARY

This unit has been mainly concerned with series of arbitrary real numbers. A very important example of such a series is an Alternating Series. To test convergence of an Alternating Series, we apply the most useful test known as Leibnitz Test, which we have discussed in

Section 7.2. In Section 7.3, we dealt with another category, the series of arbitrary terms, the one which does not follow any pattern of its terms. Such series may be **Absolutely Convergent Series** and conditionally convergent series. Absolutely Convergent Series are stable under any rearrangement, in the sense, that no rearrangement can disturb the convergence or sum of an absolutely convergent series. On the other hand, you can make a conditionally convergent series behave as you wish by a suitable rearrangement which we defined and demonstrated in Section 7.4.

Precisely speaking, in this unit we have studied three notions related to the infinite series of arbitrary terms namely

i) the convergence of the Alternating Series

ii) the absolute and conditional convergence of the series

iii) the rearrangement convergence, where every rearrangement of the series converge.

7.6 ANSWERS/HINTS/SOLUTIONS

E 1) The alternating series is –

$-u_1 + u_2 - u_3 + u_4$ ....

Where $u_1, u_2, u_3,$ .... are positive numbers satisfying

$u_1 \geq u_2 \geq u_3 \geq$ ....

and

$\lim\limits_{n\to\infty} u_n = 0.$

Let (t_n) be the sequence of the partial sums of this series. Also, let (s_n) be the sequence of the partial sums of the series

$u_1 - u_2 + u_3 - u_4 +$ ....

It is obvious that $t_n = -s_n$ for each n.

By Theorem 1, (s_n) converges. Hence (t_n) = ($-s_n$) also converges.

E 2) Here $u_n = (-1)^{n+1}\dfrac{n}{3n+2}$.

Therefore $|u_n| = \dfrac{n}{3n+2}$. Also $\lim\limits_{n\to\infty} |u_n| = \lim\limits_{n\to\infty} \dfrac{n}{3n+2} = \dfrac{1}{3}$.

Since $\lim\limits_{n\to\infty} u_n$ is not equal 0, therefore the series is divergent.

E 3) Case (i) Let p > 0. Then

$\dfrac{1}{1^p} > \dfrac{1}{2^p} > \dfrac{1}{3^p} >$ ....

and

$\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0.$

Hence the series converges.

Case (ii) Let p = 0. Then the series is

1–1+1–1+ ....

But $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} (-1)^n$ does not exist.

Therefore, the series diverges.

Case (iii) Let p < 0. Again

$\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{(-1)^{n-1}}{n^p}$

does not exist. Hence the series diverges.

REVIEW

In this Block, you have studied the notion of a sequence and its convergence. Also, you have been introduced to the infinite series and their convergence. The infinite series have been classified into two types of series namely positives term series i.e. the series with positive terms only and the series with a mix of both positive as well as negative terms. Accordingly, various tests for the convergence of the corresponding series have been discussed. You should now attempt the following self-test questions to ascertain whether or not you have achieved the main objectives of learning the material in this block. You may compare your solutions/ answers with the ones given at the end.

1 **Given below are the sequences whose nth term is given. Write the range of each of these sequences and determine which of these are bounded and unbounded.**

(i) $\frac{1}{n^2}$

(ii) $(-1)^n$

(iii) $n^{\frac{1}{n}}$

(iv) $\cos \frac{n\pi}{3}$

(v) $(1 + \frac{1}{n})^n$

2 **For each of the following sequences determine whether it converges or not. If it converges, then give its limit.**

(i) $s_n = \frac{n}{n+1}$

(ii) $s_n = \frac{n^2+3}{n^2-3}$

(iii) $s_n = 2^{-n}$

(iv) $s_n = 3 + (-1)^n$

(v) $s_n = \frac{(-1)^n}{n}$

3 **Determine which of the following sequences are monotonic ?**

(i) $\frac{3n+2}{2n-5}$

(ii) $\frac{n^2+1}{n}$

(iii) $\frac{2n^2-1}{2n^2+1}$

(iv) $1 + \frac{1}{n^2}$

(v) $n + (-1)^n$

4 **Find the sum of the first n terms of the following series and hence decide whether each series converges or diverges. If the series converges, find its sum :**

(i) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \ldots\ldots\ldots\ldots$

(ii) $2 - 4 + 6 - 8 + \ldots\ldots\ldots\ldots\ldots\ldots$

(iii) $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots\ldots\ldots\ldots$

(iv) $\log 2 + \log \frac{3}{2} + \log \frac{4}{3} + \ldots\ldots\ldots$

REVIEW

In this Block, you have studied the notion of a sequence and its convergence. Also, you have been introduced to the infinite series and their convergence. The infinite series have been classified into two types of series namely positives term series i.e. the series with positive terms only and the series with a mix of both positive as well as negative terms. Accordingly, various tests for the convergence of the corresponding series have been discussed. You should now attempt the following self-test questions to ascertain whether or not you have achieved the main objectives of learning the material in this block. You may compare your solutions/ answers with the ones given at the end.

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(iii) $s_n = 2^{-n}$

(iv) $s_n = 3+(-1)^n$

(v) $s_n = \frac{(-1)^n}{n}$

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(iv) $1+\frac{1}{n^2}$

(v) $n+(-1)^n$

4 **Find the sum of the first n terms of the following series and hence decide whether each series converges or diverges. If the series converges, find its sum :**

(i) $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+$ ....

(ii) $2-4+6-8+$ ....

(iii) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+$ ....

(iv) $\log 2+\log\frac{3}{2}+\log\frac{4}{3}+$ ....

(v) $\frac{1}{3.4} + \frac{2}{5.6} + \frac{3}{7.8} + \ldots\ldots$

5 **Test the convergence of the following series by using various tests of convergence:**

(i) $\Sigma \frac{2^n - 1}{5^n - 1}$

(ii) $\Sigma \frac{1}{n.3^n}$

(iii) $\Sigma \frac{2^n}{n!}$

(iv) $2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \ldots\ldots$

6 **Discuss the convergence of the following series :**

(i) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ (ii) $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right)$

(ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan \frac{1}{n}$

7 **Test each of the following series for convergence :**

(i) $\sum_{n=1}^{\infty} u_n$. where $u_n = \frac{1}{5^n + 10}$

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n^2}$

8 **Give an example of a series $\sum_{n=1}^{\infty} u_n$ of positive terms such that $\frac{u_{n+1}}{u_n} < 1$ for each n, but the series diverges. Why does your example not contradict the Ratio Test ?**

9 **Show that, for any $p > 0$, the series $\sum_{n=1}^{\infty} \frac{1}{n^{p+1} + 1}$ converges.**

10 **Discuss the convergence of the series $\sum_{n=1}^{\infty} n\, e^{-n}$ by using Cauchy's Integral Test.**

ANSWERS/HINTS

1 (i) $\{1, \frac{1}{4}, \frac{1}{9}, \ldots\ldots\}$ unbounded

(ii) $\{-1, 1\}$ bounded

(iii) $\{1, \sqrt{2}, 3^{1/3}, 4^{1/4}, \ldots\}$ unbounded

(iv) $\{\frac{1}{2}, \frac{-1}{2}, -1, 1\}$ bounded

(v) $\{2, (3/2)^2, (4/3)^3, (5/4)^4, \ldots\ldots\}$ unbounded

2 (i) converges to 1

(ii) converges to 1

(iii) converges to 0

(iv) does not converge

(v) converges to 0

3 (i) decreasing for $n \geq 3$

(ii) increasing

(iii) increasing

(iv) decreasing

(v) not monotonic

4 (i) $1 - (\frac{2}{3})^n$; converges to 1

(ii) $s_{2n-1} = 2n$, $s_{2n} = -2n$; infinite oscillation

(iii) $1 - \frac{1}{n+1}$; converges to 1

(iv) $\log(n + 1)$; diverges

(v) $\frac{n}{(2n+1)(2n+2)}$, converges to $\frac{1}{4}$

5 (i) use Comparison Test

(ii) use Cauchy's Root Test

(iii) use D'Alembert's Ratio Test

(iv) use D'Alembert's Ratio Test

(v) divergent

6 (i) convergent

(ii) divergent as $\lim_{n \to \infty} \sin \frac{n\pi}{2}$ does not exist.

(iii) convergent. Take $v_n = \frac{1}{n^{3/2}}$ and use the comparison test.

7 (i) $\frac{1}{5^n+10} < \frac{1}{5^n}$ and $\Sigma \frac{1}{5^n} = \Sigma (\frac{1}{5})^n$ which is a Geometric series with common ratio $\frac{1}{5} < 1$. Hence, the series is convergent.

(ii) Since $\frac{1}{n^2 \log n} < \frac{1}{n^2}$ for $n \geq 3$, therefore $\Sigma \frac{1}{n^2 \log n}$ is convergent by comparison test.

(iii) rationalise and then use Comparison Test. The series is convergent.

8 Let $\Sigma u_n = \Sigma \frac{1}{n}$, then

$$\frac{u_{n+1}}{u_n} = \frac{n}{n+1} < 1 \ \forall\ n.$$

But the series $\Sigma \frac{1}{n}$ is divergent.

This does not contradict the Ratio Test because when we take the limits as $n \to \infty$, we find that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$$

9 Use Integral Test.

10 Convergent.

NOTES

NOTES

NOTES

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UGMM - 09
Real Analysis

Block

3

LIMIT AND CONTINUITY

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BLOCK 3 LIMIT AND CONTINUITY

PREVIEW

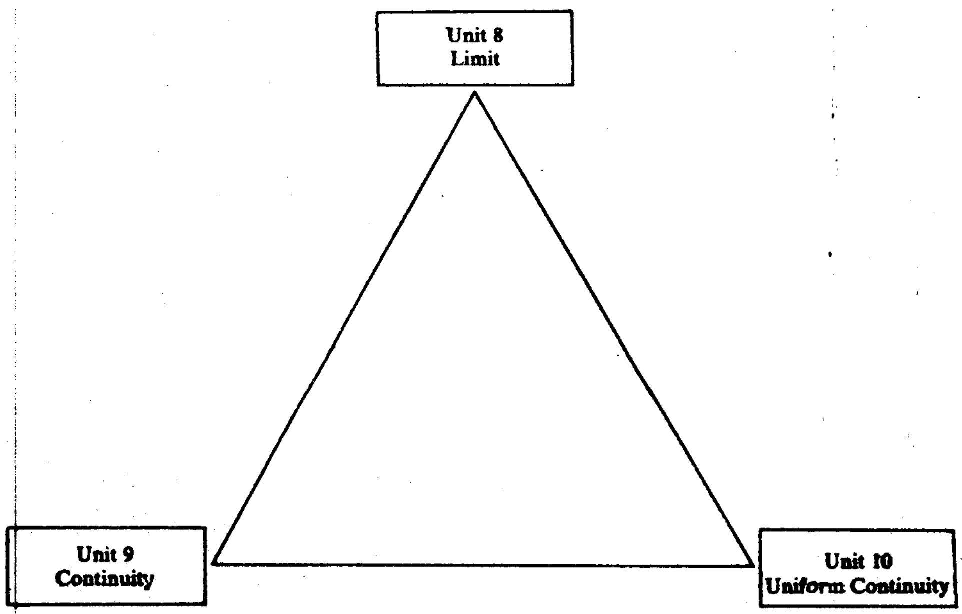
In Block 1, you were introduced to the system of real numbers and the limit point of a set of real numbers. Also, you were introduced to some real functions in this Block. In Block 2, we considered a special function called the sequence and began the study of limiting processes with the notion of convergence of infinite sequences and series.

In this block, we shall study the limit concept as applied to arbitrary real functions. The limit of a function, in general, is an abstract notion in the sense that the function never attains its value at a point but tries to approach a value called the limiting value. This limiting value of the function f(x) serves as an approximate value of the function f(x) for values of x near a (if the limit is obtained as x tends to a) or for large values of x (if the limit is obtained as x tends to infinity). In short, the precise notion of a limit covers the vague notions of approximate values.

The limit concept is fundamental to all further ideas in Real Analysis. Therefore, we shall develop it in this block and then use it to discuss the differentiability of functions in Block 4. This block contains three units. In the first unit of this block i.e. in Unit 8, we have reviewed the notion of the **limit of a function** to which you are already familiar from your study of Calculus. We have tried to illustrate certain basic facts about limits through a number of examples. We have also attempted to help you to appreciate the rigorous notion of **epsilon-delta definition** of the limit of a function and its **geometrical meaning**. Closely related to the limit of a function is the notion of sequential limits which also we shall introduce in this Unit. Finally, we discuss the algebra of limits.

In Unit 9, we introduce the notion of the **continuity** of a function at a point and extend it to the continuity of a function on an interval or on a non-empty set of real numbers. Also, we discuss some continuous and discontinuous functions as well as the algebra of continuous functions.

We use the results of Unit 9 to discuss the properties of continuous functions in Unit 10. Also, in Unit 10, we shall introduce the notion of uniform continuity of a function.



NOTATIONS AND SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	is a member of (belongs to)
$\notin$	is not a member of (does not belong to)
$\subset$	is a subset of (is contained in)
$\not\subset$	is not a subset of (is not contained in)
$\supset$	is a superset
$\cup$	Union
$\cap$	intersection
ϕ	empty set
$\Rightarrow$	implies
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\sim$	equivalence relation
$\forall$	for all
$\exists$	there exists
.	multiplication
$+$	addition
$-$	subtraction
sup	supremum
inf	infimum
min	minimum
max	maximum
$\circ$	composition
f'	derivative of f
f^{-1}	inverse of a function f
exp	exponential
log	logarithm
In	natural logarithm
sgn	signum
[x]	greatest integer not exceeding x
$\|x\|$	absolute value of x or Modulus of x
R^+	set of positive real numbers
R	set of real numbers
I	Set of irrational numbers
Q	set of rational numbers
Z	set of integers
N	set of natural numbers
F	field
C	set of complex numbers
[a, b]	closed interval
]a, b[	open interval
]a, b]	semi-open interval (open at left)—semi-closed interval
[a, b[	semi-open interval (open at right)—semi-closed interval
$+\infty$	infinity
$-\infty$	minus infinity
Σ	sum
$\sum_{n=1}^{\infty} u_n$	infinite series
(s_n)	sequence
S^c	complement of S
S'	derived set of S
$\bar{S}$	closure of S

Greek Alphabets

α	Alpha
β	Beta
γ	Gama
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
i	Iota
λ	Lambda
μ	Mu
ν	Nu
ξ	exi
π	Pi
Π	(capital Pi)
ρ	Rho
σ (Σ)	Sigma (capital Sigma)
τ	Tou
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

UNIT 8 LIMIT OF A FUNCTION

STRUCTURE

8.1 INTRODUCTION

In Unit 5, we dealt with sequences and their limits. As you know, sequences are functions whose domain is the set of natural numbers. In this unit, we discuss the limiting process for the real functions with domains as subsets of the set R of real numbers and range also a subset of R. What is the **precise meaning for the intuitive idea of the values f(x) of a function f tending to or approaching a number A as x approaches the number a?** The search for an answer to this question shall enable you to understand the concept of the limit which you have used in calculus. We shall give a rigorous meaning to the intuitive idea of the limit of a function in Section 8.2. The relation between the limit of a function and the limit of a sequence is established in Section 8.3. The effect of algebraic operations of addition, subtraction, multiplication and division on the limits of functions is examined in Section 8.4. It will then be extended to study the effect of these algebraic operations on the continuity of a function in Unit 9.

Objectives

Thus after studying this unit, you should be able to

- define limit of a function at a point and find its value
- know sequential approach to limit of a function
- find the limit of sum, difference, product and quotient of functions.

8.2 NOTION OF LIMIT

The intuitive idea of limit was used both by Newton and Leibnitz in their independent invention of Differential Calculus around 1675. Later this notion of limit was also developed by D'Alembert. **"When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all the others."**

Consider a simple example in which a function f is defined as

$$f(x) = 2x + 3, \qquad \forall\, x \in R, x \neq 1.$$

Give x the values which are near to 1 in the following way:

When x = 1.5, 1.4, 1.3, 1.2, 1.1, 1.01, 1.001

f(x) = 6, 5.8, 5.6, 5.4, 5.2, 5.02, 5.002

When x = .5, .6, .7, .8, .9, .99, .999

f(x) = 4, 4.2, 4.4, 4.6, 4.8, 4.98, 4.998

You can form a table for these values as follows:

x	.5	.6	.7	.8	.9	.99	.999	1.001	1.01	1.1	1.2	1.3	1.4	1.5
f(x)	4	4.2	4.4	4.6	4.8	4.98	4.998	5.002	5.02	5.2	5.4	5.6	5.8	6

You see that as the values of x approach 1, the values of f(x) approach 5. This is expressed by saying that limit of f(x) is 5 as x approaches 1. You may note that when we consider the limit of f(x) as x approaches 1, we do not consider the value of f(x) at x = 1.

Thus, in general, we can say as follows:
Let f be a real function defined in a neighbourhood of a point x = a except possibly at a. Suppose that as x approaches a, the values taken by f approach more and more closely a value A. In other words, suppose that the numerical difference between A and the values taken by f can be made as small as we please by taking values of x sufficiently close to a. Then we say that f tends to the limit A as x tends to a. We write

$$f(x) \to A \text{ as } x \to a$$

or

$$\lim_{x \to a} f(x) = A.$$

This intuitive idea of the limit of a function can be expressed mathematically as formulated by the German mathematician Karl Weierstrass in the late 18th Century. Thus, we have the following definition:

DEFINITION 1 : Limit of a Function
Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. The function f is said to tend to or approach a number A as x tends to or approaches a number 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \quad \text{for } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = A.$$

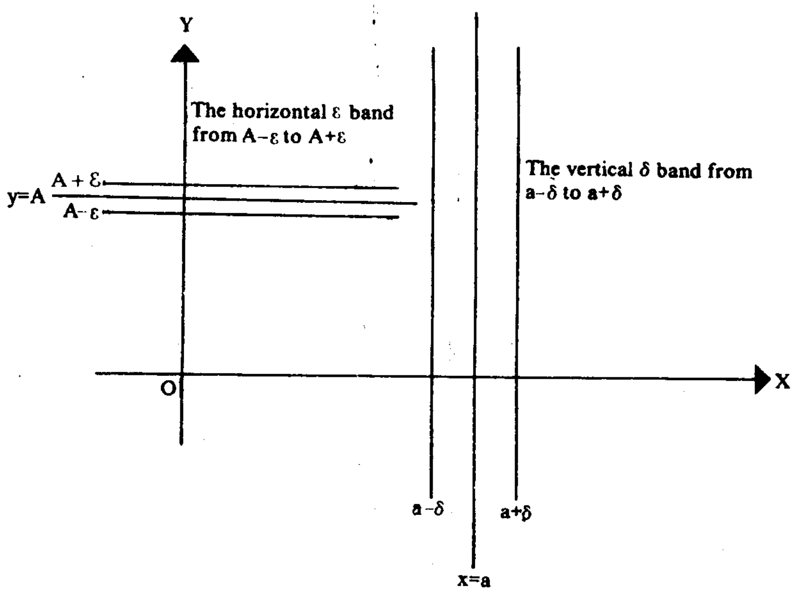
You may note that

$$|f(x) - A| < \varepsilon \quad \text{for } 0 < |x - a| < \delta.$$

can be equivalently written as (see Unit 3)

$$f(x) \in \,]A - \varepsilon, A + \varepsilon[\text{ for } x \in \,]a - \delta, a + \delta[\text{ and } x \neq a.$$

This is shown geometrically in Fig. 1. The inequality $0 < |x - a| < \delta$ determines the interval $]a - \delta, a + \delta[$ minus the point 'a' along the X-axis and the inequality $|f(x) - A| < \varepsilon$ determines the interval $]A - \varepsilon, A + \varepsilon[$ along the Y-axis.



EXAMPLE 1 : Let a function f: R → R be defined as

$$f(x) = x^2, \forall x \in R.$$

Find its limit when x → 2.

SOLUTION : By intuition, it follows that

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} x^2 = 4.$$

Let us verify this with the help of $\varepsilon - \delta$ definition. In other words, we have to show that for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| \ \varepsilon.$$

Suppose that an $\varepsilon > 0$ is fixed. Then consider the quantity $|f(x) - 4|$.

$$|f(x) - 4| = |x^2 - 4| = |(x - 2)(x + 2)|$$

Note that the term $|x - 2|$ is exactly the same that appears in the δ-inequality in the definition. Therefore this term should be less than δ. In other words,

$$|x - 2| < \delta$$
$$\Rightarrow 2 - \delta < x < 2 + \delta$$
$$\Rightarrow x \in]\, 2 - \delta, 2 + \delta\, [.$$

We restrict δ to a value 2 so that x lies in the interval $]\, 2 - \delta, 2 + \delta\, [\subset]\, 0, 4\, [$. Accordingly, then $|x + 2| < 6$. Thus, if $\delta \leq 2$, then

$$|x - 2| < 2 \Rightarrow 0 < |x + 2| < 6$$

and further that

$$|x - 2| < \delta \leq 2 \Rightarrow |x + 2|\,|x - 2| < 6\,|x - 2| < 6\,\delta.$$

If δ is small then so is $6\,\delta$. In fact it can be made less than ε by choosing δ suitably.
Let us, therefore, select δ such that $\delta = \min.(2, \varepsilon/6)$. Then

$$0 < |x - 2| < \delta \Rightarrow |f(x) - 4| < 6\,|x - 2| < 6.\,\delta \leq 6.\,\varepsilon/6 = \varepsilon.$$

This completes the solution.

Note that the first step is to manipulate the term $|f(x) - A|$ by using algebra. The second step is to use a suitable strategy to manipulate $|f(x) - A|$ into the form

$$|x - a| \text{ (trash)}$$

where the 'trash' is some expression which has the property that it is bounded provided that δ is sufficiently small. Why we use the term 'trash' for the expression as a multiple of $|x - a|$? The reason is that once we know that it is bounded, we can replace it by a number and forget about it.

In example 1, the number 6 arose by virtue of this 'trash'. If you take $\delta \leq 3$ (instead of $\delta \leq 2$), you can still show that 6 will be replaced by 7. In that case you can set δ as

$$\delta = \min.(3, \varepsilon/7)$$

and the proof will be complete. Thus, there is nothing special about 6. The only thing is that such a number (whether 6 or 7) has to be evaluated by the restriction placed on δ.

Finally, note that in general, δ will depend upon ε.
Now you should be able to solve the following exercises:

EXERCISE 1

For a function f: R → R defined by $f(x) = x^2$, find its limit when x tends to 1 by the $\varepsilon - \delta$ approach.

EXERCISE 2

Show that $\lim_{x \to 2} \dfrac{x^2 - x + 18}{3x - 1} = 4$ **using the $\varepsilon - \delta$ definition.**

In Unit 5, we proved that a convergent sequence cannot have more than one limit. In the same way, a function cannot have more than one limit at a single point of its domain. We prove it in the following theorem:

THEOREM 1 : If $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} f(x) = B$, then A = B.

PROOF : In short, we have to show that if $\lim_{x \to a} f(x)$ has two values say A and B, then A = B. Since $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} f(x) = B$, given a number $\varepsilon > 0$, there exists numbers $\delta_1, \delta_2 > 0$ such that

$$|f(x) - A| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta_1$$

and

$$|f(x) - B| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta_2.$$

If we take δ equal to minimum of δ_1 and δ_2, then we have

$$|f(x) - A| < \varepsilon/2 \text{ and } |f(x) - B| < \varepsilon/2 \text{ whenever } 0 < |x - a| < \delta.$$

Choose an x_0 such that $0 < |x_0 - a| < \delta$. Then

$$|A - B| = |A - f(x_0) + f(x_0) - B| \le |A - f(x_0)| + |f(x_0) - B|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

ε is arbitrary while A and B are fixed. Hence $|A - B|$ is less than every positive number ε which implies that $|A - B| = 0$ and hence A = B. (For otherwise, if $A \neq B$ then $A - B = C \neq 0$ (say). We can choose $\varepsilon < |C|$ which will be a contradiction to the fact that $|A - B| < \varepsilon$ for every $\varepsilon > 0$.)

In the example considered before defining limit of a function, we allowed x to assume values both greater and smaller than 1. If we consider values of x greater than 1 that is on the right of 1, we see that values of f(x) approaches 5. We say that f(x) tends to 5 as x tends to 1 from the right. Similarly you see that values of f(x) approach 5 as x tends to 1 from the left i.e. through values smaller than 1. This leads us to define right hand and left hand limits as under :

DEFINITION 2 : Right hand limits and Left hand limits
Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. It is said to tend to a number A as x tends to a number 'a' from the right or through values greater than 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for } a < x < a + \delta.$$

We write it as

$$\lim_{x \to a+} f(x) = A \text{ or } \lim_{x \to a+0} f(x) = A \text{ or } f(a+) = A.$$

See figure 2(a) .

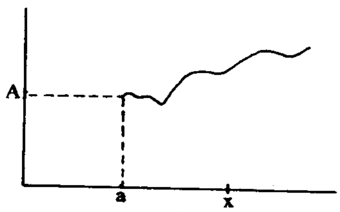


Fig. 2(a)

The function f is said to tend to a number A as x tends to 'a' from the left or through values smaller than 'a' if given a number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for } a - \delta < x < a$$

We write it as

$$\lim_{x \to a-} f(x) = A \text{ or } \lim_{x \to a-0} f(x) = A \text{ or } f(a-) = A.$$

See figure 2(b).

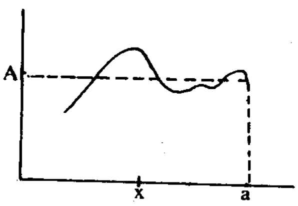


Fig. 2(b)

Since the definition of limit of a function employs only values of x different from 'a' it is totally immaterial what the value of the function is at x = a or whether f is defined at x = a at all. Also it is obvious that $\lim_{x \to a} f(x) = A$ if and only if $f(a+) = A$, $f(a-) = A$.

This we prove in the next theorem. First we consider the following example to illustrate it.

EXAMPLE 2 : Find the limit of the function f defined by

$$f(x) = \frac{x^2 + 3x}{2x} \text{ for } x \neq 0$$

when $x \to 0$

SOLUTION : The given function is not defined at x = 0 since $f(0) = \frac{0}{0}$ which is meaningless.

If $x \neq 0$, then $f(x) = \dfrac{x + 3}{2}$. Therefore

$$\text{Right Hand Limit} = \lim_{x \to 0+0} f(x)$$

$$= \lim_{h \to 0} \frac{(0 + h) + 3}{2} \quad (h > 0)$$

$$= 3/2.$$

$$\text{Left Hand Limit} = \lim_{x \to 0-0} f(x)$$

$$= \lim_{h \to 0} f(x) = \frac{(0 - h) + 3}{2} \quad (h > 0)$$

$$= 3/2.$$

Since both the right hand and left hand limits exist and are equal,

$$\lim_{x \to 0} f(x) = 3/2.$$

You can similarly solve the following exercise.

EXERCISE 3

Find the limit of the function f defined as

$f(x) = \dfrac{2x^2 + x}{3x}$, $x \neq 0$ when x tends to 0.

We, now, discuss the theorem concerning the existence of limit and that of right and the left hand limits.

THEOREM 2 : Let f be a real function. Then

$\lim_{x \to a} f(x) = A$ if and only if $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$

both exist and are equal to A.

PROOF : If $\lim_{x \to a+} f(x) = A$, then corresponding to any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - A| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

i.e. $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a + \delta$, $x \neq a$

This implies that $|f(x) - A| < \varepsilon$ whenever $a - \delta < x < a$

and $|f(x) - A| < \varepsilon$ whenever $a < x < a + \delta$.

Hence both the left hand and right hand limits exist and are equal to A. Conversely, if $f(a+)$ and $f(a-)$ exist and are equal to A say, then corresponding to $\varepsilon > 0$, there exist positive numbers δ_1 and δ_2 such that

$$|f(x) - A| < \varepsilon \text{ whenever } a < x < a + \delta_1$$

and

$$|f(x) - A| < \varepsilon \text{ whenever } a - \delta_2 < x < a.$$

Let δ be the minimum of δ_1 and δ_2. Then

$$|f(x) - A| < \varepsilon \text{ whenever } a - \delta < x < a + \delta, x \neq a$$

i.e. $|f(x) - A| < \varepsilon$ if $0 < |x - a| < \delta$

which proves that

$$\lim_{x \to a} f(x) \text{ exists and } \lim_{x \to a} f(x) = A.$$

EXAMPLE 3 : Consider the function f defined by

$$f(x) = \frac{x^2 - 1}{x - 1}, x \in R, x \neq 1$$

Find its limit as $x \to 1$.

SOLUTION : Note that f(x) is not defined at x = 1. (Why ?).

For any $x \neq 1$, $f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1$.

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (x + 1) = 2$$

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (x + 1) = 2$$

Since $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x)$, by Theorem 2, $\lim_{x \to 1} f(x) = 2$

$\lim_{x \to 1} f(x) = 2$ can be seen by ε–δ definition as follows :

Corresponding to any number $\varepsilon > 0$, we can choose $\delta = \varepsilon$ itself.
Then, it is clear that

$$0 < |x - 1| < \delta = \varepsilon \Rightarrow$$

$$|f(x) - 2| = \left| \frac{x^2 - 1}{x - 1} - 2 \right| = |x + 1 - 2| = |x - 1| < \varepsilon.$$

From Theorem 2, it follows that f(1+) and f(1–) also exist and are both equal to 2.

EXAMPLE 4 : Let f: R $\to$ R be defined as

$$f(x) = \begin{cases} |x|, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

Find its limit when $x \to 0$.

SOLUTION : You are familiar with the graph of f as given in Unit 4. It is easy to see that $\lim_{x \to 0} f(x) = 0 = f(0+) = f(0-)$. The fact that f(0) = 3 has neither any bearing on the existence of the limit of f(x) as x tends to 0 nor on the value of the $\lim_{x \to 0} f(x)$.

Now try the following exercise:

EXERCISE 4

Find, if possible, the limit of the following functions.

(i) $f(x) = \dfrac{|x - 2|}{x - 2}, x \neq 2$

when x tends to 2.

(ii) $f(x) = \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0$

when x tends to 0.

EXAMPLE 5 : Define f on the whole of the real line as follows:

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Find its limit when x tends to 0.

SOLUTION : Since f(x) = 1 for all x > 0,

$$f(0+) = \lim_{x \to 0+} f(x) = +1.$$

Similarly f (0–) = –1.

Since $\lim_{x \to 1+} f(x) \neq \lim_{x \to 1-} f(x)$,

$\lim_{x \to 0} f(x)$ does not exist.

We give another proof using ε - δ definition.
If $\lim_{x \to 0} f(x) = A$, for a given ε > 0, there must exist some δ > 0,

such that |f(x) - A| < ε.

Choose, $x_1 > \delta$, $x_2 < 0$ such that $|x_1| < \delta$ and $|x_2| < \delta$. Then

$2 = |f(x_1) - f(x_2)| \leq |f(x_1) - A| + |A - f(x_2)| < 2\varepsilon$

for every ε which is clearly impossible if ε < 1. Non-existence of $\lim_{x \to 0} f(x)$ also follows from Theorem 2, since f (0+) ≠ f(0–).

The above example shows clearly that the existence of both f(a+) and f(a–) alone is not sufficient for the existence of $\lim_{x \to 0} f(x)$.

They should also be equal for $\lim_{x \to a} f(x)$ to exist.

Now consider, the function f defined by $f(x) = \frac{1}{x}$ for x ≠ 0.

No graph of f looks as shown in the Figure 3. You know that it is a rectangular hyperbola. Here none of the $\lim_{x \to 1+} f(x)$ and $\lim_{x \to 1-} f(x)$ exists. Hence $\lim_{x \to 0} f(x)$ does not exist.

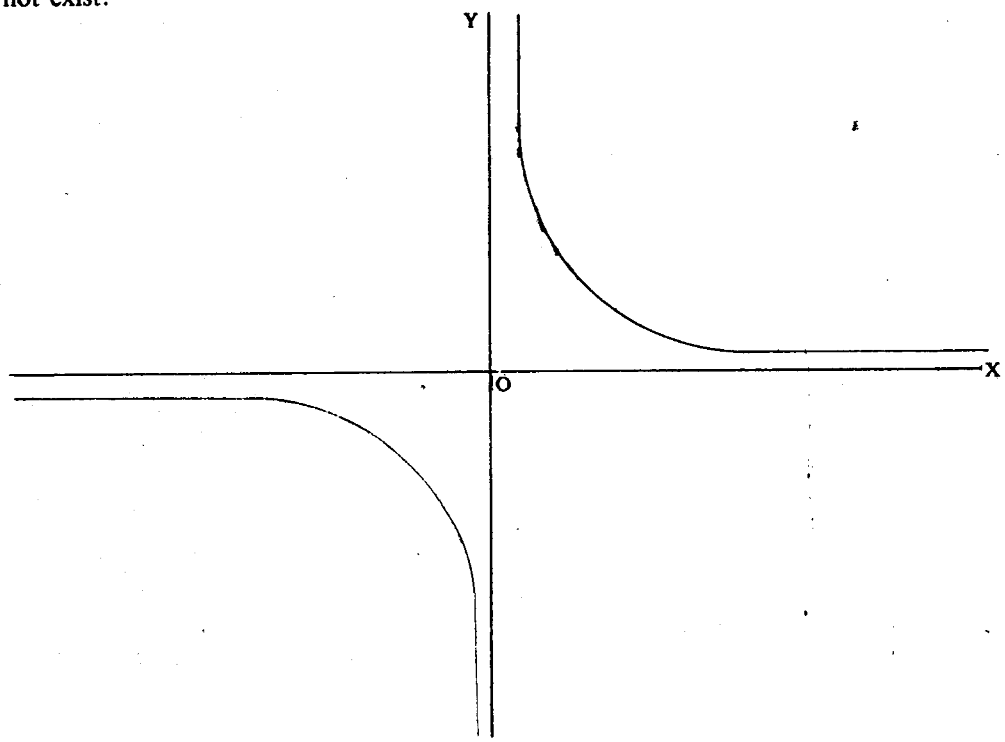


Fig. 3

This can be easily seen from the fact that 1/x becomes very large numerically as x approaches 0 either from the left or from the right. If x is positive and takes up larger and larger values, then value of 1/x i.e. f(x) is positive and becomes smaller and smaller. This is expressed by saying that f(x) approaches 0 as x tends to ∞. Similarly if x is negative and numerically takes up larger and larger values, the values of f(x) is negative and numerically becomes smaller and smaller and we say that f(x) approaches 0 as x tends to -∞. These two observations are related to the notion of the limit of a function at infinity.

Let us now discuss the behaviour of a function f when x tends to ∞.

Let a function f be defined for all values of x greater than a fixed number c. That is to say that f is defined for all sufficiently large values of x. Suppose that as x increases indefinitly, f(x) takes a succession of values which approach more and more closely a value A. Further suppose that the numerical difference between A and the values f(x) taken by the function can be made as small as we please by taking values of x sufficiently large. Then we say f tends to the limit A as x tends to infinity. More precisely, we have the following definition:

DEFINITION 3 : A function f tends to a limit A, as x tends to infinity if having chosen a positive number ε, there exists a positive number k such that

$$|f(x-A)| < \varepsilon \;\; \forall x \geq k.$$

The number ε can be made as small as we like. Indeed, however small ε we may take, we can always find a number k for which the above inequality holds. We rewrite this definition in the following way:

A function $f(x) \to A$ as $x \to \infty$ if for every $\varepsilon > 0$, there exists $k > 0$ such that

$$|f(x) - A| < \varepsilon \text{ for all } x \geq k.$$

We write it as

$$\lim_{x \to \infty} f(x) = A.$$

This notion of the limit of a function needs a slight modification when x tends to $-\infty$. This is as follows:

We say that $\lim\limits_{x \to -\infty} f(x) = A$, if for a given $\varepsilon > 0$, there exists a number $k < 0$ such that

$$|f(x) - A| < \varepsilon \text{ whenever } x \leq k.$$

We write it as $\lim\limits_{x \to -\infty} f(x) = A$.

Instead of f(x) approaching a real number A as x tends to $+\infty$ or $-\infty$, we may also have f(x) approaching $+\infty$ or $-\infty$ as x tends to a real number 'a'. For example, if $f(x) = 1/x^2$, $x \neq 0$ and x takes values near 0, the values of f(x) becomes larger and larger. Then we say that f(x) is tending to $+\infty$ as x tends to 0. We can also have f(x) tending to $+\infty$ or $-\infty$ as x tends to $+\infty$ or $-\infty$. For example $f(x) = x$ tends to $+\infty$ or $-\infty$ as x tends to $+\infty$ or $-\infty$. Again, the function $f(x) = -x$ tends to $+\infty$ or $-\infty$ as x tends to $-\infty$ or $+\infty$. We formulate the following definition to cover all such cases of infinite limits.

DEFINITION 4 : Infinite Limits of a Function
Suppose a is a real number. We say that a function f tends to $+\infty$ when x tends to a, if for a given positive real number M there exists a positive number δ such that

$$f(x) > M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = +\infty.$$

In this case we say that the function becomes unbounded and tends to $+\infty$ as x tends to a.

In the same way, f is said to $-\infty$ as x tends to a if for every real number -M, there is a positive number δ such that

$$f(x) < -M \text{ whenever } 0 < |x - a| < \delta.$$

We write it as

$$\lim_{x \to a} f(x) = -\infty.$$

In this case also f(x) is unbounded and tends to $-\infty$ as x tends to a. You can give similar definitions for $f(a+) = +\infty$, $f(a-) = +\infty$, $f(a+) = -\infty$, $f(a-) = -\infty$.

Now we define $\lim_{x \to \infty} f(x) = \infty$.

f is said to tend to ∞ as x tends to ∞ if given a number $M > 0$, there exists a number $k > 0$ such that

$f(x) > M$ for $x \geq k$.

We may similarly define

$$\lim_{x \to -\infty} f(x) = +\infty, \quad \lim_{x \to +\infty} f(x) = -\infty, \quad \lim_{x \to -\infty} f(x) = -\infty.$$

In all such cases we say that the function f becomes unbounded as x tends to $+\infty$ or $-\infty$ as the case may be.

It is easy to see from the definition of limit of a function that the limit of a constant function at any point in its domain is the constant itself. Similarly if $\lim_{x \to a} f(x) = A$, then $\lim_{x \to a} cf(x) = cA$ for any constant c where c is a real number.

EXAMPLE 6 : Justify that

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty.$$

SOLUTION : You have to verify that corresponding to a given positive number M, there exists a positive number δ, such that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < \delta.$$

Indeed for $x \neq 2$,

$$\frac{1}{(x-2)^2} > M \Rightarrow (x-2)^2 < \frac{1}{M}$$

$$\Rightarrow |x-2| < \frac{1}{\sqrt{M}}.$$

Take $\delta = \frac{1}{\sqrt{M}}$. Then you can see that

$$\frac{1}{(x-2)^2} > M \text{ whenever } 0 < |x-2| < \delta.$$

Hence

$$\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty.$$

Now try the following exercise.

EXERCISE 5

(i) Consider $f(x) = |x|$, $x \in R$. Show that $\lim_{x \to +\infty} f(x) = +\infty$.
and $\lim_{x \to -\infty} f(x) = +\infty$
and $f(0+) = f(0-) = 0 = f(0)$

(ii) Let $f(x) = -|x|$, $x \in R$. Prove that $\lim_{x \to +\infty} f(x) = -\infty$
and $\lim_{x \to -\infty} f(x) = +\infty$
and $f(0) = f(0+) = f(0-) = 0$.

We have already stated that if a function f is defined by $f(x) = 1/x$, $x \neq 0$, then the limits $f(0+)$ and $f(0-)$ and $\lim_{x \to 0} f(x)$ do not exist. It simply means that these limits do not exist as real numbers. In other words, there is no (finite) real number A such that $f(0+) = A$ $f(0-) = A$, or $\lim_{x \to 0} f(x) = A$.

You can easily solve the following exercise:

EXERCISE 6

(i) Let $f(x) = \frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x \to 0+} f(x) = +\infty$, $\lim_{x \to 0-} f(x) = \infty$ and $\lim_{x \to 0} f(x) = +\infty$.

(ii) Let $f(x) = -\frac{1}{|x|}$, $x \neq 0$. Show that $\lim_{x \to 0+} f(x) = -\infty$, $\lim_{x \to 0-} f(x) = -\infty$ and $\lim_{x \to 0} f(x) = -\infty$.

(iii) Let $f(x) = \frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x \to 0+} f(x) = +\infty$, $\lim_{x \to 0-} f(x) = -\infty$

(iv) Let $f(x) = -\frac{1}{x}$, $x \neq 0$. Prove that $\lim_{x \to 0+} f(x) = -\infty$, $\lim_{x \to 0-} f(x) = \infty$.

8.3 SEQUENTIAL LIMITS

In Unit 5, you studied the notion of the limit of a sequence. You also know that a sequence is also a function but a special type of function. What is special about a sequence? Do you remember it? Recall it from Unit 5. Naturally, you would like to know the relationship of a sequence and an arbitrary real function in terms of their limit concepts. Both require us to find a fixed number A as a first step. Both assume a small positive number ε as a test for closeness. For functions we need a positive number δ corresponding to the given positive number ε and for sequences we need a positive integer m which depends on ε. So, then what is the difference between the two notions? The only difference is in their domains in the sense that the domain of a sequence is the set of natural numbers whereas the domain of an arbitrary function is any subset of the set of real numbers. In the case of a sequence, there are natural numbers only which exceed any choice of m. But for a function with a domain as an arbitrary set of real numbers, this is not necessary the case. Thus in a way, the notion of the limit of a function at infinity is a generalization of that of limit of a sequence.

Let us now, therefore, examine the connection between the limit of a function and the limit of a sequence called the sequential limit. We state and prove the following theorem for this purpose:

THEOREM 3 : Let a function f be defined in a neighbourhood of a point 'a' except possibly at 'a'. Then f(x) tends to a limit A as x tends to 'a' if and only if for every sequence (x_n), $x_n \neq a$ for any natural number n, converging to 'a', $f(x_n)$ converges to A.

PROOF : Let $\lim_{x \to a} f(x) = A$. Then for a number $\varepsilon > 0$, there exists a $\delta > 0$ such that for $0 < |x - a| < \delta$ we have

$$|f(x) - A| < \varepsilon$$

Let (x_n) be a sequence ($x_n \neq a$ for any $n \in N$) such that (x_n) converges to a i.e. $x_n \to a$.

Then corresponding to $\delta > 0$, there exists a natural number m such that for all $n \geq m$

$$|x_n - a| < \delta.$$

Consequently, we have

$$|f(x_n) - A| < \varepsilon, \forall n \geq m.$$

This implies that $f(x_n)$ converges to A.

Conversely, let $f(x_n)$ converge to A for every sequence x_n which converges to a, $x_n \neq a$ for any n.

Suppose $\lim_{x \to a} f(x) \neq A$.

Then there exists at least one ε, say $\varepsilon = \varepsilon_0$ such that for any $\delta > 0$ we have an x_δ such that

$$0 < |x_\delta - a| < \delta$$

and

$$|f(x_\delta) - A| \geq \varepsilon_0.$$

Let $\delta = \frac{1}{n}$, $n = 1, 2, 3......$

We get a a sequence (x_n) such that $x_n = x_\delta$ where $\delta = 1/n$ and

$$0 < |x_n - a| < \frac{1}{n} \text{ for } n = 1, 2,.....$$

and

$$|f(x_n) - A| \geq \varepsilon_0.$$

$$0 < |x_n - a| \Rightarrow x_n \neq a \text{ for any } n.$$

Since $\frac{1}{n} \to 0$ and $|x_n - a| < \frac{1}{n}$, it follows that $x_n \to a$.

But $|f(x_n) - A| \geq \varepsilon_0 \Rightarrow f(x_n) \not\to A$ i.e. $f(x_n)$ does not tend to A.

Therefore $x_n \neq a \;\forall\; n$ and x_n tends to a as n tends to ∞ whereas $f(x_n)$ does not converge to A, contradicting our hypothesis. This completes the proof of the theorem.

You may note that the above theorem is true even when either a or A is infinite or both a and A are infinite (i.e. $+\infty$ or $-\infty$).

By applying this theorem, we can decide about the existence or non-existence of limit of a function at a point. Consider the following examples:

EXAMPLE 7 : Let $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

Show that at no point a in the real line R $\lim_{x \to a} f(x)$ exists.

SOLUTION : Consider any point 'a' of the real line. Let (p_n) be a sequence of rational numbers converging to the point 'a'. Since p_n is a rational number, $f(p_n) = 0$ for all n and consequently $\lim f(p_n) = 0$. Now consider a sequence (q_n) of irrational numbers converging to 'a'. Since q_n is an irrational number, $f(q_n) = 1$ for all n and consequently $\lim f(q_n) = 1$. So for two sequences (p_n) and (q_n) converging to 'a', sequences $(f(p_n))$ and $(f(q_n))$ do not converge to the same limit. Therefore $\lim_{x \to a} f(x)$ cannot exist for if it exists and is equal to A, then both $(f(p_n))$ and $(f(q_n))$ would have converged to the same limit A.

EXAMPLE 8 : Show that for the function f: R → R defined by $f(x) = x \;\forall\; x \in R$, $\lim_{x \to a} f(x)$ exists for every $a \in R$.

SOLUTION : Consider any point $a \in R$. Let (x_n) be a sequence of points of R converging to 'a'. Then $f(x_n) = x_n$ and consequently $\lim f(x_n) = \lim (x_n) = a$. So for every sequence $<x_n>$ converging to 'a' $(f(x_n))$ converges to 'a'. So by Theorem 3, $\lim_{x \to a} f(x) = a$. Consequently $\lim_{x \to a} f(x)$ exists for every $a \in R$.

Now try the following exercises.

EXERCISE 7

Show that for the function f: R → R defined by

$$f(x) = x^2,$$

$\lim_{x \to a} f(x)$ exists for every $a \in R$.

EXERCISE 8

Show that $\lim_{x \to 1} 2^x = 2$ by proving that for any sequence (x_n), $x_n \neq 1$, converging to 1, 2^{x_n} converges to 2.

8.4 ALGEBRA OF LIMITS

We discussed the algebra of limits of sequences in Unit 5. In this section we apply the same algebraic operations to limits of functions. This will enable us to solve the problem of finding limits of functions. In other words we discuss limits of sum, difference, product and quotient of functions. Before we do this, let us first recall the meanings of the sum, difference, product, quotient of two functions which you have studied in Unit 4.

DEFINITION 5 : Algebraic Operations on Functions
Let f and g be two functions with domain $D \subset R$. Then the sum, difference, product, quotient of f and g denoted by $f + g$, $f - g$, fg, f/g are functions with domain D defined by

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x).\ g(x)$$
$$(f/g)(x) = f(x)/g(x)$$

provided in the last case $g(x) \neq 0$ for all x in D.

Now we prove the theorem.

THEOREM 4
If $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where A and B are real numbers,

(i) $\lim_{x \to a} (f + g)(x) = A + B = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$,

(ii) $\lim_{x \to a} (f - g)(x) = A - B = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$,

(iii) $\lim_{x \to a} (f \cdot g)(x) = A \cdot B = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.

(iv) If further $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} f/g\,(x)$ exists and

$$\lim_{x \to a} \frac{f}{g}(x) = A/B = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

PROOF : Since $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, corresponding to a number $\varepsilon > 0$. There exist numbers

$\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2 \qquad (1)$$
$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon/2 \qquad (2)$$

Let δ = minimum (δ_1, δ_2). Then from (1) and (2) we have that

$$0 < |x - a| < \delta \Rightarrow |f(x) + g(x) - (A + B)| \leq |f(x) - A| + |g(x) - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Which shows that $\lim_{x \to a} (f + g)(x) = \lim_{x \to a} f(x) + g(x) = A + B$
This proves part (i).
The proof of (ii) is exactly similar. Try it yourself.

(iii) $|f(x)\, g(x) - AB| = |(f(x) - A)\, g(x) + A\,(g(x) - B)|$
$\leq |f(x) - A|\, |g(x)| + |A|.\, |(g(x) - B)| \qquad (3)$

Since $\lim_{x \to a} g(x) = B$ corresponding to 1, there exists a number $\alpha_0 > 0$

such that

$0 < |x - a| < \alpha_0 \Rightarrow |g(x) - B| < 1.$

which implies that $|g(x)| \leq |g(x) - B| + |B| \leq 1 + |B| = K$ (say) (4)

Since $\lim_{x\to a} f(x) = A$, corresponding to $\varepsilon > 0$, there exists a number $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon/2K \quad (5)$$

Since $\lim_{x\to a} f(x) = B$, corresponding to a number $\varepsilon > 0$, there exists a number $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \Rightarrow |f(x) - B| < \frac{\varepsilon}{2(|A|+1)} \quad (6)$$

Let $\delta = \min(\alpha_0, \delta_1, \delta_2)$. Then using (4), (5) and (6) in (3), we have for $0 < |x - a| < \delta$,

$$\begin{aligned} |f(x)\,g(x) - AB| &\leq |f(x) - A|\,|(g(x)| + |A|\,|g(x) - B| \\ &\leq |g(x) - A|.\,K + |A|\,|(g(x) - B)| \\ &< \frac{\varepsilon}{2K}.\,K + \frac{\varepsilon}{2(|A|+1)}\,|A| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore $\lim_{x\to a} g(x) = AB$ i.e. $\lim_{x\to a} (fg)(x) = AB = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x)$, which proves part (iii) of the theorem.

(iv) First we show that g does not vanish in a neighbourhood of a.

$\lim_{x\to a} g(x) = B$ and $B \neq 0$. Therefore $|B| > 0$. Then corresponding to $\frac{|B|}{2}$ we have a number $\mu > 0$ such that for $0 < |x - a| < \mu$, $|g(x) - B| < \frac{|B|}{2}$.

Now by triangle inequality, we have

$$||g(x)| - |B|| \leq |g(x) - B| < \frac{|B|}{2}.$$

i.e. $|B| - \frac{|B|}{2} < |g(x)| < |B| + \frac{|B|}{2}$.

In other words $0 < |x - a| < \mu \Rightarrow |g(x)| > \frac{|B|}{2}$. (7)

Again since $\lim_{x\to a} g(x) = B$, for a given number $\varepsilon > 0$, we have a number $\mu' > 0$ such that $0 < |x - a| < \mu'$ implies that

$$|g(x) - B| < \frac{|B|^2}{2}. \quad (8)$$

Let $\delta = \min(\mu, \mu')$. Then if $0 < |x - a| < \delta$, from (7) and (8) we have

$$\left|\frac{1}{g(x)} - \frac{1}{B}\right| = \frac{|B - g(x)|}{|g(x)|\,|B|} < \frac{2|B - g(x)|}{|B|^2} < \frac{2|B|^2}{2|B|^2} = \varepsilon$$

This proves that $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{B}$.

Now by part (iii) of this theorem, we get that

$$\begin{aligned} \lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} f(x).\,\frac{1}{g(x)} = \lim_{x\to a} f(x).\ \lim_{x\to a} \frac{1}{g(x)} \\ &= A.\,\frac{1}{B} = A/B. \end{aligned}$$

$$\text{i.e. } \lim_{x\to a} \left(\frac{f}{g}\right)(x) = A/B = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

This completes the proof of the theorem. You may note the theorem is true even when $a = \pm\infty$. You may also see that while proving (iv), we have proved that if

$\lim_{x \to a} g(x) = B \neq 0$, then $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{B}$.

Before we solve some examples, we prove two more theorems.

THEOREM 5 : Let f and g be defined in the domain D and let $f(x) \leq g(x)$ for all x in D. Then if $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist,

$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

PROOF : Let $\lim_{x \to a} f(x) = A$, $\lim_{x \to a} g(x) = B$. If possible, let $A > B$.

for $\varepsilon = \frac{A - B}{2}$, there exist $\delta_1, \delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \varepsilon$$

and $0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \varepsilon$.

If $\delta = \min. (\delta_1, \delta_2)$, then for $0 < |x - a| < \delta$, $g(x) \in]B - \varepsilon, B + \varepsilon[$ and $f(x) \in]A - \varepsilon, A + \varepsilon[$. But $B + \varepsilon = A - \varepsilon = \frac{A + B}{2}$. Therefore $g(x) < f(x)$ for $0 < |x - a| < \delta$ which contradicts the given hypothesis. Thus $A \leq B$.

THEOREM 6 : Let S and T be non-empty subsets of the real set R, and let $f: S \to T$ be a function of S onto T. Let $g: U \to R$ be a function whose domain $U \subset R$ contains T. Let us assume that $\lim_{x \to a} f(x)$ exists and is equal to b and $\lim_{y \to b} g(y)$ exists and is equal to c. Then $\lim_{x \to a} g(f(x))$ exists and is equal to c.

PROOF : Since $\lim_{y \to b} g(y) = c$, given a number $\varepsilon > 0$, there exists a number $\alpha_0 > 0$ such that

$$0 < |y - b| < \alpha_0 \Rightarrow |g(y) - c| < \varepsilon.$$

Since $\lim_{x \to a} f(x) = b$, corresponding to $\alpha_0 > 0$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - b| < \alpha_0.$$

Hence taking $y = f(x)$ and combining the two we get that for

$$0 < |x - a| < \delta, \ |g(f(x)) - c| = |g(y) - c| < \varepsilon$$

$$\text{(since } |f(x) - b| < \alpha_0\text{)}.$$

This completes the proof of the theorem. Finally we give one more result without proof.

RESULT : If $\lim_{x \to a} f(x) = A$, $A > 0$ and $\lim_{x \to a} g(x) = B$ where A and B are finite real numbers then

$$\lim_{x \to a} f(x)^{g(x)} = A^B.$$

Now we discuss some examples. You will see how the above results help us in reducing the problem of finding limit of complicated functions to that of finding limits of simple functions.

EXAMPLE 9

Find $\lim_{x \to \infty} \frac{(2x + 7)(3x - 11)(4x + 5)}{4x^3 + x - 1}$

SOLUTION

$$\lim_{x \to \infty} \frac{(2x + 7)(3x - 11)(4x + 5)}{4x^3 + x - 1}$$

$$= \lim_{x \to \infty} \frac{x^3 \left[\left(2 + \frac{7}{x}\right)\left(3 - \frac{11}{x}\right)\left(4 + \frac{5}{x}\right)\right]}{x^3 \left(4 + \frac{1}{x^2} - \frac{1}{x^3}\right)}$$

We divide the numerator and denominator by x^3 since x^3 is neither zero nor ∞.

$$= \lim_{x \to \infty} \frac{(2x + 7)(3x - 11)(4x + 5)}{4x^3 + x - 1}$$

$$= \lim_{x \to \infty} \frac{\left(2 + \frac{7}{x}\right)\left(3 - \frac{11}{x}\right)\left(4 + \frac{5}{x}\right)}{4 + \frac{1}{x^2} - \frac{1}{x^3}} = \frac{2 \times 3 \times 4}{4} = 6.$$

EXAMPLE 10 : **Find** $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 4x + 3}$

SOLUTION : $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x - 1)}$

Hence $\lim_{x \to 3} \frac{x^2 - 9}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{x + 3}{x - 1}$

$$= \frac{\lim_{x \to 3} (x + 3)}{\lim_{x \to 3} (x - 1)} = \frac{6}{2} = 3$$

The function $f(x) = \frac{x^2 - 9}{x^2 - 4x + 3}$ is not defined at $x = 3$. But we are considering only the values of the function at those points x in a neighbourhood of 3 for which $x \neq 3$ and hence we can cancel $x - 3$ factor

EXAMPLE 11 : Evaluate $\lim_{x \to 0} \frac{(1 + x)^{1/2} - 1}{(1 + x)^{1/3} - 1}$

SOLUTION : To make the problem easier, we make a substitution which enables us to get rid of fractional powers 1/2 and 1/3. L.C.M. of 2 and 3 is 6. So, we put $1 + x = y^6$

Then we have

$$\lim_{x \to 0} \frac{(1 + x)^{1/2} - 1}{(1 + x)^{1/3} - 1} = \lim_{y \to 0} \frac{y^3 - 1}{y^2 - 1} = \lim_{y \to 1} \frac{(y - 1)(y^2 + y + 1)}{(y - 1)(y + 1)}$$

$$= \lim_{y \to 1} \frac{y^2 + y + 1}{y + 1} = \frac{3}{2}.$$

Try the following exercises:

EXERCISE 9

Find

(i) $\lim_{x \to \infty} \frac{(2x + 3)^3 (3x - 2)^2}{x^5 + 5}$

(ii) $\lim_{x \to \infty} \frac{(x^3 + 1)^{1/3}}{x + 1}$

EXERCISE 10

If $g(x) = \begin{cases} 2x & \text{for } 0 \le x < 1 \\ 4 & \text{for } x = 1 \\ 5-3x & \text{for } 1 < x \le 2. \end{cases}$

find $\lim_{x \to 1} g(x)$

EXERCISE 11

Find

(i) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$

(ii) Find $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

(iii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(iv) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$

(v) $\lim_{x \to \infty} \left(\frac{x - 1}{x + 1}\right)^x$

(vi) $\lim_{x \to 0} \left(\frac{\sin 2x}{x}\right)^{1+x}$

(vii) $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1}\right)^{x^2}$

8.5 SUMMARY

In this unit, you have been introduced to the concept of a limit of a function. In Section 8.2, we started with the intuitive idea of a limit of a function. Then we derived the rigorous definition of the limit of a function, popularly called $\varepsilon - \delta$ definition of a limit. Further, we gave the notion of right and left hand limits of a function. It has been proved that $\lim_{x \to a} f(x) = A$ if and only if both right hand and left hand limits are equal to A i.e. $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = A$. In the same section we discussed the limit of a function as x tends to $+\infty$ or $-\infty$. Also we discussed the infinite limit of a function. In Section 8.3, we studied the idea of sequential limit of a function by connecting the idea of limit of an arbitrary function with the limit of a sequence. It has been shown how this relationship helps in finding the limits of functions. In Section 8.4, we defined the algebraic operations of sum, difference, product, quotient of two functions. We proved that the limit of the sum, difference, product and quotient of two functions at a point is equal to the sum, difference, product and quotient of the limits of the functions at the point provided in the case of quotient, the limit of the function in the denominator is non-zero. Finally in the same section, the usefulness of the algebra of limits in finding the limits of complicated functions has been illustrated.

8.6 ANSWERS/HINTS/SOLUTIONS

E1) We claim that $\lim_{x \to 1} f(x) = 1$. To verify this, let $\varepsilon > 0$ be a fixed real number. Then

$$|f(x) - 1| = |x^2 - 1| = |x - 1|\,|x + 1|.$$

Suppose $0 < \delta \leq 1$. Then $|x - 1| < 1 \Rightarrow 0 < |x + 1| < 3$ and also

$$|x - 1| < 1 \Rightarrow |x + 1|.\,|x - 1| < 3.\,|x - 1|.$$

Choose $\delta = \text{Min}\{1, \varepsilon/3\}$. Then

$$\begin{aligned} 0 < |x - 1| < \delta \Rightarrow |f(x) - 1| &= |x^2 - 1| \\ &< 3.\,|x - 1| \\ &< 3.\,\delta \\ &\leq 3.\,\varepsilon/3 = \varepsilon \end{aligned}$$

which proves the claim.

E2) $\frac{x^2 - x + 18}{3x - 1} - 4 = \frac{x^2 - 13x + 22}{3x - 1} = (x - 2)\frac{x - 11}{3x - 1}$

If x is near 2 then $(x - 2)$ is near zero. If x is near 2 and away from 1/3,

then $\left|\frac{x - 11}{3x - 1}\right|$ is not very large. If $\delta \leq 1$ and $0 < |x - 2| < \delta$ then

$2 - \delta < x < 2 + \delta, x \neq 2$ i.e. $1 < x < 3, x \neq 2$. Then $-10 < x - 11 < -8$ and $2 < 3x - 1 < 8$ so that $|x - 11| < 10$ and $|3x - 1| > 2$

Thus $\left|\frac{x - 11}{3x - 1}\right| < 5$. Now if $\varepsilon > 0$ is given and if simultaneously

$|x - 2| < \varepsilon/5$ and $\left|\frac{x - 11}{3x - 1}\right| < 5$ then $\left|\frac{x^2 - x + 18}{3x - 1} - 4\right| < \varepsilon$.

Hence we can choose $\delta = \min(\varepsilon/5, 1)$

Then for $0 < |x - 2| < \delta$ we have

$$\left|\frac{x^2 - x + 18}{3x - 1} - 4\right| < \varepsilon.$$

In fact, in this problem, f(2) is defined and takes the value 4.

E3) When $x \neq 0$, $f(x) = \frac{2x + 1}{3}$. Therefore

Right hand limit $= \lim_{x \to 0+} f(x)$

$$= \lim_{h \to 0} \frac{2(0 + h) + 1}{3} \quad (h > 0)$$

$$= \frac{1}{3}.$$

Left hand limit $= \lim_{x \to 0} f(x)$

$$= \lim_{h \to 0} \frac{2(0 - h) + 1}{3} \quad (h > 0)$$

$$= \frac{1}{3}.$$

Since both the right hand and left hand limits exist and are equal, therefore $\lim_{x \to 0} f(x) = \frac{1}{3}$.

E4) (i) Right hand limit $= f(2+) = \lim_{h \to 0} \frac{|0 + h - 2|}{2 + h - 2}$ (h > 0)

$= 1.$

Similarly left hand limit $= -1$

$\therefore \lim_{x \to 2} f(x)$ does not exist.

(ii) $f(0+) = \lim_{h \to 0} \frac{e^{1/h} - 1}{e^{1/h} + 1}$ (h > 0)

$$= \lim_{h \to 0} \frac{1 - \frac{1}{e^{1/h}}}{1 + \frac{1}{e^{1/h}}} = 1$$

$$f(0-) = \lim_{h \to 0} \frac{e^{-1/h} - 1}{e^{-1/h} + 1} = -1$$

Since $f(0+) \neq f(0-)$, $\lim_{x \to 0} f(x)$ does not exist.

E5) (i) When x is positive or zero $f(x) = x$, and when x is negative, $f(x) = -x$.

$f(0+) = \lim_{x \to 0+} x = 0$ and $f(0-) = \lim_{x \to 0-} -x = 0$. Also $f(0) = 0$

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} |x| = \lim_{x \to \infty} x = \infty$. In fact for any $M > 0$, $f(x) > M$ if $x \geq k$ with $k = M + 1$.

Similarly $\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} |x| = \lim_{x \to -\infty} -x = \infty$.

(ii) It is similar to (i).

E6) (i) Let $h > 0$

$$\lim_{x \to 0+} f(x) = \lim_{h \to 0} \frac{1}{|0 + h|} = \lim_{h \to 0} \frac{1}{h} = \infty.$$

$$\lim_{x \to 0-} f(x) = \lim_{h \to 0} \frac{1}{|0 - h|} = \lim_{h \to 0} \frac{1}{h} = \infty.$$

Hence it follows that $\lim_{x \to 0} f(x) = \infty$. It can also be proved as follows:

If $M > 0$ is any number, $f(x) = \frac{1}{|x|} > M$ if $0 < |x| < \frac{1}{M}$ that is

$f(x) > M$ if $0 < |x| < \delta$ where $\delta = \frac{1}{M}$.

Hence $\lim_{x \to 0} f(x) = \infty$.

(ii) is similar to (i).

(iii) $\lim_{x \to 0+} f(x) = \lim_{h \to 0} \frac{1}{h} = \infty$ and $\lim_{x \to 0-} f(x) = \lim_{h \to 0} \frac{1}{-h} = -\infty$

(iv) is similar to (iii).

E7) If (x_n) be a sequence converging to 'a', $f(x_n) = x_n^2 \to a^2$ and so by Theorem 3,

$\lim_{x \to a} f(x) = a^2$.

E8) Let (x_n) be any sequence belonging to the domain of definition of f converging to 1 and such that $x_n \neq 1$ for any n. Given $\varepsilon > 0$ we want to find an M such that for all $n \geq M$

$$2 - \varepsilon < 2^{x_n} < 2 + \varepsilon$$

Choose $\varepsilon_1 = \log_2 (1 + \varepsilon/2)$. It is clear that $\varepsilon_1 > 0$.
Since $\lim$, $x_n = 1$, therefore corresponding to $\varepsilon_1 > 0$, there exists a positive integer M such that

$$1 - \varepsilon_1 < x_n < 1 + \varepsilon_1 \text{ for } n \geq M.$$

Thus for $n \geq M$, we have

$$2^{x_n} < 2^{1+\varepsilon_1} = 2.2^{\varepsilon_1} = 2.2^{\log_2(1+\varepsilon/2)} = 2(1 + \varepsilon/2) = 2 + \varepsilon.$$

and $2^{x_n} > 2^{1-\varepsilon_1} = 2.2^{-\varepsilon_1} = \frac{2}{2^{\log_2(1+\varepsilon/2)}} = \frac{2}{1 + \varepsilon/2} = \frac{4}{2 + \varepsilon}$

$$> \frac{4 - \varepsilon^2}{2 + \varepsilon} = 2 - \varepsilon.$$

$\therefore 2 - \varepsilon < 2^{x_n} < 2 + \varepsilon$ for $n \geq M$

i.e. $|2^{x_n} - 2| < \varepsilon$ for $n \geq M$.

This proves that 2^{x_n} tends to 2. From theorem 3, it follows that $\lim_{x \to 1} 2^x = 2$.

E9) (i) $\lim_{x \to \infty} \frac{(2x + 3)^3 (3x - 2)^2}{x^5 + 5}$

$$= \lim_{x \to \infty} \frac{(2 + 3/x)^3 (3 - 2/x)^2}{1 + 5/x^5} = \frac{2^3.3^2}{1} = 72.$$

(ii) $\lim_{x \to \infty} \frac{\sqrt[3]{x^3 + 1}}{x + 1} = \lim_{x \to \infty} \frac{\sqrt[3]{1 + 1/x^3}}{1 + 1/x} = 1.$

E10) $g(x) = \begin{cases} 2x & \text{for } 0 \leq x < 1 \\ 4 & \text{if } x = 1 \\ 5 - 3x & \text{for } 1 < x \leq 2. \end{cases}$

$\lim_{x \to 1+} g(x) = \lim_{x \to 1+} (5 - 3x) = 5 - \lim_{x \to 1+} 3x = 5 - 3 = 2$

and $\lim_{x \to 1-} g(x) = \lim_{x \to 1-} 2x = 2$

Hence $\lim_{x \to 1} g(x) = 2$.

E11) (i) $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)}$

$= \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}$

(ii) $\lim_{x \to 2} \frac{3x^2 - x - 10}{x^2 + 5x - 14}$

$= \lim_{x \to 2} \frac{(3x + 5)(x - 2)}{(x + 7)(x - 2)} = \lim_{x \to 2} \frac{3x + 5}{x + 7}$

$= \frac{11}{9}$.

(iii) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} \quad \lim_{x \to 0} \frac{2 \sin^2 x/2}{4.x^2/4} = \lim_{x \to 2} \frac{1}{2} \frac{\sin^2 x/2}{(x/2)^2}$

$= \frac{1}{2}$, since $\lim_{x \to 0} \frac{\sin^2 x/2}{(x/2)^2} = \lim_{x \to 0} \left(\frac{\sin x/2}{x/2} \right)^2 = 1$

(iv) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = \lim_{x \to a} \frac{2 \cos \frac{x + a}{2} \sin \frac{x - a}{2}}{x - a}$

$= \lim_{x \to a} \cos \frac{x + a}{2} \lim_{x \to a} \frac{\sin \frac{x - a}{2}}{\frac{x - a}{2}}$

$= \cos a.$

(v) $\lim_{x \to \infty} \left(\frac{x - 1}{x + 1} \right)^x$

$\lim_{x \to \infty} \frac{x - 1}{x + 1} = \lim_{x \to \infty} \frac{1 - 1/x}{1 + 1/x} = 1.$

$\lim_{x \to \infty} \left(\frac{x - 1}{x + 1} \right)^x = \lim_{x \to \infty} \left[1 + \frac{x - 1}{x + 1} - 1 \right]^x$

$= \lim_{x \to \infty} \left\{ \left[1 + \left(\frac{-2}{x + 1} \right) \right]^{-\frac{x + 1}{2}} \right\}^{-\frac{2x}{1 + x}}$

$= \lim_{x \to \infty} e^{-\frac{2x}{1+x}} = e^{-2}$

since $\lim_{x \to \infty} \left[1 + \left(\frac{-2}{x + 1} \right) \right]^{-\frac{x + 1}{2}} = e.$

or

$\lim_{x \to \infty} \left(\frac{x - 1}{x + 1} \right)^x = \frac{\lim_{x \to \infty} (1 - 1/x)^x}{\lim_{x \to \infty} (1 + 1/x)^x}$

$= \frac{\lim_{x \to \infty} \left[\left(1 - \frac{1}{x} \right)^{-x} \right]^{-1}}{\lim_{x \to \infty} (1 + 1/x)^x} = \frac{e^{-1}}{e} = e^{-2}$

(vi) $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)^{1+x}$

$\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right) = 2$ and $\lim_{x \to 0} (1 + x) = 1$

Hence $\lim_{x \to 0} \left(\frac{\sin 2x}{x} \right)^{1+x} = 2^1 = 2.$

(vii) $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right)^{x^2}$

$\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right) = \frac{1}{2}$ and $\lim_{x \to \infty} x^2 = +\infty$

Hence $\lim_{x \to \infty} \left(\frac{x + 1}{2x + 1} \right)^{x^2} = 0.$

UNIT 9 CONTINUITY

STRUCTURE

9.1 INTRODUCTION

Suppose that you have functions which are defined on an interval, either open or closed. If you draw the graph of these functions, you will observe that some of these can be sketched down in one smooth '**continuous**' sweep of your pen, while others have many breaks or jumps. For example, draw the graphs of the following two functions:

(a) $f(x) = x^2, x \in [-2, 2]$

(b) $f(x) = \begin{cases} \frac{1}{x}, x \in [-2, 2], x \neq 0 \\ 0, x = 0 \end{cases}$

You will see that the graphs are as shown in figures 1(a) and 1(b).

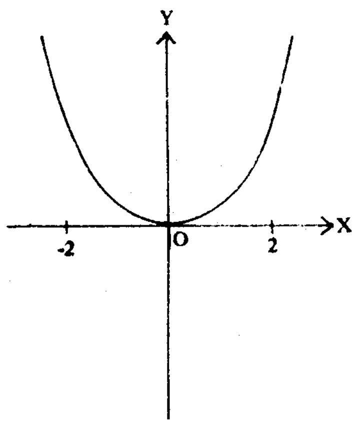


Fig. 1(a)

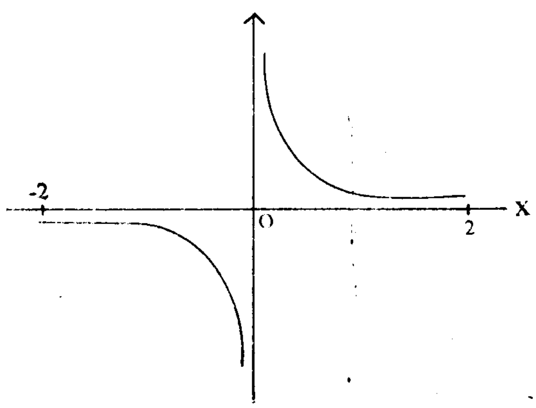


Fig. 1(b)

You can see that while the graph of the first function can be drawn in one '**continuous**' motion without lifting the pen from the paper while the graph of the other function cannot be drawn in this manner. This is an interesting property of the first function which is not possessed by the second function. It is, therefore, natural to wonder if it can be given some mathematical meaning. In fact, mathematicians of the past several centuries did confront this question, namely:

"Is there a way to specify those curves which can be drawn with a single stroke of one's pen?"

The answer is yes and the functions representing such curves are given the names as **Continuous Functions. What is, then, the mathematical meaning of a continuous function? What are the functions such as the one in figure 1(b)?** We shall try to answer these questions and a few more in this unit.

In Unit 8, we made clear our intuitive idea of the values of a function f(x) approaching a number A as the variable x approaches a given point a. In continuous graphs of functions, as you have seen in the figures 1(a) and (b) that as x approaches a, the functional values approach f(a). When there is break (or jump) in the graph, then this property fails at that point. This idea of continuity is, therefore, connected with the value of $\lim_{x \to a} f(x)$ and the value of the function f at the point a. We define in this unit the continuity of a function at a given point a in precise mathematical language. Therefore extend it to the continuity of a function on a non-empty subset of the domain of f which could be the whole of the domain of f also. We study the effect of the algebraic operations of addition, subtraction, multiplication and division on continuous functions.

We shall use these results in Unit 10 to discuss the properties of continuous functions and the concept of uniform continuity.

Objectives

After studying this unit, you should be able to:

- define the continuity of a function at a point of its domain,
- determine whether a given function is continuous or not,
- construct new continuous functions from a given class of continuous functions.

9.2 CONTINUOUS FUNCTIONS

We have seen that the limit of a function f as the variable x approaches a given point a in the domain of a function f does not depend at all on the value of the function at that point a but it depends only on the values of the function at the points near a. In fact, even if the function f is not defined at a then $\lim_{x \to a} f(x)$ may exist.

For example $\lim_{x \to 1} f(x)$ exists when

$$f(x) = \frac{x^2 - 1}{x - 1} \text{ though f is not defined at } x = 1.$$

We have also seen that $\lim_{x \to a} f(x)$ may exist, still it need not be the same as f(a) when it exists (see example 2, Unit 8). Naturally, we would like to examine the special case when both $\lim_{x \to a} f(x)$ and f(a) exist and are equal. If a function has these properties, then it is called a continuous function at the point a. We give the precise definition as follows:

DEFINITION 1 : Continuity of a Function at a Point
A function f defined on a subset S of the set R is said to be continuous at a point $a \in S$, if

i) $\lim_{x \to a} f(x)$ exists and is finite

ii) $\lim_{x \to a} f(x) = f(a)$.

Note that in this definition, we assume that S contains some open interval containing the point a. If we assume that there exists a half open (semi-open) interval [a, c[contained in S for some $c \in R$, then in the above definition , we can replace $\lim_{x \to a} f(x)$ by $\lim_{x \to a+} f(x)$ and say that the function is continuous from the right of a or f is right continuous at a.

Similarly, you can define left continuity at a, replacing the role of $\lim_{x \to a} f(x)$ by $\lim_{x \to a-} f(x)$. Thus, f is continuous from the right at a if and only if

$$f(a+) = f(a)$$

It is continuous from the left at a if and only if

$f(a-) = f(a)$.

From the definition of continuity of a function f at a point a and properties of limits it follows that $f(a+) = f(a-) = f(a)$ if and only if, f is continuous at a. If a function is both continuous from the right and continuous from the left at a point a, then it is continuous at a and conversely.

The definition 1 is popularly known as **the Limit-Definition of Continuity.**

Since $\lim_{x \to a} f(x)$ is also defined in terms of ε and δ, we have an equivalent formulation of the definition 1 in terms of ε and δ. Note that whenever we talk of continuity of a function f at a in S, we always assume that S contains a neighbourhood containing a. Also remember that if there is one such neighbourhood there are infinitely many such neighbourhoods. An equivalent definition of continuity in terms of ε and δ is given as follows:

DEFINITION 2 : (ε, δ)-Definition of Continuity
A function f is continuous at $x = a$ if f is defined in a neighbourhood of a and corresponding to a given number $\varepsilon > 0$, there exists some number $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Note that unlike in the definition of limit, we should have

$|f(x) - f(a)| < \varepsilon$ for $|x - a| < \delta$.

The two definitions are equivalent. Though this fact is almost obvious, it will be appropriate to prove it.

THEOREM 1 : The limit definition of continuity and the (ε, δ)-definition of continuity are equivalent.

PROOF : Suppose f is continuous at a point a in the sense of the limit definition. Then give $\varepsilon > 0$, we have a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$. When $x = a$, we trivially have

$|f(x) - f(a)| = 0 < \varepsilon$.

Hence, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$
which is the (ε, δ)-definition.

Conversely we now assume that f is continuous in the sense of (ε, δ)-definition. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Leaving the point 'a', we can write it as

$0 < |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

This implies the existence of $\lim_{x \to a} f(x)$ and that $\lim_{x \to a} f(x) = f(a)$.

Note that δ in the definition 2, in general, depends on the given function f, ε and the point a. Also $|x - a| < \delta$ if and only if $a - \delta < x < a + \delta$ and $]a - \delta, a + \delta[$ is an open interval containing a. Similarly $|f(x) - f(a)| < \varepsilon$ if and only if

$f(a) - \varepsilon < f(x) < f(a) + \varepsilon$.

We see that f is continuous at a point a, if corresponding to a given (open) ε-neighbourhood U of f(a) there exists a (open) δ-neighbourhood V of a such that $f(V) \subset U$. Observe that this is the same as $x \in V \Rightarrow f(x) \in U$. This formulation of the continuity at a is more useful to generalise this definition to more general situations in Higher Mathematics.

A function f is said to be continuous on a set S if it is continuous at every point of the set S. It is clear that a constant function defined on S is continuous on S.

Let us, now, study some examples and exercises:

EXAMPLE 1 : Examine the continuity of the following functions:
i) The absolute value (Modulus) function,
ii) The signum function.

SOLUTION

i) You know from Unit 4, that the absolute value function $f: R \to R$ is defined as $f(x) = |x|, \forall x \in R$.
The function is continuous at every point $x \in R$. For given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ itself. If $a \in R$ be any point them $|x - a| < \delta = \varepsilon$ implies that

$$|f(x) - f(a)| = |\,|x| - |a|\,| \leq |x - a| < \varepsilon.$$

ii) The signum function, as you know from Unit 4, is a function $f : R \to R$ defined as

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

This function is **not** continuous at the point $x = 0$. We have already seen in Unit 8 that $f(0+) = 1$, $f(0-) = -1$. Since $f(0+) \neq f(0-)$, $\lim_{x \to 0} f(x)$ does not exist and consequently the function is not continuous at $x = 0$. For every point $x \neq 0$ the function f is continuous. This is easily seen from the graph of the function f as described in Unit 4. There is a jump at the point $x = 0$ in the values of $f(x)$ defined in a neighbourhood of 0.

Note that if $f : R \to R$ is defined as,

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0. \\ -1 & \text{if } x < 0. \end{cases}$$

then, it is easy to see that this function is continuous from the right at $x = 0$ but not from the left. It is continuous at every point $x \neq 0$.

Similarly, if f is defined by $f(x) = 1$ if $x > 0$
$= -1$ if $x \leq 0$

then f is continuous from the left at $x = 0$ but not from the right.

EXERCISE 1
Examine the continuity of the following functions:

i) The function $f: R \sim \{0\} \to R$ defined as

$$f(x) = \frac{|x|}{x},$$

at the point $x = 0$

ii) The function $f: R \sim \{1\} \to R$ defined as

$$f(x) = \frac{x^2 - 1}{x - 1},$$

iii) The function $f: R \sim \{0\} \to R$ defined as

$$f(x) = \frac{1}{x}.$$

EXAMPLE 2 : Discuss the continuity of the function sin x on the real line R.

SOLUTION : Let $f(x) = \text{Sin } x \ \forall x \in R$.
We show by the (ε, δ)-definition that f is continuous at every point of R.

Consider an arbitrary point $a \in R$. We have

$$|f(x) - f(a)| = |\sin x - \sin a| = \left|2 \sin \frac{x - a}{2} \cos \frac{x + a}{2}\right|$$

$$= 2\left|\text{Sin } \frac{x - a}{2}\right| \left|\cos \frac{x + a}{2}\right|$$

$$\leq 2 \left|\text{Sin } \frac{x - a}{2}\right| \left(\text{since } \left|\cos \frac{x + a}{2}\right| \leq 1\right)$$

From Trigonometry, you know that $|\text{Sin } \theta| \leq |\theta|$.

Therefore $\left|\text{Sin } \frac{x - a}{2}\right| \leq \left|\frac{x - a}{2}\right| = \frac{|x - a|}{2}$

Consequently $|f(x) - f(a)| \leq |x - a|$
$< \varepsilon$ if $|x - a| < \delta$ where $\delta = \varepsilon$.

So f is continuous at the point a. But a is any point of R. Hence Sin x is continuous on the real line R.

EXERCISE 2
Discuss the continuity of cos x on the real line R.

In Unit 8, we have connected the limit of a function with the limit of a sequence of real numbers. In the same way, we can discuss the continuity of a function in the language of the sequence of real numbers in the domain of the function. This is explained in the following theorem.

THEOREM 2 : A function f : S → R is continuous at point a in S if and only if for every sequence (x_n), ($x_n \in S$) converging to a, f(x_n) converges to f(a).

PROOF : Let us suppose that f is continuous at a. Then $\lim_{x \to a} f(x) = f(a)$.

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

If x_n is a sequence converging to 'a', then corresponding to $\delta > 0$, there exists a positive integer M such that

$|x_n - a| < \delta$ for $n \geq M$.

Thus, for $n \geq M$, we have $|x_n - a| < \delta$ which, in turn, implies that

$|f(x_n) - f(a)| < \varepsilon$,

proving thereby $f(x_n)$ converges to f(a).

Conversely, let us suppose that whenever x_n converges to a, $f(x_n)$ converges to f(a). Then we have to prove that f is continuous at a. For this, we have to show that corresponding to an $\varepsilon > 0$, there exists some $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$, whenever $|x - a| < \delta$.

If not, i.e., if f is not continuous at a, then there exists an $\varepsilon > 0$ such that whatever $\delta > 0$ we take there exists an x_δ such that

$|x_\delta - a| < \delta$ but $|f(x_\delta) - f(a)| \geq \varepsilon$.

By taking $\delta = 1, 1/2, 1/3, \ldots$ in succession we get a sequence $\{x_n\}$, where $x_n = x_\delta$ for $\delta = 1/n$, such that $|f(x_n) - f(a)| \geq \varepsilon$. The sequence $\{x_n\}$ converges to a. For, if $m > 0$, these exists M such that $1/n < m$ for $n \geq M$ and therefore $|x_n - a| < m$ for $n \geq M$. But $f(x_n)$ does not converge to f(a), a contradiction to our hypothesis. This completes the proof of the theorem.

Theorem 2 is sometimes used as a definition of the continuity of a function in terms of the convergent sequences. This is popularly known as the Sequential Definition of Continuity which we state as follows:

DEFINITION 3 : Sequential Continuity of a Function
Let f be a real-valued function whose domain is a subset of the set R. The function f is said to be continuous at a point a if, for every sequence (x_n) in the domain of f converging to a, we have,

$$\lim_{n \to \infty} f(x_n) = f(a)$$

The next example illustrates this definition.

EXAMPLE 3 : Let f : R → R be defined as

$$f(x) = 2x^2 + 1, \forall x \in R$$

Prove that f is continuous on R by using the sequential definition of the continuity of a function.

SOLUTION : Suppose (x_n) is a sequence which converges to a point 'a' of R. Then, we have

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (2x_n^2 + 1) = 2 (\lim_{n\to\infty} x_n)^2 + 1 = 2a^2 + 1 = f(a)$$

This shows that f is continuous at a point a ∈ R. Since a is an arbitrary element of R, therefore, f is continuous everywhere on R.

EXERCISE 3

Prove by sequential definition of continuity that the function f: R → R defined by $f(x) = \sqrt{x}$ is continuous at x = 0.

9.3 ALGEBRA OF CONTINUOUS FUNCTIONS

As, in Unit 8, we proved limit theorems for sum, difference, product etc. of two functions, we have similar results for continuous functions also. These algebraic operations on the class of continuous functions can be deduced from the corresponding theorems on limits of functions in Unit 8, using the limit definition of continuity. We leave this deduction as an exercise for you. However, we give a formal proof of these algebraic operations by another method which illustrates the use of Theorem 2. We prove the following theorem:

THEOREM 3 : Let f and g be any real functions both continuous at a point a ∈ R. Then,

i) αf defined by $(\alpha f)(x) = \alpha f(x)$, is continuous for any real number α,

ii) f + g defined by $(f + g)(x) = f(x) + g(x)$ is continuous at a,

iii) f − g defined by $(f - g)(x) = f(x) - g(x)$ is continuous at a,

iv) fg defined by $(fg)(x) = f(x) g(x)$ is continuous at a,

v) f/g defined by $(f/g)(x) = \dfrac{f(x)}{g(x)}$, is continuous at a provided $g(a) \neq 0$.

PROOF : Let x_n be an arbitrary sequence converging to a. Then the continuity of f and g imply that the sequences $f(x_n)$ and $g(x_n)$ converge to f(a) and g(a) respectively. In other words, $\lim f(x_n) = f(a)$, $\lim g(x_n) = g(a)$.

Using the algebra of sequences discussed in Unit 5, we can conclude that

$$\lim \alpha f(x_n) = \alpha f(a),$$
$$\lim (f + g)(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a),$$
$$\lim (f - g)(x_n) = \lim f(x_n) - \lim g(x_n) = f(a) - g(a),$$
$$\lim (f \cdot g)(x_n) = \lim f(x_n) \lim g(x_n) = f(a)\, g(a).$$

This proves the parts (i), (ii), (iii) and (iv). To prove the part (v) we proceed as follows:

Since $g(a) \neq 0$, we can find $\alpha > 0$ such that the interval $]g(a) - \alpha, g(a) + \alpha[$ is either entirely to the right or to the left of zero depending on whether $g(a) > 0$ or $g(a) < 0$. Corresponding to $\alpha > 0$, there exists a $\delta_1 > 0$ such that $|x - a| < \delta_1$ implies $|g(x) - g(a)| < \alpha$, i.e., $g(a) - \alpha < g(x) < g(a) + \alpha$. Thus, for x such that $|x - a| < \delta_1$, $g(x) \neq 0$. If (x_n) converges to a, omitting a finite number of terms of the sequence if necessary, then we can assume that $g(x_n) \neq 0$, for all n. Hence, $\dfrac{f(x_n)}{g(x_n)}$ converges to $\dfrac{f(a)}{g(a)}$ and so $\dfrac{f}{g}$ is continuous at a. This completes the proof of the theorem.

> If infinite number of x_n's are such that $g(x_n) = 0$, then $g(x_n) \to g(a)$ implies that $g(a) = 0$, a contradiction.

In part (v) if we define f by f(x) = 1, then it follows that if g is continuous at 'a' and g(a) $\neq$ 0, then its reciprocal function 1/g is continuous at 'a'.

Now, we prove another theorem, which shows that **a continuous function of a continuous function is continuous.**

THEOREM 4 : Let f and g be two real functions such that the range of g is contained in the domain of f. If g is continuous at x = a, f is continuous at b = g(a) and h(x) = f(g(x)) for x in the domain of g, then h is continuous at a.

PROOF : Given $\varepsilon > 0$, the continuity of f at y = b = g(a) implies the existence of an $\eta > 0$ such that for

$$|y - b| < \eta, \ |f(y) - f(b)| < \varepsilon \qquad \ldots(1)$$

Corresponding to $\eta > 0$, from the continuity of g at x = a, we get a $\delta > 0$ such that

$$|x - a| < \delta \text{ implies } |g(x) - g(a)| < \eta \qquad \ldots(2)$$

Combining (1) and (2) we get that

$$|x - a| < \delta \text{ implies that } |h(x) - h(a)| = |f(g(x)) - f(g(a))|$$
$$= |f(y) - f(b)| < \varepsilon$$

where we have taken y = g(x).
Hence h is continuous at a which proves the theorem.

Let us now study the following example:

EXAMPLE 4 : Examine for continuity the following functions:

i) **The polynomial function (Refer to Unit 4) f: R → R defined by**

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots + a_xx^n$$

ii) **The rational function (Refer to Unit 4) f: R → R defined as**

$$f(x) = \frac{p(x)}{q(x)} \ \forall \ x \text{ for which } q(x) \neq 0.$$

SOLUTION

i) It is obvious that the function f(x) = x, x $\in$ R, is continuous on the whole of the real line. It follows from theorem 3(iv) that the functions x^2, x^3, ... are all continuous. Again from theorem 3(i) and 3(ii) and the fact that constant functions are continuous, we get that any polynomial f(x) in x, i.e., the function f defined by

$$f(x) = a_0 + a_1x + a_2 x^2 + \ldots + a_n x^n,$$

is continuous on R.

ii) It follows from theorem 3(v) that a rational function f, defined by,

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_\sigma + a_1x + \ldots + a_n x^n}{b_0 + b_1x + \ldots + b_mx^m}$$

is continuous at every point a $\in$ R for which q(a) $\neq$ 0.

Try the following exercise.

EXERCISE 4

Examine the continuity of the function f: R → R defined as,

i) **f(x) = x^3 at a point a $\in$ R**

ii) $$f(x) \begin{cases} = \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ = 1 & \text{if } x = 2 \end{cases}$$

9.4 NON-CONTINUOUS FUNCTIONS

You have seen that a function may or may not be continuous at a point of the domain of the function. Let us now examine why a function fails to be continuous.

A function f: S → R fails to be continuous on its domain S if it is not continuous at a particular point of S. This means that there exists a point a ∈ S such that, either

i) $\lim_{x \to a} f(x)$ does not exist

or ii) $\lim_{x \to a} f(x)$ exists but is not equal to f(a).

But you know that a function f is continuous at a point a if and only if

$f(a+) = f(a-) = f(a)$.

Thus, if f is not continuous at a, then one of the following will happen:

i) either f(a+) or f(a–) does not exist (this includes the case when both f(a+) and f(a–) do not exist).

ii) both f(a+) and f(a–) exist but $f(a+) \neq f(a-)$.

iii) both f(a+) and f(a–) exist and $f(a+) = f(a-)$ but they are not equal to f(a).

If a function f: S → R is discontinuous for each b ∈ S, then we say that f is totally discontinuous on S. Functions which are totally discontinuous are not often encountered but by no means rare. We give an example.

EXAMPLE 5 : Examine whether or not the function f: R → R defined as,

$$f(x) = \begin{cases} 1, & \text{if x is irrational} \\ 0, & \text{if x is rational} \end{cases}$$

is totally discontinuous.

SOLUTION : Let b be an arbitrary but fixed real number. Choose $\varepsilon = 1/2$. Let $\delta > 0$ be fixed. Then the interval defined by

$x - b| < \delta$

is $\{x : b - \delta < x < b + \delta\}$

or $]b - \delta , b + \delta[$

This interval contains both rational as well as irrational numbers. Why? (Refer to Unit 2 for the answer.)

If b is rational, then choose x in the interval to be irrational. If b is irrational then choose x in the interval to be rational. In either case,

$0 < |x - b| < \delta$

and

$|f(x) - f(b)| = 1 > \varepsilon$.

Thus, f is not continuous at b. Since b is an arbitrary element of S, f is not continuous at any point of S and hence is totally discontinuous.

Now you should be able to try the following exercise:

EXERCISE 5

Show that the function f: R → R defined by

$$f(x) = \begin{cases} 1, & \text{if x is rational} \\ 0, & \text{if x is irrational} \end{cases}$$

is totally discontinuous. Does f(a+) and f(a–) exist at any point a ∈ R?

There are certain discontinuities which can be removed. These are known as removable discontinuities. **A discontinuity a of a given function $f : S \to R$ is said to be removable if the limit of f(x) as x tends to a exists and that**

$$\lim_{x \to a} f(x) \neq f(a)$$

In other words, f has removable discontinuity at $x = a$ if $f(a+) = f(a-)$ but none is equal to f(a).

The removable discontinuities of a function can be removed simply by changing the value of the function at the point a of discontinuity. For this a function with removable discontinuities can be thought of as being almost continuous. We discuss the following example to illustrate a few cases of removable discontinuities.

EXAMPLE 6 : Discuss the nature of the discontinuities of the following functions:

i) $f(x) = \frac{x^2 - 4}{x - 2}, \quad x \neq 2$

$\quad = 1 \quad x = 2$

at $x = 2$.

ii) $f(x) = 3, \quad x \neq 3$

$\quad = 1 \quad x = 3$

at $x = 3$.

iii) $f(x) = x^2, \quad x \in]-2, 0 (U) 0, 2[$

$\quad = 1 \quad x = 0$

at $x = 0$.

SOLUTION

i) This function is discontinuous at $x = 2$. This is a removable discontinuity, for if we redefine $f(x) = 4$, then we can restore the continuity of f at $x = 2$.

ii) This is again a case of removable discontinuity at 3. Therefore, if f is defined by $f(x) = 3 \; \forall \; x \in R$, then it is continuous at $x = 3$.

iii) This function is discontinuous at $x = 0$. Why? This is a case of discontinuity which is removable. To remove the discontinuity, set $f(0) = 0$. In other words, define f as

$$f(x) = x^2, \; x \in]-2, 0 [U] 0, 2[$$
$$= 0, \quad x = 0$$

This is continuous at $x = 0$. Verify it.

EXAMPLE 7 : Let a function $f : R \to R$ be defined as,

i) $f(x) = \frac{1}{x}, \quad x \neq 0$

$\quad = 0, \quad x = 0$

ii) $f(x) = \frac{1}{x}, \quad$ if $x > 0$

$\quad = 1, \quad$ if $x < 0$

iii) $f(x) = \frac{1}{x}, \quad$ if $x < 0$

$\quad = 1, \quad$ if $x > 0$

Test the continuity of the function. Determine the type of discontinuity if it exists.

SOLUTION

i) Here $f(0+)$ and $f(0-)$ both do not exist (as finite real numbers) and so function is discontinuous. This is not a case of removable discontinuity.

ii) In this case, f(0) does not exist whereas $f(0+)$ exists and $f(0-) = f(0) = 1$. This is not a case of removable discontinuity.

Now try the following exercises:

EXERCISE 6

Prove that the function f defined by $f(x) = x \sin 1/x$ if $x \neq 0$ and $f(0) = 1$ has a removable discontinuity at $x = 0$.

EXERCISE 7

Prove that the function $|f|$ defined by $|f|(x) = |f(x)|$ for every real x is continuous on R whenever f is continuous on R.

EXERCISE 8

i) Find the type of discontinuity at $x = 0$ of the function f defined by $f(x) = x + 1$ if $x > 0$, $f(x) = -(x + 1)$ if $x < 0$ and $f(0) = 0$.

ii) The function f is defined by

$$f(x) = \sin \frac{1}{x}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

Is f continuous at 0?

9.5 SUMMARY

In this unit you have been introduced to the concept of the continuity of a function at a point of its domain and on a subset of its domain. The limit definition and (ε, - δ)-definition of continuity have been given in Section 9.2. It has been proved that both the definitions are equivalent. In the same section, sequential definition of continuity has been discussed and illustrations regarding its use for solving problems have been given. In Section 9.3, the algebra of continuous functions is considered and it has been proved that the sum, difference, product and quotient of two continuous functions at a point is also continuous at the point provided in the case of quotient, the function occurring in the denominator is not zero at the point. In the same section, we have proved that a continuous function of a continuous function is continuous. Finally in Section 9.4, discontinuous and totally discontinuous functions are discussed. Also in this section, one kind of discontinuity that is removable discontinuity has been studied.

9.6 ANSWERS/HINTS/SOLUTIONS

E1) i) The function f is not defined at $x = 0$. Therefore, f is not continuous at $x = 0$.

ii) This function is continuous at every point $x \neq 1$, since f(a) is defined at every point $a \neq 1$ and $\lim_{x \to a} f(x) = f(a)$. It is not defined at $x = 1$ and so not continuous at $x = 1$. If we define $f(1) = 2$, then the function is continuous at $x = 1$.

iii) In this case again, f is continuous at all $x \neq 0$. See the graph in the figure 1(b). Justify it by (ε, δ)-definition. f is not defined at 0 and so not continuous at 0.

E2) $$|\cos x - \cos a| = \left|2 \sin \frac{x + a}{2} \sin \frac{a - x}{2}\right|$$

$$= 2 \left|\sin \frac{x + a}{2} \sin \frac{x - a}{2}\right|$$

$$= 2 \left|\sin \frac{x + a}{2}\right| \left|\sin \frac{x - a}{2}\right|$$

$\leq 2 \left| \sin \frac{x-a}{2} \right|$

$= |x - a|$.

Then proceed as in example 2.
Hence cos x is continuous on R.

E3) Suppose (x_n) is a sequence which converges to 0. Then,

$\lim f(x_n) = \lim (\sqrt{x_n})$

$= \sqrt{\lim x_n} = \sqrt{0} = 0 = f(0)$

which shows that f is continuous at x = 0.

E4) i) The function $f(x) = x^3$ is continuous at x = a, for if (x_n) is a sequence which converges to a, then $\lim f(x_n) = \lim x_n^3 = (\lim x_n)^3 = a^3 = f(a)$ which shows that f is continuous at a.

ii) This function is not continuous at x = 2 because $\lim_{x \to 2} f(x) = 4$ whereas f(2) = 1.

E5) Let a be rational in R. Then f(a) = 1. We have irrationals x as close to a as we want, i.e., there exist points x in every neighbourhood of a such that $|f(x) - f(a)| = 1$ and so if $\varepsilon < 1$, we cannot find a $\delta > 0$ such that for $|x - a| < \delta$, $|f(x) - f(a)| < \varepsilon$, i.e., the function is not continuous at a. Similar argument holds good when a is irrational. Hence, the function f is discontinuous everywhere. It is clear from the above argument that f(a+) and f(a–) also do not exist at any point a.

E6) $f(x) = x \sin 1/x$ if $x \neq 0$ and $f(0) = 1$

$|f(x)| = |x \sin 1/x| \leq |x|$

since sine function is a bounded function with absolute value bounded by 1.

For $\varepsilon > 0$, if $\delta = \varepsilon$, $0 < |x| < \delta \Rightarrow |f(x)| < \varepsilon$, i.e., $\lim_{x \to 0} f(x) = 0$.

Hence, $f(0+) = f(0-) = \lim_{x \to 0} f(x) = 0$.

If we redefine f(0) = 0 instead of 1, we see that f is continuous at 0. Hence 0 is a removable discontinuity.

E7) Since f is given to be a continuous function on R, f is continuous at any point a in R. Hence, given $\varepsilon > 0$, there exists a $\delta > 0$ such that,

$|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

Now by triangle inequality for $|\ |$ we get,

$\big| |f(x)| - |f(a)| \big| \leq |f(x) - f(a)| < \varepsilon$

which proves that $|f| : x \to |f(x)|$ is continuous at a. a being arbitrary, |f| is continuous on R.

E8) i) $\lim_{x \to 0+} f(x) = \lim_{x \to 0} (x + 1) =$ i.e. $f(0+) = 1$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} (-(x + 1)) = -1$ i.e. $f(0-) = -1$

$f(0+) \neq f(0-)$

Hence, 0 is a discontinuity which is not a removable discontinuity.

ii) The function $f(x) = \sin 1/x$ for $x \neq 0$, $f(0) = 0$ has an irremovable discontinuity at x = 0 since neither f(0+) nor f(0–) exists.

UNIT 10 PROPERTIES OF CONTINUOUS FUNCTIONS

STRUCTURE

10.1 INTRODUCTION

Having studied in the last two units limit and continuity of a function at a point, algebra of limits and continuous functions, the connection between limits and continuity etc., we now take up the study of the behaviour of continuous functions on bounded closed intervals on the real line. In Section 10.2 you will learn that continuous functions on such intervals are bounded and attain their bounds; they take all values in between any two values taken at points of such intervals. In Section 10.3 you will also be introduced to the concept of uniform continuity and further you will see that a continuous function on a bounded closed interval is uniformly continuous. This means that continuous functions are well-behaved on bounded closed intervals. Thus, we will see that bounded closed intervals form an important subclass of the class of subsets of the real line which are known as compact subsets of the real line. You will study more about this in higher mathematics at a later stage. We will henceforth call bounded closed intervals of R as compact intervals.

The results of this unit play an important and crucial role in Real Analysis and so for further study in analysis, you must understand clearly the various theorems given in this unit. Some of the deep theorems of Block 3 are contained in this unit.

It may be noted that an interval of **R** will not be a compact interval if it is not a bounded or closed interval.

Objectives

After the completion of the study of this unit, you should be able to

- distinguish between the properties of continuous functions on bounded closed intervals and those on intervals which are not closed or bounded.
- understand the important role played by bounded closed intervals in Real Analysis.
- know the concept of uniform continuity and its relationship with continuity.

10.2 CONTINUITY ON BOUNDED CLOSED INTERVALS

We now consider functions continuous on bounded closed intervals. They have properties which fail to be true when the intervals are not bounded or closed. Firstly, we prove the properties and then with the help of examples we will show the failures of these properties. To prove these properties, we need an important property of the real line that was discussed in Unit 1. This property called the **completeness property of R** states as follows:

Any non-empty subset of the Real line R which is bounded above has the least upper bound. Or equivalently, any non-empty subset of R which is bounded below has the greatest lower bound.

In the following theorems we prove the properties of functions continuous on bounded closed intervals. In the first two theorems we show that a continuous function on a bounded closed interval is bounded and attains its bounds in the interval. Recall that f is bounded on a set S, if there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in S$. Note also that a real function f defined on a domain D (whether bounded or not) is bounded if and only if its range f(D) is a bounded subset of **R**.

THEOREM 1 : A function f continuous on a bounded and closed interval [a, b] is necessarily a bounded function.

PROOF : Let S be the collection of all real numbers c in the interval [a, b] such that f is bounded on the interval [a, c]. That is, a real number c in [a, b] belongs to S if and only if there exists a constant M_c such that $|f(x)| \leq M_c$ for all x in [a, c]. Clearly, $S \neq \phi$ since $a \in S$ and b is an upper bound for S. Hence, by completeness property of R, there exists a least upper bound for S. Let it be k (say). Clearly, $k \leq b$. W prove that $k \in S$ and $k = b$ which will complete the proof of the theorem.

Corresponding to $\varepsilon = 1$, by the continuity of f at $k(\leq b)$ there exists a $d > 0$ such that

$$|f(x) - f(k)| < \varepsilon = 1 \text{ whenever } |x - k| < d, \ x \in [a, b].$$

By the triangle inequality we have

$$|\ |f(x)| - |f(k)|\ | \leq |f(x) - f(k)| < 1$$

Hence, for all x in [a. b] for which $|x - k| < d$, we have that

$$|f(x)| < |f(k)| + 1 \qquad \ldots(1)$$

Since k is the least upper bound of S, k–d is not an upper bound of S. Therefore, there is a number $c \in S$ such that

$$k - d < c \leq k$$

Consider any t such that $k \leq t < k + d$. If x belongs to the interval [c, t] then $|x - k| < d$. For,

$$x \in [c, t] \Rightarrow c \leq x \leq t \Rightarrow k - d < c \leq x \leq t < k + d \qquad \ldots(2)$$

Now $c \in S$ implies that there exists $M_c > 0$ such that for all

$$x \in [a, c], \ |f(x)| \leq M_c \qquad \ldots(3)$$

$$x \in [a, t] = [a, c] \cup [c, t] \Rightarrow \text{either } x \in [a, c] \text{ or } x \in [c, t].$$

If $x \in [a, c]$, by (3) we have

$$|f(x)| \leq M_c < M_c + |f(k)| + 1.$$

If, however, $x \in [c, t]$ then by (1) and (2) we have

$$|f(x)| < |f(k)| + 1 < M_c + |f(k)| + 1$$

In any case we get that $x \in [a, t]$ implies that

$$|f(x)| < M_c + |f(k)| + 1$$

This shows that f is bounded in the interval [a, t] thus proving that $t \in S$ whenever $k \leq t < k + d$. In particular $k \in S$. In such a case $k = b$. For otherwise we can choose a 't' such that $k < t < k + d$ and $t \in S$ which will contradict the fact that k is an upper bound. This completes the proof of the theorem.

Having proved the boundedness of the function continuous on a bounded closed interval, we now prove that the function attains its bounds that is it has the greatest and the smallest values.

THEOREM 2 : If f is a continuous function on the bounded closed interval [a, b] then there exists points x_1 and x_2 in [a, b] such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a, b]$ (i.e. f attains its bounds).

PROOF : From Theorem 1, we know that f is bounded on [a,b].
Therefore there exists M such that $|f(x)| \leq M \ \forall \ x \in [a, b]$.
Hence, the collection $\{f(x) : a \leq x \leq b\}$ has an upper bound,
since $f(x) \leq |f(x)| \leq M \ \forall \ x \in [a, b]$.
So by the completeness property of R, the set $\{f(x) : a \leq x \leq b\}$ has a least upper bound.

Let us denote by K the least upper bound of $\{f(x) : a \leq x \leq b\}$.
Then $f(x) \leq K$ for all x such that $a \leq x \leq b$. We claim that there exists x_2 in [a, b] such that $f(x_2) = K$. If there is no such x_2, then $K - f(x) > 0$ for all $a \leq x \leq b$. Hence, the function g given by,

$$g(x) = \frac{1}{K - f(x)}$$

is defined for all x in [a, b] and g is continuous since f is continuous (Refer Unit 9). Therefore by Theorem 1, there exists a constant $M' > 0$ such that

$$|g(x)| \leq M' \ \forall \ x \in [a, b]$$

Thus, we get

$$|g(x)| = \frac{1}{|K - f(x)|} = \frac{1}{K - f(x)} \leq M'$$

i.e., $$f(x) \leq K - \frac{1}{M} \ \forall \ x \in [a, b].$$

But this contradicts the choice of K as the least upper bound of the set $\{f(x) : a \leq x \leq b\}$. This contradiction, therefore, proves the existence of an x_2 in [a,b] such that $f(x_2) = K \geq f(x)$ for $a \leq x \leq b$. The existence of x_1 in [a,b] such that $f(x_1) \leq f(x)$ for $a \leq x \leq b$ can be proved on exactly similar lines by taking the g.l.b. of $\{f(x) : a \leq x \leq b\}$ instead of the l.u.b. or else by considering $-f$ instead of f. (Try it).

Theorems 1 and 2 are usually proved using what is called the **Heine-Borel property** on the real line or other equivalent properties. The proofs given in this unit straightaway appeal to the completeness property of the real line (Unit 2) namely that any subset of the real line bounded above has least upper bound. These proofs may be slightly longer than the conventional ones but it does not make use of any other theorem except the property of the real line stated above.

As remarked earlier, the properties of continuous functions fail if the intervals are not bounded or closed, that is, the intervals of the type

$]a, b[$, $]a, b]$, $[a, b[$, $[a, \infty[$, $]a, \infty[$, $]-\infty, a]$, $]-\infty, a[$ or $]-\infty, \infty[$.

We illustrate them with the help of the following examples and exercises.

EXAMPLE 1 : Show that the function f defined by $f(x) = x^2 \ \forall \ x \in [0, \infty[$ is continuous but not bounded.

SOLUTION : The function f being a polynomial function is continuous in $[0, \infty[$. The domain of the function is an unbounded closed interval. The function is not bounded since the set of values of the function that is the range of the function is $\{x^2 : x \in [0, \infty[\} = [0, \infty[$ which is not bounded.

EXAMPLE 2 : Show that the function f defined by $f(x) = \frac{1}{x} \ \forall \ x \in]0, 1[$ is continuous but not bounded.

SOLUTION : The function f is continuous being the quotient of continuous functions $F(x) = 1$ and $G(x) = x$ with

$G(x) \neq 0, x \in]0, 1[$ (Refer Unit 9).

Domain of f is bounded but not a closed interval. The function is not bounded since its range is $\{1/x : x \in]0, 1[\} =]1, \infty[$ which is not a bounded set.

EXERCISE 1
Show that the function f defined by $f(x) = x \ \forall \ x \in]-\infty, \infty[$ is continuous but not bounded.

EXERCISE 2 : Show that the function f given by $f(x) = \frac{1}{(x-2)^2}$ $\forall x \in]2, 3[$ is continuous but not bounded.

EXAMPLE 3 : Show that the function f such that $f(x) = x \; \forall x \in]0, 1[$ is continuous but does not attain its bounds.

SOLUTION : As mentioned in Example 2, the identity function f is continuous in $]0, 1[$. Here the domain of f is bounded but is not a closed interval. The function f is bounded with least upper bound (l.u.b) = 1 and greatest lower bound (g.l.b) = 0 and both the bounds are not attained by the function, since range of $f =]0, 1[$.

EXAMPLE 4 : Show that the function f such that

$$f(x) = \frac{1}{x^2} \; \forall x \in]0, 1[.$$

is continuous but does not attain its g.l.b.

SOLUTION : The function G given by $G(x) = x^2 \; \forall x \in]0, 1[$ is continuous and $G(x) \neq 0 \; \forall x \in]0, 1[$ therefore its reciprocal function $f(x) = 1/x^2$ is continuous in $]0, 1[$ (Refer Unit 9). Here the domain f is bounded but is not a closed interval. Further l.u.b. of f does not exist whereas its g.l.b. is 1 which is not attained by f.

EXERCISE 3

Show that the function f given by $f(x) = \sin x$, $x \in]0, \pi/2[$ is continuous but does not attain any of its bounds.

EXERCISE 4

Prove that the function f given by $f(x) = x^2 \; \forall x \in]-\infty, 0[$ is continuous but does not attain its g.l.b.

We next prove another important property known as the **intermediate value property** of a continuous function on an interval I. We do not need the assumption that I is bounded and closed. This property justifies our intuitive idea of a continuous function namely as a function f which cannot jump from one value to another since it takes on between any two values f(a) and f(b) all values lying between f(a) and f(b).

THEOREM 3 : (Intermediate Value Theorem). Let f be a continuous function on an interval containing a and b. If K is any number between f(a) and f(b) then there is a number c, $a \leq c \leq b$ such that $f(c) = K$.

PROOF : Either $f(a) = f(b)$ or $f(a) < f(b)$ or $f(b) < f(a)$. If $f(a) = f(b)$ then $K = f(a) = f(b)$ and so c can be taken to be either a or b. We will assume that $f(a) < f(b)$. (The other case can be dealt with similarly.) We can, therefore, assume that $f(a) < K < f(b)$.

Let S denote the collection of all real numbers x in [a, b] such that $f(x) < K$. Clearly S contains a, so $S \neq \phi$ and b is an upper bound for S. Hence, by completeness property of R, S has least upper bound and let us denote this least upper bound by c. Then $a \leq c \leq b$. We want to show that $f(c) = K$.

Since f is continuous on [a, b], f is continuous at c. Therefore, given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever x is in [a, b] and $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$,

i.e., $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. ...(4)

If $c \neq b$, we can clearly assume that $c + \delta < b$. Now c is the least upper bound of S. So $c - \delta$ is not an upper bound of S. Hence, there exists a y in S such that $c - \delta < y \leq c$. Clearly $|y - c| < \delta$ and so by (4) above, we have

$$f(c) - \varepsilon < f(y) < f(c) + \varepsilon.$$

Since y is in S, therefore $f(y) < K$. Thus, we get

$$f(c) - \varepsilon < K$$

If now $c = b$ then $K - \varepsilon < K < f(b) = f(c)$, i.e., $K < f(c) + \varepsilon$. If $c \neq b$, then $c < b$; then there exists an x such that $c < x < c + \delta$, $c + \delta$, $x \in [a, b]$ and for this x, $f(x) < f(c) + \varepsilon$ by (4) above. Since $x > c$, $K \leq f(x)$, for otherwise x would be in S which will imply that c is not an upper bound of S. Thus, again we have $K \leq f(x) < f(c) + \varepsilon$.

In any case,

$$K < f(c) + \varepsilon \qquad \ldots(6)$$

Combining (5) and (6), we get for every $\varepsilon > 0$

$$f(c) - \varepsilon < K < f(c) + \varepsilon$$

which proves that $K = f(c)$, since ε is arbitrary while K, f(c) are fixed. In fact, when $f(a) < K < f(b)$ and $f(c) = K$, then $a < c < b$.

COROLLARY 1 : If f is a continuous function on the closed interval [a, b] and if f(a) and f(b) have opposite signs (i.e., $f(a) f(b) < 0$), then there is a point x_0 in]a, b[at which f vanishes. (i.e., $f(x_0) = 0$).

Corollary follows by taking $K = 0$ in the theorem.

COROLLARY 2 : Let f be a continuous function defined on a bounded closed interval [a, b] with values in [a, b]. Then there exists a point c in [a, b] such that $f(c) = c$. (i.e., there exists a fixed point c for the function f on [a, b]).

PROOF : If $f(a) = a$ or $f(b) = b$ then there is nothing to prove. Hence, we assume that $f(a) \neq a$ and $f(b) \neq b$.

Consider the function g defined by $g(x) = f(x) - x$, $x \in [a, b]$. The function g, being the difference of two continuous functions, is continuous on [a, b]. Further, since f(a), f(b) are in [a, b], $f(a) > a$ (since $f(a) \neq a$, $f(a) \in [a, b]$) and $f(b) < b$. (Since $f(b) \neq b$, $f(b) \in [a, b]$). So, $g(a) > 0$ and $g(b) < 0$. Hence, by Corollary 1, there exists a c in]a, b[such that $g(c) = 0$, i.e., $f(c) = c$. Hence, there exists a c in [a, b] such that $f(c) = c$.

The above Corollary 1 helps us sometimes to locate some of the roots of polynomials. We illustrate this with the following example.

EXAMPLE 5 : The equation $x^4 + 2x - 11 = 0$ has a real root lying between 1 and 2.

SOLUTION : The function $f(x) = x^4 + 2x - 11$ is a continuous function on the closed interval [1, 2], $f(1) = -8$ and $f(2) = 7$. Hence, by Corollary 1, there exists an $x_0 \in]1, 2[$ such that $f(x_0) = 0$, i.e., x_0 is a real root of the equation $x^4 + 2x - 11 = 0$ lying in the interval]1, 2[.

Try the following exercises:

EXERCISE 5

Show that the equation $16x^4 + 64x^3 - 32x^2 - 117 = 0$ has a real root > 1.

EXERCISE 6

Prove that the equation $\cos x - x = 0$ possesses a root lying in the interval]0, π [.

EXERCISE 7

Prove that any polynomial of odd power with real coefficients has at least one real root.

EXERCISE 8

Show that the equation $4x^3 - 9x^2 - 6x + 2 = 0$ has a real root in each of the intervals] -1, 0 [,] 0, 1 [and] 2, 3 [.

10.3 POINTWISE CONTINUITY AND UNIFORM CONTINUITY

In this section, you will be introduced with the concept of uniform continuity of a function. The concept of uniform-continuity is given in the whole domain of the function whereas the concept of continuity is pointwise that is it is given at a point of the domain of the function. If a function f is continuous at a point a in a set A, then corresponding to a number $\varepsilon > 0$, there exists a positive number $\delta(a)$ (we are denoting δ as $\delta(a)$ to stress that δ in general depends on the point a chosen) such that $|x - a| < \delta(a)$ implies that $|f(x) - f(a)| < \varepsilon$. The number $\delta(a)$ also depends on ε. When the point a varies $\delta(a)$ also varies. We may or may not have a δ which serves for all points a in A. If we have such a δ common to all points a in A, then we say that f is uniformly continuous on A. Thus, we have the following definition of uniform continuity.

DEFINITION : Uniform Continuity of a Function
Let f be a function defined on a subset A contained in the set R of all reals It corresponding to any number $\varepsilon > 0$, there exists a number $\delta > 0$ (depending only on ε) such that

$$|x - y| < \delta, x, y \in A \Rightarrow |f(x) - f(y)| < \varepsilon$$

then we say that f is uniformly continuous on the subset A.

An immediate consequence of the definition of uniform continuity is that uniform continuity in a set A implies pointwise continuity in A. This is proved in the following theorem.

THEOREM 4 : If a function f is uniformly continuous in a set A, then it is continuous in A.

PROOF : Since f is uniformly continuous in A, given a positive number ε, there corresponds a positive number δ such that

$$|x - y| < \delta;\ x, y \in A \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \ldots(7)$$

Let a be any point of A. In the above result (1), take $y = a$. Then we get,

$$|x - a| < \delta;\ x \in A \Rightarrow |f(x) - f(a)| < \varepsilon$$

which shows that f is continuous at 'a'. Since 'a' is any point of A, it follows that f is continuous in A.

Now we consider some examples.

EXAMPLE 6 : Show that the function f: R → R given by

$$f(x) = x \ \forall\ x \in R.$$

is uniformly continuous on R

SOLUTION : For a given $\varepsilon > 0$, δ can be chosen to be ε itself so that

$$|x - y| < \delta = \varepsilon \Rightarrow |f(x) - f(y)| = |x - y| < \varepsilon.$$

EXAMPLE 7 : Show that the function f: R → R given by

$$f(x) = x^2 \ \forall\ x \in R$$

is not uniformly continuous on R.

SOLUTION : Let ε be any positive number. Let $\delta > 0$ be any arbitrary positive number. Choose $x > \varepsilon/\delta$ and $y = x + \delta/2$. Then

$$|x - y| = \frac{\delta}{2} < \delta.$$

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| = |x + y|\,|x - y| \\ &= \left(\frac{\delta}{2}\right)|x + y| = \left(\frac{\delta}{2}\right)\left|2x + \frac{\delta}{2}\right| \\ &> \frac{\delta}{2}\left(\frac{2\varepsilon}{\delta} + \frac{\delta}{2}\right) = \varepsilon + \frac{\delta^2}{4} > \varepsilon. \end{aligned}$$

That is whatever $\delta > 0$ we choose, there exist real numbers x, y such that $|x - y| < \delta$ but $|f(x) - f(y)| > \varepsilon$ which proves that f is not uniformly continuous. But we know that f is a continuous function on R.

EXAMPLE 8 : In the above example if we restrict the domain of f to be the closed interval [-1, 1], then show that f is uniformly continuous on [-1, 1].

SOLUTION : Given $\varepsilon > 0$, choose $\delta < \frac{\varepsilon}{2}$. If $|x - y| < \delta$ and $x, y \in [-1, 1]$, then using the triangle inequality for | | we get,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| = |x + y|\,|x - y| \\ &< \delta\,(|x| + |y|) \\ &\leq 2\delta \text{ (since } |x| \leq 1, |y| \leq 1) \\ &< \varepsilon. \end{aligned}$$

You should be able to solve the following exercises:

EXERCISE 9

Show that $f(x) = x^n$, $n > 1$ is not uniformly continuous on R even though for each $n > 1$, it is a continuous function on R.

EXERCISE 10

Show that the function $f(x) = \frac{1}{x}$ for $0 < x < 1$ is continuous for every x but not uniformly on]0, 1[.

EXERCISE 11

Show that the function $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on the interval]0, 1[even though it is continuous in that interval.

EXERCISE 12

Show that $f(x) = cx$ where c is a fixed non-zero real number is a uniformly continuous function on R.

We have seen in Exercise 10 that the function defined by $f(x) = 1/x$ on the open interval]0, 1[is not uniformly continuous on]0, 1[even though it is a continuous function on]0, 1[. Similarly, in Example 7, the function f defined as $f(x) = x^2$ is continuous on the entire real line R but is not uniformly continuous on **R**. However, if we restrict the domain of this function to the bounded closed interval [-1, 1], then it is uniformly continuous. This property is not a special property of the function f, where $f(x) = x^2$ but is common to all continuous functions defined on bounded closed intervals of the real line. We prove it in the following theorem.

THEOREM 5 : If f is a continuous function on a bounded and closed interval [a, b] then f is uniformly continuous on [a, b].

PROOF : Let f be a continuous function defined on the bounded closed interval [a, b]. Let S be the set of all real numbers c in the interval [a, b] such that for a given $\varepsilon > 0$, there exists positive number d_c such that for points x_1, x_2 belonging to closed interval [a, c],

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < d_c.$$

(In other words f is uniformly continuous on the interval [a, c]. Clearly $a \in S$ so that S is non-empty. Also b is an upper bound of S. From completeness property of the real line S has least upper bound which we denote by k. $k \leq b$.

f is continuous at k. Hence given $\varepsilon > 0$, there exists positive real number d_k such that

$$|f(x) - f(k)| < \varepsilon/2 \text{ whenever } |x - k| < d_k \qquad \ldots(8)$$

Since k is the least upper bound of S, $k - \frac{1}{2} d_k$ is not an upper bound of S. Therefore there exists a point $c \in S$ such that

$$k - 1/2\, d_k < c \leq k. \qquad \ldots(9)$$

Since $c \in S$, from the definition of S we see that there exists d_c such that

$$|f(x_1) - f(x_2)| < \varepsilon \text{ whenever } |x_1 - x_2| < d_c,\ x_1, x_2 \in [a, c] \qquad \ldots(10)$$

Let $d = \min((1/2)\, d_k, d_c)$ and $b' = \min.(k + (1/2)\, d_k, b)$.

Now let $x_1, x_2 \in [a, b']$ and $|x_1 - x_2| < d$. Then if $x_1, x_2 \in [a, c]$, $|x_1 - x_2| < d \le d_c$ by the choice of d and d_c, then $|f(x_1) - f(x_2)| < \varepsilon$ by (10). If one of x_1, x_2 is not in [a, c], then both x_1, x_2 belong to the interval $]k - d_k, k + d_k[$. For $x_1 \notin [a, c]$, implies $b' \ge x_1 > c > k - (1/2)\, d_k > k - d_k$ by (9) above. This means $x_1 \le b'$ implies $x_1 \le k + (1/2)\, d_k < k + d_k$ by the choice of b'. i.e.

$$k - d_k < k - (1/2)\, d_k < x_1 < k + (1/2)\, d_k < k + d_k \qquad ...(11)$$

$|x_1 - x_2| < d$ implies that $x_1 - (1/2)\, d_k < x_2 < x_1 + (1/2)\, d_k$ since $d \le (1/2)\, d_k$ by the choice of d. Thus we get from (11) above that

$$k - d_k < x_1 - (1/2)\, d_k < x_2 < x_1 + (1/2)\, d_k < k + \left(\frac{1}{2}\right) d_k + \frac{1}{2}\, d_k = k + d_k \qquad ...(12)$$

Then (11) and (12) show that $x_1, x_2 \in\,]k - d_k, k + d_k[$.

Thus we get that $|x_1 - k| < d_k$ and $|x_2 - k| < d_k$, which in turn implies, by (8) above, that $|f(x_1) - f(k)| < \varepsilon/2$ and $|f(x_2) - f(k)| < \varepsilon/2$.

Thus $|f(x_1) - f(x_2)| \le |f(x_1) - f(k)| + |f(k) - f(x_2)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. In other words, if $|x_1 - x_2| < d$ and x_1, x_2 are in $[a, b']$ then $|f(x_1) - f(x_2)| < \varepsilon$ which proves that $b' \in S$ i.e. $b' \le k$. But $k \le b'$ by the choice of b' since $k \le k + (1/2)\, d_k$ and $k \le b$. Thus we get that $k = b'$. This can happen only when $k = b$. For if $k < b$. i.e. $k = b' = \min(k + (1/2)\, d_k, b) < b$, then it implies that $\min(k + (1/2)\, d_k, b) = (k + (1/2)\, d_k = b'$, where $b' \in S$ i.e. $k + (1/2)\, d_k$ is in S and is greater than k which is a contradiction to the fact that k is the l.u.b of S. Thus we have shown that $k = b \in S$. In other words there exists a positive number d_b (corresponding to b) such that $|x_1 - x_2| < d_b$, $x_1, x_2 \in [a, b]$ implies $|f(x_1) - f(x_2)| < \varepsilon$. Therefore f is uniformly continuous in [a, b].

You may note that uniform continuity always implies continuity but not conversely (see Exercise 10). Converse is true when continuity is in the bounded closed interval.

Before we end this unit, we state a theorem without proof regarding the continuity of the inverse function of a continuous function.

THEOREM 6 : Inverse Function Theorem
Let f: I → J be a function which is both one-one and onto. If f is continuous on I, then $f^{-1} : J \to I$ is continuous on J.

For example the function

$$f: \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] \text{ defined by}$$

$$f(x) = \sin x,$$

is both one-one and onto. Besides f is continuous on $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. Therefore, by Theorem 6, the function

$$f^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \text{ defined by}$$

$$f^{-1}(x) = \sin^{-1} x$$

is continuous on [-1, 1].

10.4 SUMMARY

In this unit you have been introduced to the properties of continuous functions on bounded closed intervals and you have seen the failure of these properties if the intervals are not bounded and closed. In Section 10.2, these properties have been studied. It has been proved that if a function f is continuous on a bounded and closed interval, then it is bounded and it also attains its bounds. In the same section we proved the **Intermediate Value Theorem** that is if f is continuous on an interval containing two points a and b, then f takes every value between f(a) and f(b). In Section 10.3, the notion of uniform continuity is discussed. We have proved that if a function f is uniformly continuous in a set A, then it is continuous in A. But converse is not true. It has been proved that if a function is continuous on a bounded and closed interval, then it is uniformly continuous in the interval. These properties fail if the intervals are not bounded and closed. This has been illustrated with a few examples.

10.5 SOLUTIONS/HINTS/ANSWERS

E1) Continuity at any point c of $]-\infty, \infty[$ follows easily $|f(x) - f(c)| = |x - c| < \varepsilon$ if $|x - c| < \delta$ where $\delta = \varepsilon$.
Range of $f =]-\infty, \infty[$ which is not bounded and so f is not bounded.

E2) The functions F and G given by $F(x) = 1$ and $G(x) = (x-2)^2 \ \forall \ x \in]2, 3[$ are continuous and $G(x) \neq 0$ in $]2, 3[$ and so

$\frac{F(x)}{G(x)} = \frac{1}{(x-2)^2}$ i.e. $f(x) = \frac{1}{(x-2)^2}$ is continuous in $]2, 3[$. Its range $]1, \infty[$ is not bounded.

E3) Continuity of f is proved in Unit 9.

Range of $f =]0, 1[$.
g.l.b. $f = 0$ & l.u.b. $f = 1$ and they are not attained.

E4) Continuity of f can be proved easily, (Refer Unit 9) g.l.b. of f is 0 which is not attained by f.

E5) $f(x) = 16x^4 + 64x^4 - 32x^2 - 117$
being a polynomial is a continuous function on the interval
$[1, 2]$. $f(1) = 16 + 64 - 32 - 117 = 69 < 0$.
$f(2) = 256 + 512 - 128 - 117 = 523 > 0$
Hence by Corollary 1 of Theorem 3 there exists an x_0 in $]1, 2[$ such that $f(x_0) = 0$. i.e. there exists a root x_0, $1 < x_0 < 2$ of the equation
$16x^4 + 64x^3 - 32x^2 - 117 = 0$.

E6) Let $f(x) = \cos x - x$. Then f is a continuous function on $[0, \pi]$.
$f(0) = 1 > 0$ and $f(\pi) = -(1 + \pi) < 0$. Hence there exists an x in $]0, \pi[$ such that $f(x) = 0$. i.e. there exists a real root for $\cos x - x = 0$ between 0 and π.

E7) Let $f(x) = a_{2n+1} x^{2n+1} + ... + a_0$ be a polynomial of odd degree, $a_{2n+1} \neq 0$. It is a continuous function on the whole of the real line R. We will suppose without loss of generality that $a_{2n+1} > 0$.

Then $\lim_{x \to \infty} \frac{f(x)}{x^{2n+1}} = a_{2n+1} > 0$.

Hence we can find a real number b large enough so that $f(b) > 0$.
(Justify this) $\lim_{x \to -\infty} \frac{f(x)}{x^{2n+1}} = + a_{2n+1} > 0$. Hence we can find a real number a such that $f(a) < 0$ (when x is negative, x^{2n+1} is negative). f is continuous on the interval $[a, b]$, $f(a) < 0$ and $f(b) > 0$. Hence, by Corollary 1 of Theorem 3, there exists a real number x_0 in $]a, b[$ such that $f(x_0) = 0$. x_0 is then a real root of the polynomial f.

E8) If $f(x) = 4x^3 - 9x^2 - 6x + 2$,

then f is a continuous function on the whole real line R and hence on the intervals $[-1, 0]$, $[0, 1]$ and $[2, 3]$ also.

$f(-1) = -4 - 9 + 6 + 2 < 0$ and $f(0) = 2 > 0$.

Hence there exists a root x_0 in the interval $]-1, 0[$

$f(0) = 2 > 0$

$f(1) = 4 - 9 - 6 + 2 < 0$.

Hence there exists a root x_1 in the interval]0, 1[

$f(2) = 32 - 36 - 12 + 2 < 0$

$f(3) = 108 - 81 - 18 + 2 < 0.$

Therefore again there exists a root x_2 in the interval]2, 3[.

E9) $f(x) = x^4$, $n > 1$. Already we have proved in Example 7, that $f(x) = x^2$ is continuous on R but not uniformly continuous.
The proof for a general $n > 1$ is very much similar.
Let now $f(x) = x^n$, $\delta > 0$ be arbitrarily chosen and kept fixed. Let ε be any positive number. Choose $x > 1$, where

$$1 > \left(\frac{2\varepsilon}{\delta n}\right)^{1/n-1}$$

i.e. $n > \frac{2\varepsilon}{\delta}$. Take $y = x + \delta/2$, then $|x-y| = \delta/2 < \delta$ and $x, y > 1$.

$$|f(x) - f(y)| = |x^n - y^n| = |x-y|\,|x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots xy^{n-2} + y^{n-1}|$$

$$> (\delta/2)\, n > (\delta/2)\frac{2\varepsilon}{\delta} = \varepsilon.$$

Therefore f is not uniformly continuous.

E10) $f(x) = \frac{1}{x}$ for $0 < x < 1$.

Let a be fixed such that $0 < a < 1$. Then x_n $(0 < x_n < 1)$ converges to a implies that $\frac{1}{x_n}$ converges to $\frac{1}{a}$, hence f is continuous at a. a being arbitrary, f is continuous on]0, 1[. Let $\varepsilon > 0$ be given. Let $\delta > 0$ be an arbitrarily chosen positive number and kept fixed. Choose M large enough so that

$M > (\varepsilon/\delta)(1+\delta)$. Take a y such that $0 < y < \frac{1}{M}$.

Put $x = y + \frac{\delta}{M}$. Then $x < \frac{1}{M} + \frac{\delta}{M} = \frac{1}{M}(1 + \delta)$

Hence

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x-y|}{xy} > \frac{\delta}{M} \cdot M\frac{M}{1+\delta} = \frac{M\delta}{1+\delta} > \varepsilon$$

by the choice of M, whereas $|x-y| = \delta/M < \delta$.
This proves that f is not uniformly continuous.

E11) $f(x) = \sin 1/x$, $0 < x < 1$.

The function $g(x) = \frac{1}{x}$ is a continuous function on]0, 1[and also $h(x) = \sin x$ is a continuous function on]0, 1[. Hence $f(x) = h(g(x))$ is a continuous function on]0, 1[. We will now prove that it is not uniformly continuous on]0, 1[. Let $0 < \varepsilon < 2$ and $\delta > 0$ be any positive number. Take

$$x = \frac{2}{(4k+3)\pi},\ y = \frac{2}{(4k+1)\pi}.$$

Let k be chosen large enough so that

$|x-y| < \delta$. Then

$$|f(x) - f(y)| = \left|\sin\frac{1}{x} - \sin\frac{1}{y}\right| = \left|\sin\frac{(4k+3)\pi}{2} - \sin\frac{(4k+1)\pi}{2}\right| = 2 > \varepsilon.$$

Hence f is not uniformly continuous.

E12) Let $\varepsilon > 0$. Choose $\delta < \varepsilon/|c|$. Then whenever $|x-y| < \delta$, we have $|f(x) - f(y)| = |cx - cy| = |c|\,|x-y| < |c|\,\delta < \varepsilon$ and so f is uniformly continuous.

REVIEW

In this block, you have been introduced to the concept of the limit of a function f(x) as x tends to a point 'a'. You were also acquainted with the notion of the sequential limit. Subsequently, the notion of continuity and uniform continuity of a function has been discussed. Further the properties of functions continuous on bounded closed intervals have been proved. You have also seen the failure of these properties if the functions are continuous on intervals which are not bounded or closed. You should now attempt the following self-test questions to ascertain whether or not you have achieved the main objectives of learning the material in this block. You may compare your solutions/answers with those given at the end.

1. **Find the limits of the following functions:**

   (i) $f(x) = x \cos \frac{1}{x}, x \neq 0$, as $x \to 0$.

   (ii) $f(x) = \frac{|x|}{x}, x \neq 0$, as $x \to \infty$

   (iii) $f(x) = \frac{\sin x}{x}, x \neq 0$, as $x \to \infty$

2. **For the following functions, find the limit, if it exists:**

   (i) $f(x) = \frac{\sqrt{x} - \sqrt{b}}{x - b}$ for $x \neq b$ where $b > 0$, as $x \to b$

   (ii) $f(x) = \frac{1}{1 + e^{-1/x}}$ for $x \neq 0$, as $x \to 0^+$

   (iii) $f(x) = \begin{cases} 1 - x \text{ when } x \leq 1 \\ 2x \text{ when } x > 1. \end{cases}$, as $x \to 1$.

3. **Test whether or not the limit exists for the following:**

   (i) $f(x) = \begin{cases} 3 - x \text{ when } x > 1 \\ 1 \text{ when } x = 1 \\ 2x \text{ when } x < 1. \end{cases}$, as $x \to 1$.

   (ii) $f(x) = \frac{x^2 - 4}{x^2 + 4}, x \in R$, as $x \to 1$.

   (iii) $f(x) = \frac{\sqrt{4 + x} - 2}{x}, x \neq 0$ as $x \to 0$

   (iv) $f(x) = \frac{1}{x - 1}\left(\frac{1}{x + 3} - \frac{2}{3x + 5}\right)$ as $x \to 1$.

4. **Discuss the continuity of the following functions at the points noted against each.**

   (i) $f(x) = \begin{cases} x^2 \text{ for } x \neq 1. \\ 0 \text{ for } x = 1. \end{cases}$

   as $x \to 1$.

   (ii) $f(x) = \begin{cases} 1 \text{ for } 0 \leq x < 1 \\ 0 \text{ otherwise} \end{cases}$

   as $x \to 1$.

   (iii) $f(x) = \frac{x^2 - 4}{x - 1}$ when $x \neq 1$.

   $f(1) = 2$

   as $x \to 1$.

(iv) $f(x) = \begin{cases} (1 + x)^{1/x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

as $x \to 0$.

(v) $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

as $x \to 0$.

5. Show that the function $f : R \to R$ defined as

$$f(x) = \frac{1}{1 + |x|}$$

does not attain its infimum.

6. Show that the function $f : R \to R$ such that $f(x) = x$ is not bounded but is continuous in $[1, \infty[$.

7. Which of the following functions are uniformly continuous in the interval noted against each? Give reasons.

   (i) $f(x) = \tan x$, $x \in [0, \pi/4]$

   (ii) $f(x) = \frac{1}{x^2 - 3}$ on $[1, 4]$.

8. Which of the following functions have removable discontinuity at $x = 0$?

   (i) $f(x) = \begin{cases} (1 + x)^{1/x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

   (ii) $f(x) = \begin{cases} \frac{|x|}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$

   (iii) $f(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 2 & \text{for } x = 0 \end{cases}$

   (iv) $f(x) = \begin{cases} \frac{3x}{|x| + x^2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$

9. Give an example of the following:

   (i) A function which is nowhere continuous but its absolute value is everywhere continuous.

   (ii) A function which is continuous at one point only.

   (iii) A linear function which is continuous and satisfies the equation $f(x + y) = f(x) + f(y)$.

   (iv) Two uniform continuous functions whose product is not uniformly continuous.

10. State whether or not the following are true or false.

    (i) A polynomial function is continuous at every point of its domain.

    (ii) A rational function is continuous at every point at which it is defined.

    (iii) If a function is continuous, then it is always uniformly continuous.

    (iv) The functions e^x and $\log x$ are inverse functions for $x > 0$ and both are continuous for each $x > 0$.

    (v) The functions $\cos x$ and $\cos^{-1} x$ are continuous for all real x.

    (vi) Every continuous function is bounded.

(vii) A continuous function is always monotonic.

(viii) The function sin x is monotonic as well as continuous for $x \in \left[0, \frac{\pi}{2}\right]$.

(ix) The function cos x is continuous as well as monotonic for every $x \in R$.

(x) The function $|x|$, $x \in R$ is continuous.

ANSWERS/HINTS

1. (i) Limit is zero, since $\left|x \cos \frac{1}{x}\right| \leq |x|$

   and limit of $|x|$ as x tends to 0 is zero.

   (ii) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x}{x} = 1.$

   (iii) $\left|\frac{\sin x}{x}\right| \leq \frac{1}{|x|}$ for $x \neq 0$ and $\lim_{x \to \infty} \frac{1}{|x|} = 0.$

   So $\lim_{x \to \infty} f(x) = 0.$

2. (i) $\frac{1}{2\sqrt{b}}.$

   (ii) 1.

   (iii) $\lim_{x \to 1-} f(x) = 0$ and $\lim_{x \to 1+} f(x) = 2$. So $\lim_{x \to 1} f(x)$ does not exist.

3. (i) $\lim_{x \to 1-} f(x) = 2$ and $\lim_{x \to 1+} f(x) = 2.$

   $\therefore \lim_{x \to 1} f(x)$ exists and is 2.

   (ii) $-\frac{3}{5}.$

   (iii) $\frac{1}{4}.$

   (iv) $\frac{1}{32}.$

4. (i) $\mathrm{Lim}_{x \to 1} f(x) = 1$ but $f(1) = 0$. So f is discontinuous at 1.

   (ii) $\mathrm{Lim}_{x \to 1+} f(x) = 1$ and $\mathrm{Lim}_{x \to 1-} f(x) = 0$. So $\mathrm{Lim}_{x \to 1} f(x)$ does not exist.

   So f is discontinuous at $x = 1$.

   (iii) $\mathrm{Lim}_{x \to 1} f(x)$ does not exist. So f is discontinuous.

   (iv) $\mathrm{Lim}_{x \to 0} f(x) = e$ but $f(0) = 1$. So f is discontinuous at 0.

   (v) $\mathrm{Lim}_{x \to 0} f(x) = 0$ but $f(0) = 1$. So f is discontinuous at 0.

5. Inf. $f = 0$ which is not attained by f.

6. Range of $f = [1, \infty[$ which is not bounded.

7. Both the functions are uniformly continuous since they are continuous in bounded closed intervals.

8. (i) and (iii).

9. (i) $\begin{cases} f(x) = 1 & \text{if x is rational} \\ \quad\;\;\, = -1 & \text{if x is irrational} \end{cases}$

(ii) $\begin{cases} f(x) = x & \text{if x is rational} \\ \quad\;\; = -x & \text{if x is irrational} \end{cases}$

the only point of continuity is 0.

(iii) $f(x) = Cx, \forall x \in R$ where C is a fixed constant.

(iv) $f(x) = x, g(x) = \sin x, \forall x \in R$

Both f(x) and g(x) are uniformly continuous but their product

$f(x) g(x) = x \sin x$

is not uniformly continuous on R.

10. (i) True
 (ii) True
 (iii) False
 (iv) True
 (v) True
 (vi) False
 (vii) False
 (viii) True
 (ix) False
 (x) True

NOTES

NOTES

NOTES

Uttar Pradesh
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UGMM - 09
Real Analysis

Block

4

DIFFERENTIABILITY

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BLOCK 4 DIFFERENTIABILITY

PREVIEW

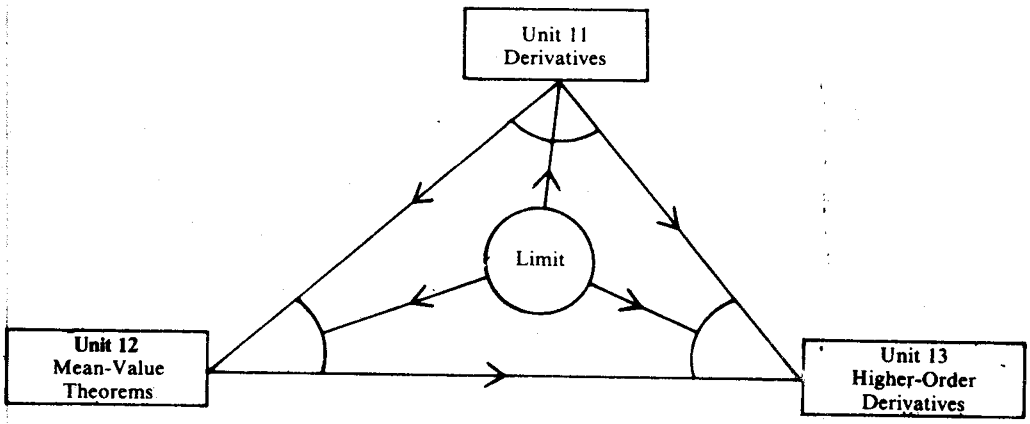
Isaac Newton

G.W. Leibniz

We live in a world of change. Our values, ideals, policies, hopes, institutions etc. are undergoing a constant change. Certain changes are happening too rapidly, while other changes are slow. Although the idea of change is important, yet it is the rate of change that is more relevant. For example, in the study of population growth of India, it is not enough to know that the population is increased to double than what it was 30 years before. It is equally important to know the rate at which this increase took place because several aspects of the country's development are linked with it. One of the important mathematical tools that is used to measure such rates is given by Calculus — one of the most beautiful areas of Mathematics.

Calculus, even since its discovery in the late 17th century, has been dominating the study of Mathematics because of its wide applications. Historically, as you know that, **Calculus** has been divided into two branches — **Differential Calculus** and **Integral Calculus**. The Differential Calculus is used to find rates of change and slopes of tangents to curves while Integral Calculus is used to find areas of the regions which are bounded by the curves. The basic idea of Differential Calculus is the differentiation of a function and that of Integral Calculus is the integration of a function. The differentiation and integration are two fundamental limiting processes of Calculus which are closely related to the study of Real Analysis. Isolated instances of these processes in Calculus were considered in the ancient times but the systematic development of calculus was started in 17th century by the two great pioneers, Newton and Leibniz. The key to this systematic development is the intimate inverse relationship between the two concepts. Many persons such as Fermat, Galileo, Kepler contributed to the foundations of Calculus. In the 19th Century, with the careful formulation of the concept of limit and with the analysis of number system, the notions of the derivative and integral of a function were strongly founded. The explicit formulation of continuity and differentiability in terms of limits was the main contributions by Cauchy. This set the pattern for the subsequent expositions of the subject by Rolle and Lagrange among others.

In this block, we shall deal with the notion of the derivative of a function and certain basic results given by Rolle, Lagrange and Cauchy. We shall conclude this block by discussing some applications of the derivative of a function. There are three units in this block namely Unit 11, Unit 12 and Unit 13 as depicted in the following picture.



In Unit 11, we introduce the notion of the derivative of a function and give its geometrical interpretation. Also, we discuss its relationship with the continuity of a function and then we define the algebraic operations of addition, subtraction, multiplication and division on the differentiable functions.

Unit 12 deals with the important contributions made by Rolle, Lagrange and Cauchy in the form of mean-value theorems. We also discuss the generalized mean-value theorem, intermediate-value theorem and Darboux theorem.

In Unit 13, we confine our discussion to Taylor's and Maclaurin's theorems and discuss applications of differentiability to evaluate some indeterminate forms of the functions as well as their extreme-values.

NOTATIONS AND SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	is a member of (belongs to)
$\notin$	is not a member of (does not belong to)
$\subset$	is a subset of (is contained in)
$\not\subset$	is not a subset of (is not contained in)
$\supset$	is a superset
$\cup$	Union
$\cap$	intersection
ϕ	empty set
$\Rightarrow$	implies
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\sim$	equivalence relation
$\forall$	for all
$\exists$	there exists
.	multiplication
$+$	addition
$-$	subtraction
sup	supremum
inf	infimum
min	minimum
max	maximum
$\circ$	composition
f'	derivative of f
f^{-1}	inverse of a function f
exp	exponential
log	logarithm
In	natural logarithm
sgn	signum
[x]	greatest integer not exceeding x
$\|x\|$	absolute value of x or Modulus of x
$\mathbf{R}^+$	set of positive real numbers
$\mathbf{R}$	set of real numbers
$\mathbf{I}$	Set of irrational numbers
$\mathbf{Q}$	set of rational numbers
$\mathbf{Z}$	set of integers
$\mathbf{N}$	set of natural numbers
$\mathbf{F}$	field
$\mathbf{C}$	set of complex numbers
[a, b]	closed interval
]a, b[	open interval
]a, b]	semi-open interval (open at left)—semi-closed interval
[a, b[	semi-open interval (open at right)—semi-closed interval
$+\infty$	infinity
$-\infty$	minus infinity
Σ	sum
$\sum_{n=1}^{\infty} u_n$	infinite series
(s_n)	sequence
S^c	complement of S
S'	derived set of S
$\bar{S}$	closure of S

Greek Alphabets

α	Alpha
β	Beta
γ	Gama
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
i	Iota
λ	Lambda
μ	Mu
ν	Nu
ξ	exi
π	Pi
Π	(capital Pi)
ρ	Rho
σ (Σ)	Sigma (capital Sigma)
τ	Tou
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

UNIT 11 DERIVATIVES

Structure

11.1 INTRODUCTION

You have been introduced to the limiting process in various ways. In Block I, this process was discussed in terms of the limit point of a set. The limit concept as applied to sequences was studied in Block 2. In Block 3, the limit concept was formalized for any function in general. It was used to define the continuity of a function. We now consider another important aspect of the limiting process. This is in relation to the development of the derivative of a function.

You may think for a while that perhaps there is some chronological order in the historical development of the limiting process. However, this is, perhaps not the case. As a matter of fact historically Differential Calculus was created by Newton and Leibnitz long before the structure of real members was put on the firm foundation.

Moreover, the concept of limit as discussed in Unit 8 was framed much later by Cauchy in 1821. **How, then, is the limit concept used in the development of the definition of the derivative of a function?** This is the first and the foremost question, we have to tackle in this unit. Besides, we have to answer a few more related questions viz. **What is the geometrical meaning of the derivative of a function?** The answer to this question will help you in appreciating the geometrical significance of some important theorems to be discussed in Unit 12.

The limit concept is common to both continuity and differentiability of a function. **Does it indicate some connection between the notions of continuity and differentiability? If so, then what is the relationship between the two notions?** We shall find suitable answers to these questions. Also, we shall discuss the characterization of the monotonic functions (refer to Unit 4) with the help of the derivative of the function

OBJECTIVES

Therefore, after studying this unit, you should be able to

- define the derivative of a function at a point and give its geometrical meaning
- apply the algebraic operations of addition, subtraction, multiplication and division on the derivatives of functions
- obtain a relationship between the continuity and differentiability of a function
- characterise the monotonic functions with the help of their derivatives.

11.2 DERIVATIVE OF A FUNCTION

The well-known British Mathematician Issac Newton (1642-1727) and the eminent German mathematician G.W. Leibnitz (1646-1716) share the credit of initiating Calculus towards the end of seventeenth century. To some extent, it was an attempt to answer problems already tackled by ancient Greeks but primarily Calculus was created to treat some major problems viz.

i) To find the velocity and acceleration at any instant of a moving object, given a function describing the position of the object with respect to time.

ii) To find the tangent to a curve at a given point.

iii) To find the maximum or minimum value of a function.

These were some of the problems among others which led to the development of the derivative of a function at a point. We define it in the following way :

DEFINITION 1 : DERIVATIVE AT A POINT

Let f be a real function defined on an open interval]a, b[. Let c be a point of this interval so that $a < c < b$. The function f is said to be differentiable at the point $x = c$ if

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

exists and is finite.

We denote it by $f'(c)$ and say that 'f is **derivable** at $x = c$' or 'f has **derivative** at $x = c$' or simply that $f'(c)$ exists. Further $f'(c)$ is called the **derivative** or the **differential co-efficient** of the function f at the point c.

Note that in the definition of the derivative, to evaluate the limit of the expression $\frac{f(x) - f(c)}{x - c}$ at the point c, the expression must be defined in a NBD of the point c. In other words, the function f must be defined in a NBD of the point c. It is because of this reason that we have to define the derivative of a function at a point c in an open interval]a, b[.

If

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

exists and is finite, then we say that f is **derivable from the right** at c. It is denoted by $f'(c+)$ or R $f'(c)$. Also it is called the right hand derivative of f at c.

Similarly, if

$$\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$$

exists and is finite, then we say that f is **derivable from the left** at c. It is denoted by $f'(c-)$ or $Lf'(c)$. It is also called the left hand derivative of f at c.

From the definition of limits in Unit 8, if follows that $f'(c)$ exists if and only if $Lf'(c)$ and $Rf'(c)$ exist, and
$Lf'(c) = Rf'(c)$
i.e.
$f'(c)$ exists $\Longleftrightarrow Lf'(c)$ & $Rf'(c)$ exist and $Lf'(c) = Rf'(c)$.
For example, consider the function f defined on]a, b[as
$f(x) = x^2 \; \forall \; x \in]a, b[$.
Let c be an interior point of]a, b[i.e. $a < c < b$. Then

$$Lf'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{h \to 0} \frac{f(c - h) - f(c)}{c - h - c} \; (h > 0)$$

$$= \lim_{h \to 0} \frac{(c - h)^2 - (c)^2}{-h} = 2c$$

Similarly, you can calculate $Rf'(c)$ and obtain

$Rf'(c) = 2c$.

This shows that $Lf'(c) = Rf'(c) = 2c$. Hence $f'(c)$ exists and is equal to 2c.

We have taken the point c as a point of the interval]a, b[. What happens if f is defined in a closed interval [a, b] and either $c = a$ or $c = b$ or c takes any value in the interval? To answer these questions, we give the following definitions :

DEFINITION 2 : DERIVATIVE IN AN INTERVAL

Let the function f be defined on the closed interval [a, b]. Then

(i) f is said to be derivable at the end point a i.e. $f'(a)$ exists, if

$\lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$ exists. In other words

$$f'(a) = \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

(ii) Likewise we say f is derivable at the end point b

$$\lim_{x \to b-} \frac{f(x) - f(b)}{x - b}$$

exists and

$$f'(b) = \lim_{x \to b-} \frac{f(x) - f(b)}{x - b}$$

(iii) If the function f is derivable at each point of the interval]a, b[, then it is said to be derivable in the open interval]a, b[.

(iv) If f is derivable at each point of the open interval]a, b[and also at the end points a and b, then f is said to be derivable in the closed interval [a, b].

We can similarly define the derivability in [a, b[or]a, b] or] − ∞, a[or] − ∞, a[or]a, ∞[or [a, ∞[or R =] − ∞, ∞[.

Note that for finding $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$, generally we write x = c + h, so that x → c is equivalent to h → 0. Accordingly, then we have

$$\lim_{x \to c+} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

and $f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$

Now let us discuss the following examples :

EXAMPLE 1 : Let f : R → R be a function defined as

(i) $f(x) = x^n, \forall x \in R$
where n is a fixed positive integer, and

(ii) $f(x) = k, \forall x \in R$
where k is any fixed real number.

Discuss the differentiability of f at any point x ∈ R.

SOLUTION : (i) Let c be any point of **R**. Then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^n - c^n}{x - c}$$

$$= \lim_{x \to c} (x^{n-1} + x^{n-2}c + x^{n-3}c^2 + ... + c^{n-1})$$

$$= n c^{n-1}$$

$$\Longrightarrow f'(c) = nc^{n-1}$$

Since c is any point of R, therefore f′(x) exists for all x ∈ **R**. It is given by $f'(x) = nx^{n-1}, \forall x \in R$.

(ii) If c is any point of **R**, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{k - k}{x - c} = 0$$

$\Longrightarrow f'(c) = 0.$

Since c is any point of **R**, this means that
$f'(x) = 0 \; \forall x \in R.$

EXAMPLE 2 : Let a function, f : [0, 5] → R be defined as

$$f(x) = \begin{cases} 2x \times 1 \text{ when } 0 \le x < 3 \\ x^2 - 2 \text{ when } 3 \le x \le 5 \end{cases}$$

Is f derivable at x = 3?

SOLUTION : $f'(3-) = \lim_{x \to 3-} \frac{f(x) - f(3)}{x - 3}$

$$= \lim_{x \to 3-} \frac{(2x + 1) - (9 - 2)}{x - 3}$$

$$= \lim_{x \to 3-} \frac{2(x - 3)}{x - 3} = 2$$

and $f'(3+) = \lim_{x \to 3+} \frac{f(x) - f(3)}{x - 3}$

$$= \lim_{x \to 3+} \frac{(x^2 - 2) - 7}{x - 3}$$

$$= \lim_{x \to 3+} (x + 3) = 6$$

$f'(3-) \neq f'(3+)$

$\Longrightarrow f'(3)$ does not exist i.e. f is not derivable at $x = 3$.

Now, you should try the following exercises :

EXERCISE 1

Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} x \text{ if } x < 0 \\ 0 \text{ if } x \geq 0. \end{cases}$$

show that $f'(0+) \neq f'(0-)$

EXERCISE 2

(i) Find the points at which the function $f: R \to R$ defined by $f(x) = |x - 1| + |x - 2|, \forall x \in R$ is not derivable.

(ii) Prove that $f : R \to R$ defined by $f(x) = x|x|, \forall x \in R$ is derivable at the origin.

EXAMPLE 3 : **Let $f : R \to R$ be defined as $f(x) = x^2 \cos(1/x)$ if $x \neq 0$ and $f(0) = 0$. Find the derivative at $x = 0$, if it exists.**

SOLUTION : $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos(1/x)}{x} = 0,$

$$= \lim_{x \to 0} x \cos\left(\frac{1}{x}\right).$$

Also $\cos \frac{1}{x}$ takes values between -1 and 1 and thus is bounded i.e. $|\cos \frac{1}{x}| \leq 1$. Hence

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} x \cos \frac{1}{x} = 0.$$

So that $f'(0)$ exists and is equal to 0.

EXERCISE 3

Let $f : R \to R$ be defined as

$f(x) = x \sin \frac{1}{x}$, if $x \neq 0$

$= 0$, if $x = 0$

Is f derivable at $x = 0$?

EXAMPLE 4 : For the function, f defined by

$f(x) = |\log x|$ $(x > 0)$,

determine $f'(1+)$ and $f'(1-)$.

SOLUTION : $f'(1+) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1}$

$$= \lim_{h \to 0+} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0+} \frac{|\log(1+h)| - |\log 1|}{h}$$

$$\doteq \lim_{h \to 0+} \frac{\log(1+h)}{h}$$

$$= \lim_{h \to 0+} \log(1+h)^{1/h}$$

$$= \log e = 1.$$

Also $f'(1-) = \lim_{h \to 0-} \frac{\log(1-h)}{h} = -1.$

EXERCISE 4

(i) Given :

$f(x) = x. \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ if $x \neq 0$ and if $f(0) = 0$.

Determine $f'(0+)$ and $f'(0-)$.

(ii) Let f be a function defined by

$f(x) = \frac{x}{1+|x|}, \forall x \in R.$

Show that f is differentiable everywhere.

(iii) If the function given by

$$f(x) = \begin{cases} ax^2 + b; x \leq 0 \\ x^2 \log x > 0 \end{cases}$$

possesses derivative at $x = 0$, then find a and b.

(iv) Let f be an even function defined on R.
If $f'(0)$ exists, then find its value.

11.2.1 Geometrical Interpretation of the Derivative

One of the important problems of Geometry is that of finding or drawing the tangent at any point on a given curve. The tangent describes the direction of the curve at the point and to define it, we have to use the notion of limit. A convenient measure of the direction of the curve is provided by the gradient or the slope of the tangent. This slope varies from point to point on the curve. You will see that the problem of finding the tangent and its gradient (slope) at any point on the curve is equivalent to the problem of finding the derivative of the function $y = f(x)$ which represents the curve. Thus, the **tangent to the curve $y = f(x)$ at the point with abscissa x exists if the function has a derivative at the point x and the tangent slope** $= f'(x)$. This is what is called the geometrical interpretation of the derivative of a function at a point of the domain of the function. We explain it is follows :.

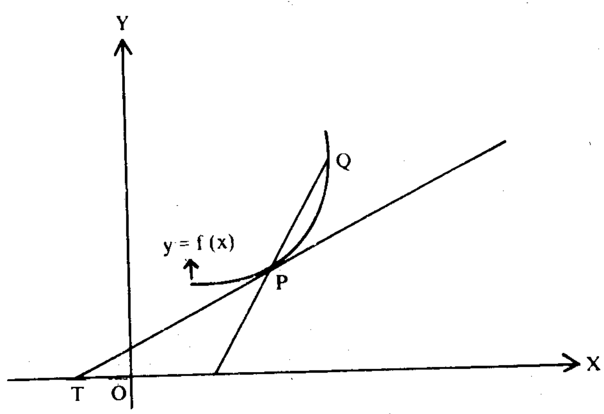


Fig. 1

Let f be a differentiable function on an interval I. The graph of f is the set

$\{(x, y)/y = f(x), x \in I\}$.

Let c, c + h ∈ I, so that P(c, f(c)) and Q(c + h, f(c + h)) are two points on the graph of f.

Therefore, the slope of the line PQ is the number

$\frac{f(c + h) - f(c)}{(c + h) - h}$, i.e., $\frac{f(c + h) - f(c)}{h}$

Also as $h \to 0$, $Q \to P$.

By definition, the derivative of f at c is

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

$$= \lim_{h \to 0} (\text{slope of PQ})$$

$$= \lim_{Q \to P} (\text{slope of PQ})$$

In the limit when $Q \to P$, the line PQ becomes the tangent at P. Therefore

$f'(c) = \lim_{Q \to P} (\text{slope of PQ})$

= slope of the tangent to the curve, y = f(x) at P. Thus when f′(c) exists, it gives the slope of the tangent line to the graph of f at the point (c, f(c)).

That is f′(c) is the tangent of the angle which this tangent line at (c, f(c)) makes with the positive direction of the axis of x.

If f′(c) = 0 the tangent line to the graph of f at x = c is parallel to the axis of x and if f′(c) exists and does not have finite value, then the tangent line is parallel to the axis of y.

11.3 DIFFERENTIABILITY AND CONTINUITY

You have seen that the notion of limit is essential and common for both the continuity and the differentiability of a function at a point. Obviously then there should be some relation between the continuity of a function and its derivative. This relation is same as the one between the curve, the graph of the function and the existence of a tangent to the curve. **A curve may have tangents at all points on it. It may have no tangent at some points on it.** For instance in the Figure 2(a), the curve has tangents at points on it while the curve in Figure 2(b) has a point P, **a sharp point** where no tangent exists.

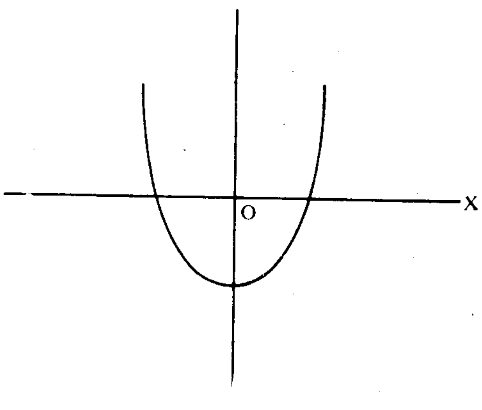


Fig. 2(a)

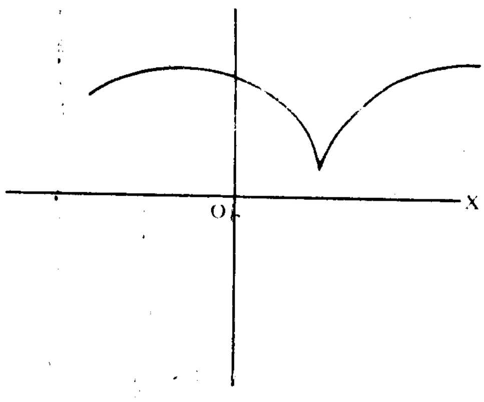


Fig. 2(b)

The fact that a curve is continuous does not necessarily imply that a tangent exists at all points on the curve. However, intuitively it follows that if a curve has a tangent at a point, then the curve must be continuous at that point. Thus, it follows that the existence of a derivative (tangent to a curve) of a function at a point implies that the function is continuous at that point. Hence differentiability of a function implies the continuity of the function. However, a continuous function may not be always differentiable. For example, the absolute value function $f : R \to R$ defined as $f(x) = |x| \ \forall \ x \in R$ is continuous at every point of its domain but it is not differentiable at the point $x = 0$. This is evident from the graph of this function which you can easily see in Unit 4.

Now we prove it in the form of the following theorem.

THEOREM 1 : Let a function f be defined on an interval I. If f is derivable at a point $c \in I$, then it is continuous at c.

PROOF : Since f is derivable at $x = c$, therefore,

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists and is equal to $f'(c)$.

Now $f(x) - f(c) = \frac{f(x) - f(c)}{x - c} \cdot (x - c)$ for $x \neq c$.

$$\lim_{x \to c} [f(x) - f(c)] = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c)$$

$$= f'(c).0 = 0$$

$\Longrightarrow \lim_{x \to c} f(x) = f(c)$

$\Longrightarrow$ f is continuous at $x = c$.

We have given the proof for the case when c is not an end point of the interval I. If c is an end point of the interval, then $\lim_{x \to c}$ is to be replaced by $\lim_{x \to c^+}$ or $\lim_{x \to c^-}$ according as c is left end point or the right end point of the interval.

Thus, it follows that continuity is a necessary condition for derivability at a point. However it is not sufficient. Many functions are readily available which are continuous at a point but not derivable there at. We give below examples of two such functions.

EXAMPLE 5 : Left $f : R \to R$ be the function given by

$f(x) = |x| \ \forall \ x \in R$.

Then f is continuous at $x = 0$ but it is not derivable there at.

SOLUTION : Recall from Unit 4 that f(x) is of the form

$$f(x) = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0. \end{cases}$$

We claim that f is continuous at $x = 0$, for

$\lim_{h \to 0+} f(x) = \lim_{h \to 0-} f(x) = 0 = f(0)$. (See the graph in Unit 4).

Now,

$$f'(0+) = \lim_{h \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{h \to 0+} \frac{x - 0}{x} = 1,$$

and $f'(0-) = \lim_{h \to 0-} \frac{-x - 0}{x - 0} = -1$.

Thus f is not derivable at $x = 0$.

EXERCISE 5

Justify that $f : R \to R$ defined as

(i) $f(x) = |x| + |x - 1|$ is continuous but not derivable at $x = 0$ and $x = 1$.

(ii) $f(x) = |x| + |x - 1| + |x - 2|$ is continuous but not derivable at $x = 0, 1, 2$.

EXAMPLE 6 : Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} x \text{ for } 0 \leq x < 1 \\ 1 \text{ for } x \geq 1. \end{cases}$$

Then f is not derivable at x = 1 but it is continuous at x = 1.

SOLUTION : Clearly $\lim_{x \to 1+} f(x) = \lim_{x \to 1-} f(x) = 1 = f(1)$.

This shows that f is continuous at x = 1.

$$f'(1+) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \to 1+} \frac{1 - 1}{x - 1} = 0$$

and $f'(1-) = \lim_{x \to 1-} \frac{x - 1}{x - 1} = 1$

i.e. $f'(1+) \neq f'(1-)$,
which shows that f is not derivable at x = 1.

From the above examples, it is clear that derivability is a more restrictive property than continuity. One might visualise that if a function is continuous on an interval, then it might fail to be derivable at finitely many points at the most in the said interval. This, however, is not true; there exists functions which are continuous on **R** but which are not derivable at any point whatsoever. In 1872, German Mathematician, K. Weierstrass, first gave an example of such a function. Here we mention an example due to Van der Waerden. The function is defined as

$$f(x) = \Sigma \frac{|10^n x - [10^n x + a]|}{10^n}$$

where a = 1/2 or − 1/2 according as $x \geq 0$ or $x < 0$. This function is known to be continuous everywhere but derivable nowhere.

Now try the following exercise.

EXERCISE 6

Prove that a function f : R → R defined as

$$f(x) = x \sin \frac{1}{x}, x \neq 0$$
$$= 0 \quad x = 0$$

is continuous but not derivable at the origin.

11.4 ALGEBRA OF DERIVATIVES

You have seen that whenever we have a new limit-definition a natural question arises. How does it behave with respect to the algebraic operations of addition, subtraction, multiplication and division? We discussed the algebra of convergent sequences in Unit 5. We also discussed algebra of limits and continuous functions in Block 3.

In this section, we shall discuss some theorems regarding the derivability of the sum, product, quotient and composite of a pair of derivable functions.

I SUM OF TWO DERIVABLE FUNCTIONS

Let f and g be two functions both defined on an interval I. If these are derivable at $c \in I$ then f + g is also derivable at x = c and

$$(f + g)'(c) = f'(c) + g'(c).$$

PROOF : By definition, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c).$$

Then

$$\lim_{x \to c} \frac{(f + g)(x) - (f + g)(c)}{x - c} = \lim_{x \to c} \frac{f(x) + g(x) - f(c) - g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{\{ f(x) - f(c) \} + \{ g(x) - g(c) \}}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} + \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$= f'(c) + g'(c).$$

$\Longrightarrow (f + g)'(c) = f'(c) + g'(c).$
Thus $f + g$ is derivable at $x = c$.

In the same way you can also prove that $f - g$ is also derivable at $x = c$ and $(f - g)'(c) = f'(c) - g'(c)$.

II PRODUCT OF TWO DERIVABLE FUNCTIONS

Let f and g be two functions both defined on an interval I. If these are derivable at $c \in I$, then f.g. is also derivable at $x = c$ and

$$(fg)'(c) = f'(c).g(c) + f(c).g'(c).$$

PROOF : By definition, you have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

and $\lim_{x \to c+} \frac{g(x) - g(c)}{x - c} = g'(c)$

Now $\frac{(fg)(x) - (fg)(c)}{x - c} = \frac{f(x)\, g(x) - f(c)\, g(c)}{x - c}$

$$= \frac{\{ f(x) - f(c) \}\, g(x) + f(c)\, \{ g(x) - g(c) \}}{x - c}$$

$$= \frac{f(x) - f(c)}{x - c} \cdot g(x) + f(c) \cdot \frac{g(x) - g(c)}{x - c}$$

By using the above two definitions of $f'(c)$ and $g'(c)$ as well as the algebra of limits (refer to unit 8), we have

$\lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c}$ exists and is equal to

$f'(c) \,.\, g(c) + f(c) \,.\, g'(c).$
$\Longrightarrow (fg)'(c) = f'(c) \,.\, g(c) + f(c) \,.\, g'(c)$
Hence fg is derivable at $x = c$.

If a function f is derivable at a point c, then for each real number k, the function kf is also derivable at c and

$$(kf)'(c) = k \,.\, f'(c).$$

For the proof, take $f = k$, $g = f$ in Result II and use the fact that derivative of a constant function is zero everywhere.

III QUOTIENT OF TWO DERIVABLE FUNCTIONS

Let f and g be two functions both defined on an interval I. If f and g are derivable at a point $c \in I$ and $g(c) \neq 0$, then the function f/g is also derivable at c and

$$(f/g)'(c) = \frac{g(c) \,.\, f'(c) - f(c) \,.\, g'(c)}{\{ g(c) \}^2}$$

PROOF : By definitions, we have

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

and

$$\lim_{x \to c} \frac{g(x) - g(c)}{x - c} = g'(c)$$

Now

$$\frac{(f/g)(x) - (f/g)(c)}{x - c} = \frac{f(x)/g(x) - f(c)/g(c)}{x - c}$$

$$= \frac{f(x)\,g(c) - g(x)\,f(c)}{(x - c)\,g(x)\,g(c)}$$

$$= \frac{g(c)\{f(x) - f(c)\} - f(c)\{g(x) - g(c)\}}{(x - c)\,g(x)\,g(c)}$$

$$= \frac{g(c)\left\{\dfrac{f(x) - f(c)}{x - c}\right\} - f(c)\left\{\dfrac{g(x) - g(c)}{x - c}\right\}}{g(x)\,g(c)}$$

Proceeding to limits as $x \to c$, keeping in mind that f and g are derivable and $g(c) \neq 0$, we get

$$(f/g)'(c) = \frac{g(c)\,f'(c) - f(c)\,g'(c)}{\{g(c)\}^2}$$

which proves the result.

Let f be derivable at c and let $f(c) \neq 0$, then the function $\frac{1}{f}$ is derivable at c and

$(1/f)'(c) = -f'(c)/\{f(c)\}^2$.

This is known as the **Reciprocal Rule** for Derivatives. For its proof,

take $f(x) = 1$ $g = f$ in result III and use the fact that derivative of a constant function is zero everywhere.

IV CHAIN RULE

Let f and g be two functions such that the range of f is contained in the domain of g. If f is derivable at c and g is derivable at f(c), then gof is derivable at c and

$(g \circ f)'(c) = g'(f(c)).\,f'(c)$.

PROOF : The range of f is contained in the domain of g. This implies that the domain of g o f is the domain of f.

To show that $\lim\limits_{h \to 0} \frac{(g \circ f)(c + h) - (g \circ f)(c)}{h}$

exists and is equal to $g'(f(c)).\,f'(c)$, let us define a new function $\phi : R \to R$ as

$$\phi(h) = \begin{cases} \dfrac{g(f(c + h)) - g(f(c))}{f(c + h) - f(c)}, & \text{if } f(c + h) - f(c) \neq 0 \\ g'(f(c)), & \text{if } f(c + h) - f(c) = 0 \end{cases}$$

Also let $\psi : R \to R$ be a function defined as

$$\psi(h) = \frac{(g \circ f)(c + h) - (g \circ f)(c)}{h}$$

$$= \phi(h) \cdot \frac{f(c + h) - f(c)}{h}, \text{ if } h \neq 0. \text{ (How?)} \qquad (1)$$

Since f is derivable at c, therefore

$\lim\limits_{x \to c} \frac{f(c + h) - f(c)}{h}$

exists and is equal to $f'(c)$.

This, along with (1) implies that the proof of the theorem will be complete if we can show that $\lim\limits_{h \to 0} \phi(h)$ exists and is equal to $g'(c)$ for if this is shown, then from (1), we find that.

$\lim\limits_{h \to 0} \psi(h)$ exists and equal $g'(f(c)).\,f'(c)$.

Thus we have to show that $\lim\limits_{h \to 0} \psi(h) = g'(f(c))$.

As g is derivable at f(c),

$\lim\limits_{k \to 0} \frac{g(f(c) + k) - g(f(c))}{k}$ exists and equals $g'(f(c))$.

which implies that given $\epsilon > 0$, $\exists\ \delta > 0$, such that

$0 < |k| < \delta \implies \left| \frac{g(f(c) + k) - g(f(c))}{k} - g'(f(c)) \right| < \epsilon$ (2)

f is derivable at c
$\implies$ f is continuous at c
$\implies \exists\, \delta' > 0$ such that
$|h| < \delta' \implies |f(c + h) - f(c)| < \delta$ (3)

Let us consider a number h such that $|h| < \delta'$. We have two cases :

(i) $f(c + h) = f(c)$ (ii) $f(c + h) \neq f(c)$.

In case (i),
$|\phi(h) - g'(f(c))| < \epsilon$ (4)
In case (ii), let us write
$f(c + h) - f(c) = k \neq 0$
and by (1).

$$\phi(h) = \frac{g(f(c + h)) - g(f(c))}{f(c + h) - f(c)}$$

$$= \frac{g(f(c) + k) - g(f(c))}{k} \quad (5)$$

Now by (3),

$|h| < \delta' \implies |f(c + h) - f(c)| < \delta$
$\implies 0 < |k| < \delta$, by definition of k
$\implies |\phi(h) - g'(f(c))| < \epsilon$, (6)

by using (2) and (5). By (4) and (6), we get

$|h| < \delta' \implies |\phi(h) - g'(f(c))| < \epsilon$
$\implies \lim_{h \to 0} \phi(h) = g'(f(c))$.

This completes the proof.

Alternately, we can say that if
$y = g(u)$ and $u = f(x)$,
and if both $\frac{dy}{du}$ and $\frac{du}{dx}$ exist, then $\frac{dy}{dx}$ exists and is given by

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Recall that this form of **chain rule** is generally used in problems of Calculus.

For example, to find the derivative of the function
$f(x) = (x^3 + x^2 + 2)^{25}$
let $y = h(u) = u^{25}$ where $u = x^3 + x^2 + 2$. Then

$$\frac{dy}{du} = 25\, u^{24}$$
$$= 25\,(x^3 + x^2 + 2)^{24},$$

and

$$\frac{du}{dx} = 3x^2 + 2x$$

Therefore, $f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
$= 25\,(x^3 + x^2 + 2)^{24} . (3x^2 + 2x)$.

We now show how to differentiate the inverse of a differentiable function. Let f be a one-one differentiable function on an open interval I. Then f is strictly increasing or decreasing and the range f(I) of f is an interval J. Then the inverse function $g = f^{-1}$ has the domain J and
$f \circ g = i_J,\ g \circ f = i_I$

where i_I and i_J are the identity functions on I and J respectively. Then you know that
$f(x) = y \iff g(y) = x\ \forall\ x \in I, y \in J$.

Consider any point c of I. We have assumed that f is derivable at c. A natural question arises; **Is it possible for g to be derivable at f(c)? If it is so, then under what conditions?** We discuss this question as follows.:

Now f is derivable at c. If g is derivable at f(c), then by the chain rule for derivatives, g o f is derivable at c and

$(g \circ f)'(c) = g'(f(c))\, f'(c)$

But $(g \circ f)(x) = g(f(x)) = x \ \forall\ x \in I$. Therefore

$(g \circ f)'(x) = 1 \ \forall\ x \in I$

In particular for x = c, we get

$(g \circ f)'(c) = 1$

$\Rightarrow g'(f(c)) \cdot f'(c) = 1$

$\Rightarrow f'(c) \neq 0$

Thus for g to be derivable it is necessary that $f'(c) \neq 0$ i.e. the condition for the inverse of f to be derivable at a point c is that its derivative must not be zero at that point i.e. $f'(c) \neq 0$. In other words, we can say that, if $f'(c) = 0$, then the inverse of f is not derivable at c. Thus we find that a necessary condition for the derivability of the inverse function of f at c is that $f'(c) \neq 0$. Is this condition sufficient also? To answer this question, we state and prove the following theorem :

THEOREM 2 : INVERSE FUNCTION THEOREM

Suppose f is one-one continuous function on an open interval I and let J = f(I). If f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0) \in J$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

PROOF : Note that J is also an open interval.

Since f is differentiable at x_0 e 1, therefore

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Since $f'(x_0) \neq 0$ and f being one-one, $f(x) \neq f(x_0)$ for $x \neq x_0$, we have

$$\lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

$$\text{i.e. } \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$$

So given ϵ 0, there exists $\delta > 0$ such that

$$\left| \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} \right| < \epsilon \quad \text{for } 0 < |x - x_0| < \delta \qquad (7)$$

Let $g = f^{-1}$. Since f is one-one continuous function on I, therefore by inverse function theorem for continuous functions, the inverse function g is continuous on J. In particular, g is continuous at y_0. Also g is one-one. Hence there exists $n > 0$ such that

$$0 < |g(y) - g(y_0)| < \delta \text{ for } 0 < |y - y_0| < n$$

$$\text{i.e. } 0 < |g(y) - x_0| < \delta \text{ for } 0 < |y - y_0| < n \qquad (8)$$

From (1) and (2), we get

$$\left| \frac{g(y) - x_0}{f(g(y)) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \epsilon \text{ for } 0 < |y - y_0| < n$$

It follows that

$$\lim_{y \to y_0} \frac{g(y) - x_0}{f(g(y)) - f(x_0)} = \frac{1}{f'(x_0)}$$

Now $y_0 = f(x_0) \Leftrightarrow x_0 = g(y_0)$ and $f(g(y)) = y$

$$\text{Therefore } \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}$$

Hence g is differentiable at y_0 and $g'(y_0) = \dfrac{1}{f'(x_0)}$

Replacing g by f^{-1}, we can say that f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

To illustrate the above theorem, consider the following example.

EXAMPLE 7 : Find the derivative at a point y_0 of the domain of the inverse function of the function f where $f(x) = \sin x$, $x \in]-\pi/2, \pi/2[$.

SOLUTION : You know that the inverse function g of f is denoted by $\sin^{-1}$. **Domain of g is** $]-1, 1[$. Since f is one-one continuous function on $]-\pi/2, \pi/2[$ and it is **differentiable at** all points of $]-\pi/2, \pi/2[$, using the above theorem, you can see that g is **differentiable in** $]-1, 1[$ and if $y_0 = \sin x_0$ be any pt. of $]-1, 1[$ where $x_0 \in]-\pi/2, \pi/2[$, **we have**

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{\cos x_0}.$$

$$\cos x_0 = \sqrt{1 - \sin^2 x_0} = \sqrt{1 - y_0^2}$$

Hence $g'(y_0) = \dfrac{1}{\sqrt{1 - y_0^2}}$ i.e. $(\sin^{-1})'(y_0) = \dfrac{1}{\sqrt{1 - y_0^2}}$

Try the following exercise.

EXERCISE 7

Find the derivative at a point y_0 of the domain of the inverse function of the function f where

$f(x) = \log x$, $x \in]0, \infty[$.

11.5 SIGN OF A DERIVATIVE

In this section, we shall discuss the meaning of the derivative of a function at a point being positive or negative. For this, we have to recall the idea of monotonic or monotone functions which functions have already been discussed in Unit 4. But here we require the concept of increasing or decreasing function at a point of the domain of the function. So we give all these concepts in the following definition.

DEFINITION 3 : MONOTONIC FUNCTIONS

Let f be a function with domain as interval I and let $c \in I$. We say that f is an increasing function at $x = c$ if $\exists\ \delta > 0$ such that

$$x \in]c - \delta, c + \delta[\implies f(c - \delta) \leq f(x) \leq f(c + \delta).$$

Again we say that f is a decreasing function at $x = c$ if $\exists\ \delta > 0$ such that

$$x \in]c - \delta, c + \delta[\implies f(c - \delta) \geq f(x) \geq (c + \delta).$$

Further f is said to be an increasing (or a decreasing) function in the interval I if for $x_1, x_2 \in I$;
$x_1 < x_2 \implies f(x_1) \leq f(x_2)$ (or $f(x_1) \geq f(x_2)$).

Also f is said to be strictly increasing (or decreasing) in I, if
for $x_1, x_2 \in I$
$x_1 < x_2 \implies f(x_1) < f(x_2)$ (or $f(x_1) > f(x_2)$).

f is said to be monotone or monotonic in I if either it is increasing in I or it is decreasing in I.

We can similarly define strictly monotone (or monotonic) functions.

Obviously the function f defined by
$f(x) = x^2$ in $[0, 1]$
is an increasing function.
and the function f defined by
$f(x) = 1/x$ in $[1, 2]$
is a decreasing function.

Now we give the significance of the sign of the derivative of a function at a point.

MEANING OF THE SIGN OF THE DERIVATIVE AT A POINT

It is often possible to obtain valuable information about a function from the knowledge of the sign of the derivative of a function.

We discuss the two according as the derivative is positive or negative i.e.
$f'(x) > 0$ and $f'(x) < 0$.
for some x in the domain of f.

Case (i) Let c be any interior point of the domain [a, b] of a function f.

Let $f'(c)$ exist. Suppose $f'(c) > 0$

which means $\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$.

Thus for a given ϵ, $(0 < \epsilon < f'(c)$, $\exists\, \delta > 0$ such that

$$0 < |x - c| < \delta \implies \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon$$

i.e. $x \in]c - \delta, c + \delta[, x \neq c \implies f'(c) - \epsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \epsilon$

$\implies \frac{f(x) - f(c)}{x - c} > 0$, by the choice of ϵ which is less than $f'(c)$.

Therefore for all $x \in]c, c + \delta[$, $f(x) > f(c)$
and for all $x \in]c - \delta, c[$, $f(x) < f(c)$.

Thus f is increasing at $x = c$.
Now let $f'(c) < 0$.

Define a function ϕ as
$\phi(x) = -f(x) \ \forall\ x \in [a, b]$
So $\phi'(c) = -f'(c) > 0$.

Therefore using the above proved result, there exists $\delta > 0$ such that
$\forall\ x \in]c, c + \delta[$, $\phi(x) > \phi(c) \implies f(x) < f(c)$.
and $x \in]c - \delta, c[$, $\phi(x) < \phi(c) \implies f(x) > f(c)$.

Thus f is decreasing at $x = c$.
We now consider the end points of the interval, [a, b].

Case (ii) Consider the end point 'a'. You can show as in case (i),
if $f'(a)$ exists, there exists $\delta > 0$ such that

$f'(a) > 0 \implies f(x) > f(a)$ for $x \in]a, a + \delta[$
and $f'(a) < 0 \implies f(x) < f(a)$ for $x \in]a, a + \delta[$

Case (iii) Consider the end point 'b'. You can show that there exist $\delta > 0$ such that

$f'(b) > 0 \implies f(x) < f(b)$ for $x \in]b - \delta, b[$
and $f'(b) < 0 \implies f(x) > f(x) > f(b)$ for $x \in]b - \delta, b[$

Consider the following examples to make the idea clear.

EXAMPLE 8 : Show that the function f, defined on R by

$f(x) = x^3 - 3x^2 + 3x - 5 \ \forall\ x \in R$
is increasing in every interval.

SOLUTION : Now $f(x) = x^3 - 3x^2 + 3x - 5$
$\therefore\ f'(x) = 3x^2 - 6x + 3$
$= 3(x - 1)^2$.
$\implies f'(x) > 0$ when $x \neq 1$.

Let c be any real number less than 1. Then f is continuous in [c, 1] and $f'(x) > 0$ in]c, 1[;
$\implies$ f is increasing in [c, 1].

Similarly f is increasing in every interval [1, d], where d is any real number greater than 1.
We find that f is increasing in every interval.

EXAMPLE 9 : Separate the intervals in which the function f defined on R by $f(x) = 2x^3 - 15x^2 + 36x + 5 \forall x \in R$. is increasing or decreasing.

SOLUTION : Now $f(x) = 2x^3 - 15x^2 + 36x + 5$

$$\therefore f'(x) = 6x^2 - 30x + 36$$
$$= 6(x^2 - 5x + 6)$$
$$= 6(x - 2)(x - 3)$$

so that $f'(x) > 0$, whenever $x > 3$ or $x < 2$.
thus f is increasing in the intervals,
$]-\infty, 2]$ and $[3, \infty[$.
Also $f'(x) < 0$ for $2 < x < 3$
Therefore f is decreasing in the interval [2, 3].

Now try the following exercises.

EXERCISE 8
Separate the intervals in which the function, f, defined on R by
$f(x) = x^3 - 6x^2 + 9x + 4 \forall x \in R$
is increasing or decreasing.

EXERCISE 9
Show that the function, f, defined on R by
$f(x) = 9 - 12x + 6x^2 - x^3 \forall x \in R$
is decreasing in every interval.

Let f be a function with domain as an interval $I \subset R$

Let $I_1 = \{ x_0 \in I / f'(x_0)$ exists $\}$. If $I_1 \neq \phi$, we get a function f′ with domain I_1. We call f′ the derivative or the first derivative of f. We also denote the first derivative of f by f_1, or Df.

If we write $y = f(x), x \in I$, then the first derivative of f or y is also written as $\frac{dy}{dx}$ or y_1 or Dy.

Again let $I_2 = \{ t \in I_1 \mid (f')'(t)$ exists $\}$. If $I_2 \neq \phi$, we get a function $(f')'$ with domain I_2 which we call second order derivative of f and denote it by f″ or f_2. We can define higher order derivative of f in the same way. We shall discuss higher order derivatives in Unit 13. In the meantime, you should study the following example:

EXAMPLE 10 : Let f : R → R be defined as

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$$

Show that f″(0) exists. Find its value.

SOLUTION : For $x \neq 0$, clearly

$$f'(x) = 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right)$$

while

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) = 0.$$

Thus we get

$$f'(x) = 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), \text{ if } x \neq 0$$

$$f'(0) = 0.$$

Now $$f''(0) = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{4x^3 \sin(1/x) - x^2 \cos(1/x)}{x}$$

$$= \lim_{x \to 0} [4x^2 \sin(1/x) - x \cos(1/x)] = 0.$$

Now try the following exercise.

EXERCISE 10

If $f: R \to R$ is defined as

$f(x) = \sin(\sin x) \forall x \in R$,

then show that

$f''(x) + \tan x \, f'(x) + \cos^2 x \, f(x) = 0$.

11.6 SUMMARY

In this unit, we have discussed the differentiability of a function. In Section 11.2, we defined the derivative of a function f at a point c of its domain, an open interval]a, b[. If $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists, then the limit is called the derivative of f at 'c' and is denoted by $f'(c)$. If we consider the right hand limit, $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$ and it exists, then it is called the right hand derivative of f at 'c' and is denoted by $Rf'(c)$. Likewise $\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$, if it exists, is called the left hand derivative of f at c and is denoted by $L f'(c)$. From the definition of limit it follows that $f'(c)$ exists $\Leftrightarrow$ $Lf'(c)$ and $Rf'(c)$ both exist and $Lf'(c) = Rf'(c)$. If f is derivable at each point of the open interval]a, b[, then it is said to be derivable in]a, b[. If the function f is defined in the closed interval [a, b], then f is said to be derivable at the left end point 'a' if $\lim_{x \to b-} \frac{f(x) - f(a)}{x - a}$ exists and the limit is called derivative of f at 'a' and denoted by $f'(a)$.

Similarly, if $\lim_{x \to b-} \frac{f(x) - f(b)}{x - b}$ exists, that f is said to be derivable at 'b' and the limit is denoted by $f'(b)$ and is called the derivative of f at 'b'. The function f is said to be derivable in [a, b] if it is derivable in the open interval]a, b[and also at the end points 'a' and 'b'. In the same section, geometrical interpretation of the derivative is discussed and you have seen that the derivative $f'(c)$ of a function f at a point 'c' of it domain represents the slope of the tangent at the point (c, f(c)) on the graph of the function f. In Section 11 3, the relationship between the differentiability and continuity is discussed. We have proved that a function which is derivable at a point is continuous these at and illustrated that the converse is not true always. In Section 11.4, the algebra of derivatives was considered. It has been proved that if f and g are derivable at a point c, then $f \pm g$, fg are derivable at 'c' and $(f \pm g)'(c) = f'(c) \pm g'(c)$, $(fg)'(c) = f(c) g'(c) + f'(c) g(c)$. Further if $g(c) \neq 0$, then f/g is also derivable at c and

$$(f/g)'(c) = \frac{g(c) f'(c) - f(c) g'(c)}{[g(c)]^2}$$

Also in this section the chain rule for differentiation is proved, that is, if f and g are two functions such that the range of f is contained in the domain of g and f, g, are derivable respectively at c, f(c) then g o f derivable at c and $(gof)'(c) = g'(f(c)).\, f'(c)$. Result concerning the differentiation of inverse function is discussed in the same section. If f is one-one continuous function on an open interval I and f(I) = J and if f is differentiable at $x_0 \in I$, $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0) \in J$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$. Finally in Section 11.5, you have seen that a function f is increasing or decreasing at a point 'c' of its domain if its derivative $f'(c)$ at the point is positive or negative.

11.7 ANSWERS/HINTS/SOLUTIONS

E 1) $f'(0-) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{x}{x} = \lim_{x \to 0-} 1 = 1$

$f'(0+) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{0}{x} = \lim_{x \to 0+} 0 = 0$

So $f'(0-) \neq f'(0+)$

E 2) (i) The given function, f, is

$$f(x) = \begin{cases} -2x + 3 \text{ if } x < 1 \\ 1 \text{ if } 1 \leq x < 2 \\ 2x - 3 \text{ if } x \geq 2. \end{cases}$$

critical points are 1 and 2. At other points f is derivable. Show that $f'(1-) = -2$. $f'(1+) = 0$, $f'(2-) = 0$, $f'(2+) = 2$.

The only points, where f is not derivable are $x = 1$ and 2.

(ii) The given function is

$$f(x) = \begin{cases} -x^2 \text{ if } x < 0 \\ x^2 \text{ if } x \geq 0. \end{cases}$$

$$f'(0+) = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} \frac{x^2}{x} = \lim_{x \to 0^+} (x) = 0$$

$$f'(0-) = \lim_{x \to 0^-} \frac{-x^2}{x} = \lim_{x \to 0^-} (-x) = 0$$

Since $f'(0+) = f'(0-)$, therefore f is derivable at 0.

E 3) $f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0} \sin \frac{1}{x}$$

Limit does not exist.

Hence f is not derivable at $x = 0$.

E 4) (i) $f'(0+) = 1$, $f'(0-) = -1$.

(ii) The given function is

$$f(x) = \frac{x}{1 + |x|} = \begin{cases} \frac{x}{1 + x} \text{ if } x \geq 0 \\ \frac{x}{1 - x} \text{ if } x < 0. \end{cases}$$

Since $\frac{x}{1 + x} (x > 0)$ and $\frac{x}{1 - x} (x < 0)$ have non-zero polynomials in their denominators, they are derivable in their respective domains. Check directly at $x = 0$ and find that

$f'(0+) = f'(0-) = 1$.

So f is derivable $\forall\ x \in \mathbb{R}$.

(iii) Given that f has derivative at $x = 0$

Therefore $f'(0+) = f'(0-)$

i.e. $\lim_{x \to 0} \frac{x^2 \log x - b}{x} = \lim_{x \to 0} \frac{(ax^2 + b) - b}{x}$

$$= \lim_{x \to 0} (ax) = a \cdot 0 \qquad (9)$$

Since f is derivable at 0.

therefore f is continuous at $x = 0$.

Then $f(0+) = f(0-)$

$$\Rightarrow \lim_{x \to 0^+} x^2 \log x = b$$

It is known that $\lim_{x \to 0+} x^2 \log x = 0$
So $b = 0$.
Then from (1),
$a \cdot 0 = \lim_{x \to 0} x \log x$

$= 0$, which holds $\forall\ a \in \mathbf{R}$.
Hence a is arbitrary real number and $b = 0$.

(iv) Since f is an even function,
$f(-x) = f(x) \forall\ x \in \mathbf{R}$.
Also $f'(0)$ exists implies
$f'(0+) = f'(0-) = f'(0)$

$$\text{Therefore } f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{f(-x) - f(0)}{x - 0} \text{ (f is an even function } f(x) = f(-x))$$

$$= -\lim_{(-x) \to 0} \frac{f(-x) - f(0)}{-x - 0} = -f'(0)$$

which gives $2f'(0) = 0 \implies f'(0) = 0$.

E 5) (i) Now $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

and $|x - 1| = \begin{cases} x - 1 \text{ if } x \geq 1 \\ 1 - x \text{ if } x < 1 \end{cases}$

f(x) can be written as

$$f(x) = \begin{cases} 1 - 2x \text{ if } x < 0 \\ 1 \quad \text{if } 0 \leq x < 1 \\ 2x - 1 \text{ if } x \geq 1 \end{cases}$$

$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} 1 = 1$ and $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (1 - 2x) = 1$

Thus $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x)$

Therefore f is continuous at 0.

Now $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} (2x - 1) = 1$ and $\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 1 = 1$

Thus $\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = f(1)$

Therefore f is continuous at 1.

$$f'(0+) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{0}{x} = 0$$

$$f'(0-) = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{1 - 2x - 1}{x} = -2$$

$f'(0+) \neq f'(0)$ which implies that f is not derivable at 0.

$$f'(1+) = \lim_{x \to 1+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+} \frac{2x - 1 - 1}{x - 1} = \lim_{x \to 1+} 2 = 2$$

$$f'(1-) = \lim_{x \to 1-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 1-} \frac{1 - 1}{x - 1} = 0$$

$f'(1+) \neq f'(1+)$ which implies that f is not derivable at 1

(ii) Proceed as in (i).

E 6) $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin \frac{1}{x} = 0$

($\sin \frac{1}{x}$ is bounded and $\lim_{x \to 0} x = 0$)

So f is continuous at $x = 0$

$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \sin \frac{1}{x}$ which does not exist.

E 7) The inverse function g of f is given by $g(y) = e^y$. Domain of g is R, the set of real numbers. Since f is one-one continuous function on $]0, \infty[$ and it is differentiable at all points of $]0, \infty[$, so g is differentiable in R and if $y_0 = \log x_0$ be any point of R where $x_0 \in]0, \infty[$, we have

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{1/x_0} = x_0$$

Now $y_0 = \log x_0 \Rightarrow x_0 = e^{y_0}$

Hence $g'(y_0) = e^{y_0}$.

E 8) Here $f(x) = x^3 - 6x^2 + 9x + 4 \; \forall \; x \in R$

$\therefore \; f'(x) = 3(x^2 - 4x + 3)$
$\quad = 3(x - 1)(x - 3)$

So that $f'(x) > 0$, whenever $x > 3$ or $x < 1$

Therefore f is increasing in the intervals, $]-\infty, 1]$ and $[3, \infty[$.
Also $f'(x) < 0$ for $1 < x < 3$.

Therefore f is decreasing in the interval $[1, 3]$.

E 9) $f(x) = 9 - 12x + 6x^2 - x^3$

Therefore $f'(x) = -12 + 12x - 3x^2$
$\quad = -3(x^2 - 4x + 4)$
$\quad = -3(x - 2)^2 < 0$, for $x \neq 2$.

Let c be any member less than 2.

Then $f'(x) < 0$ in $]c, 2[$.

Therefore f is decreasing in $[c, 2]$.

Similarly f is decreasing in $[2, d]$, where d is any real number greater than 2.

Hence we get that f is decreasing in every interval.

E 10) $f'(x) = \cos(\sin x) \cos x$,
$f''(x) = -\sin(\sin x)\cos^2 x - \cos(\sin x) \, . \, \sin x$

$$= -f(x)\cos^2 x - \frac{f'(x)}{\cos x} \sin x$$

Hence it follows that
$f''(x) + \tan x \, f'(x) + \cos^2 x \, f(x) = 0$.

UNIT 12 MEAN-VALUE THEOREMS

Structure

12.1 INTRODUCTION

In Unit 11, you were introduced to the notion of derivable functions. Some interesting and very useful properties are associated with the functions that are continuous on a closed interval and derivable in the interval except possibly at the end points. These properties are formulated in the form of some theorems, called Mean Value Theorems which we propose to discuss in this unit. Mean value theorems are very important in Analysis because many useful and significant results are deducible from them. First we shall discuss the well-known Rolle's theorem. This theorem is one of the simplest, yet the most fundamental theorem of real analysis. It is used to establish the mean-value theorems. Finally, we shall illustrate the use of these theorems in solving certain problems of Analysis.

Objectives

After studying this unit, you should be able to

- know Rolle's theorem and its geometrical meaning
- deduce the mean value theorems of differentiability by using Rolle's Theorem
- give the geometrical interpretation of the mean value theorems
- apply Mean Value Theorems to various problems of Analysis
- understand the Intermediate Value Theorem for derivatives and the related Darboux Theorem.

12.2 ROLLE'S THEOREM

The first theorem which you are going to study in this unit is Rolle's theorem given by Michael Rolle (1652-1719), a French mathematician. This theorem is the foundation stone for all the mean value theorems. First we discuss this theorem and give its geometrical interpretation. In the subsequent sections you will see its application to various types of problems. We state and prove the theorem as follows :

THEOREM 1 : (ROLLE'S THEOREM)
If a function $f : [a, b] \to R$ is

(i) continuous on $[a, b]$
(ii) derivable on $]a, b[$,
and (iii) $f(a) = f(b)$,
then there exists at least one real number $c \in]a, b[$ such that $f'(c) = 0$.

PROOF : Since the function f is continuous on the closed interval [a, b], it is bounded and attains its bounds (refer to Unit 10). Let sup. f = M and inf. f = m. Then there are points c, $d \in [a, b]$ such that
$f(c) = M$ and $f(d) = m$.
Only two possibilities arise :
Either $M = m$ or $M \neq m$.
Case (i) When $M = m$.

Then M = m $\Longrightarrow$ f is constant over [a, b]
$\Longrightarrow f(x) = k \; \forall \; x \in [a, b]$, for some fixed real number k.
$\Longrightarrow f'(x) = 0 \; \forall \; x \in [a, b]$.

Case (ii) : When $M \neq m$. Then we proceed as follows :

Since f(a) = f(b), therefore at least one of the numbers M and m, is different from f(a) (and also different from f(b)).

Suppose that M is different from f(a) i.e. $M \neq f(a)$. Then it follows that $f(c) \neq f(a)$ which implies that $c \neq a$.

Also $M \neq f(b)$. This implies that $f(c) \neq f(b)$ which means $c \neq b$. Since $c \neq a$ and $c \neq b$, therefore $c \in]a, b[$.

Again, f(c) is the supremum of f on [a, b]. Therefore
$f(x) \leq f(c) \; \forall \; x \in [a, b]$
$\Longrightarrow f(c - h) \leq f(c)$,
for any positive real numbers h such that $c - h \in [a, b]$. Thus
$$\frac{f(c - h) - f(c)}{-h} \geq 0,$$
for a positive real number h such that $c - h \in [a, b]$.

Taking limit as $h \to 0$ and observing that f′(x) exists at each point x of]a, b[, in particular at x = c, we have
$f'(c -) \geq 0$
Again $f(x) \leq f(c)$ also implies that
$$\frac{f(c + h) - f(c)}{h} \leq 0$$
for a positive real number h such that $c + h \in [a, b]$. Again on taking limits as $h \to 0$, we get $f'(c +) \leq 0$.

But
$f'(c -) = f'(c +) = f'(c)$.
Therefore $f'(c -) \geq 0$ and $f'(c +) \leq 0$ imply that
$f'(c) \leq 0$ and $f'(c) \geq 0$
which gives $f'(c) = 0$, where $c \in]a, b[$.
You can discuss the case, $m \neq f(a)$ and $m \neq f(b)$ in a similar manner.

Note that under the conditions stated, Rolle's theorem guarantees the existence of at least one c in]a, b[such that $f'(c) = 0$. It does not say anything about the existence or otherwise of more than one such number. As we shall see in problems, for a given f, there may exist several numbers c such that $f'(c) = 0$.

Next we give the geometrical significance of the theorem.

Geometrical Interpretation of Rolle's Theorem

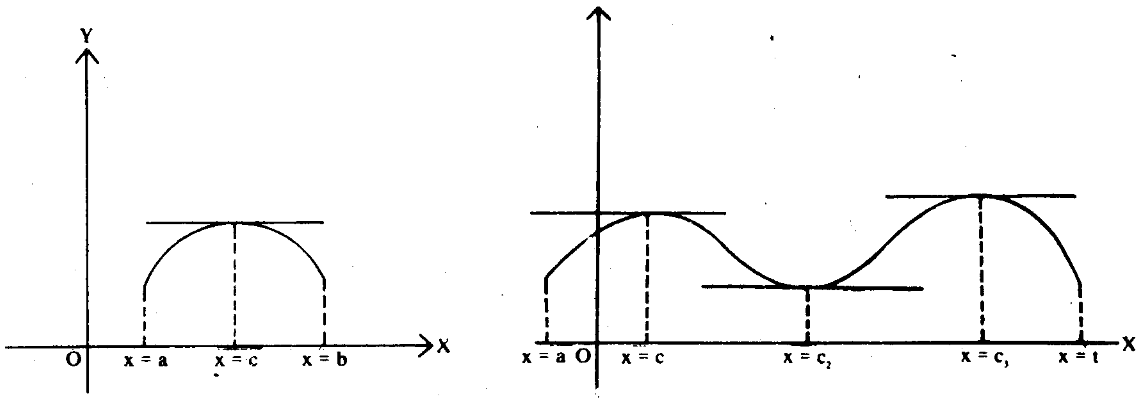


Fig. 1

You know that $f'(c)$ is the slope of the tangent to the graph of f at $x = c$. Thus the theorem simply states that between two end points with equal ordinates on the graph of f, there exists at least one point where the tangent is parallel to the axis of X, as shown in the Figures 1.

After the geometrical interpretation, we now give you the algebraic interpretation of the theorem.

Algebraic Interpretation of Rolle's Theorem

You have seen that the third condition of the hypothesis of Rolle's theorem is that $f(a) = f(b)$. If for a function f, both f(a) and f(b) are zero that is a and b are the roots of the equation $f(x) = 0$, then by the theorem there is a point c of]a, b[, where $f'(c) = 0$ which means that c is a root of the equation $f'(x) = 0$.

Thus Rolle's theorem implies that between two roots a and b of $f(x) = 0$, there always exists at least one root c of $f'(x) = 0$ where $a < c < b$. This is the algebraic interpretation of the theorem.

Before we take up problems to illustrate the use of Rolle's theorem you may note that the hypothesis of Rolle's theorem cannot be weakened. To see this, we consider the following three cases :

Case (i) Rolle's theorem does not hold if f is not continuous in [a, b].

For example, consider f where

$$f(x) = \begin{cases} x \text{ if } 0 \leq x < 1 \\ 0 \text{ if } x = 1. \end{cases}$$

Thus f is continuous everywhere between 0 and 1 except at $x = 1$. So f is not continuous in [0, 1]. Also it is derivative in]0, 1[and $f(0) = f(1) = 0$. But $f'(x) = 1 \;\forall\; x \in]0, 1[$ i.e. $f'(x) \neq 0 \;\forall\; x \in]0, 1[$.

Case (ii) The theorem no more remains true if f' does not exist even at one point in]a, b[. Consider f where

$f(x) = |x| \;\forall\; x \in]-1, 1[$.

Here f is continuous in [– 1, 1], $f(-1) = f(1)$,
but f is derivable $\forall\; x \in]-1, 1[$ except at $x = 0$.

$$\text{Also } f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1. \end{cases}$$

Hence there is not point $c \in]-1, 1[$ such that $f'(c) = 0$.

Case (iii) The theorem does not hold if $f(a) \neq f(b)$. For example if f is the function such that $f(x) = x$ in [1, 2], then
$f(1) = 1 \neq 1 \neq 2 = f(2)$.

Also $f'(x) = 1 \;\forall\; x \in]1, 2[$ i.e. there is no point $c \in]1, 2[$ such that $f'(c) = 0$.

Now we consider one example which illustrates the theorem :

EXAMPLE 1 : Verify Rolle's theorem for the function f defined by

(i) $f(x) = x^3 - 6x^2 + 11x - 6 \;\forall\; x \in [1, 3]$.

(ii) $f(x) = (x - a)^m (x - b)^n \;\forall\; x \in [a, b]$ where m and n are positive integers.

SOLUTION : (i) Being a polynomial function, f is continuous on [1, 3] and derivable in]1, 3[.

Also $f'(1) = f(3) = 0$.
Now $f'(x) = 3x^2 - 12x + 11 = 0$

$$\Rightarrow x = 2 + \frac{1}{\sqrt{3}}, 2 - \frac{1}{\sqrt{3}}$$

Clearly both of them lie in]1, 3[.

(ii) $f(x) = (x - a)^m (x - b)^n$

Obviously f is continuous in [a, b] and derivable in]a, b[.

Also $f(a) = f(b) = 0$.

Now $f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1} = 0$ implies that

$(x-a)^{m-1}(x-b)^{n-1}[m(x-b) + n(x-a)] = 0$

i.e. $m(x-b) + n(x-a) = 0$.

(As $x \neq a$ or b : we want those points which are in $]a, b[$).

Thus $x = \dfrac{na + mb}{m + n}$

This is point c and it clearly lies in $]a, b[$. You may note from Example 1(i) that **point c is not** unique.

Now you should be able to try the following exercises :

EXERCISE 1

Verify Rolle's Theorem for the function f where
$f(x) = \sin x, x \in [-2\pi, 2\pi]$.

EXERCISE 2

Examine the validity of the hypothesis and the conclusion of Rolle's theorem for the function f defined by

(a) $f(x) = \cos x \ \forall\ x \in [-\pi/2, \pi/2[$

(b) $f(x) = 1 + (x-1)^{2/3} \ \forall\ x \in [0, 2]$.

Next we give an example which shows application of Rolle's Theorems to the theory of equations.

EXAMPLE 2 : Show that there is no real number λ for which the equation $x^3 - 27x + \lambda = 0$ has two distinct roots in $[0, 2]$.

SOLUTION : Let $f(x) = x^3 - 27x + \lambda$.

Suppose for some value of λ, $f(x) = 0$ has two distinct root α and β that is f has two zeros' α and β, $\alpha \neq \beta$ in $[0, 2]$.

Without any loss of generality, we can suppose, $\alpha < \beta$.

Therefore $[\alpha, \beta] \subset [0, 2]$.

Now f is clearly continuous on $[\alpha, \beta]$, derivable in $]\alpha, \beta[$ and $f(\alpha) = f(\beta) = 0$.

Therefore by Rolle's theorem, $\exists\ c \in]\alpha, \beta[$ such that

$f'(c) = 0$

$\Longrightarrow 3c^2 - 27 = 0$

$\Longrightarrow c^2 - 9 = 0 \Longrightarrow c = \pm 3$.

Clearly none of 3 or -3 lies in $]0, 2[$, whence
-3 or $3 \notin]\alpha, \beta[$.

Thus we arrive at a contradiction. Hence the result.

EXERCISE 3

Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

EXERCISE 4

Prove that if $a_0, a_1, ..., a_n \in R$ be such that

$\dfrac{a_0}{n+1} + \dfrac{a_1}{n} + \cdots + \dfrac{a_{n-1}}{n} + a_n = 0$, then there exists at least one real number x between 0 and 1 such that

$a_0x^n + a_1x^{n-1} + ... + a_n = 0$.

Next examples show how Rolle's Theorem helps in solving some difficult problems.

EXAMPLE 3 : If f and g are continuous in [a, b] and derivable in]a, b[with $g'(x) \neq 0 \; \forall \; x \in]a, b[$; prove that there exists $c \in]a, b[$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(c) - f(a)}{g(b) - g(c)}$$

SOLUTION : The result to be proved can be written as

$$f(c)\, g'(c) + f'(c)\, g(c) - f(a)\, g'(c) - g(b)\, f'(c) = 0$$

the left hand side of which is the derivative of the function $f(x)\,g(x) - f(a)\,g(x) - g(b)\,f(x)$ at $x = c$. This suggests that we should apply Rolle's Theorem to the function ϕ where

$$\phi(x) = f(x)\,g(x) - f(a)\,g(x) - g(b)\,f(x), \forall\; x \in [a, b].$$

Since f and g are continuous in [a, b] and derivable in]a, b[, therefore ϕ is continuous in [a, b] and derivable in]a, b[. Also $\phi(a) = -g(b)\,f(a) = \phi(b)$. So ϕ satisfies all the conditions of Rolle's Theorem. Thus there is a point c in]a, b[such that $\phi'(c) = 0$ that is

$$f(c)\,g'(c) + f'(c)\,g(c) - f(a)\,g'(c) - g(b)f'(c) = 0$$

i.e. $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$.

which proves the result.

EXAMPLE 4 : If a function f is such that its derivative, f′ is continuous on [a, b] and derivable on]a, b[, then show that there exists a number $c \in]a, b[$ such that

$$f(b) = f(a) + (b - a)\,f'(a) + \frac{1}{2}(b - a)^2 f''(c).$$

SOLUTION : Clearly the functions f and f′ are continuous and derivable on [a, b].

Consider the function ϕ where

$\phi(x) = f(b) - f(x) - (b - x)\,f'(x) - (b - x)^2 A, \forall\; x \in [a, b]$ where A is a constant to be determined such that

$\phi(a) = \phi(b)$.

$$\therefore\; f(b) - f(a) - (b - a)\,f'(a) - (b - a)^2 A = 0 \qquad (1)$$

Now ϕ, being the sum of continuous and derivable functions, is itself continuous on [a, b] and derivable on]a, b[and also $\phi(a) = \phi(b)$.

Thus ϕ satisfies all the conditions of Rolle's theorem.
Therefore there exists $c \in]a, b[$ such that $\phi'(c) = 0$.

Now $\phi'(x) = -f'(x) + f'(x) - (b - x)\,f''(x) + 2(b - x)A$
This gives $0 = \phi'(c) = -(b - c)\,f''(c) + 2(b - c)A$

which means $A = \frac{1}{2} f''(c)$ since $b \neq c$.

Putting the value of A in (1), you will get

$$f(b) = f(a) + (b - a)\,f'(a) + \frac{1}{2}(b - a)^2 f''(c).$$

EXERCISE 5

Assuming f″ to be continuous on [a, b], show that

$$f(c) - f(a)\cdot\frac{b - c}{b - a} - f(b)\cdot\frac{c - a}{b - a} = \frac{1}{2}(c - a)(c - b)\,f''(d)$$

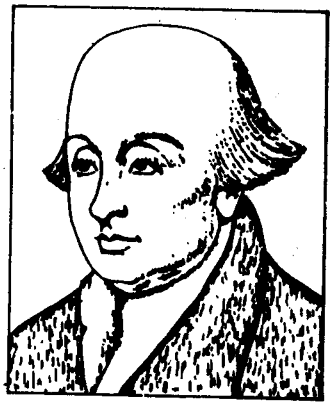
where both c and d lie in [a, b].

Note that the key to our proof of the above examples 3 and 4 and Exercise 5 and many more such situations, is the judicious choice of the function, ϕ, and many students compare it with the magician's trick of pulling a rabbit from a hat. If one can hit at a proper choice of ϕ, the problems are more than half done.

12.3 MEAN VALUE THEOREM

In this section, we discuss some of the most useful results in Differential Calculus known as the mean-value theorems given again by the two famous French mathematicians Cauchy and

Lagrange. Lagrange proved a result only by using the first two conditions of Rolle's theorem. Hence it is called Lagrange's Mean-Value Theorem. Cauchy gave another mean-value theorem in which he used two functions instead of one function as in the case of Rolle's theorem and Lagrange's Mean-Value Theorem. You will see later that Lagrange's theorem is a particular of Cauchy's Mean Value Theorem. Finally, we discuss the generalized form of these two theorems. We begin with Mean-Value Theorem given by J.L. Lagrange [1736-1813]



J.L. Lagrange

THEOREM 2 : LAGRANGE'S MEAN VALUE THEOREM

If a function $f : [a, b] \to R$ is
(i) continuous on $[a, b]$
and (ii) derivable on $]a, b[$,
then there exists at least one point $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(This is also known as the First Mean Value Theorem of Differential Calculus.)

PROOF : We define a new function ϕ as follows :
$\phi(x) = f(x) + Ax \; \forall \; x \in [a, b]$
where A is a constant to be chosen such that $\phi(a) = \phi(b)$.
$\phi(a) = f(a) + Aa$ and $\phi(b) = f(b) + Ab$.
$\phi(a) = \phi(b)$ gives

$$A = -\frac{f(b) - f(a)}{b - a}.$$

Now the function ϕ, being the sum of two continuous and derivable functions is itself

(i) continuous on $[a, b]$
(ii) derivable on $]a, b[$,
and (iii) $\phi(a) = \phi(b)$.

Therefore by Rolle's theorem $\exists$ a real number $c \in]a, b[$
such that $\phi'(c) = 0$.
But $\phi'(x) = f'(x) + A$
So $0 = \phi'(c) = f'(c) + A$

which means that $f'(c) = -A = \dfrac{f(b) - f(a)}{b - a}$

In the statement of the above theorem, sometimes b is replaced by $a + h$, so that the number c between a and b can be taken as $a + \theta h$ where $0 < \theta < 1$. Accordingly then, the theorem can be restated as follows :

Let f be defined and continuous on $[a, a + h]$ and derivable on $]a, a + h[$, then there exists θ, $0 < \theta < 1$ such that
$f(a + h) = f(a) + hf'(a + \theta h)$.

Certain important and useful results can be deduced from Lagrange Mean-Value Theorem. We state and prove these results as follows :

You already know that derivative of a constant function is zero. Conversely if the derivative of a function is zero, then it is a constant function. This can be formalized in the following way :

I. If a function f is continuous on $[a, b]$, derivable on $]a, b[$ and $f'(x) = 0 \; \forall \; x \in]a, b[$, then $f(x) = k \; \forall \; x \in [a, b]$, where k is some fixed real number.

To prove it, let λ be any point of $[a, b]$.

Then $[a, \lambda] \subset [a, b]$.

Thus f is

i) continuous on $[a, \lambda]$

ii) derivable on $]a, \lambda[$

Therefore, by Lagrange's mean value theorem, $\exists \; c \in]a, \lambda[$ such that

$$f'(c) = \frac{f(\lambda) - f(a)}{\lambda - a}$$

Now $f'(x) = 0 \ \forall \ x \in \,]a, b[$
$\Longrightarrow f'(x) = 0 \ \forall \ x \in \,]a, \lambda[$
$\Longrightarrow f'(c) = 0$
$\Longrightarrow f(\lambda) = f(a) \ \forall \ \lambda \in [a, b]$

But λ is any arbitrary point of $[a, b]$. Therefore
$f(x) = f(a) = k$ (say) $\forall \ x \in [a, b]$.

Note that if the derivatives of two functions are equal, then they differ by a constant. We have the following formal result :

II. If two functions f and g are (i) continuous in [a, b], (ii) derivable in]a, b[and (iii) $f'(x) = g'(x) \ \forall \ x \in \,]a, b[$, then $f - g$ is a constant function.

PROOF : Define a function ϕ as
$\phi(x) = f(x) - g(x) \ \forall x \in [a, b]$
Therefore $\phi'(x) = 0 \ \forall \ x \in \,]a, b[$ because it is given that
$f'(x) = g'(x)$ for each x in $]a, b[$
Also ϕ is continuous in $[a, b]$, therefore,
$\phi(x) = k, \ \forall \ x \in [a, b]$,
where k is some fixed real number. This means that
$f(x) - g(x) = k \ \forall \ x \in [a, b]$
i.e. $(f - g)(x) = k \ \forall \ x \in [a, b]$.
Thus $f - g$ is a constant function in $[a, b]$
The next two results give us method to test whether the given function is increasing or decreasing.

III. If a function f is (i) continuous on [a, b] (ii) derivable on]a, b[and (iii) $f'(x) > 0 \ \forall \ x \in \,]a, b[$, then f is strictly increasing on [a, b].

For the proof, let x_1, x_2 $(x_1 < x_2)$ be any two points of $[a, b]$. Then f is continuous in $[x_1, x_2]$ and derivable in $]x_1, x_2[$, so by Lagrange's mean value theorem,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0, \text{ for } x_1 < c < x_2$$

which implies that
$f(x_2) - f(x_1) > 0 \Longrightarrow f(x_2) > f(x_1)$ for $x_2 > x_1$
Thus $f(x_2) > f(x_1)$ for $x_2 > x_1$.
Therefore f is strictly increasing on $[a, b]$.
If the condition (iii) is replaced by $f'(x) \geq 0 \ \forall \ x \in [a, b]$, then f is increasing in $[a, b]$ since you will get $f(x_2) \geq f(x_1)$ for $x_2 > x_1$.

IV. If a function f is (i) continuous on [a, b] (ii) derivable on]a, b[and (iii) $f'(x) < 0 \ \forall \ x \in \,]a, b[$ then f is strictly decreasing on [a, b].

Proof is similar to that of III. Prove it yourself. If condition (iii) in IV is replaced by $f'(x) \leq 0 \ \forall \ x \in \,]a, b[$, then f is decreasing in $[a, b]$.

The result III and IV remain true if instead of $[a, b]$ we have the intervals $[a, \infty[$, $]-\infty, b]$, $]-\infty, \infty[$, $]a, \infty[$, $]-\infty, b[$, etc.
Note that the conditions of Lagrange's mean value theorem cannot be weakened. To see this, consider the following examples :

(1) Let f be the function defined on $[1, 2]$ as follows :

$$f(x) = \begin{cases} 1 \text{ if } x = 1 \\ x^2 \text{ if } 1 < x < 2 \\ 2 \text{ if } x = 2 \end{cases}$$

Clearly f is continuous on $[1, 2[$ and derivable on $]1, 2[$, it is not continuous only at $x = 2$ i.e. the first condition of Lagrange's Mean Value Theorem is violated.

Also $\dfrac{f(2) - f(1)}{2 - 1} = 2 - 1 = 1$.

and $f'(x) = 2x$ for $1 < x < 2$.

If this theorem is to be true then

$f'(x) = 1$ i.e. $2x = 1$ i.e. $x = 1/2$ must lie in]1, 2[, which is clearly false.

(2) Let f be the function defined on [− 1, 2] as

$f(x) = |x|$.

Here f is continuous on [− 1, 2] and derivable at all point of]− 1, 2[except at x = 0, so that the second condition of Lagrange's Mean Value Theorem is violated.

As

$$f(x) = \begin{cases} x \text{ if } 0 \le x \le 2 \\ -x \text{ if } -1 \le x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1 \text{ if } 0 < x < 2 \\ -1 \text{ if } -1 < x < 0 \end{cases}$$

Also $\dfrac{f(2) - f(-1)}{2 - (-1)} = \dfrac{2 - (+1)}{2} = \dfrac{1}{3}$

so that $\dfrac{f(2) - f(-1)}{2 - (-1)} \neq f'(x)$ for any x in]− 1, 2[.

We may remark that the conditions of Lagrange's mean value theorem are only sufficient. They are not necessary for the conclusion. This can be seen by considering the function on [0, 2] defined as :

$$f(x) = \begin{cases} 0 \text{ if } 0 \le x < \frac{1}{4} \\ x \text{ if } \frac{1}{4} \le x < \frac{1}{2} \\ \frac{x}{2} + 1 \text{ if } \frac{1}{2} \le x \le 2 \end{cases}$$

For $\dfrac{1}{4} < x < \dfrac{1}{2}$, $f'(x) = 1$.

In particular, $f'(3/8) = 1$.

Also $\dfrac{f(2) - f(0)}{2 - 0} = \dfrac{2 - 0}{2 - 0} = 1 = f'(3/8)$

even though f is neither continuous in the interval [0, 2] nor it is derivable on]0, 2[, since f is neither continuous nor derivable at 1/4 and 1/2.

Now you will see the geometrical significance of Lagrange's Mean Value Theorem.

Geometrical Interpretation of Lagrange's Mean Value theorem

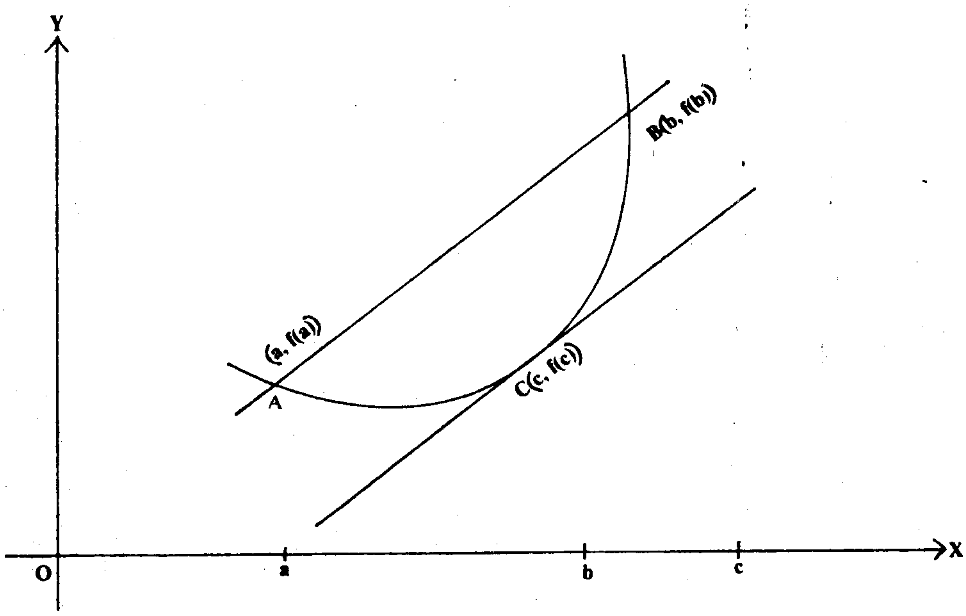


Fig. 2

Draw the graph of the function f between the two points A(a, f(a)) and B(b, f(b)). The number $\frac{f(b) - f(a)}{b - a}$ gives the slope of the chord AB. Also f′(c) gives the slope of the tangent to the graph, at the point P(c, f(c)). Thus the geometrical meaning of Lagrange's Mean Value theorem is stated as above :

If the graph of f is continuous between two points A and B and possesses a unique tangent at each point of the curve between A and B, then there is at least one point on the graph lying between A and B, where the tangent is parallel to the chord AB.

Before considering example, we have another interpretation of the theorem.

We know that f(b) − f(a) is the change in the function f as x changes from a to b so that

$\{f(b) - f(a)\}/(b - a)$

is the **average rate of change** of the function over the interval [a, b]. Also f′(c) is the **actual** rate of change of the function for x = c. Thus, the Lagrange's mean value theorem states that the average rate of change of a function over an interval is also the actual rate of change of the function at some point of the interval.

This interpretation of the theorem justifies the name '**Mean Value**' for the theorem.

Now we consider an example which verifies Lagrange's Mean Value Theorem.

EXAMPLE 5 : Verify the hypothesis and conclusion of Lagrange's mean value theorem for the functions defined as :

i) $f(x) = \frac{1}{x} \; \forall \, x \in [1, 4]$.

ii) $f(x) = \log x \; \forall x \in [1, 1 + \frac{1}{e}]$.

SOLUTION : (i) Here $f(x) = 1/x$; $x \in [1, 4]$.

Clearly f is continuous in [1, 4] and derivable in]1, 4[. So f satisfies the hypothesis of Lagrange's mean value theorem. Hence there exists a point c ∈]1, 4[satisfying

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

Putting the values of f and f′, you get

$$-\frac{1}{c^2} = \frac{(1/4) - 1}{3}$$

which gives $c^2 = 4$ i.e. $c = \pm 2$.
Of these two values of c, c = 2 lies in]1, 4[.

(ii) Here $f(x) = \log x$; $x \in [1, 1 + e^{-1}]$.
Clearly f is continuous in $[1, 1 + e^{-1}]$ and derivable in $]1, 1 + e^{-1}[$.

Therefore the hypothesis of Lagrange's mean value theorem is satisfied by f. Therefore there exists a point

$c \in \,]1, 1 + e^{-1}[$ such that

$$f'(c) = \frac{f(1 + e^{-1}) - f(1)}{(1 + e^{-1}) - 1}$$

Putting the values of f and f′, you get

$$\frac{1}{c} = \frac{\log (1 + e^{-1}) - \log 1}{e^{-1}}$$

which gives $c = [\, e \log (1 + e^{-1})\,]^{-1}$.

You can use the inequality

$\frac{x}{1 + x} < \log (1 + x) < x \; (x > 1)$ to see that $c \in \,]1, 1 + e^{-1}[$.

Now try the following exercises.

EXERCISE 6
Verify Lagrange's Mean Value theorem for the function f defined in $[0, \pi/2]$ where $f(x) = \cos x \;\forall\; x \in [0, \pi/2]$.

EXERCISE 7
Find 'c' of the Lagrange's Mean Value Theorem for the function f defined as $f(x) = x(x-1)(x-2) \;\forall\; x \in [0, 3]$.

Now you will be given examples showing the use of Lagrange's Mean Value Theorem in solving different types of problems.

EXAMPLE 6 : Prove that for any quadratic function, $lx^2 + mx + n$, the value of θ in Lagrange's Mean Value theorem is always $\frac{1}{2}$, whatever l, m, n, a and h may be.

SOLUTION : Let $f(x) = lx^2 + mx + n$; $x \in [a, a+h]$.
f being a polynomial function is continuous in $[a, a+h]$ and derivable in $]a, a+h[$. Thus f satisfies the conditions of Lagrange's Mean Value theorem.

Therefore there exists θ $(0 < \theta < 1)$ such that
$f(a+h) = f(a) + hf'(a+\theta h)$

Putting the values of f and f′ you will get
$l(a+h)^2 + m(a+h) + n = la^2 + ma + n + h[2l(a+\theta h) + m]$
i.e. $\Longrightarrow lh^2 = 2l\,\theta h^2$
which gives $\theta = 1/2$, whatever a, h, l, m, n may be.

EXAMPLE 7 : If a and b $(a < b)$ are real numbers, then there exists a real number c between a and b such that

$$c^2 = \frac{1}{3}(a^2 + ab + b^2).$$

SOLUTION : Consider the function, f, defined by
$f(x) = x^3 \;\forall\; x \in [a, b]$.

Clearly f satisfies the hypothesis of Lagrange's mean value theorem. Therefore there exists $c \in]a, b[$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which gives

$$3c^2 = \frac{b^3 - a^3}{b - a} = b^2 + ba + a^2$$

i.e. $c^2 = \frac{1}{3}(a^2 + ab + b^2)$ where $a < c < b$.

EXERCISE 8
Show that on the curve, $y = ax^2 + bx + c$, $(a, b, c \in R\; a \neq 0)$, the chord joining the points whose abscissae are $x = m$ and $x = n$, is parallel to the tangent at the point whose abscissa is given by $x = (m+n)/2$.

EXERCISE 9
Let f be defined and continuous on $[a-h, a+h]$ and derivable on $]a-h, a+h[$. Prove that there exists a real number θ $(0 < \theta < 1)$ for which
$f(a+h) + f(a-h) - 2f(a) = h[f'(a+\theta h) - f'(a-\theta h)]$.

With the help of Lagrange's Mean Value Theorem we can prove some inequalities in Analysis. We consider the following example.

EXAMPLE 8 : Prove that $\sin x < x$ for $0 < x \leq \pi/2$.

SOLUTION : Let $f(x) = x - \sin x$; $0 \leq x \leq \pi/2$.
f is continuous in $[0, \pi/2]$ and derivable in $]0, \pi/2[$.

Also $f'(x) = 1 - \cos x > 0$ for $0 < x < \pi/2$.

Therefore f is strictly increasing in $[0, \pi/2]$ which means that
$f(x) > f(0)$ for $0 < x \leq \pi/2$ (Using corollary III of Lagrange's Mean Value Theorem)
i.e. $x - \sin x > 0$ for $0 < x \leq \pi/2$.
i.e. $\sin x < x$ for $0 < x \leq \pi/2$.

We can also start with the function $g(x) = \sin x - x$ for $0 \leq x \leq \pi/2$. Then we have to use corollary IV of Lagrange's Mean Value Theorem to arrive at the desired result.

EXAMPLE 9 : Prove that x > x, whenever $0 < x < \pi/2$.

SOLUTION : Let c be any real number such that $0 < c < \pi/2$. Consider the function, f, defined by

$f(x) = \tan x - x \; \forall \; x \in [0, c]$.

The function f is continuous as well as derivable on [0, c].
Also, $f'(x) = \sec^2 x - 1 = \tan^2 x > 0 \; \forall \; x \in \;]0, c[$
Thus f is strictly increasing in [0, c].
Consequently $f(0) < f(c) \implies 0 < f(c)$
which shows that $0 < \tan x - x$, when $x = c$
$\implies \tan x > x$, when $x = c$
since c is any real number such that $0 < c < \pi/2$, therefore
$\tan x > x$ whenever $0 < x < \pi/2$.

EXAMPLE 10 : Show that $\frac{x}{1+x} < \log(1 + x) < x \; \forall \; x > 0$.

SOLUTION : Let $f(x) = x - \log(1 + x), x \geq 0$.

Therefore $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

Clearly $f'(x) > 0$ for $x > 0$.

Therefore f is strictly increasing in $[0, \infty[$.

$\therefore \; f(x) > f(0) = 0 \; \forall \; x > 0$

i.e. $x > \log(1 + x) \; \forall \; x > 0$

i.e. $\log(1 + x) < x \; \forall \; x > 0$

Again, let $g(x) = \log(1 + x) - \frac{x}{1+x}$, $x \geq 0$. Then

$$g'(x) = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{1}{(1+x)^2}$$

Clearly $g'(x) > 0 \; \forall \; x > 0$

i.e. $\log(1 + x) > \frac{x}{1+x} \; \forall \; x > 0$

i.e. $\frac{x}{1+x} < \log(1 + x) \; \forall \; x > 0$

Hence $\frac{x}{1+x} < \log(1 + x) < x \; \forall \; x > 0$

Now try the following exercises .

EXERCISE 10
Prove that

i) $x - x^3 < \tan^{-1} x$ if $x > 0$

ii) $e^{-x} > 1 - x$ if $x > 0$

Cauchy generalized Lagrange's Mean Value Theorem by using two functions.

THEOREM 3 : CAUCHY'S MEAN VALUE THEOREM

Let f and g be two functions defined on [a, b] such that

i) f and g are continuous on [a, b],

ii) f and g are derivable on]a, b[, and

iii) $g'(x) \neq 0 \; \forall \; x \in \,]a, b[$,
then there exists at least one real number $c \in \,]a, b[$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(This is also known as **Second Mean Value Theorem** of Differential Calculus.)

PROOF : Let us first observe that the hypothesis implies $g(a) \neq g(b)$

(Since $g(a) = g(b)$, combined with the other two conditions g has, means g satisfies the hypothesis of Rolle's Theorem. Thus there exists $c \in \,]a, b[$ such that $g'(c) = 0$, which violates condition (iii)).

Let a function ϕ be defined by

$\phi(x) = f(x) + A\,g(x) \; \forall \; x \in [a, b]$,
where A is a constant to be chosen such that
$\phi(a) = \phi(b)$
i.e. $f(a) + Ag(a) = f(b) + Ag(b)$
which gives
$A = -\{f(b) - f(a)\} / \{g(b) - g(a)\}$.
(As proved above, $g(b) - g(a) \neq 0$).
Now (1) ϕ is continuous on [a, b], since f and g are so,
(2) ϕ is derived on]a, b[, since f and g are so,
and (3) $\phi(a) = \phi(b)$.

Thus ϕ satisfies the conditions of Rolle's Theorem. Therefore there is a point $c \in \,]a, b[$ such that $\phi'(c) = 0$
which means that $f'(c) + Ag'(c) = 0$

i.e. $$\frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Alternative statement of Cauchy's Mean Value Theorem
If in the statement of above theorem, b is replaced by a + h, then the number $c \in \,]a, b[$ can be written as $a + \theta h$ where $0 < \theta < 1$. The above theorem then can be restated as :

Let f and g be defined and continuous on [a, a + h], derivable on]a, a + h[and $g'(x) \neq 0$ $\forall x \in \,]a, a + h[$, then there exists a real number $\theta (0 < \theta < 1)$ such that

$$\frac{f'(a + \theta h)}{g'(a + \theta h)} = \frac{f(a + h) - f(a)}{g(a + h) - g(a)}$$

As remarked earlier, **Lagrange's Mean Value Theorem can be deduced from Cauchy's Mean Value Theorem in the following way**
In Cauchy's mean value theorem, take $g(x) = x$. Then $g'(x) = 1$ and have $g'(c) = 1$. Also $g(a) = a$, $g(b) = b$. Result of Cauchy's mean value theorem becomes

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

This holds if (i) f is continuous in [a, b] and (ii) f is derivable in]a, b[which is nothing but Lagrange's mean value theorem.

Note that you might be tempted to prove Cauchy's mean value theorem by applying Lagrange's mean value theorem to the two functions f and g separately and then dividing. The desired result cannot be obtained in this manner. In fact, we will obtain

$$\frac{f'(c_1)}{f'(c_2)} = \frac{f(b) - f(a)}{g(b) - g(a)},$$

where $c_1 \in \,]a, b[$ and $c_2 \in \,]a, b[$. Note that here c_1 is not necessarily equal to c_2.

As in the case of Rolle's Theorem and Lagrange's Mean Value Theorem we give geometrical significance of Cauchy's Mean Theorem

Geometrical Interpretation of Cauchy's Mean Value Theorem

The conclusion of Cauchy's mean value theorem may be written as

$$\left\{\frac{f(b)-f(a)}{b-a}\right\} \Big/ \left\{\frac{g(b)-g(a)}{b-a}\right\} = \frac{f'(c)}{g'(c)}$$

This means

$$\frac{\text{slope of the chord joining (a, f(a)) and (b, f(b))}}{\text{slope of the chord joining (a, g(a)) and (b, g(b))}}$$

$$= \frac{\text{slope of the tangent to } y = f(x) \text{ at } (c, f(c))}{\text{slope of the tangent to } y = g(x) \text{ at } (c, g(c))}$$

Suppose that two curves $y = f(x)$ and $y = g(x)$ are continuously drawn between the two ordinates $x = a$ and $x = b$ as shown in the Figure 3. Suppose further that the tangent can be drawn to each of the curves at each point lying between these abscissae and no where does the tangent to the curve, $y = g(x)$, between these abscissae become parallel to the X-axis. Then there exists a point c between a and b such that the ratio of the slopes of the chords joining the end points of the curves is equal to ratio of the slopes of the tangents to the curves at the points obtained by the intersection of the curves and the ordinate at c.

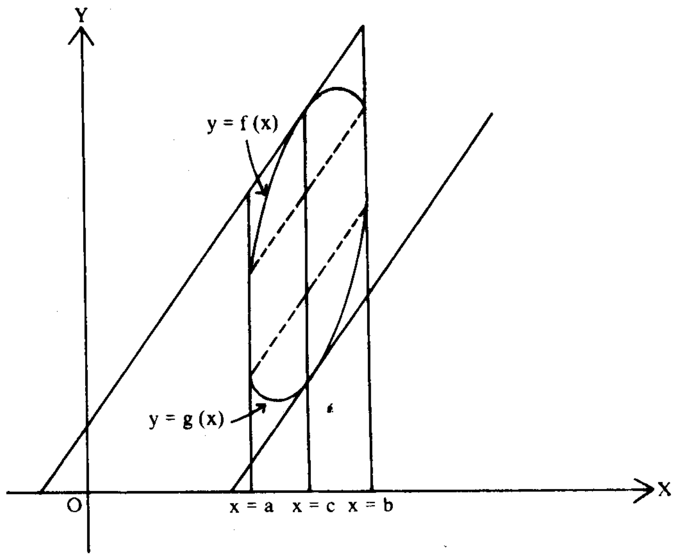


Fig. 3

As in the case of Rolle's Theorem and Lagrange's Mean Value Theorem, we now give examples concerning the verification and application of Cauchy's Mean Value Theorem.

EXAMPLE 11 : Verify Cauchy's Mean Value Theorem for the functions f and g defined as $f(x) = x^2, g(x) = x^4 \ \forall \ x \in [2, 4]$.

SOLUTION : The function f and g, being polynomial functions, are continuous in [2, 4] and derivable in]2, 4[. Also $g'(x) = 4x^3 \neq 0 \ \forall \ x \in]2, 4[$. All the conditions of Cauchy's Mean Value Theorem are satisfied. Therefore there exists a point $c \in]2, 4[$ such that

$$\frac{f(4)-f(2)}{g(4)-g(2)} = \frac{f'(c)}{g'(c)}$$

i.e. $\frac{12}{240} = \frac{2c}{4c^3}$

i.e. $c = \pm\sqrt{10}$

$c = \sqrt{10}$ lies in]2, 4[

So Cauchy's Mean Value Theorem is verified.

EXAMPLE 12 : Apply Cauchy's Mean Value Theorem to the functions f and g defined as $f(x) = x^2, g(x) = x \ \forall \ x \in [a, b]$,
and show that 'c' is the arithmatic mean of 'a' and 'b'.

SOLUTION : Clearly the function f and g satisfy the hypothesis of Cauchy's Mean Value Theorem. Therefore

$\exists$ c $\in$]a, b[such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Putting the values of f, g, f', g' we get

$$\frac{2c}{1} = \frac{b^2 - a^2}{b - a} = b + a$$

$$\Longrightarrow c = \frac{1}{2}(a + b).$$

which shows that c is the arithmetic mean of 'a' and 'b'.

EXAMPLE 13 : Show that $\dfrac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta.$

where $0 < \alpha < \theta < \beta < \pi/2$.

SOLUTION : Let f(x) = sin x and g(x) = cos x.

where $x \in [\alpha, \beta] \subset]0, \frac{\pi}{2}[$.

Now f'(x) = cos x and g'(x) = − sin x

Functions f and g are both continuous on $[\alpha, \beta]$, derivable on $]\alpha, \beta[$, and $g'(x) \neq 0 \ \forall \ x \in]\alpha, \beta[$.

By Cauchy's mean value theorem, there exists $\theta \in]\alpha, \beta[$ such that

$$\frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} = \frac{\cos \theta}{- \sin \theta}$$

$$\Longrightarrow \frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta.$$

Try the following exercise.

EXERCISE 11

Verify the Cauchy's mean value theorem for the functions, f(x) = sin x, g(x) = cos x in the interval $[-\pi/2, 0]$.

EXERCISE 12

Let the functions f and g be defined as :

$f(x) = e^x$ and $g(x) = e^{-x} \ \forall \ x \in [a, b]$.

Show that 'c' obtained from Cauchy's mean value theorem is the arithmetic mean of a and b.

EXERCISE 13

Let $f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x} \ \forall \ x \in [a, b]$ given that $0 \notin [a, b]$. Verify Cauchy's mean value theorem and show that c obtained thus is the geometric mean of a and b.

EXERCISE 14

Two functions f and g are defined as :

$f(x) = x^{-1}$ and $g(x) = x^{-2} \ \forall \ x \in [a, b]$, given that $0 \notin [a, b]$.

Apply Cauchy's mean value theorem and show that c thus obtained is the harmonic mean between a and b.

The following theorem generalises both Lagrange's and Cauchy's mean value theorems. In this theorem, three functions f, g, h are involved. Both Lagrange's and Cauchy's mean value theorems are its special cases.

THEOREM 4 : GENERALISED MEAN VALUE THEOREM

If three functions, f, g and h are continuous in [a, b] and derivable in]a, b[, then there exists a real number c $\in$]a, b[such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

PROOF : Define the function, ϕ, as

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

for all x in [a, b].

Since each of the functions f, g and h is continuous on [a, b] and derivable on]a, b[, therefore ϕ is also continuous on [a, b] and derivable on]a, b[.

$$\phi(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0,$$ since two rows of the determinant are identical.

Similarly $\phi(b) = 0$.

Thus $\phi(a) = \phi(b)$.

Therefore ϕ satisfies all the conditions of Rolle's theorem.

So there exists c ∈]a, b[such that

$\phi'(c) = 0$.

$$\phi'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \quad \forall\ x \in\ ]a, b[.$$

$$\text{So } \phi'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

which proves the theorem.

Now we show that Lagrange's and Cauchy's mean value theorems are deducible from this theorem by choosing the functions f and g specially.

i) First we deduce Lagrange's Mean Value Theorem from the Generalized Mean Value Theorem.

Take $g(x) = x$ and $h(x) = 1\ \forall\ x \in [a, b]$,
so that

$$\phi(x) = \begin{vmatrix} f(x) & x & 1 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix}$$

$$\Longrightarrow \phi'(x) = \begin{vmatrix} f'(x) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = f'(x)(a - b) - [f(a) - f(b)]$$

Now $\phi'(c) = 0$ gives $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ which is Lagrange's mean value theorem.

ii) Next we deduce Cauchy's mean value theorem from the Generalized Mean-Value Theorem

Take $h(x) = 1\ \forall\ x \in [a, b]$.

$$\text{So that } \phi(x) = \begin{vmatrix} f(x) & g(x) & 1 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix}$$

$$\Longrightarrow \phi'(x) = \begin{vmatrix} f'(x) & g'(x) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = f'(x)[g(a) - g(b)] - g'(x)[f(a) - f(b)]$$

Now $\phi'(c) = 0 \Longrightarrow f'(c)[g(a) - g(b)] - g'(c)[f(a) - f(b)] = 0$

$\Longrightarrow \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$ provided $g'(x) \neq 0$ for x ∈]a, b[.

which is the Cauchy's mean value theorem.

12.4 INTERMEDIATE VALUE THEOREM

We end this unit by discussing Intermediate Value Theorem for derivatives. Just as you studied Intermediate Value Theorem for continuous functions in Unit 10, there is an Intermediate value theorem for derivable functions which we now state and prove.

THEOREM 5 : INTERMEDIATE VALUE THEOREM FOR DERIVATIVES
If a function f is derivable on [a, b] and $f'(a) \neq f'(b)$, then for each k lying between $f'(a)$ and $f'(b)$, there exists a point $c \in]a, b[$ such that $f'(c) = k$.

PROOF : Consider a function, $g : [a, b] \to R$ defined as

$g(x) = f(x) - kx \ \forall\ x \in [a, b]$.

Then $g'(x) = f'(x) - k \ \forall\ x \in [a, b]$.

$\therefore\ g'(a) = f'(a) - k$ and $g'(b) = f'(b) - k$.

Since k lies between $f'(a)$ and $f'(b)$, we get that

$g'(a)$ and $g'(b)$ are of opposite signs.
Assume : $g'(a) > 0$ and $g'(b) < 0$

which means that $g'(a) = \lim_{h \to 0} \dfrac{g(a+h) - g(a)}{h} > 0$ and

$$g'(b) = \lim_{h \to 0} \frac{g(b-h) - g(b)}{-h} < 0.$$

This implies that there exist $\delta_1, \delta_2 > 0$ such that

$$0 < h < \delta_1 \implies \left| \frac{g(a+h) - g(a)}{h} - g'(a) \right| < g'(a)$$

$$\implies 0 < \frac{g(a+h) - g(a)}{h} < 2g'(a)$$

$$\implies g(a) < g(a+h),$$

and $0 < h < \delta_2 \implies \left| \dfrac{g(b-h) - g(b)}{-h} - g'(b) \right| < -g'(b)$

$$\implies 2g'(b) < \frac{g(b-h) - g(b)}{-h} < 0$$

$$= g(b) < g(b-h).$$

Since g is derivable on [a, b], therefore it is continuous on [a, b] and it attains its supremum on [a, b].

But the supremum is not attained at a or b because

$g(a) < g(a + h)$

and $g(b) < g(b - h)$.

Hence, there exists $c \in]a, b[$ such that $g(c) = \text{Sup. } g$.

We shall show that $g'(c) = 0$ i.e. $f'(c) = k$.

If possible, suppose $g'(c) \neq 0$. Then either $g'(c) > 0$ or $g'(c) < 0$.

Suppose $g'(c) > 0$.

Since g is derivable at c, for $\epsilon = g'(c) > 0$, there exists some $\delta_3 > 0$ such that

$$0 < h < \delta_3 \implies \left| \frac{g(c+h) - g(c)}{h} - g'(c) \right| < g'(c)$$

$\implies g(c) < g(c + h)$
$\therefore\ g(c) \neq \text{Sup. } g$. which is a contradiction.

Therefore $g'(c) \not> 0$.
Similarly $g'(c) \not< 0$.
Hence $g'(c) = 0$ i.e. $f'(c) = k$

In case, $g'(a) < 0$ and $g'(b) > 0$, then
$-g'(a) > 0$ and $-g'(b) < 0$
Therefore at some point $c \in]a, b[$,
$-g'(c) = 0$ or $-f'(c) + k = 0$ or $f'(c) = k$.

Another French mathematician, J.G. Darboux [1842-1917], gave a theorem which is useful in determining the maximum or minimum values of a function. This is popularly known as Darboux Theorem. This is infact a particular case of Intermediate Value Theorem.

THEOREM 6 : DARBOUX'S THEOREM :
Let f be derivable [a, b]. If $f'(a)$ and $f'(b)$ are of opposite signs, then there exists a point $c \in]a, b[$ such that $f'(c) = 0$.

PROOF : Since $f'(a)$ and $f'(b)$ are of opposite signs, therefore one of $f'(a)$, $f'(b)$ is positive and other is negative. Take $k = 0$ in the Intermediate Value Theorem. You get a point $c \in]a, b[$ such that $f'(c) = 0$.

A deduction from Darboux theorem is that if the derivative of a function does not vanish for any point x in]a, b[, then the derivative has the same sign for all x in]a, b[.
This is proved in the following example.

EXAMPLE 14 : If f is derivable in]a, b[and $f'(x) \neq 0 \; \forall \; x \in]a, b[$, then $f'(x)$ retains the same sign, positive or negative for all x in]a, b[.

SOLUTION : If possible, suppose $x_1, x_2 \in]a, b[$, $x_1 < x_2$, such that $f'(x_1)$, $f'(x_2)$ have opposite signs. By Darboux Theorem there exists a point $c \in]x_1, x_2[\subset]a, b[$ such that $f'(c) = 0$ which is a contradiction. Hence $f'(x)$ retains the same sign for all x in]a, b[.

12.6 SUMMARY

In this unit mean value theorems of differentiability have been proved. In Section 12.2, Rolle's theorem, the fundamental theorem of Real Analysis is proved. According to this theorem if $f : [a, b] \rightarrow R$ is a function, continuous in [a, b], derivable in]a, b[and $f(a) = f(b)$, then there is at least one point $c \in]a, b[$ such that $f'(c) = 0$. The geometric significance of the theorem is also given. Geometrically, on the graph of the function f, there is atleast one point between the end points, where the tangent is parallel to the x-axis. Using Rolle's theorem, Lagrange's Mean Value Theorem is proved in Section 12.3. It states that if a function $f : [a, b] \rightarrow R$ is continuous in [a, b] and derivable in]a, b[, there is atleast one point c in]a, b[such that $\frac{f(b) - f(a)}{b - a} = f'(c)$. An important consequence of the theorem is that if f is continuous on [a, b] and derivable on]a, b[with $f'(x) = 0$ on]a, b[, then f is a constant function on [a, b]. Another important deduction from the theorem is that if f is continuous in [a, b] and derivable in]a, b[then (i) f is increasing or decreasing on [a, b] according as $f'(x) \geq 0 \; \forall \; x \in]a, b[$ or $f(x) \leq 0 \; \forall \; x \in]a, b[$ (ii) f is strictly increasing or strictly decreasing in [a, b] according as $f'(x) > 0 \; \forall \; x \in]a, b[$ or $f'(x) < 0 \; \forall \; x \in]a, b[$. Applying these results, some inequalities in real analysis are established. With the help of Rolle's theorem, Cauchy's theorem is proved in Section 12.4. It states that if f and g be two functions from [a, b] to R such that they are continuous in [a, b], derivable in]a, b[and $g'(x) \neq 0 \; \forall \; x \in]a, b[$, then there exists at least one point c in]a, b[such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$. Lagrange's Mean Value Theorem is a particular case of Cauchy's mean value theorem if we choose the function g as $g(x) = x \; \forall \; x \in [a, b]$. A more general theorem, known as generalised mean value theorem is given in Section 12.5. You have seen that it is also established with the help of Rolle's Theorem. According to this theorem, if f, g, h be three functions from [a, b] to R such that they are continuous in [a, b] derivable in]a, b[, then there exists at least one point $c \in]a, b[$ such that

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Both Lagrange's and Cauchy's theorem are particular cases of this theorem. If you take $g(x) = x$ and $h(x) = 1 \ \forall \ x \in [a, b]$, then you get Lagrange's theorem from it. Cauchy's mean value theorem follows from this general theorem if you take only $h(x) = 1 \ \forall \ x \in [a, b]$. Finally, in this section, Intermediate Value Theorem for derivatives is given according to which if f is derivable in [a, b], $f'(a) \neq f(b)$ and k is any number lying between $f'(a)$ and $f'(b)$, then there exists a point $c \in]a, b[$ such that $f'(c) = k$. From this follows Darboux Theorem namely if f is derivable in [a, b] and $f'(a).\ f'(b) < 0$, then there is a point c in]a, b[such that $f'(c) = 0$.

12.6 ANSWERS/HINTS/SOLUTIONS

E 1) f is continuous in $[-2\pi, 2\pi]$ and derivable in $]-2\pi, 2\pi[$.
$f(-2\pi) = f(2\pi) = 0$. All conditions of Rolle's Theorem are satisfied. Therefore there exists a point c in $]-2\pi, 2\pi[$ such that

$f'(c) = 0$
$\Longrightarrow \cos c = 0$
$\Longrightarrow c = \pm \pi/2$
Both the points $\pm \pi/2 \in]-2\pi, 2\pi[$.

E 2) (a) Clearly $f(x) = \cos x$ satisfies the hypothesis of Rolle's theorem over $[-\pi/2, \pi/2]$. So conclusion of the theorem will also be true.

Thus there is point $c \in]-\pi/2, \pi/2]$ such that $f'(c) = 0$
Here $f'(x) = -\sin x$ and $f'(x) = 0$ implies
$x = 0$. So $c = 0 \in]-\pi/2, \pi/2[$.

(b) Conditions of Rolle's Theorem are not satisfied, as $f'(x) = \frac{2}{3}(x-1)^{-1/3}$

for $x \neq 1$ and f is not derivable at $x = 1$ a point of]0, 2[. So f is not derivable in]0, 2[. So hypothesis of Rolle's theorem is not valid. As $f'(x) \neq 0$ for any x in]0, 2[, so conclusion of the theorem is not true.

E 3) Let a and b, $a \neq b$, be any two roots of
$e^x \sin x = 1 \Longleftrightarrow \sin x = e^{-x}$
$\Longleftrightarrow e^{-x} - \sin x = 0.$
$\therefore\ e^{-a} - \sin a = 0$ and $e^{-b} - \sin b = 0$

Let $f(x) = e^{-x} - \sin x \ \forall \ x \in [a, b]$.
Clearly f is continuous in [a, b] and derivable in]a, b[. Also $f(a) = f(b) = 0$.
Therefore the hypothesis of Rolle's Theorem is satisfied by f over [a, b].

Therefore there exists $c \in]a, b[$ such that $f'(c) = 0$ which implies
$-e^{-c} - \cos c = 0 \Longrightarrow e^c \cos c + 1 = 0.$
So $e^x \cos x + 1 = 0$ has a root c for some $c \in]a, b[$.

E 4) Consider the function, f, defined as :

$$f(x) = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \cdots + \frac{a_n x}{1} \ \forall \ x \in [0, 1].$$

Here f is continuous on [0, 1], derivable on]0, 1[and $f(0) = 0$

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + a_n = 0 \text{ (given condition)}$$

$\therefore\ f(0) = f(1)$

So f satisfies the hypothesis of Rolle's theorem and therefore there is a point $c \in]0, 1[$ such that $f'(c) = 0$ that is there is $x \in]0, 1[$ such that $f'(x) = 0$ that is

$a_0x^n + a_1x^{n-1} + \ldots . . a_nx = 0.$

E 5) We have to show that

$$(b-a)\, f(c) - (b-c)\, f(a) - (c-a)\, f(b) = \frac{1}{2}(b-a)(c-a)\, f''(d) \text{ for some}$$

c, d, $\in [a, b]$.
Consider the function ϕ, defined by

$\phi(x) = (b - a) f(x) - (b - x) f(a) - (x - a) f(b) - (b - a)(x - a)(x - b) A$
where the constant A is to be determined such that $\phi(c) = 0$.
This implies

$$(b - a) f(c) - (b - c) f(a) - (c - a) f(b) - (b - a)(c - a)(c - b) A = 0. \quad (2)$$

It is given that f'' is continuous on [a, b] which implies that f_1, f', f'' are continuous on [a, b].

$\phi(a) = \phi(b) = 0$ and ϕ is differentiable in [a, b]. So ϕ satisfies all the conditions of Rolle's theorem on each of the intervals [a, c] and [c, b]
Thus there exists two numbers c_1, c_2 respectively in]a, c[and]c, b[such that
$\phi'(c_1) = 0$ and $\phi'(c_2) = 0$.
Again $\phi'(x) = (b - a) f'(x) + f(a) - f(b) - (b - a)[2x - (a + b)] A$ which is continuous and derivable in [a, b] and in particular on $[c_1, c_2]$. Also
$\phi'(c_1) = \phi'(c_2) = 0$.
By Rolle's theorem, $\exists\, d \in\,]c_1, c_2[$ such that
$\phi''(d) = 0$
Now $\phi''(x) = (b - a) f''(x) - (b - a). 2A$

$$\phi''(d) = (b - a) f''(d) - (b - a) 2A = 0 \implies A = \frac{1}{2} f''(d) \quad (3)$$

where $a < c_1 < d < c_2 < b$ and the result follows from (2) and (3).

E 6) Here $f(x) = \cos x, x \in [0, \pi/2]$. f is continuous in $[0, \pi/2]$ and derivable in $]0, \pi/2[$.

By Lagrange's Mean Value Theorem, there exists a pt, c in $]0, \pi/2[$ such that

$$f'(c) = \frac{f(\pi/2) - f(0)}{\pi/2 - 0}$$

i.e. $-\sin c = -\frac{2}{\pi}$

i.e. $\sin c = \frac{2}{\pi}$

i.e. $c = \sin^{-1} \frac{2}{\pi} \in\,]0, \pi/2[$.

E 7) Here $f(x) = x^3 - 3x^2 + 2x$
Therefore $f'(x) = 3x^2 - 6x + 2$

Let us solve the equation $f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(3) - f(0)}{3 - 0}$

i.e. $3c^2 - 6c + 2 = 2$
i.e. $3c^2 - 6c = 0$
i.e. $c = 0, 2$

Since 0 does not lie in]0, 3[, this value of c is rejected. So the required value of c which lies in]0, 3[is c = 2.

E 8) Apply Lagrange's Mean Value theorem to the function f, given by
$f(x) = ax^2 + bx + d \;\forall \in [m, n]$.
You will get a $c \in\,]m, n[$ satisfying

$$f'(c) = \frac{f(n) - f(m)}{n - m} \text{ (Assume : } n > m)$$

$$\implies 2ac + b = \frac{(an^2 + bn + d) - (am^2 + bm + d)}{n - m}$$

$$= a(n + m) + b$$

$$\implies c = \frac{m + n}{2} \text{ and } c \in\,]m, n[$$

which implies that at $x = \frac{m + n}{2}$, the tangent to the given curve is parallel to the chord joining the points whose abscissae are x = m and x = n.

E 9) Define a function, ϕ, by setting
$\phi(x) = f(a + hx) + f(a - hx) \;\forall\, x \in [0, 1]$.

As x varies over [0, 1], a − hx varies over [a − h, a] and a + hx varies over [a, a + h].

Therefore ϕ is continuous in [0, 1] and derivable in]0, 1[. By Lagrange's mean value theorem $\exists\ \theta(0 < \theta < 1)$ such that

$$\phi'(\theta) = \frac{\phi(1) - \phi(0)}{1 - 0}$$

$$\Longrightarrow f(a + h) + f(a - h) - 2f(a) = h\,[f'(a + \theta h) - f'(a - \theta h)].$$

E 10) i) Consider $F(x) = \tan^{-1} x - (x - x^3/3),\ x \geq 0.$

$$F'(x) = \frac{1}{1 + x^2} - (1 - x^2) = \frac{x^4}{1 + x^2} > 0 \text{ for } x > 0.$$

Thus F is strictly increasing in $[0, \infty[$.

$\therefore\ F(x) > F(0)$ for $x > 0$

i.e. $\tan^{-1} x - (x - \frac{x^3}{3}) > 0$ for $x > 0$

i.e. $\tan^{-1} x > x - x^3/3$ for $x > 0$.

ii) Consider $F(x) = e^{-x} - (1 - x)$ for $x \geq 0$ and proceed as in (i).

E 11) The given functions satisfy the hypothesis of Cauchy's mean value theorem.

$\therefore\ \exists\ \theta \in]-\pi/2, 0[$ such that

$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(-\pi/2) - f(0)}{g(-\pi/2) - g(0)}$$

$$\Longrightarrow \frac{\cos\theta}{-\sin\theta} = \frac{-1 - 0}{0 - 1} \Longrightarrow \cot\theta = -1 \Longrightarrow \theta = -\pi/4$$

which clearly lies in $]-\pi/2, 0[$.

E 12) We find c from

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

i.e. $\dfrac{e^c}{-e^{-c}} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}}$

i.e. $e^{2c} = e^a \cdot e^b = e^{a+b}$

Thus $2c = a + b \Longrightarrow c = (a + b)/2$.

which means that c is the arithmetic mean of a & b.

E 13) We find c from

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

i.e. $\dfrac{1/2\sqrt{c}}{-1/2c\sqrt{c}} = \dfrac{\sqrt{b} - \sqrt{a}}{(1/\sqrt{b}) - (1/\sqrt{a})}$

which gives $c = \sqrt{a}\sqrt{b} = \sqrt{ab}$

Thus c is the geometric mean of a and b.

E 14) We find c from

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

i.e. $\dfrac{-1/c^2}{-2/c^3} = \dfrac{\left(\frac{1}{b}\right) - \left(\frac{1}{a}\right)}{\left(\frac{1}{b^2}\right)\left(-\frac{1}{a^2}\right)}$

which gives $\dfrac{c}{2} = \dfrac{ab}{b + a} \Longrightarrow \dfrac{2}{c} = \dfrac{1}{a} + \dfrac{1}{b}$

thus c is the harmonic mean between a and b

UNIT 13 HIGHER ORDER DERIVATIVES

Structure

13.1 INTRODUCTION

In Unit 12, you have learnt Rolle's theorem and have seen how to apply this theorem in proving mean value theorems. In these theorems only the first derivative of the functions are involved. In this unit, you will study the application of Rolle's Theorem in proving theorems involving the higher order derivatives of functions.

Given a real function f(x), can we find an infinite series of real-numbers say of the form
$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots$
whose sum is precisely the given function?
To answer this question we have to approximate a function with an infinite series of the above form which is also known as the infinite polynomial or **power series.** This approach of approximating a function was known to Newton around 1676 but it was developed late by the two British mathematicians Brook Taylor [1685-1731] and S.C. Maclaurin [1698-1746]. The functions which can be represented as infinite series of the above form are some of the very special functions.

Such a representation of a function requires a number of derivatives of the function i.e. the derivatives of higher orders particularly at x = 0 which we intend to discuss in this Unit. Some work done by Taylor in this direction has found recent applications in the mathematical treatment of **Photogrammetry**—*the science of surveying by means of photographs taken from an aeroplane.*

Besides, we shall also demonstrate the use of derivatives for finding the limits of indeterminate forms and the maximum and minimum values of functions in this unit.

Objectives

After studying this unit, you should, therefore, be able to
- know theorems involving higher order derivatives viz. Taylor's Theorem
- expand functions in a power series viz. Maclaurin's series
- evaluate the limits of indeterminate forms
- find the maximum and minimum values of functions.

13.2 TAYLOR'S THEOREM

In this section, we shall discuss the use of Rolle's theorem in proving theorems involving higher order derivatives of functions. Before proving these theorems, you will be introduced to the idea of higher derivatives through the following definition :

DEFINITION 1 : HIGHER DERIVATIVES
Let f be a function with domain D as a subset of R. Let $D_1 \neq \phi$ be the set of points of D at which f is derivable. We get another function with domain D_1 such that its value at any point c of D_1 is $f'(c)$. We call this function the derivative of f or first derivative of f and denote it by f'. If the derivative of f' at any point c of its domain D_1 exists, then it is called second derivative of f at c and is denoted by $f''(c)$. If $D_2 \neq \phi$ be the set of all those points of D_1 at which f' is derivable, we get a function with domain D_2 such that its value at any point c of

D_2 is $f''(c)$. We call this function second derivative of f and denote it by f''. Similarly we can define 3rd derivative f''' and in general, the nth derivative f^n of the function f.

The following example will make the definition clear :

EXAMPLE 1 : Find the nth derivative f^n of the function $f : R \to R$ defined by $f(x) = |x| \ \forall \ x \in R$.

SOLUTION : You already know that this function f is derivable everywhere in R except at $x = 0$.

$$\text{Now } f(x) = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

$$\text{and } f'(x) = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \end{cases}$$

So the first derivative f' is a function with domain $R \sim \{0\}$. Since $f'(x) = 1$ for $x > 0$, f' is a constant function on $]0, \infty[$. Since derivative of a constant function is 0, therefore f' is derivable at all points in $]0, \infty[$ and $f''(x) = 0 \ \forall \ x \in \ ]0, \infty[$.

Likewise $f''(x) = 0 \ \forall \ x \in \ ]-\infty, 0[$.

So the second derivative f'' is a function with domain $R \sim \{0\}$. Continuing like this, you will get

$f'''(x) = 0$ and in general for $n > 1$, $f^n(x) = 0 \ \forall \ x \in R \sim \{0\}$.

So you find that f'' and in general for $n > 1$, f^n is a function with domain $R \sim \{0\}$.

Try the following exercise.

EXERCISE 1

Find the nth derivative f^n of the function $f : R \to [-1, 1]$ defined by $f(x) = \sin x$.

Now we give a theorem known as Taylor's theorem which involves the higher derivatives of a function.


Taylor

THEOREM 1 : TAYLOR'S THEOREM WITH SCHLOMILCH AND ROCHE FORM OF REMAINDER :

If a function $f : [a, b] \to R$ is such that

i) its $(n-1)$th derivative, $f^{(n-1)}$ is continuous on $[a, b]$

ii) its $(n-1)$th derivative is derivable on $]a, b[$,
then there exists atleast one real number $c \in \ ]a, b[$ such that

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots\dots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p.(n-1)!} f^{(n)}(c),$$

p being a positive integer.

PROOF : By hypothesis, $f, f', \dots\dots\dots, f^{(n-1)}$ are all continuous in $[a, b]$ and derivable in $]a, b[$.

We define a function, ϕ, on $[a, b]$ as follows :

$$\phi(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!} f''(x) \dots\dots\dots$$

$$\dots\dots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x) - A\frac{(b-x)^p}{(b-a)^p}, \qquad (1)$$

where A is a constant to be determined such that

$\phi(a) = \phi(b)$. It is obvious from (1) that

$\phi(b) = 0$. Now

$$\phi(a) = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(a) \dots\dots$$

$$- \frac{(b-a)^{n-1} f^{n-1}(a)}{(n-1)!} - A$$

Therefore, $\phi(a) = \phi(b) = 0 \implies$

$$A = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!} f''(a) - \ldots\ldots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \qquad (2)$$

Now

i) ϕ is continuous in [a, b], since f, f', ... , $f^{(n-1)}$ and $(b-x)^p$, for all positive integers p, are all continuous in [a, b].

ii) ϕ is derivable in]a, b[, since f, f', ... , $f^{(n-1)}$ and $(b-x)^p$, for all positive integers p, are all derivable in]a, b[.

iii) $\phi(a) = \phi(b)$.

Therefore by Rolle's theorem, $\exists\, c \in]a, b[$ such that $\phi'(c) = 0$.

$$\text{Now } \phi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) + Ap\frac{(b-x)^{p-1}}{(b-a)^p}$$

$$\phi'(c) = -\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) + Ap\frac{(b-c)^{p-1}}{(b-a)^p} = 0$$

which gives $A = \dfrac{(b-c)^{n-p}(b-a)^p}{p\,.\,(n-1)!} f^{(n)}(c)$.

Substituting this value of A in (2), we obtain,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^p(b-c)^{n-p}}{p\,.\,(n-1)!}f^{(n)}(c) \qquad (3)$$

This completes the proof of the theorem.

The expression,

$$R_n = \frac{(b-a)^p(b-c)^{n-p}}{p\,.\,(n-1)!}f^{(n)}(c) \qquad (4)$$

which occurs in (3), after n terms, is called Taylor's remainder after n terms. The form (4) is called Schlomilch and Roche form of remainder.

From this we deduce two special forms of remainder after n terms.

i) Take p = n in (4),

$$R_n = \frac{(b-a)^n}{n!}f^{(n)}(c).$$

This is called Lagrange's form of remainder.

ii) Take p = 1 in (4),

$$R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!}f^{(n)}(c).$$

This is called Cauchy's form of remainder.

The Taylor's theorem with Lagrange's form of remainder states :

If a function f defined on [a, b] be such that $f^{(n-1)}$ is continuous on [a, b] and derivable on]a, b[then $\exists$ a real number $c \in]a, b[$ satisfying

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(c) \qquad (5)$$

Alternative form of Taylor's theorem with Lagrange's form of remainder is obtained if instead of interval [a, b], we have the interval [a, a + h].

If we put b = a + h then we can write c = a + θ h for some θ between 0 and 1 and the theorem can be restated as :

If $f^{(n-1)}$ is continuous on [a, a + h] and derivable on]a, a + h[, then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^n(a+\theta h), \quad (6)$$

for some real θ satisfying $0 < \theta < 1$.

Now let x be any point of [a, b]. If f satisfies the condition of Taylor's theorem on [a, b], then it also satisfies the conditions of Taylor's theorem on [a, x] where $x > a$.
Therefore, from (5)

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^n(c), \quad (7)$$

where c is some real number in]a, x[.
Note that (7) also holds when $x = a$ because, then (7) reduces to the identity $f(a) = f(a)$ as the remaining terms on the right hand side of (7) vanish.

You may note that we can have forms similar to (5), (6), (7) for Taylor's theorem with Cauchy's form of Remainder.

If in Taylor's theorem, we take $a = 0$, then we get a theorem known as Maclaurin's theorem. We give below Maclaurin's theorem with Lagrange's and Cauchy's form of remainder. You can also write Schlomilch and Roche form of remainder.

THEOREM 2 : MACLAURIN'S THEOREM WITH LAGRANGE'S FORM OF REMAINDER

If f be a function defined on [0, b] such that $f^{(n-1)}$ is continuous on [0, b] and derivable on]0, b[, then for each x in [0, b] there exists a real number c ($0 < c < x$) such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(c).$$

PROOF : Take $a = 0$ in (7) above.
We can similarly write Maclaurin's theorem with Cauchy's form of remainder as :

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x(x-c)^{n-1}}{(n-1)!} f^{(n)}(c).$$

You may note that $R_n(x) = \frac{x^n}{n!} f^{(n)}(c)$

$$= \frac{x^n}{n!} f^n(\theta x) \; (0 < \theta < 1)$$

in case of Lagrange's form of Remainder and

$$R_n(x) = \frac{x(x-c)^{n-1}}{(n-1)!} f^n(c) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) \; (0 < \theta < 1),$$

in case of Cauchy's form of Remainder.

By applying Taylor's theorem or Maclaurin's theorem also we can prove some inequalities of real analysis. Earlier, in the last unit, you were given a method of proving the inequalities by examining the sign of derivative of some function. Consider the following example :

EXAMPLE 2 : Using Taylor's theorem, prove that

$$\cos x \geq 1 - \frac{x^2}{2} \; \forall \; x \in R$$

SOLUTION : For $x = 0$, result is obvious. Now let $x > 0$ and consider $f(t) = \cos t$. f has derivatives of all orders for all t in R. By Taylor's theorem (or Maclaurin's theorem) with remainder after two terms applied to f in [0, x],

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x) \text{ where } 0 < \theta < 1.$$

Putting the values of f, f′, f″ we have

$$\cos x = 1 - \frac{x^2}{2} \cos(\theta x).$$

Now $\cos \theta x \le 1$ and so $1 - \frac{x^2}{2} \cos \theta x \ge 1 - \frac{x^2}{2}$ i.e. $\cos x \ge 1 - \frac{x^2}{2}$

If $x < 0$, then $-x > 0$ and therefore $\cos(-x) \ge 1 - (1-x)^2$

that is $\cos x \ge 1 - \frac{x^2}{2}$. Hence $\cos x \ge 1 - \frac{x^2}{2} \forall x \in R$.

You should be able to solve the following exercise.

EXERCISE 2

Using Taylor's theorem, show that

$$x - \frac{x^3}{3!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ for } x \ge 0,$$

$$\text{and } x - \frac{x^3}{3!} \ge \sin x \ge x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ for } x < 0.$$

MACLAURIN'S EXPANSION

Now you will see how to find the Maclaurin's expansion of certain elementary functions of the type, e^x, sin x, cos x, $(1 + x)^m$ and log (1 + x) in terms of an infinite series (power series) as

$a_0 + a_1x + a_2x^2 + \ldots..$

with the help of Taylor's and Maclaurin's theorems.

We have seen before that

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots..$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n(x)$$

where $R_n(x)$ is the Taylor's remainder after n terms.

Put $S_n = f(a) + (x - a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \ldots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a)$

Then $f(x) = S_n + R_n(x)$ (8)

A natural question arises as to whether we can express f(x) in the form of the infinite series

$f(a) + (x - a) f'(a) + \ldots.. + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \ldots..$ (9)

and if so under what conditions? This question can be split up in the following situations:

i) Under what conditions on f is each term of the series defined?

ii) Under what conditions does the series (9) converge?

iii) Under what conditions is the sum of the series (9), f(x)?
We examine these now one by one.
   i) Each term of the series (2) is defined iff $f^n(a)$ exists for all positive integers n.
   ii) Assuming $f^n(a)$ exists $\forall$ n, we have from (8),
   $S_n = f(x) - R_n(x)$ (assuming the conditions for Taylor's theorem are satisfied in some interval [a, a + h])

From this, it follows that $< S_n >$ converges iff $\lim_{n\to\infty} R_n(x)$ exists and the series (9) converges iff $\lim_{n\to\infty} R_n(x)$ exists.

iii) Assuming the series (9) converges, we find from above that its sum is $f(x) - \lim_{n\to\infty} R_n(x)$.

Now $f(x) - \lim_{n\to\infty} R_n(x) = f(x)$ iff $\lim_{n\to\infty} R_n(x) = 0$, showing that the series (9) converges to f(x) iff $\lim_{n\to\infty} R_n(x) = 0$.

Summing up the above discussion, we have the following results.

THEOREM 3 : If $f : [a, a + h] \to R$ be a function such that

i) **$f^{(n)}(x)$ exists for each positive integer n, for all $x \in [a, a + h]$.**

ii) **$\lim_{n\to\infty} R_n(x) = 0 \ \forall\ x \in [a, a + h]$,**

then

$$f(x) = f(a) + (x - a) f'(a) + \frac{(x - a)^2}{2!} f''(a) + \ldots + \frac{(x - a)^{n-1}}{(n - 1)!} + \ldots$$

for every $x \in [a, a + h]$.

This is called Taylor's infinite series expansion of f(x). We also sometimes call it the expression for f(x) as a power series in (x − a).

We give an example to illustrate Taylor's series for a function.

EXAMPLE 3 : Assuming the validity of expansion, show that

$$\tan^{-1} x = \tan^{-1} \frac{\pi}{4} + \frac{(x - \pi/4)}{1 + \pi^2/16} - \frac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + \ldots \ \forall\ x \in R$$

SOLUTION : Let $f(x) = \tan^{-1} x$

$= \tan^{-1}(\pi/4 + x - \pi/4)$

Here $a = \pi/4$, $h = x - \pi/4$.

$f^n(x)$ exists $\forall$ x and $\forall$ n.

Now $f'(x) - \frac{1}{1 + x^2}$, $f''(x) = -\frac{2x}{(1 + x^2)^2}, \ldots$

$f'(\pi/4) = \frac{1}{1 + \pi^2/16}$, $f''(\pi/4) = -\frac{\pi}{2(1 + \pi^2/16)^2}, \ldots$

By Taylor's series,

$$f(a + h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \ldots$$

Putting the values of f, f′, f″..... we obtain

$$\tan^{-1} x = \tan^{-1} \frac{\pi}{4} + \frac{x - \pi/4}{1 + \pi^2/16} - \frac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + \ldots (x \in R)$$

EXERCISE 3

Assuming the validity of expansion expand cos x in powers of $(x - \pi/4)$.

If you put a = 0 in the Taylor's series you get the following result:

THEOREM 4 : Let $f : [0, h] \to R$ be a function such that

i) **$f^n(x)$ exists for every positive integer n and for each $x \in [0, h]$**

ii) **$\lim_{n\to\infty} R_n(x) = 0$ for each $x \in [0, h]$,**

Then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^n}{n!} f^{(n)} f(0) + \ldots \text{ for every } x \in [0, h]$$

This series is called the Maclaurin's infinite series expansion of f(x).

Note that Taylor's series remains valid in the interval [a − h, a + h] and Maclaurin's series remains valid in the interval [− h, h] also provided the requirements of the expansion are satisfied in the intervals.

You may also note that one may consider any form of remainder $R_n(x)$ in the above discussion. We shall now consider Maclaurin's series expansions of the functions e^x, cos x and log (1 + x).

EXAMPLE 4 : Find the Maclaurin series expansion of e^x, cos x and log (1 + x).

SOLUTION 1 : Expansion of e^x

Let $f(x) = e^x \; \forall \; x \in R$

Then $f^{(n)}(x) = e^x \; \forall \; x \in [-h, h], h > 0$

and for all positive integers, n. In other words, $f^n(x)$ exists for each n and for all x in R.

Let us now consider the limit of the remainder, $R_n(x)$. Taking Lagrange's form of remainder, we have

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} e^{\theta x} \; (0 < \theta < 1)$$

$$\lim_{n\to\infty} R_n(R) = \lim_{n\to\infty} \frac{x^n}{n!} e^{\theta x}$$

$\lim_{n\to\infty} \frac{x^n}{n!} = 0$ as shown below:

Let $u_n = \frac{|x|^n}{n!}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 \text{ if } x \neq 0.$$

So by Ratio test, $\Sigma \, | u_n |$ is convergent and therefore, Σu_n is convergent and consequently

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ if } x \neq 0$$

If $x = 0$, then also $\lim_{n\to\infty} \frac{x^n}{n!} = 0$

$\therefore \; \lim_{n\to\infty} R_n(x) = 0$

Thus the conditions of Maclaurin's infinite expansion are satisfied.

Also $f(0) = 1$ and $f^{(n)}(0) = 1, n = 1, 2, \ldots\ldots$

$$\therefore \; f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots$$

$$\Longrightarrow e^x = 1 + xf'(0) + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots, \forall \; x \in R.$$

2. EXPANSION OF cos x

Let $f(x) = \cos x \; \forall \; x \in R$.

Then $f^{(n)}(x) = \cos(x + \frac{n\pi}{2}) \; n = 1, 2, \ldots\ldots$

Therefore $f(0) = 1$ and $f^{(n)}(0) = \cos(n\pi/2) \; \forall \; n$.

Clearly f and all its derivatives exists for all real x.

Taking Lagrange's form of the remainder;

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!} \cos(\theta x + \frac{n\pi}{2})$$

$$\text{Therefore } | R_n(x) | = \left| \frac{x^n}{n!} \right| \cdot \left| \cos(\theta x + \frac{n\pi}{2}) \right|$$

$\leq \left| \frac{x^n}{n!} \right| \to 0$ as $n \to \infty, \forall \; x \in R$ (Proved in the expansion of e^x).

which implies $\lim_{n\to\infty} R_n(x) = 0 \; \forall \; x \in R$.

Thus the conditions of Maclaurin's infinite expansion are satisfied.

From $f^n(0) = \cos\left(\frac{n\pi}{2}\right)$, we get

$$\overset{2m+1}{f(0)} = \cos(2m+1)\frac{\pi}{2} = 0 \text{ and } \overset{2m}{f(0)} = \cos(2m)\,\pi/2 = \cos m\pi = (-1)^m.$$

Substituting these values in the expansion, we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots + (-1)^n \frac{x^{2n}}{(2n)!} \ldots\ldots \forall \; x \in R$$

3. EXPANSION OF log (1 + x)

Let $f(x) = \log(1+x)$ for $-1 < x \leq 1$.

Then $f^{(n)}(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$, $x > -1$.

We shall consider the following cases:

i) $0 \leq x \leq 1$.

Taking Lagrange's form of remainder after n terms, we have

$$R_n(x) = \frac{x^n}{n!} f^{(n)}(\theta x)$$

$$= \frac{x^n}{n!} \cdot \frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$= \frac{(-1)^{n-1}}{n} \cdot \left(\frac{x}{1+\theta x}\right)^n.$$

Since $0 \leq x \leq 1, 0 < \theta < 1$, therefore

$$0 \leq \frac{x}{1+\theta x} < 1.$$

$$\therefore \left(\frac{x}{1+\theta x}\right)^n \to 0 \text{ as } n \to \infty$$

Also $\dfrac{1}{n} \to 0$ as $n \to \infty$

Hence $\lim_{n\to\infty} R_n(x) = 0$.

So the conditions of Maclaurin's infinite expansion are satisfied for $0 \leq x \leq 1$.

ii) $-1 < x < 0$.

In this case, x may or may not be numerically less than $1 + \theta x$; so that nothing can be said about the limit of $\left(\dfrac{x}{1+\theta x}\right)^n$ as $n \to \infty$. Thus Lagrange's form of remainder does not help to draw any definite conclusion. We now take the help of Cauchy's form of remainder, which is

$$R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$$

$$= \frac{(-1)^{n-1} x^n (1-\theta)^{n-1}}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \cdot x^n \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \frac{1}{1+\theta x}$$

Now $0 < 1 - \theta < 1 + \theta x$ (for $-1 < x < 0; 0 < \theta < 1$)

Therefore $\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1} \to 0$ as $n \to \infty$.

Also $x^n \to 0$ as $n \to \infty$,

and $\dfrac{1}{1+\theta x} < \dfrac{1}{1-|x|}$ and it is independent of n.

Thus $\lim_{n\to\infty} R_n(x) = 0$.

Hence the conditions of Maclaurin's series expansion are satisfied also when $-1 < x < 0$.

Thus substituting $f(0) = 0$, $f^{(n)}(0) = (-1)^{n-1}(n-1)!$ in the expansion, we get

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots \text{ for } -1 < x \leq 1.$$

EXERCISE 4

Prove that $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\ldots + \dfrac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \ldots\ldots \ \forall x \in R$.

EXERCISE 5

Prove that $(1+x)^m = 1 + mx + \dfrac{m(m-1)}{2!} x^2 + \ldots\ldots$

for all integers m and when $|x| < 1$.

EXERCISE 6

Assuming the validity of expansion, expand log $(1 + \sin x)$ in powers of x, upto 4th power of x.

13.3 INDETERMINATE FORMS

We have proved in Unit 8 (Block 3) that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

provided $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist and $\lim_{x \to a} g(x) \neq 0$. It may sometimes happen that $\lim_{x \to a} \{ f(x)/g(x) \}$ exists even though $\lim_{x \to a} g(x) = 0$. One can easily see that if $\lim_{x \to a} g(x) = 0$, then a necessary condition for $\lim_{x \to a} \frac{f(x)}{g(x)}$ to exist and be finite is that $\lim_{x \to a} f(x) = 0$.

In fact, if $\lim_{x \to a} \{ f(x)/g(x) \} = k$,

$$\begin{aligned} \text{then } \lim_{x \to a} f(x) &= \lim_{x \to a} [f(x)/g(x) . g(x)] \\ &= \lim_{x \to a} \{ f(x)/g(x) \} . \lim_{x \to a} g(x) \\ &= k . 0 = 0. \end{aligned}$$

In this section we propose to discuss the method of evaluating $\lim_{x \to a} \{ f(x)/g(x) \}$ when both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are zero or infinite. In these cases $\frac{f(x)}{g(x)}$ are said to assume indeterminate forms 0/0 or ∞/∞ respectively as $x \to a$.

DEFINITION 2 : INDETERMINATE FORM $\frac{0}{0}$

If $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$ then $\frac{f(x)}{g(x)}$ is said to assume the indeterminate form $\frac{0}{0}$ as x tends to 'a'

DEFINITION 3 : INDETERMINATE FORM $\frac{\infty}{\infty}$

If $\lim_{x \to a} f(x) = \infty$, $\lim_{x \to a} g(x) = \infty$, then $\frac{f(x)}{g(x)}$ is said to assume the indeterminate form $\frac{\infty}{\infty}$ as x tends to 'a'. Other indeterminate forms are $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0 which can be similarly defined. Ordinary methods of evaluating the limits are of little help. Some special methods are required to evaluate these peculiar limits. This special method, generally called, L, Hopital's Rule is due to the French mathematician, L' Hopital (1661-1704). In fact, this method is due to J. Bernoulli, who happened to be a teacher of L' Hopital and his (Bernoulli's) lectures were published by the latter in the book form in 1696, but subsequently Bernoulli's name almost disappeared. Let us consider the indeterminate from $\frac{0}{0}$ and discuss some related theorems. Note the differences in the hypothesis of these theorems and the line of proof should be very carefully noted.

THEOREM 5 : Let f and g be two functions such that

i) $\lim_{x \to a} f(x) = 0$, $\lim_{x \to a} g(x) = 0$,

ii) $f'(a)$ and $g'(a)$ exist, and (iii) $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

PROOF : By hypothesis, f and g are derivable at $x = a$

$\Longrightarrow$ they are continuous at $x = a$

$\Longrightarrow \lim_{x \to a} f(x) = f(a)$, and $\lim_{x \to a} g(x) = g(a)$.

Therefore by condition (i), $f(a) = 0 = g(a)$.

Also $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a}$

and $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{g(x)}{x - a}$.

$\therefore \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x) / (x - a)}{g(x) / (x - a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$.

We may remark that condition (i) in the above theorem can be replaced by $f(a) = g(a) = 0$.

THEOREM 6 : (L' Hopital's rule for $\frac{0}{0}$ form)

If f and g are two functions such that

i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

ii) $f'(x)$ and $g'(x)$ exist and $g'(x) \neq 0$ for all x in $]a - \delta, a + \delta[$, $\delta > 0$, except possibly at a, and

iii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

PROOF : Define two functions ϕ and ψ such that

$$\phi(x) = \begin{cases} f(x) \ \forall x \in]a - \delta, a + \delta[\text{ and } x \neq a \\ \lim_{x \to a} f(x) \text{ at } x = a, \end{cases}$$

and,

$$\psi(x) = \begin{cases} g(x) \ \forall x \in]a - \delta, a + \delta[\text{ and } x \neq a \\ \lim_{x \to a} g(x) \text{ at } x = a, \end{cases}$$

Since $f'(x)$ and $g'(x)$ exist $\forall x \in]a - \delta, a + \delta[$ except possibly at a, ϕ and ψ are continuous and derivable on $]a - \delta, a + \delta[$ except possibly at a.

Also since $\lim_{x \to a} \phi(x) = \lim_{x \to a} f(x) = \phi(a)$

and $\lim_{x \to a} \psi(x) = \lim_{x \to a} g(x) = \psi(a)$,

therefore ϕ and ψ are continuous at $x = a$, as well.

Let x be a point of $]a - \delta, a + \delta[$ such that $x \neq a$.

For $x > a$, ϕ and ψ satisfy the conditions of Cauchy's mean value theorem on $[a, x]$ so that

$$\frac{\phi(x) - \phi(a)}{\psi(x) - \psi(a)} = \frac{\phi'(c)}{\psi'(c)} \text{ for some } c \in]a, x[.$$

But $\phi(a) = \lim_{x \to a} f(x) = 0$ & $\psi(a) = \lim_{x \to a} g(x) = 0$

$$\therefore \frac{\phi(x)}{\psi(x)} = \frac{\phi'(c)}{\psi'(c)}.$$

Proceeding to limits

$$\lim_{x \to a+} \frac{\phi(x)}{\psi(x)} = \lim_{x \to a+} \frac{\phi'(c)}{\psi'(c)} = \lim_{x \to a+} \frac{\phi'(x)}{\psi'(x)}$$

$$\Longrightarrow \lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}.$$

For $x < a$, we can similarly prove that

$$\lim_{x \to a-} \frac{f(x)}{g(x)} = \lim_{x \to a-} \frac{f'(x)}{g'(x)}.$$

But $\lim_{x \to a+} \frac{f'(x)}{g'(x)} = \lim_{x \to a-} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

You may note that if the expression $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ represents the indeterminate form $\frac{0}{0}$ and the functions f′(x) and g′(x) satisfy the conditions of the above theorem, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}=\lim_{x\to a}\frac{f''(x)}{g''(x)}$$

In fact the above rule can be generalised as follows :

If f and g are two functions such that

i) $f^{(n)}(x)$, $g^{(n)}(x)$ exist and $g^{(r)}(x)\neq 0$ $(r = 1, 2, \ldots\ldots, n)$ for any x in $]a-\delta, a+\delta[$ except possibly at $x = a$,

ii) $\left.\begin{array}{l}\lim f(x)=\lim f'(x)=\ldots\ldots=\lim f^{(n-1)}(x)=0\\ \lim g(x)=\lim g'(x)=\ldots\ldots=\lim g^{(n-1)}(x)=0\end{array}\right\}$ as $x\to a$,

iii) $\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

This is known as **Generalised L′ Hopital's Rule for $\frac{0}{0}$ form.**

Note that L′ Hopital's Rule is valid even if $x\to\infty$.
In fact, we have

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=\lim_{z\to 0+}\frac{f(\frac{1}{z})}{g(\frac{1}{z})},\text{ where } x = x = \frac{1}{z},$$

$$=\lim_{z\to 0+}\frac{f'(\frac{1}{z})-\frac{1}{z^2}}{g'(\frac{1}{z})-\frac{1}{z^2}}\text{ (by L′ Hopital's Rule)}$$

$$=\lim_{z\to 0+}\frac{f'(\frac{1}{z})}{g'(\frac{1}{z})}$$

$$=\lim_{x\to\infty+}\frac{f'(x)}{g'(x)}.$$

Now we give examples to illustrate the use of L′ Hopitals's rule in evaluating the limits of indeterminate form $\frac{0}{0}$.

EXAMPLE 5 : Evaluate each of the following limits :

i) $\lim_{x\to 0}\frac{\sqrt{2}-2\cos(\pi/4+x)}{x}$

ii) $\lim_{x\to 0}\frac{\tan x - x}{x^2\sin x}$

iii) $\lim_{x\to 0}\frac{x\cos x-\log(1+x)}{x^2}$

SOLUTION :

i) Let us write $\frac{\sqrt{2}-2\cos(\pi/4+x)}{x}=\frac{f(x)}{g(x)}$,

where $f(x)=\sqrt{2}-2\cos(\pi/4+x)$ and $g(x) = x$.

$\lim_{x\to 0} f(x)=\sqrt{2}-2\cos\pi/4=0$ and $\lim_{x\to 0} g(x) = 0$.

f(x)/g(x) is, therefore, of the form 0/0 as $x\to 0$.

Applying L′ Hopital's rule,

$$\lim_{x\to 0}\frac{\sqrt{2}-2\cos(\pi/4+x)}{x}=\lim_{x\to 0}\frac{2\sin(\pi/4+x)}{1}=2\sin\frac{\pi}{4}=\sqrt{2}$$

ii) Writing

$$\frac{\tan x - x}{x^2 \sin x} = \frac{\tan x - x}{x^3} \cdot \frac{x}{\sin x}$$

we find that

$$\lim_{x \to 0} \frac{\tan x - x}{x^2 \sin x} = \lim_{x \to 0} \frac{\tan x - x}{x^3} \cdot \lim_{x \to 0} \frac{x}{\sin x}$$

$$= \lim_{x \to 0} \frac{\tan x - x}{x^3} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} \text{(By L' Hopital's Rule)}$$

$$= \frac{1}{3} \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^2 = \frac{1}{3}$$

iii) $\lim_{x \to 0} \frac{x \cos x - \log(1 + x)}{x^2} \left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x \to 0} \frac{\cos x - x \sin x - \frac{1}{1 + x}}{2x} \quad \left(\text{again } \frac{0}{0} \text{ form}\right)$$

$$= \frac{1}{2} \cdot \lim_{x \to 0} \frac{-\sin x - (\sin x + x \cos x) + 1/(1 + x)^2}{1}$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2}.$$

EXAMPLE 6 : Determine the values of a and b for which

$\lim_{x \to 0} \{ x(a - \cos x) + b \sin x \} / x^3$

exists and equals 1/6.

SOLUTION : The given function is of the form (0/0) for all values of a and b when $x \to 0$.

$$\therefore \lim_{x \to 0} \frac{x(a - \cos x) + b \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{(a - \cos x) + x \sin x + b \cos x}{3x^2}.$$

The denominator tends to 0 as x tends to 0, the fraction will tend to a finite limit only if the numerator also tends to zero as $x \to 0$.

This requires

$a - 1 + b = 0$ (10)

Supposing (10) is satisfied, we have

$$\lim_{x \to 0} \frac{a + (b - 1) \cos x + x \sin x}{3x^2}$$

$$= \lim_{x \to 0} \frac{-(b - 1) \sin x + x \cos x + \sin x}{6x}$$

$$= \lim_{x \to 0} \frac{x \cos x + (2 - b) \sin x}{6x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{-x \sin x + \cos x + (2 - b) \cos x}{6}$$

$$= \frac{3 - b}{6} = \frac{1}{6} \text{ (given)}$$

$\Longrightarrow b = 2$

From (10), $a = -1$.

Now you should be able to solve the following exercises.

EXERCISE 7

Evaluate the following limits :

i) $\lim\limits_{x \to 0} \dfrac{\sin 3x^2}{\log \cos (2x^2 - x)}$

ii) $\lim\limits_{x \to 0} \dfrac{\sinh x - \sin x}{x \tan^2 x}$

iii) $\lim\limits_{x \to 0} \dfrac{(1 + x)^{\frac{1}{x}} - e + \frac{1}{2} ex}{x^2}$

EXERCISE 8

If the limit $\dfrac{\sin 4x + a \sin 2x}{x^3}$ as $x \to 0$ is finite, find the value of 'a' and the limit.

EXERCISE 9

What is wrong with the following application of L' Hopital's rule :

$$\lim_{x \to 1} \frac{x^3 - 4x + 3}{x^2 + x - 2} = \lim_{x \to 1} \frac{3x^2 - 4}{2x + 1} = \lim_{x \to 1} \frac{6x}{2} = 3.$$

Find also the correct limit.

Next we consider the indeterminate form $\frac{\infty}{\infty}$, L' Hopital's rule for $\frac{\infty}{\infty}$ form is similar to that for 0/0 form. We only state the result for $\frac{\infty}{\infty}$ form without proof.

THEOREM 7 : (L' Hopital's rule for $\frac{\infty}{\infty}$ form)

If f and g be two functions such that

i) $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x) = \infty$,

ii) $f'(x)$ and $g'(x)$ exist, $g'(x) \neq 0$, $\forall\, x \in \,]a - \delta, a + \delta[$, $\delta > 0$ except possibly at a, and

iii) $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ exists; then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The above theorem tells us that $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$, when f(x) and g(x) both tend to infinity as $x \to a$, can be dealt with in the same way as $\left(\frac{0}{0}\right)$ form. In fact forms $\left(\frac{0}{0}\right)$ and $\left(\frac{\infty}{\infty}\right)$ can be interchanged and care should be taken to select the form which would enable us to evaluate the limit quickly.

The above theorem also holds in the case of infinite limits.

Now we consider examples to illustrate the application of L' Hopital's rule for finding the limi of indeterminate form $\frac{\infty}{\infty}$.

EXAMPLE 7

Evaluate the following limits :

i) $\lim\limits_{x \to 0^+} \dfrac{\log \tan 2x}{\log \tan x}$

ii) $\lim\limits_{x \to \infty} \dfrac{\log x}{x^\alpha}$ $(\alpha > 0)$

SOLUTION : (i) Writing $\dfrac{\log \tan 2x}{\log \tan x} = \dfrac{f(x)}{g(x)}$, where f(x) = log tan 2x and g(x) = log tan x, we find that the given expression is of the form $\frac{\infty}{\infty}$ as $x \to 0^+$.

$$\therefore \lim_{x\to 0+} \frac{\log \tan 2x}{\log \tan x} = \lim_{x\to 0+} \frac{2 \cot 2x \sec^2 2x}{\cot x \sec^2 x}$$

$$= \lim_{x\to 0+} \frac{2 \sin x \cos x}{\sin 2x \cos 2x} = \lim_{x\to 0+} \frac{1}{\cos 2x} = 1.$$

ii) $\lim_{x\to\infty} \frac{\log x}{x^a} (a > 0)$ is $\frac{\infty}{\infty}$ form

Therefore its value is equal to $\lim_{x\to\infty} \frac{1/x}{\alpha . x^{\alpha-1}}$.

$$= \lim_{x\to\infty} \frac{1}{\alpha . x^{\alpha}} = 0.$$

EXERCISE 10

Evaluate the following limits :

i) $\lim_{x\to\pi/2+} \frac{\log (x - \frac{\pi}{2})}{\tan x}$

ii) $\lim_{x\to 0-} \frac{\log \sin x}{\cot x}$

Now we consider the indeterminate forms $0.\infty$ and $\infty - \infty$. These can be converted to 0/0 or ∞/∞ forms as shown below :

i) $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$\lim_{x\to a} f(x) . g(x)$ is $0 . \infty$ form.

We can write

$$f(x) . g(x) = \frac{f(x)}{1/g(x)} \text{ or } \frac{g(x)}{1/f(x)},$$

which are respectively $\frac{0}{0}$ or $\frac{\infty}{\infty}$ forms and hence can be evaluated by L' Hopital's rule.

ii) If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$, then

$\lim_{x\to a} \{ f(x) - g(x) \}$ is $\infty - \infty$ form.

This can be reduced to $\frac{0}{0}$ form by writing

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\, g(x)}}$$

and then we can apply L'Hopital's rule.

The following example will clarify the procedure. First we consider $0.\infty$ form

EXAMPLE 8

Evaluate:

i) $\lim_{x\to 0+} x \log x$

ii) $\lim_{x\to 1} \sec \frac{\pi x}{2} \log (1/x)$.

SOLUTION :

i) Take f(x) = x and g(x) = log x.

Then $\lim_{x\to 0+} f(x) = 0$ and $\lim_{x\to 0+} g(x) = -\infty$,

so that the given form is $0 \times \infty$.

We can write it as

$$\lim_{x\to 0+} x \log x = \lim_{x\to 0+} \frac{\log x}{1/x} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= \lim_{x\to 0+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0+} x = 0.$$

ii) Taking $f(x) = \log(1/x)$ and $g(x) = \sec(\pi x/2)$,
We get that the given form is $0 \times \infty$ as $x \to 1$.

$$\therefore \lim_{x\to 1} \sec(\pi x/2) \log(1/x)$$

$$= \lim_{x\to 1} \frac{\log(1/x)}{\cos(\pi x/2)} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{-1/x}{-\sin(\pi x/2) \cdot \pi/2}$$

$$= 2/\pi.$$

EXERCISE 11
Evaluate the following limits :

i) $\lim_{x\to 0} \sin x \log x^2$

ii) $\lim_{x\to 1} (1 - x) \tan(\pi x/2)$.

Now we consider example for $\infty - \infty$ form.

EXAMPLE 9 : Evaluate

i) $\lim_{x\to 4} \left\{ \frac{1}{\log(x-3)} - \frac{1}{x-4} \right\}$

ii) $\lim_{x\to \pi/2} \left(\sec x - \frac{1}{1 - \sin x} \right)$.

SOLUTION

i) Let $f(x) = \frac{1}{\log(x-3)}$ and $g(x) = \frac{1}{x-4}$.

Both these tend to ∞ as $x \to 4$.
Thus the given limit is $\infty - \infty$ form.
We can write it as

$$\lim_{x\to 4} \frac{(x-4) - \log(x-3)}{(x-4)\log(x-3)} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 4} \frac{1 - \frac{1}{x-3}}{\log(x-3) + \frac{x-4}{x-3}}$$

$$= \lim_{x\to 4} \frac{x-4}{(x-3)\log(x-3) + (x-4)} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 4} \frac{1}{1 + \log(x-3) + 1} = \frac{1}{2}.$$

ii) $\lim_{x\to \pi/2} \left(\sec x - \frac{1}{1 - \sin x}\right)$ (it is $\infty - \infty$ form)

$$= \lim_{x\to \pi/2} \frac{-\cos x + \sin x}{-\sin x(1 - \sin x) - \cos^2 x}$$

$$= -\infty$$

EXERCISE 12
Evaluate the following limits :

i) $\lim_{x\to 0} \left(\frac{1}{e^x - 1} - \frac{1}{x} \right)$

ii) $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\tan^2 x} \right)$.

Finally we consider the Indeterminate forms 1^∞, ∞^0, 0^0.
For all these forms we have to evaluate

$\lim_{x \to a} [f(x)]^{g(x)}$,

where $\lim_{x \to a} f(x) = 1$, ∞ or 0 and $\lim_{x \to a} g(x) = \infty$ or 0, 0 (respectively).

We can write

$y = [f(x)]^{g(x)}$

Therefore $\log y = g(x) \log f(x)$

$$\lim_{x \to a} \log y = \lim_{x \to a} [g(x) \log f(x)]. \quad (11)$$

In each of these three cases, right hand side is $0 . \infty$ form which can be evaluated.

Let $\lim_{x \to a} [g(x) \log f(x)] = 1$.

Therefore $\lim_{x \to a} \log y = 1$

which implies

$\log [\lim_{x \to a} y] = 1$

$\Longrightarrow \lim_{x \to a} y = e^1$

$\Longrightarrow \lim_{x \to a} [f(x)]^{g(x)} = e^1$.

The following example discusses these indeterminate forms.

EXAMPLE 10 : Evaluate

i) $\lim_{x \to 0} \left(\frac{\tan x}{x} \right)^{1/x^2}$

ii) $\lim_{x \to \pi/2^-} (\sec x)^{\cot x}$

iii) $\lim_{x \to 1^-} (1 - x^2)^{2/\log (1-x)}$

SOLUTION :

i) It is of the form 1^∞.

Let $y = \left(\frac{\tan x}{x} \right)^{1/x^2}$

Therefore $\log y = \frac{1}{x^2} \log \left(\frac{\tan x}{x} \right)$

$$\lim_{x \to 0} \log y = \lim_{x \to 0} \frac{\log \left(\frac{\tan x}{x} \right)}{x^2} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x}$$

$$= \lim_{x \to 0} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{2x \sec^2 x \tan x}{2 [2x \tan x + x^2 \sec^2 x]}$$

$$= \lim_{x \to 0} \frac{\sec^2 x \tan x}{2 \tan x + x \sec^2 x}$$

$$= \lim_{x \to 0} \frac{\tan x}{\sin 2x + x} (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{\sec^2 x}{2 \cos 2x + 1} = \frac{1}{3}.$$

which gives $\lim_{x\to 0} y = e^{1/3}$

ii) It is of the form ∞^0.
Let $y = (\sec x)^{\cot x}$
So $\log y = \cot x \log \sec x$.

Therefore $\lim_{x\to\pi/2-} \log y = \lim_{x\to\pi/2-} \frac{\log \sec x}{\tan x}$ ($\frac{\infty}{\infty}$ form)

$$= \lim_{x\to\pi/2-} \frac{\frac{1}{\sec x}\cdot \sec x \tan x}{\sec^2 x}$$

$$= \lim_{x\to\pi/2-} (\sin x \cos x) = 0.$$

which implies $\log = \lim_{x\to\pi/2-} y = 0 \Longrightarrow \lim_{x\to\pi/2-} y = e^0 = 1,$

iii) It is of the form 0^0.
Let $y = (1 - x^2)^{2/\log(1-x)}$

$$\text{So } \log y = \frac{2}{\log(1-x)} \log(1-x^2)$$

$$= 2. \frac{\log(1-x^2)}{\log(1-x)}$$

$$\lim_{x\to 1-} \log y = 2. \lim_{x\to 1-} \frac{\log(1-x^2)}{\log(1-x)} \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= 2. \lim_{x\to 1-} \frac{-2x/(1-x^2)}{-1/(1-x)} \text{ (By L' Hopital's Rule)}$$

$$= 2^2 \lim_{x\to 1} \frac{x}{1+x} = 2$$

which gives $\lim_{x\to 1} y = e^2$.

EXERCISE 13
Evaluate the following limits :

i) $\lim_{x\to 0}\left[\sin^2\left(\frac{\pi}{2-ax}\right)\right]\sec^2\frac{\pi}{2-bx}$

ii) $\lim_{x\to 0+} (\cot x)^{\sin x}$

iii) $\lim_{x\to\pi/2-} (\cos x)^{\cos x}$.

13.4 EXTREME VALUES

In this section, we shall be concerned with the applications of derivatives and Mean Value theorems to the determination of the values of a function which are greatest or least in their immediate neighbourhoods; generally known as local or relative maximum and minimum values. The interest in finding the maximum or the minimum values of a function, arose from many diverse directions. During the war period, the cannon operators wanted to know if they could somehow maximize (and if so, to what extent) the distance travelled horizontally i.e. the range, when a cannon-ball is shot from the cannon. The position of the angle at which the cannon was inclinded to the ground mattered the most in such cases. Another direction was the study of motion of planets. It involved maxima and minima problems such as finding the greatest and the least distances of the planets from the sun at a particular time and so on.

We shall find below the necessary and sufficient conditions for the existence of maxima or minima. First we define extreme values of a function.

DEFINITION 4 : EXTREME VALUE OF A FUNCTION.
Let f be a function defined on an interval 1 and let c be any interior point of I.

1) f is said to have a local or relative maximum value (a local or relative maximum) at $x = c$ if $\exists$ a number $\delta > 0$ such that

   $\forall\ x \in]c - \delta, c + \delta[,\ x \neq c \implies f(x) < f(c)$

   i.e. f(c) is the greatest value of the function in tne interval $]c - \delta, c + \delta[$
   i.e. f(c) is a local maximum value of the function f if $\exists\ \delta > 0$ such that
   $f(c) > f(c + h) \iff f(c + h) - f(c) < 0$ for $0 < |h| < \delta$.

2) f is said to have a local or relative minimum value (a local or relative minimum) at $x = c$ if $\exists$ a number $\delta > 0$ such that

   $\forall\ x \in]c - \delta, c + \delta[,\ x \neq c \implies f(x) > f(c)$

   or equivalently $f(c + h) - f(c) > 0$ for $0 < |h| < \delta$.
   or f(c) is the least value of the function f in the interval $]c - \delta, c + \delta[$.

3) f is said to have an extreme value (an extremum or a turning value) at $x = c$, if it has either a local maximum or a local minimum at $x = c$.

The following simple examples will clarify your ideas about maximum and minimum values.

EXAMPLE 11 : Let f be a function defined on R as
$f(x) = x^2\ \forall\ x \in R$,
then f has a local minimum at $x = 0$. From the graph (Fig. 1), the values in the neighbourhood of the value at $x = 0$ is greater than 0.

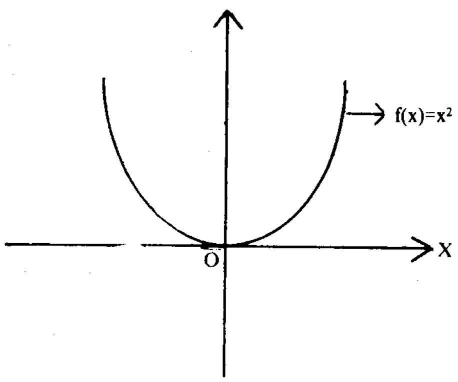


Fig. 1

EXAMPLE 12 : Let f be a function defined on R as
$f(x) = \sin x\ \forall\ x \in R$;
then f has a local minimum at $x = -\pi/2$ and a local maximum at $x = \pi/2$. In fact, f has a minimum at $x = 2n\pi - \pi/2$ and a maximum at $x = 2n\pi + \pi/2$; n being any integer as is evident from the following Figure 2:

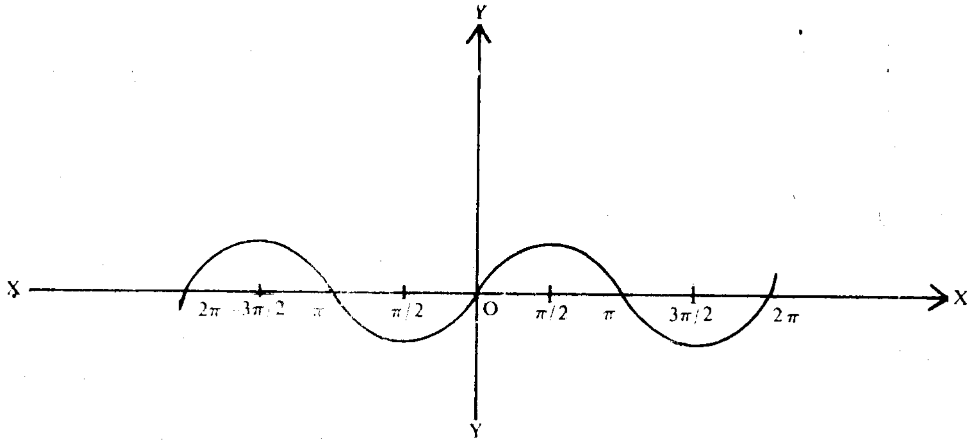


Fig. 2

EXAMPLE 13 : Let f be a function f defined as :

$f(x) = x^3 \ \forall \ x \in R$;

then f has neither a maximum nor a minimum at $x = 0$. At $x = 0$ $f(0) = 0$. If we take any interval $]-d, d[$ about the point 0, then it contains points x_1, x_2 such that $x_1 > 0$ and $x_2 < 0$. Now $f(x_1) > f(0) = 0$ and $f(x_2) < f(0) = 0$.

Note that while ascertaining whether a value f(c) is an extreme value of f or not, we compare f(c) with the values of f in any small neighbourhood of c, so that the values of the function outside the neighbourhood do not come in question.

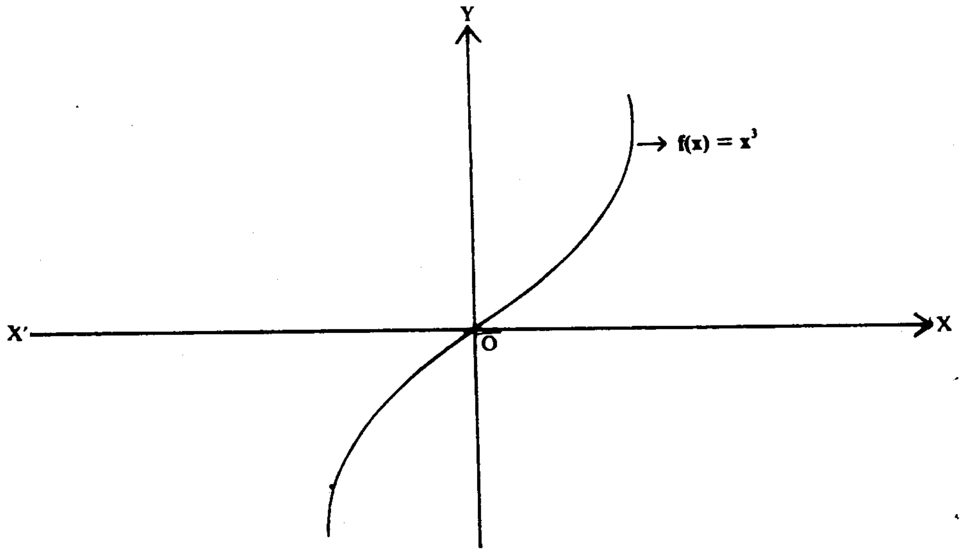


Fig. 3

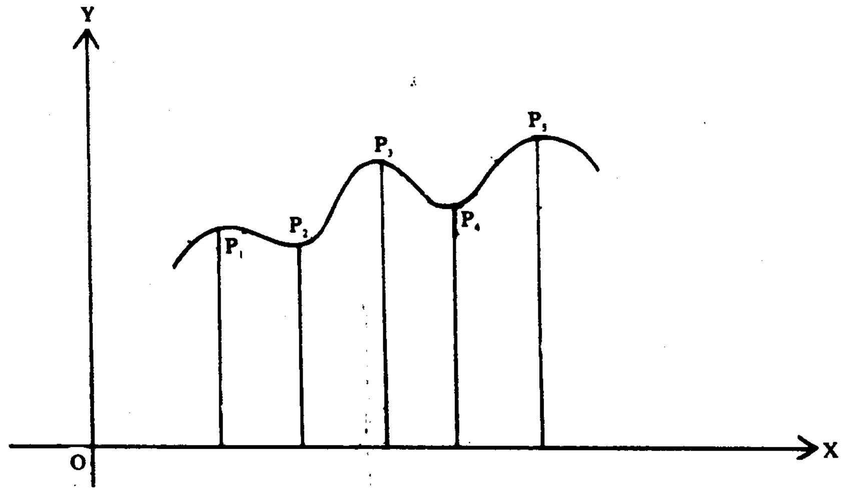


Fig. 4

Thus a local maximum (minimum) value of a function may not be the greatest (least) of all the values of the function in a finite interval. In fact, a function can have several local maximum and minimum values and a local minimum value may even be greater than a maximum value. A glance at the above Figure 4 shows that the ordinates of the points P_1, P_3, P_5 are the local maximum and the ordinates of the points P_2, P_4 are the local minimum values of the corresponding function and that the ordinate of P_4 which is a local minimum is greater than the ordinate of P_1, which is a local maximum.

Further you must have noticed that the tangents at the points P_1, P_2, P_3, P_4 P_5 in the above figure are parallel to the axis of x, so that if c_1, c_2, c_3, c_4, c_5 are the abscissae of these points, then each of $f'(c_1)$, $f'(c_2)$, $f'(c_3)$, $f'(c_4)$, $f'(c_5)$ is zero.

We proceed to establish the truth of this result below :

THEOREM 8 : A necessary condition for f(c) to be an extreme value of a function f is that $f'(c) = 0$, in case it exists.

PROOF : Here, we assume f is derivable at c. Let, further, f(c) be a local maximum value of f. Thus there exists a real number $\delta > 0$ such that

$\forall\ x \in]c-\delta, c+\delta[, x \neq c \implies f(x) < f(c).$

i.e. $\forall\ h \in]-\delta, \delta[, h \neq 0 \implies f(c+h) < f(c).$

Now for $h > 0$, we have $\frac{f(c+h)-f(c)}{h} \leq 0$ (12)

and for $h < 0$ we have $\frac{f(c+h)-f(c)}{h} \geq 0$ (13)

From (12) and (13), we have

$$\lim_{x \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0 \text{ and } \lim_{x \to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

which gives

$f'(c) \leq 0$ and $f'(c) \geq 0$.

Therefore $f'(c) = 0$

It can be similarly shown that $f'(c) = 0$, if f(c) is a local minimum value of f.

The vanishing of f′(c) is only a necessary but not a sufficient condition for f(c) to be an extreme value as we now show with the help of the following example.

Consider a function, f, defined by

$f(x) = x^3\ \forall\ x \in R$

Then

$f'(x) = 3x^2$,

$f'(0) = 0$. Also $f(0) = 0$.

Clearly for $x > 0$, $f(x) > 0 = f(0)$

and for $x < 0$, $f(x) < 0 = f(0)$

thus f(0) is not a local extreme value even though $f'(0) = 0$.

Further you can note that a function may have a local maximum or a local minimum value at a point without being derivable at that point. For example, if $f(x) = |x|\ \forall\ x \in R$, then f is not derivable at $x = 0$, but has local minimum at $x = 0$.

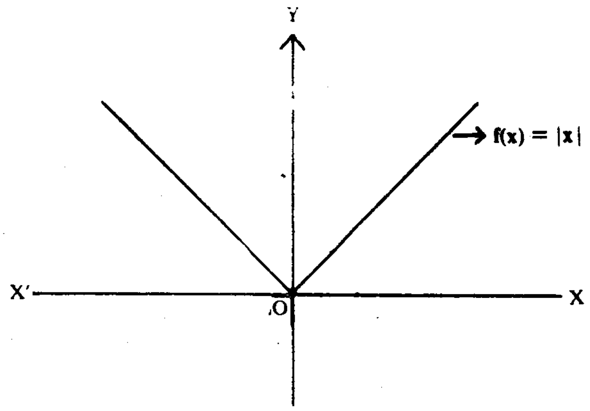


Fig. 5

We may remark that in view of the above theorem, we find that if a function f has a local extreme value at a point $x = c$, then either f is not derivable at $x = c$ or $f'(c) = 0$. Thus in order to investigate the local maxima and minima of a function f, we have to first find out the values of x for which f′(x) does not exist or if f′(x) exists, then it vanishes. (These values are generally called the **critical values of f.**) We then examine for which of these values, does the function actually have a local maximum or a local minimum. The points where first derivative of a function vanish are called stationary points.

DEFINITION 5 : STATIONARY VALUE OF A FUNCTION

$x = c$ is called a stationary point for the function f if $f'(c) = 0$. Also f(c) is then called the stationary value.

You have seen that if a function f is derivable at an interior point c of its domain and $f'(c) = 0$, then f may not have an extreme value at c. To decide whether f has an extreme value or not at such a point, we need some method. By knowing the sign of the derivative on the left and right of the point we can decide whether f has a local maximum or local minimum, at the point. This is the purpose of the next theorem.

THEOREM 9 (FIRST DERIVATIVE TEST)

Let a function f be derivable on an interval $]c - \delta, c + \delta[$, $\delta > 0$, and let $f'(c) = 0$. If

i) **$f'(x) > 0 \ \forall\ x \in\ ]c - \delta, c[$ and $f'(x) < 0\ \forall\ x \in\ ]c, c + \delta[$, then f has a local maximum at $x = c$.**

ii) **$f'(x) < 0 \ \forall\ x \in\ ]c - \delta, c[$ and $f'(x) > 0\ \forall\ x \in\ ]c, c + \delta[$, then f has a local minimum at $x = c$.**

PROOF :

i) Let b be an arbitrary point of $]c - \delta, c[$. Then f satisfies the conditions of Lagrange's mean value theorem in [b, c], so that

$f(c) - f(b) = (c - b) f'(\alpha)$ for some $\alpha \in\ ]b, c[$.

Since $f'(x) > 0 \ \forall\ x \in\ ]c - \delta, c[$,
therefore $f'(\alpha) > 0$,
and so $f(c) - f(b) > 0$.
Now b is any point of $]c - \alpha, c[$,
$\therefore\ f(c) - f(x) > 0 \ \forall\ x \in\ ]c - \delta, c[$. (14)

Let now d be an arbitrary point of $]c, c + \delta[$. Then f satisfies the conditions of Lagrange's mean value theorem in [c, d], so that

$f(d) - f(c) = (d - c) f'(\beta)$ for some $\beta \in\ ]c, d[$.
$f'(x) < 0 \ \forall\ x \in\ ]c, c + \delta[$
$\therefore\ f'(\beta) < 0$.

So $f(d) - f(c) < 0$.
Now d is any point of $]c, c + d[$,
therefore $f(x) - f(c) < 0 \ \forall\ x \in\ ]c, c + \alpha[$ (15)
From (14) and (15), we find that

$\forall\ x \in\ ]c - \delta, c + \delta[, x \neq c \implies f(x) < f(c)$
$\implies$ f has a local maximum at $x = c$

ii) You can similarly prove it.
If $\exists\ \delta > 0$ such that
$x \in\ ]c - \delta, c[\ \implies f'(x) > 0$
and $x \in\ ]c, c + \delta[\ \implies f'(x) < 0$,

then we say that f'(x) changes sign from positive to negative as x passes through c.
Similarly, if $\exists\ \delta > 0$ such that

$x \in\ ]c - \delta, c[\ \implies f'(x) < 0$
and $x \in\ ]c, c + \delta[\ \implies f'(x) > 0$,

then we say that f'(x) changes sign from negative to positive as x passes through c.

In view of this terminology, the above theorem can be stated as follows :

Let f be derivable on an open interval I and let $f'(c) = 0$ at some point $c \in I$. If f'(x) changes sign from positive to negative (negative to positive) as x passes through c, then f has a local maximum (minimum) at $x = c$.
You may note that the conditions of the above theorem are sufficient but not necessary. For example, consider the function f, defined by

$$f(x) = x^4 \left(2 + \sin \frac{1}{x}\right) \text{ when } x \neq 0,$$

and $f(0) = 0$.

This function f is derivable everywhere, f'(x) does not change sign from negative to positive as x passes through 0 and yet f has a local minimum at $x = 0$.

You may further note that if f'(x) does not change sign i.e. it has the same sign throughout the interval $]c - \delta, c + \delta[$, for some $\delta > 0$, then f is either strictly increasing or strictly decreasing throughout this neighbourhood f(c) is not an extreme value of f.

Geometrically interpreted, the above theorem states that the tangent to a curve at every point in a certain left handed neighbourhood of the point P whose ordinate is a local maximum (minimum) makes an acute (obtuse) angle and the tangent at any point in a certain right handed neighbourhood of P makes an obtuse (acute) angle with the axis of X. In case the tangent on either side of P makes an acute angle (or abtuse angle, the ordinate of P is neither a local maximum nor a local minimum.

The following example shows the application of the above theorem for finding extreme values of a function.

EXAMPLE 14 : Examine the function f given by
$f(x) = (x-2)^4 (x+1)^5; \forall\ x \in R,$
for extreme values.

SOLUTION : Here $f(x) = (x-2)^4 (x+1)^5$
Thus $f'(x) = 4(x-2)^3 (x+1)^5 + 5(x-2)^4 (x+1)^4$
$= (x-2)^3 (x+1)^4 (9x-6)$
So $f'(x) = 0$ for $x = -1, 2/3, 2$.

Thus we expect the function to have extreme values for these values of x.
Now $f'(x) > 0$ for $x < -1$,
and $f'(x) > 0$ when x is slightly greater than -1.
Therefore f has neither maximum nor minimum at $x = -1$.

Next $f'(x)$ changes sign from positive to negative at $x = 2/3$, therefore f has a local maximum at $x = 2/3$.

Also $f'(x)$ changes sign from negative to positive at $x = 2$ and therefore it has a local minimum thereat.

EXERCISE 14 : Examine the polynomial function given by
$10x^6 - 24x^5 + 15x^4 - 40x^3 + 108\ \forall\ x \in R$
for local maximum and minimum values.

We can also decide about the maximum and minimum values of a function at a point c from the sign of second derivative at c. This, you will see, in the next theorem, called the second derivative test.

THEOREM 10 : (SECOND DERIVATIVE TEST)
Let f be derivable on an interval $]c-\delta, c+\delta[$ and $f'(c) = 0$.
i) If $f''(c) < 0$, then f has a local maximum at $x = c$.
ii) If $f''(c) > 0$, then f has a local minimum at $x = c$.

PROOF : The existence of $f''(c)$ implies that f and f′ exist and are continuous at $x = c$. Continuity at c implies the existence of f and f′in a certain neighbourhood, $]c-\delta_1, c+\delta_1[$, $0 < \delta_1 < \delta$.
(i) Let $f''(c) < 0$.

This implies that f′ is a strictly decreasing function at $x = c$.

Thus there exists δ_2 $(0 < \delta_2 < \delta_1)$ such that

$$f'(x) < f'(c) = 0\ \forall\ x \in\ ]c, c+\delta_2[\quad (16)$$
$$\text{and } f'(x) > f'(c) = 0\ \forall\ x \in\ ]c-\delta_2, c[\quad (17)$$

Now (16) gives $f'(x) < 0\ \forall\ x \in\ ]c, c+\delta_2[$ which implies that f is a decreasing function in $[c, c+\delta_2]$ and (2) gives $f'(x) > 0\ \forall\ x \in\ ]c-\delta_2, c[$ which implies that f is an increasing function in $[c-\delta_2, c]$, so that at $x = c$ f has a local maximum.

(ii) You can similarly work out the proof.

We may remark that the above theorem ceases to be helpful if for some c, both $f'(c)$ and $f''(c)$ are zero. To provide for this deficiency, we need to consider higher order derivatives. We make use of the Higher Mean Value theorem i.e. Taylor's theorem to obtain generalisation of this result after the following remark.

It is not possible to draw any conclusion regarding extreme values of a function at a point $x = c$ if $f''(c) = 0$.

i) Let the function, be defined by

$f(x) = x^3, \forall\ x \in R$

Here $f'(0) = 0 = f''(0)$ and the function f has neither a local maximum nor a local minimum at $x = 0$.

ii) Let the function be defined by

$f(x) = x^4, \forall\ x \in R.$

Here $f'(0) = 0 = f''(0)$ and f has a local minimum at $x = 0$.
Similarly $f(x) = -x^4, \forall\ x \in R$ has a local maximum at $x = 0$.

Now we give general criteria for finding extreme values and the second derivative test is also special case of this.

THEOREM 11 : (GENERAL CRITERIA)
Let f be a function defined on an interval I and let c be an interior point of I. Let

(i) $f'(c) = f''(c) = \dots f^{n-1}(c) = 0$

and (ii) $f^n(c)$ exist and be different from zero,
then if n is even, f(c) is a local minimum or a local maximum value of f according as $f^n(c) > 0$ or $f^n(c) < 0$; if n is odd, f(c) is not an extreme value of f.

PROOF : Since $f^n(c)$ exists, we have that

$f, f', f'', \dots, f^{n-1}$ all exist and are continuous at $x = c$.

Also continuity at $x = c$ implies the existence of $f, f', f'', \dots f^{n-1}$ in a certain neighbourhood $]c - \delta_1, c + \delta_1[$ of c $(\delta_1 > 0)$.

As $f^n(c) \neq 0$, $\exists$ a neighbourhood $]c - \delta, c + \delta[\ (0 < \delta < \delta_1)$ such that
for $f^n(c) > 0$,
$f^{n-1}(x) < f^{n-1}(c) = 0\ \forall\ x \in]c - \delta, c[$
and $f^{n-1}(x) > f^{n-1}(c) = 0\ \forall\ x \in]c, c + \delta[$ (18)
and for $f^n(c) < 0$,
$f^{n-1}(x) > f^{n-1}(c) = 0\ \forall\ x \in]c - \delta, c[$ (19)
and $f^{n-1}(x) < f^{n-1}(c) = 0\ \forall\ x \in]c, c + \delta[$
Again for any real number h, where $|h| < \delta$, we have by Taylor's theorem with Lagrange's form of remainder after $(n - 1)$ terms,

$$f(c + h) = f(c) + hf'(c) + \frac{h^2}{2!} f''(c) + \dots \frac{h^{n-1}}{(n-1)!} f^{n-1}(c + \theta h)\ (0 < \theta < 1).$$

From which we get

$$f(c + h) - f(c) = \frac{h^{n-1}}{(n-1)!} f^{n-1}(c + \theta h) \qquad (20)$$

where $c + \theta h \in]c - \delta, c + \delta[$. (Putting $f'(c), f''(c), \dots f^{n-2}(c)$ equal to zero).

Let n be odd :
Clearly $h^{n-1} > 0$ for any real number h and further, when $f^n(c) > 0$, we deduce from (18) that for h negative $c + \theta h \in]c - \delta, c[$ and $f^{n-1}(c + \theta h) < 0$ and for h positive, $f^{n-1}(c + \theta h) > 0$.

So from (20), $f(c + h) < f(c)\ \forall\ c + h \in]c - \delta, c[$
and $f(c + h) > f(c)\ \forall\ c + h \in]c, c + \delta[$
which shows that f(c) is not an extreme value.

When $f^n(c) < 0$, it may similarly be shown that f(c) is not an extreme value.

Let n be even :
In this case, h^{n-1} is positive or negative according as h is positive or negative, we deduce from (18) and (20) as before that if $f^n(c) > 0$, then for every point

$c + h \in]c - \delta, c + \delta[$, $f(c + h) > f(c)$

which means that f has a local minimum at $x = c$.

It may similarly be shown from (19) and (20) that f has a local maximum at $x = c$ if $f^n(c) < 0$.

The second derivative test can be deduced from this general criteria by taking $n = 2$. From this theorem, we see that extreme values exist only if the first non-vanishing derivative is of even order. In the following example, you will see the application of this general criteria.

EXAMPLE 15 : Examine the function $(x - 3)^5 (x + 1)^4$ for extreme values.

SOLUTION : Let $f(x) = (x - 3)^5 (x + 1)^4$
Then $f'(x) = (x - 3)^4 (x + 1)^3 (9x - 7)$,

$$f''(x) = 8(x - 3)^3 (x + 1)^2 (9x^2 - 14x + 1),$$
$$f'''(x) = 24 (x - 3)^2 (x + 1) (21x^3 - 49x^2 + 7x + 13),$$
$$f^{iv}(x) = 24 (x - 3) (3x - 1) (21x^3 - 49x^2 + 7x + 13) + 168 (x - 3)^2 (x + 1) (9x^2 - 14x + 1),$$
and
$$f^{v}(x) = 48(3x - 5) (21x^3 - 49x^2 + 7x + 13) + 336 (x - 3) (3x - 1) (9x^2 - 14x + 1) + 336 (x - 3)^2 (x + 1) (9x - 7),$$

Now f' vanishes for $x = -1, \frac{7}{9}, 3$.

Let us now test these for extreme values.

At $x = -1$, f^{iv} is the first non-vanishing derivative and
$f^{iv}(-1) = -24.4.4.64 < 0$.
Therefore $x = -1$ is a point of local maxima.

At $x = \frac{7}{9}$, f'' is the first non-vanishing derivative

and $f''\left(\frac{7}{9}\right) = 8.\left(\frac{20}{9}\right)^3 \cdot \frac{16}{9} \cdot \frac{40}{9} > 0.$

So $x = \frac{7}{9}$ is a point of local minima.

At $x = 3$, the first non-vanishing derivative is f^v, and it is of odd order.

Thus $x = 3$ is neither a point of local maxima nor a point of local minima for the function.

EXAMPLE 16 : Show that the function sin x (1 + cos x) has a local maxima at $x = \pi/3$, $(0 \le x \le 2\pi)$.

SOLUTION : Let $f(x) = \sin x (1 + \cos x) \forall x \in [0, 2\pi]$.

Then $f'(x) = \cos x (1 + \cos x) - \sin^2 x = \cos x + \cos 2x$
and
$f''(x) = -\sin x - 2 \sin 2x$.

At $x = \pi/3$, $f'(\pi/3) = 0$, $f''(\pi/3) = -\frac{3\sqrt{3}}{2} < 0$.

Therefore f has a local maxima at $x = \frac{\pi}{3}$.

Try the following exercises.

EXERCISE 15
Find the local maximum and minimum values of the function f defined by

i) $f(x) = 4x^{-1} - (x - 1)^{-1}, \forall x \in R - \{0, 1\}$.

ii) $f(x) = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \ \forall x \in [0, \pi]$

EXERCISE 16
Show that the function f defined by

$f(x) = x^m (1 - x)^n \forall x \in R$,

where m and n are positive integers has a local maximum value at some point of its domain, whatever the values of m and n may be.

EXERCISE 17
Show that the local maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$

We end this section by giving a method of finding greatest and least values of a function in an interval provided the function is derivable at all interior points of the interval.

The greatest and the least values of a function are also its extreme values in case they are attained at points within the interval so that the derivatives must be zero at the corresponding points.

The greatest value of a function is also called global or absolute maximum. Similarly the least value of a function is also known as global or absolute minimum.

If $c_1, c_2, \ldots, c_k$ be the roots of the equation, $f'(x) = 0$ which belong to]a, b[, then the greatest and the least values of the function f in [a, b] are the greatest and the least members respectively of the finite set

$\{f(a), f(c_1), f(c_2), \ldots, f(c_k), f(b)\}$.

Consider the following example.

EXAMPLE 17 : Find the greatest and the least values of the function f defined by

$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

in the interval [0, 2].

SOLUTION : We have

$$f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$$

$$\text{Therefore } f'(x) = 12x^3 - 6x^2 - 12x + 6$$
$$= 6(x - 1)(x + 1)(2x - 1)$$
$$\Longrightarrow f'(x) = 0 \text{ for } x = 1, -1, +1/2.$$

The number -1 does not belong to the interval [0, 2] and is not to be considered. Now

$$f(1) = 2,\ f\left(\frac{1}{2}\right) = \frac{39}{16},\ f(0) = 1 \text{ and } f(2) = 21.$$

Thus the greatest value of f in [0, 2] is 21 and the least value is 1.

Try the following exercise.

EXERCISE 18

Find the least and the greatest value of the function f defined by :

$f(x) = x^4 - 4x^3 - 2x^2 + 12x + 1$

in the interval [− 2, 5].

13.5 SUMMARY

In this unit, some theorems involving higher order derivatives of a function have been proved and also the application of derivatives for finding the limits of indeterminate forms and finding the extreme values of a function has been discussed.

In Section 13.2, Taylor's Theorem has been proved with the help of Rolle's Theorem. According to this theorem, if f : [a, b] → R is a function such that its (n − 1)th derivative f^{n-1} is continuous in [a, b] and derivable on]a, b[, then there is at least one real number c ∈]a, b[such that

$$f(b) = f(a) + (b - a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$
$$+ \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^n(c)$$

p being any positive integer.

The term $\frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^n(c)$ is called Taylor's Remainder after n terms and denoted by R_n and this form of remainder in due to Schlomilch and Roche. By putting p = n and p = 1, we get respectively Lagrange's and Cauchy's form of remainder. If we put a = 0 is

Taylor's theorem, we obtain Maclaurin's theorem. In the same section, you have seen how to obtain Maclaurin's series expansion of a function. If f: [a, b] → R is a function such that $f^n(x)$ exists for any positive integer n and for each x ∈ [0, h] and $\lim_{n\to\infty} R_n(x) = 0$ for each x ∈ [0, h], then for all x in [0, h],

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^n}{n!} f^n(0) + ...$$

which is Maclaurin's series for f(x). Using this result, Maclaurin's series expansions of e^x, sin x, cos x, log (1 + x), $(1 + x)^m$ have been obtained as :

$$e^x = 1 + x + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + ... \forall x \in R,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} ..... + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + ... \forall x \in R,$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... + (-1)^n \frac{x^{2n}}{(2n)!} + ... \forall x \in R$$

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} ..... \quad -1 < x \leq 1$$

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + ..., \quad |x| < 1.$$

In Section 13.3, methods for finding limits of Indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0, ∞, ∞ − ∞, 1^∞, ∞^0, 0^0 have been given. All these are based on L'Hopital's Rule for $\frac{0}{0}$ form. If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, then $\frac{f(x)}{g(x)}$ is said to assume the indeterminate forms $\frac{0}{0}$ as x tends to 'a'.

L'Hopital's Rule for $\frac{0}{0}$ form states that if $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, f'(x), g'(x) exist and g'(x) ≠ 0 for all x in]a − δ, a + δ[(δ > 0) and $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$
L'Hopital's rule for $\frac{\infty}{\infty}$ form to similar.

In Section 13.4, application of derivatives for finding extreme values of a function is given. If f is a function defined on an open interval I and c is any interior point of I, then f is said to have a local or relative maximum at c if there exists a number δ > 0 such that x ∈]c −δ, c + δ[, x ≠ c ⟹ f(x) < f(c). Likewise, f is said to have a local or relative minimum at c if there exists a number δ > 0 such that x ∈]c − δ, c + δ[, x ≠ c ⟹ f(x) > f(c). f is said to have an extreme value at c if it is either a local maximum or a local minimum at c. You have seen that the necessary condition for f to have an extreme value at c is that f'(c) = 0 provided it exists. The condition f'(c) = 0 is not sufficient for f to have an extreme value at c. For example the function f defined by f(x) = | x | ∀ x ∈ R it has a local minimum at x = 0 but f'(0) does not exist. For deciding whether a function f has an extreme value at a point c, we have the following general test.

Suppose that f is a function defined on an interval I and c is an interior point of I such that

$f'(c) = f''(c) = ... f^{n-1}(c) = 0$ and $f^n(c) \neq 0$. Then if n is odd, then f does not have an extreme value at c and if n is even, then f has a local maximum or local minimum at c according as $f^n(c) < 0$ or $f^n(c) > 0$.

13.6 ANSWERS/HINTS/SOLUTIONS

E 1) f is derivable for all x in R and f'(x) = cos x ∀ x ∈ R. So the first derivative f' has domain R. The function cos x is also derivable at all points of R. So the second derivative f'' has also the domain R and

f''(x) = − sin x ∀ x ∈ R.

In general the nth derivative f^n has also domain R. We can write f'(x) = sin (x + π/2), f''(x) = sin (x + 2 . π/2) and in general.

$f^n(x) = \sin (x + n\frac{\pi}{2})$, ∀ x ∈ R.

E 2) (i) If x = 0, the statement is trivially true,
consider x > 0.
By applying Taylor's Theorem to the function f defined by f(t) = sin t in [0, x] and writing the remainder after 3 terms, we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(\theta x) \text{ where } 0 < \theta < 1)$$

i.e. $\sin x = x - \frac{x^3}{3!} \cos(\theta x)$ (putting the values of f, f', f'', f'''). Since $\cos \theta x \leq 1$,

whatever θx may be and x > 0,

$$\therefore \quad -\frac{x^3}{3!} \leq -\frac{x^3}{3!} \cos \theta x$$

$$\Longrightarrow x - \frac{x^3}{3!} \leq x - \frac{x^3}{3!} \cos \theta x = \sin x \tag{21}$$

Again, applying Taylor's theorem to the function f in [0, x] and writing the remainder after 5 terms, we get

$\sin x = x - \frac{x^3}{3!} + \frac{x^3}{5!} \cos(\theta_1 x)$, where $0 < \theta_1 < 1$.

Since $\cos(\theta_1 x) \leq 1$, whatever $\theta_1 x$ may be and since x > 0, therefore

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} \cos(\theta_1 x) \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\Longrightarrow \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} \tag{22}$$

From (21) and (22) we have

$$x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ whenever } x > 0 \tag{23}$$

Let now x < 0. Set y = − x, then y > 0.

From (23), we have

$$y - \frac{y^3}{3!} \leq \sin y \leq y - \frac{y^3}{3!} + \frac{y^5}{5!}$$

Put y = −x in it and simplify. You will get

$$x - \frac{x^3}{3!} \geq \sin x \geq x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ for } x < 0.$$

E 3) (i) Let f(x) = cos x

$$= \cos\left(\frac{\pi}{4} + x - \frac{\pi}{4}\right)$$

Here a = π/4, h = x − π/4.

Now $f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right)$

Therefore $f^{(n)}\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)$

Put n = 1, 2, 3, ... we get

$$f'(\pi/4) = -\sin\frac{\pi}{4},\ f''\left(\frac{\pi}{4}\right) = -\cos\frac{\pi}{4},\ f'''(\pi/4) = \sin\frac{\pi}{4}, \ldots..$$

Assuming the possibility of expansion, we have by Taylor's Series,

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots..$$

i.e. $\cos x = \cos\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\left\{-\sin\frac{\pi}{4}\right\} + \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2\left\{-\cos\frac{\pi}{4}\right\}$

$$+ \frac{1}{3!}\left(x - \frac{\pi}{4}\right) \sin\frac{\pi}{4} \ldots..$$

$$= \frac{1}{\sqrt{2}}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \ldots..\right]$$

which is the required expansion of cos x.

E 4) Proceed as in the expansion of cos x, in Example 4 by taking $f(x) = \sin x$, $f^n(x) = \sin(x + \frac{n\pi}{2})$.

E 5) Two cases arise :

(i) m is a positive integer.

Let $f(x) = (1 + x)^m \forall x \in R$.

We find that $\forall n \in N$,

$f^{(n)}(x)$ exists $\forall x \in R$ and $f^{(n)}(x) = 0 \forall n > m$ and $\forall x \in R$.

Therefore $R_n(x) = 0$ for all $n > m$.

Which implies $\lim_{n\to\infty} R_n(x) = 0$ and, we have

$$f(x) = f(0) + xf'(0) + ... + \frac{x^m}{m!}f^{(m)}(0), \forall x \in R,$$

since the other terms all vanish. Substituting the values of $f(x), f(0), f'(0), ... f^{(m)}(0)$, we have

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + ... + x^m.$$

(ii) m is not a positive integer.

In this case, we find that if we write

$f(x) = (1 + x)^m$, whenever $m \neq -1$,

then $f^{(n)}(x) = m(m-1) ..... (m-n+1)(1+x)^{m-n}$.

Thus for each positive integer n, $f^{(n)}$ is defined in $[-h, h]$ for each $h \in]0, 1[$.

If $R_n(x)$ denotes Cauchy's form of Taylor remainder after n terms we have

$$R_n(x) = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^n(\theta x);\ 0 < \theta < 1$$

$$= \frac{x^n(1-\theta)^{n-1}}{(n-1)!} \cdot m(m-1) ..... (m-n+1)(1+\theta x)^{m-n}.$$

$$= \frac{m(m-1)...(m-n+1)}{(n-1)!}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot (1+\theta x)^{m-1}$$

We know that for $|x| < 1$,

$$\frac{m(m-1)...(m-n+1)}{(n-1)!}x^n \to 0 \text{ as } n \to \infty.$$

Also $0 < \frac{1-\theta}{1+\theta x} < 1$ for $0 < \theta < 1$ and for $-1 < x < 1$.

Therefore

$$\lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0.$$

Next $(1+\theta x)^{m-1} < (1+|x|)^{m-1}$ if $m > 1$ as $0 < \theta < 1$,

and $(1+\theta x)^{m-1} = \frac{1}{(1+\theta x)^{1-m}} < \frac{1}{(1-|x|)^{1-m}}$ when $m < 1$.

Thus $R_n(x) \to 0$ as $n \to \infty$ for $|x| < 1$.

Hence the conditions of Maclaurin's infinite expansion are satisfied. Making the substitutions,

$f(0) = 1, f'(0) = m, f''(0) = m(m-1), ...$

$f^n(0) = m(m-1) ..... (m-n+1)$,

we get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + ...$$

for $|x| < 1$.

Note that when m is not a positive integer, the expansion is not possible if $|x| > 1$, for then as $n \to \infty$,

$\frac{m(m-1) ..... (m-n+1)}{(n-1)!} x^n$ and so $R_n(x)$, does not tend to zero.

E 6) Let $f(x) = \log (1 + \sin x)$

Then $f'(x) = \frac{\cos x}{1 + \sin x}$

$$f''(x) = -\frac{1}{1 + \sin x}$$

$$f'''(x) = \frac{\cos x}{(1 + \sin x)^2}$$

$$f^{iv}(x) = -\frac{\sin x + \sin^2 x + 2\cos^2 x}{(1 + \sin x)^3}$$

....

....

Putting $x = 0$, we get

$f(0) = 0, f'(0) = 1, f''(0) = -1, f'''(0) = 1, f^{iv}(0) = -2, ...$

Substituting these values in

$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ...$, we get

$$\log (1 + \sin x) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - 2\frac{x^4}{4!} .....$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} .....$$

E 7) $\lim_{x \to 0} \frac{\sin 3x^2}{\log \cos (2x^2 - x)}$ ($\frac{0}{0}$ form)

$$\lim_{x \to 0} \frac{6x \cos 3x^2}{- \tan (2x^2 - x)(4x - 1)}$$

$$= -6 \lim_{x \to 0} \cos 3x^2 . \frac{x(2x - 1)}{\tan (2x - x)^2} \cdot \frac{1}{(2x - 1)(4x - 1)}$$

$$= -6 \lim_{x \to 0} \cos 3x^2 . \lim_{x \to 0} \frac{x(2x - 1)}{\tan x (2x - 1)} \lim_{x \to 0} \frac{1}{(2x - 1)(4x - 1)}$$

$$= -6.1.1 \frac{1}{(-1)(-1)} = -6.$$

ii) $\lim_{x \to 0} \frac{\sin h\, x - \sin x}{x \tan^2 x}$ ($\frac{0}{0}$ form)

$$= \lim_{x \to 0} \frac{\sin h\, x - \sin x}{x^3} \left(\frac{x}{\tan x}\right)^2$$

$$= \lim_{x \to 0} \frac{\sin h\, x - \sin x}{x^3} \left(\frac{0}{0} \text{ form}\right)$$

$$\left(\frac{x}{\tan x} \to 1 \text{ as } x \to 0\right)$$

$$= \lim_{x \to 0} \frac{\cos h\, x - \cos x}{3x^2} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\sin h\, x + \sin x}{6x} \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\cos h\, x + \cos x}{6} = \frac{1 + 1}{6} = \frac{1}{3}.$$

iii) $\lim_{x \to 0} \frac{(1 + x)^{1/x} - e + \frac{1}{2} ex}{x^2}$ is (0/0) form as

$\lim_{x \to 0} (1 + x)^{1/x} = e.$

Let $y = (1 + x)^{1/x}$

Therefore $\log y = \frac{1}{x} \log (1 + x)$ ·

$$= \frac{1}{x}(x - \frac{x^2}{2} + \frac{x^3}{3} \cdots)$$

$$= (1 - \frac{x}{2} + \frac{x^2}{3} \cdots)$$

$$\therefore\ y = e^{(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots)}$$

$$= e.e^{(-\frac{x}{2} - \frac{x^2}{3} \cdots)}$$

$$= e. [1 + (-\frac{x}{2} + \frac{x^2}{3} - \cdots) + \frac{1}{2!}(-\frac{x}{2} + \frac{x^2}{3} - \cdots)^2 + \ldots]$$

$$= e [1 - \frac{x}{2} + \frac{11}{24} x^2 + 0 (x^3)],$$

where $0 (x^3)$ stands for those terms of x containing x^3 or higher powers.
Thus the given limit becomes

$$= \lim_{x \to 0} \frac{e [1 - \frac{x}{2} + \frac{11}{24} x^2 + 0 (x^3)] - e + \frac{1}{2} ex}{x^2}$$

$$= \lim_{x \to 0} [\frac{11}{24} e + 0(x)] = \frac{11e}{24} \cdot$$

E 8) Now

$$\lim_{x \to 0} \frac{\sin 4x + a \sin 2x}{x^3} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{4 \cos 4x + 2a \cos 2x}{3x^2},$$

its denominator tends to zero for $x \to 0$, the fraction will tend to a finite limit only if the numerator also tends to zero as $x \to 0$. This requires

$4 + 2a = 0 \implies a = -2.$

When this is satisfied, we have $\frac{0}{0}$ form and the given limit

$$= \lim_{x \to 0} \frac{-16 \sin 4x - 4a \sin 2x}{6x} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{-64 \cos 4x - 8a \cos 2x}{6}$$

$$= \frac{-64 - 8a}{6} = -8 \ (a = -2).$$

E 9) The expression $\frac{3x^2 - 4}{2x + 1}$ · is not of the form 0/0 as $x \to 1$.

Therefore it is not correct to apply L' Hopital's rule to evaluate $\lim_{x \to 1} \frac{3x^2 - 4}{2x + 1}$.

In fact $\lim_{x \to 1} \frac{3x^2 - 4}{2x + 1} = \frac{\lim_{x \to 1} (3x^2 - 4)}{\lim_{x \to 1} (2x + 1)} = \frac{-1}{3}$.

E 10) i) $\lim_{x \to \pi/2+} \frac{\log (x - \frac{\pi}{2})}{\tan x} \quad (\frac{\infty}{\infty} \text{ form})$

$$= \lim_{x \to \pi/2+} \frac{\frac{1}{x - \pi/2}}{\sec^2 x}$$

$$= \lim_{x \to \pi/2+} \frac{\cos^2 x}{x - \frac{\pi}{2}} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to \pi/2+} \frac{2 \cos x (- \sin x)}{1} = 0.$$

ii) It is $\frac{\infty}{\infty}$ form. Apply L' Hopital's Rule. The limit is 0.

E 11) i) Given form is $0.\infty$.

$$\lim_{x \to 0} \sin x \log x^2$$

$$= \lim_{x \to 0} \frac{\log x^2}{\operatorname{cosec} x} \quad (\frac{\infty}{\infty} \text{ form})$$

$$= \lim_{x \to 0} \frac{-2}{x \operatorname{cosec} x \cot x}$$

$$= \lim_{x \to 0} -2 \frac{\sin x}{x} . \tan x$$

$$= 0.$$

ii) The given form is $0.\infty$. Convert it to $\frac{0}{0}$ form and then find the limit. The limit is $\frac{2}{\pi}$.

E 12) i) The limit is $-\frac{1}{2}$.

ii)

$$\lim_{x \to 0} (\frac{1}{x^2} - \frac{1}{\tan^2 x}) \quad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \quad (\frac{x}{\tan x})^2$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \quad \lim_{x \to 0} (\frac{x}{\tan x})^2$$

$$= \lim_{x \to 0} \frac{\tan^2 x - x^2}{x^4} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{\tan x . \sec^2 x - x}{2x^3} \quad \text{(By L' Hopital's Rule)}$$

$$= \lim_{x \to 0} \frac{\tan x + \tan^3 x - x}{2x^3} \quad (\frac{0}{0} \text{ form})$$

$$= \lim_{x \to 0} \frac{\text{Sec}^2 x + 3 \tan^2 x \sec^2 x - 1}{6x^2}$$

$$= \lim_{x \to 0} (\frac{\tan x}{x})^2 \frac{1 + 3 \sec^2 x}{6} = \frac{2}{3}$$

E 13) i) $\lim_{x \to 0} [\sin^2 (\frac{\pi}{2 - ax})] \sec^2 (\frac{\pi}{2 - bx})$

is 1^∞ form.

Let $y = [\sin^2 (\frac{\pi}{2 - ax})] \sec^2 (\frac{\pi}{2 - bx})$

Then

$$\log y = 2 \sec^2 (\frac{\pi}{2 - bx}) \log \sin (\frac{\pi}{2 - ax})$$

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{2 \log \sin\left(\frac{\pi}{2-ax}\right)}{\cos^2\left(\frac{\pi}{2-bx}\right)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{2 \cot\left(\frac{\pi}{2-ax}\right) \cdot \frac{\pi a}{(2-ax)^2}}{-\sin\left(\frac{2\pi}{2-bx}\right) \frac{\pi b}{(2-bx)^2}}$$

$$= -\frac{2a}{b} \lim_{x\to 0} \frac{\cot\left(\frac{\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)} \cdot \lim_{x\to 0} \frac{(2-bx)^2}{(2-ax)^2}$$

$$= -\frac{2a}{b} \lim_{x\to 0} \frac{\cot\left(\frac{\pi}{2-ax}\right)}{\sin\left(\frac{2\pi}{2-bx}\right)} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= -\frac{2a}{b} \lim_{x\to 0} \frac{-\operatorname{cosec}^2\left(\frac{\pi}{2-ax}\right) \frac{\pi a}{(2-ax)^2}}{\cos\left(\frac{2\pi}{2-bx}\right) \frac{2\pi b}{(2-bx)^2}}$$

$$= -\frac{2a}{b} \cdot \frac{a}{2b}$$

$$= -\frac{a^2}{b^2}$$

$$\Longrightarrow \lim_{x\to 0} y = e^{-a^2/b^2}.$$

ii) 1. iii) 1.

E 14) $f'(x) = 60\, x^2(x+1)^2 (x-2)$. Apply first derivative test and show that at $x = 0$, f has neither local maxima nor local minima and at $x = 2$, f has local minima.

E 15) i) $f(x) = \frac{4}{x} - \frac{1}{x-1} \; \forall x \in R - \{0, 1\}$

$$\Longrightarrow f'(x) = \frac{-4}{x^2} + \frac{1}{(x-1)^2}$$

Now $f'(x) = 0 \Longrightarrow x = 2, 2/3$.

Also $f''(x) = \frac{8}{x^3} - \frac{2}{(x-1)^3}$

Clearly then f has local maximum at $x = 2$ and local minimum at $x = 2/3$.

ii) Maximum at $x = \pi/4$ and $3\pi/4$, minimum at $x = 2\pi/3$.

E 16) Here $f(x) = x^m (1-x)^n \; \forall x \in R$

Therefore $f'(x) = mx^{m-1}(1-x)^n - nx^m(1-x)^{n-1}$

$= x^{m-1}(1-x)^{n-1}(m - mx - nx)$

If both $m = n = 1$,

then $f'(x) = 1 - 2x = 0 \Longrightarrow x = \frac{1}{2}$,

and $f''(x) = -2 < 0$ i.e. f has a local maximum at $x = 1/2$ in this case.

Therefore we can assume either $m > 1$ or $n > 1$.

In any case, $f'(x) = 0$ yields $x = m/(m+n)$ and $f'(x)$ changes sign from positive to negative at this point.

Thus f has a local maximum at $x = m/(m+n)$.

E 17) Let $y = \left(\frac{1}{x}\right)^x$

Therefore $\log y = x \log\left(\frac{1}{x}\right) = -x \log x$

$$\Longrightarrow \frac{1}{y}\frac{dy}{dx} = -\log x - 1 \qquad (24)$$

For extreme values, $\frac{dy}{dx} = 0$, which means

$\log x + 1 = 0 \Longrightarrow x = e^{-1}$.

Now (24) $\Longrightarrow$

$$\frac{dy}{dx} = -y(\log x + 1)$$

$$\Longrightarrow \frac{d^2y}{dx^2} = -\frac{dy}{dx}(\log x + 1) - \frac{y}{x}$$

At $x = e^{-1}$, we have $\frac{d^2y}{dx^2} = -e^{1/e} \,.\, e < 0$

Therefore at $x = e^{-1}$, y has a local maximum and the maximum value of y

REVIEW

In this block, you have been introduced to the rigorous notion of the derivative of a function. Also, the relationship between the continuity and differentiability have been explained. Further, the algebra of the derivatives has been discussed. All these have been dealt with in Unit 11. In Unit 12, certain mean-value theorems have been discussed while in Unit 13, the results of these theorems have been extended to higher derivatives in the form of Taylor's theorem and Maclaurin's series. Further, these theorems have been used to give the power series expansions of some algebraic and transcendental functions as well as evaluating the limits of indeterminate forms and extreme values of some functions. Now try the following questions so as to test for yourself your conceptual understanding of the material.

1) Give an example of each of the following :
   a) A function which is not differentiable at one point of its domain.
   b) A function which is not differentiable at two points of its domain.
   c) A function which is not differentiable at three points of its domain.

2) State whether the following statements are true or false.
   a) A continuous function is always differentiable.
   b) A differentiable function is always continuous.
   c) Every monotonic function is differentiable.
   d) Every differentiable function is monotonic.

3) Give an example of each of the following :
   a) Two functions f and g such that f + g is derivable but f and g may not be derivable
   b) Two functions f and g such that f . g is differentiable but f and g may not be differentiable.
   c) Two functions f and g such that f − g is differentiable but f and g may not be differentiable.
   d) Two functions f and g such that f/g is defined and differentiable but f and g may not be differentiable.

4) Give an example of a function f such that it is not derivable but | f | is derivable at every point of the domain.

5) Give an example of each of the following :
   a) A function f to which Rolle's theorem is applicable.
   b) A function f to which Rolle's theorem is not applicable.

6) Using the Maclaurin series expansions of sin x and cos x, find the power series expansion of cos 2x and sin x cos 2x.

7) Is stationary value of a function necessarily an extreme value? Justify your answer.

8) Prove that
   i) if f is continuous in [a, ∞[and $f'(x) > 0 \ \forall x \in]a, \infty[$, then f is strictly increasing in [a, ∞[.
   ii) if f is continuous in [a, ω[and $f'(x) < 0 \ \forall x \in]a, \infty[$, then f is strictly decreasing in [a, ∞ [.

9) What is wrong with the following use of L'Hopital's Rule

$$\lim_{x\to 0} \frac{x^3 - 4x^2}{9x^2 - 2x} = \lim_{x\to 0} \frac{3x^2 - 8x}{18x - 2} = \lim_{x\to 0} \frac{6x - 8}{18} = -\frac{4}{9}$$

10) Which indeterminate form is $\lim_{x\to 0} [(4 + \frac{1}{x}) - \frac{1}{\sin x}]$. Find the limit.

ANSWERS/HINTS

1) a) f(x) = | x | ∀x ∈ R. f is not differentiable at one point '0' of its domain.
   b) f(x) = | x | + | x + 1 | ∀x ∈ R. Differentiable except at the points − 1 and 0 of the domain.
   c) f(x) = | x | + | x + 1 | + | x − 1 | ∀x ∈ R. Differentiable except at the points − 1, 0 and 1 of the domain

2) a) False. The function f given by f(x) = | x − 1 | ∀x ∈ R is continuous but not differentiable at 1

b) True.

c) False. The function $f : R \to R$ defined by

$$f(x) = \begin{cases} 1 \text{ when } x > 0 \\ -1 \text{ when } x < 0 \\ 0 \text{ when } x = 0 \end{cases}$$

is monotonic but not differentiable at 0.

d) False. The function $f : R \to R$ defined by

$$f(x) = \frac{x^3}{3} - 4x \ \forall x \in R$$

is differentiable in R but it is monotonic because $f'(x) = x^2 - 4 = (x-2)(x+2)$ which is positive if $x > 2$ or $x < -2$ and is negative for $-2 < x < 2$ and consequently f is monotonically increasing for $x \leq -2$ and $x \geq 2$ and f is monotonically decreasing in $[-2, 2]$.

3) a) Take $f(x) = |x|$ and $g(x) = -|x| \ \forall x \in R$.

b) Take $f(x) = g(x) = |x| \ \forall x \in R$.

4) Consider $f(x) = \begin{cases} 1 \text{ when x is rational} \\ -1 \text{ when x is irrational} \end{cases}$

So f is not derivable at any point of R but $|f|$ is derivable and $|f|'(x) = 0 \ \forall x \in R$.

5) a) Take $f(x) = x^4$, $a = -1$, $b = 1$

b) Take $f(x) = \sin x$; $a = -\pi/2$, $b = \pi/2$.

6) $\cos^2 2x = \left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4}[3/2 + 2\cos 4x + \frac{1}{2}\cos 8x]$

$\sin x \cos 2x = \frac{1}{2}(\sin 3x - \sin x)$

Now expand $\cos 4x$, $\cos 8x$, $\sin 3x$, $\sin x$ in power series.

7) Consider the function $f(x) = (x-1)^7 \ \forall x \in R$.
$f'(1) = 0$ and so $f(1)$ is a stationary value. But $f(x) > f(1)$ for $x > 1$ and $f(x) < f(1)$ for $x < 1$ and consequently $f(1)$ is not an extreme value.

8) Proceed exactly in the same way as that for the finite interval [a, b].

9) $\lim_{x \to 0} \frac{3x^2 - 8x}{18x - 2}$ is not of $\frac{0}{0}$ form. L Hopital's rule cannot be applied.

10) It is $\infty - \infty$ form. The given limit can be written as

$\lim_{h \to 0} \frac{4x \sin x + \sin x - x}{x \sin x}$. Now it is $\frac{0}{0}$ form. Apply L'Hopital's rule and you will get the limit as 4.

NOTES

NOTES

Uttar Pradesh
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UGMM - 09
Real Analysis

Block

5

INTEGRABILITY

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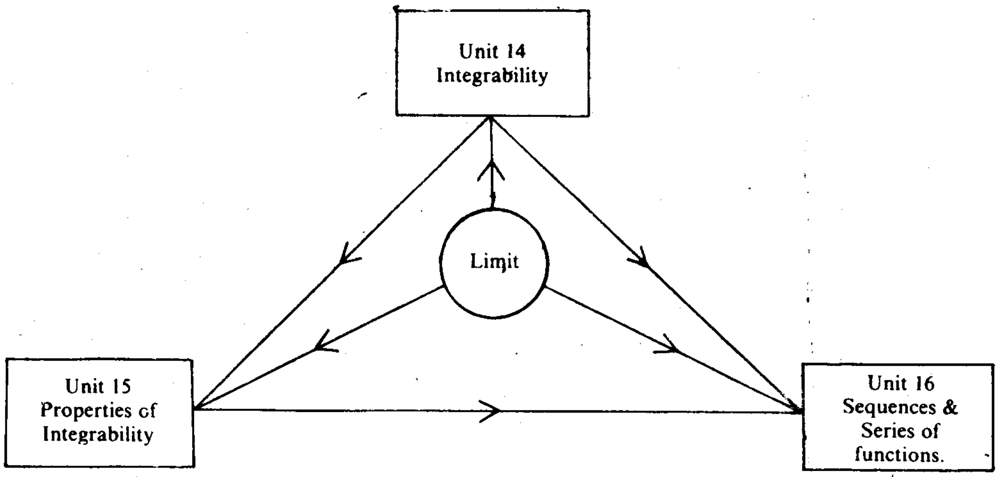
BLOCK 5 INTEGRABILITY

PREVIEW

You know that Calculus deals with two fundamental problems. The first is related to the slope of a curve at a point while the second is about the area of a region under a curve. These problems, as you know, can be easily handled by the geometric methods when the graph of a curve is either a straight line or it consists of several line segments. Calculus is needed when graph of a curve is neither straight line nor it consists of line segments. You have seen in Block 4 that the slope problem can be settled with the notion of the **derivative** of a function which represents the curve. Thus the slope-problem and other related material are covered under the branch of Calculus called the **Differential Calculus.** The area problem is connected with the **integral** of a function representing a curve which we intend to discuss in **this block.** You know that the branch of Calculus that deals with the **integrability** of a function is called the **Integral** Calculus.

Both the problems namely the slope-problem and the area problem were studied for special cases. But it was only in the 17th Century that the close link between the two problems was discovered in the sense that finding a derivative and finding an integral are inverse processes. Therefore in this block, we also intend to show that differential and integral calculus are connected by a relationship called the fundamental theorem of calculus.

This is the last block and has three units namelyUnits 14, 15 and 16. The Unit 14 deals with the notion of the integrability of a function. In Unit 15, we discuss the mean-value theorems of integrability including the fundamental theorem of calculus. Finally in the last unit, we discuss the sequences and series of functions. This topic has important applications in engineering and the physical sciences since these areas of study involve the differentiation and integration of a function which can be described by an infinite sum of a series of functions. The pictorial representation of the block in terms of its units in relation with the limiting process is given in the following diagram.



NOTATIONS AND SYMBOLS

$=$	is equal to
$\neq$	is not equal to
$>$	is greater than
$<$	is less than
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	is a member of (belongs to)
$\notin$	is not a member of (does not belong to)
$\subset$	is a subset of (is contained in)
$\not\subset$	is not a subset of (is not contained in)
$\supset$	is a superset
$\cup$	Union
$\cap$	intersection
ϕ	empty set
$\Rightarrow$	implies
$\Leftarrow$	implied by
$\Leftrightarrow$	if and only if
$\sim$	equivalence relation
$\forall$	for all
$\exists$	there exists
$\cdot$	multiplication
$+$	addition
$-$	subtraction
sup	supremum
inf	infimum
min	minimum
max	maximum
$\circ$	composition
f'	derivative of f
f^{-1}	inverse of a function f
exp	exponential
log	logarithm
In	natural logarithm
sgn	signum
$[x]$	greatest integer not exceeding x
$\|x\|$	absolute value of x or Modulus of x
$\mathbf{R}^+$	set of positive real numbers
$\mathbf{R}$	set of real numbers
$\mathbf{I}$	Set of irrational numbers
$\mathbf{Q}$	set of rational numbers
$\mathbf{Z}$	set of integers
$\mathbf{N}$	set of natural numbers
$\mathbf{F}$	field
$\mathbf{C}$	set of complex numbers
$[a, b]$	closed interval
$]a, b[$	open interval
$]a, b]$	semi-open interval (open at left)—semi-closed interval
$[a, b[$	semi-open interval (open at right)—semi-closed interval
$+\infty$	infinity
$-\infty$	minus infinity
Σ	sum
$\sum_{n=1}^{\infty} u_n$	infinite series
(s_n)	sequence
S^c	complement of S
S'	derived set of S
$\bar{S}$	closure of S

Greek Alphabets

α	Alpha
β	Beta
γ	Gama
δ	Delta
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
i	Iota
λ	Lambda
μ	Mu
ν	Nu
ξ	exi
π	Pi
Π	(capital Pi)
ρ	Rho
$\sigma\ (\Sigma)$	Sigma (capital Sigma)
τ	Tou
ϕ	Phi
χ	Chi
ψ	Psi
ω	Omega

UNIT 14 THE RIEMANN INTEGRATION

Structure

14.1 INTRODUCTION

You are quite familiar with the words **'differentiation'** and **'integration'**. You know that in ordinary language, differentiation refers to separating or distinguishing things while integration means putting things together. In Mathematics, particularly in Calculus and Analysis, differentiation and integration are considered as some kind of operations on functions. You have used these operations in your study of Calculus. You have also studied differentiation in a rigorous way in Unit 11. In this unit, you will be introduced to the operation of integration in a rigorous manner.

There are essentially two ways of describing the operation of integration. One way is to view it as the inverse operation of differentiation. The other way is to treat it as some sort of limit of a sum.

The first view gives rise to an integral which is the result of reversing the process of differentiation. This is the view which was generally considered during the eighteenth century.

Accordingly, the method is to obtain, from a given function, another function which has the first function as its derivative. This second function, if it be obtained, is called the **indefinite integral** of the first function. This is also called the **'primitive'** or **anti-derivative** of the first function. Thus, the integral of a function f(x) is obtained by finding an anti-derivative or primitive function F(x) such that $F'(x) = f(x)$. The indefinite integral of f(x), is symbolized by the notation
$\int f(x)\, dx$.

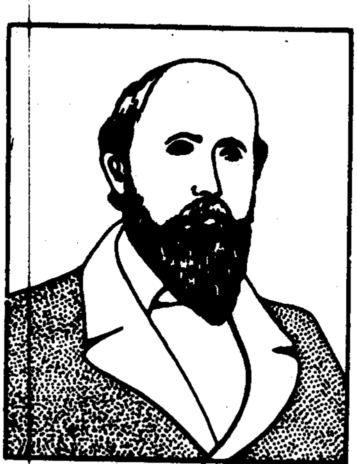
The second view is related to the limiting process. It gives rise to an integral which is the limit of all the values of a function in an interval. This is the integral of a function f(x) over an interval [a,b,]. It is called the **definite integral and is denoted by**

$$\int_a^b f(x)\, dx.$$

The definite integral is a number since geometrically it corresponds to an area of a region enclosed by the graph of a function.

Although both the notions of integration are closely related, yet, you will see later, the definite integral turns out to be a more fundamental concept. In fact, it is the starting point for some important generalizations like the double integrals, triple integrals, line integrals etc. which you may study in the Course MTE on **Advanced Calculus.**

The integral in the anti-derivative sense was given by Newton. This notion was found to be adequate so long as the functions to be integrated were continuous. But in the early 19th century, Fourier brought to light the need for making integration meaningful for the functions that are not continuous. He came across such functions in applied problems. Cauchy formulated rigorous definition of the integral of a function. He essentially provided a general theory of integration but only for continuous functions. Cauchy's theory of Integration for continuous functions is sufficient for piece-wise continuous functions as well as for the functions having isolated



Georg Riemann

discontinuities. However, it was G.B.F. Riemann [1826-1866] a German mathematician who extended Cauchy's integral to the discontinuous functions also. Riemann answered the question **"what is the meaning of $\int f(x)\,dx$?"**

The concept of definite integral was given by Riemann in the middle of the nineteenth century. That is why, it is called Riemann Integral. Towards the end of 19th Century, T.J. Stieltjes [1856-1894] of Holland, introduced a broader concept of integration replacing certain linear functions used in Riemann Integral by functions of more general forms. In the beginning of this century, the notion of the **measure** of a set of real numbers paved the way to the foundation of modern theory of Lebesgue Integral by an eminent French Mathematician H. Lebesgue [1875-1941], a beautiful generalisation of Riemann Integral which you may study in some advanced courses of Mathematics. In this unit, the Riemann Integral will be defined without bringing in the idea of differentiation. In unit 12, you will see the usual connection between the Integration and Differentiation. Just by applying the definition, it is not always easy to test the integrability of a function. Therefore, condition of integrability will be derived with the help of which it becomes easier to discuss the integrability of functions. Then just as in the case of continuity and derivability, we will also consider algebra of integrable functions. Finally, in this unit, second definition of integral as the limit of a sum will be given to you and you will be shown the equivalence of the two definitions.

Objectives

After the study of this unit, you should, therefore, be able to

→ define the Riemann Integral of a function
→ derive the conditions of Integrability and determine the class of functions which are always integrable
→ discuss the algebra of integrable functions
→ compute the integral as a limit of a sum.

14.2 RIEMANN INTEGRATION

The study of the integral began with the geometrical consideration of calculating areas of plane figures. You know that the well-known formula for computing the area of a rectangle is equal to the product of the length and breadth of the rectangle. The question that arises from this formula is that of finding the correct modification of this formula which we can apply to other plane figures. To do so, consider a function defined on a closed interval [a,b] of the real line, which assumes a constant value $K \geq O$ throughout the interval. The graph of such a function gives rise to a rectangular region bounded by the X-axis and the ordinates $x = a$, $x = b$ as shown in the Figure 1.

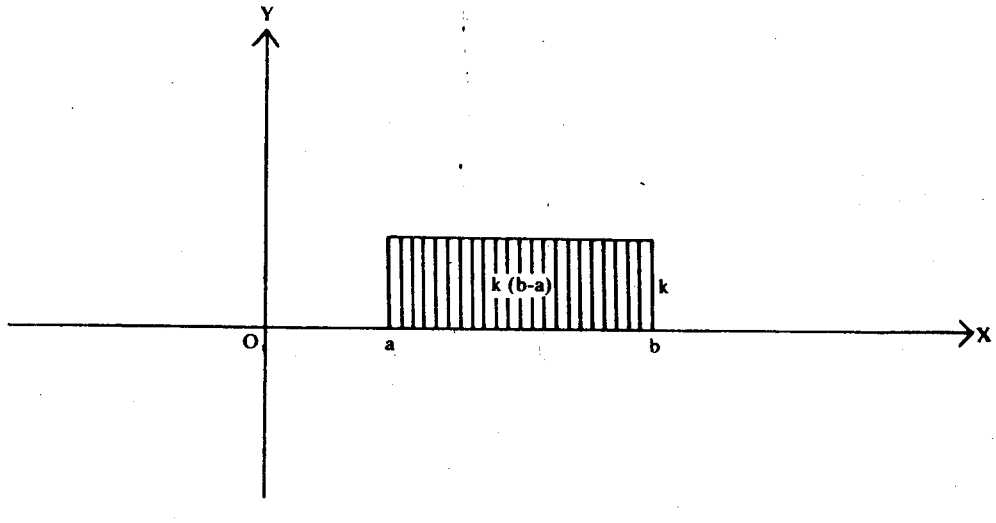


Fig. 1

Obviously, the area enclosed is k (b−a). Now, suppose that (a,b) is further divided into smaller intervals by inserting points of division say

$$a = x_0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 = b$$

and the function f is defined so as to take a constant value at each of the resulting sub-intervals say

$$\begin{aligned} f(x) &= k_1, \text{ if } x \in [x_0, x_1[\\ &= k_2, \text{ if } x \in [x_1, x_2[\\ &= k_3, \text{ if } x \in [x_2, x_3[\\ &= k_4, \text{ if } x \in [x_3, x_4[. \end{aligned}$$

and

$$f(b) = k_4.$$

Further suppose that d_i = length of the ith interval $= (x_i - x_{i-1})$ i.e.

$$d_1 = |x_1 - x_0|,\ d_2 = |x_2 - x_1|,\ d_3 = |x_3 - x_2|,\ d_4 = |x_4 - x_3|$$

Then we get 4 rectangular regions and the area of each region is

$A_1 = k_1d_1$, $A_2 = k_2d_2$, $A_3 = k_3d_3$, $A_4 = k_4d_4$ as shown in the figure 2.

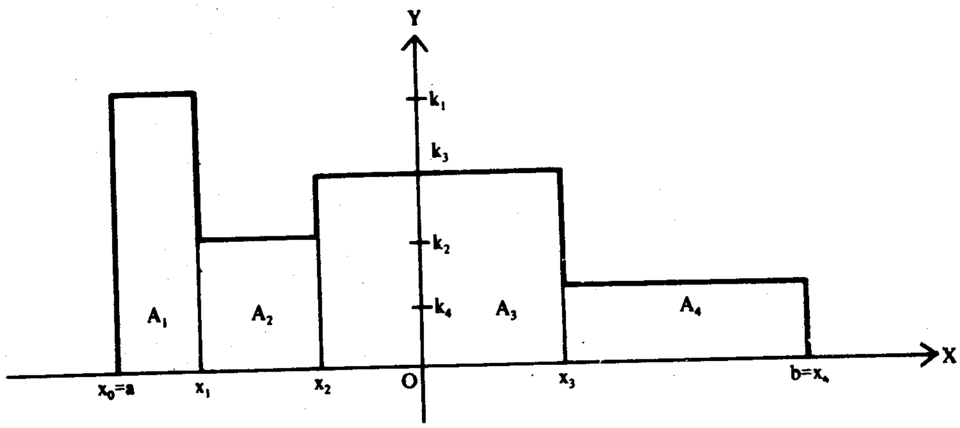


Fig. 2

The total area enclosed by the graph of the function, X-axis and the ordinates x=a, x=b is equal to the sum of these areas i.e.

$$\begin{aligned} \text{Area} &= A_1 + A_2 + A_3 + A_4 \\ &= k_1d_1 + k_2d_2 + k_3d_3 + k_4d_4. \end{aligned}$$

Note that in the last equation, we have generalized the notion of area. In other words, we are able to compute the area of a region which is not of rectangular shape. How did we get it? By breaking up the region into a series of non-overlapping rectangles which include the totality of the figure and summing up their respective areas. This is simply a slight obstraction of the same process which is used in Geometry.

Since the graph of the function in figure 2 consists of 4 different steps, such a function, as you know from unit 4, is called a **step function**. What we have obtained is the area of a region bounded by

i) a non-negative step function
ii) the vertical lines defined by x=a and x=b
iii) the X-axis.

This area is just the sum of the areas of a finite number of non-overlapping rectangles resulting from the graph of the given function. The area is nothing but a real number.

Now suppose that the graph of a given function is as shown in the figure 3.

Does it make any sense to obtain the area of the region under the graph of f? If so, how can we compute its value? To answer this question, we introduce the notion of the integral of a function as given by Riemann.

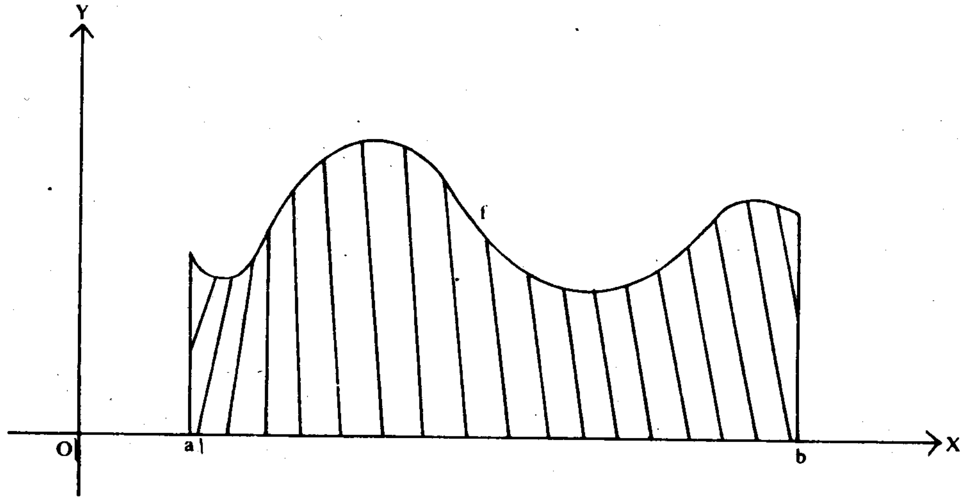


Fig. 3

To introduce the notion of an integral of a function, we will require such a real number which results from applying the function and which represents the area of the region bounded by the graph of f, the vertical lines x=a, x=b and the X-axis. This can be achieved by approximating the given function by suitable step functions. The area of the region will, then, be approximated by the areas enclosed by these step functions, which in turn are obtained as sum of the areas of non-overlapping rectangles as we have computed for the figure 2. This is precisely the idea behind the formal treatment of the integral which we discuss in this section. First, we introduce some terminology and basic notions which will be used throughout the discussion.

Let f be a real function defined and bounded on a closed interval [a,b].

Recall that a real function f is said to be bounded if the range of f is a bounded subset of R, that is, if there exist numbers m and M such that $m \leq f(x) \leq M$ for each $x \in [a,b]$. M is an upper bound and m is a lower bound of f in [a,b]. You also know that when f is bounded, its supremum and infimum exist. We introduce the concept of a partition of [a,b] and other related definitions:

DEFINITION 1: PARTITION
Let [a,b] be a given interval. By a partition P of [a,b] we mean a finite set of points $\{x_0, x_1 \ldots\ldots x_n\}$ where

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

We write $\Delta x_i = x_i - x_{i-1}$, (i=1, 2, ... n). So Δx_i is the length of the ith sub-interval given by the partition P.

DEFINITION 2: NORM OF A PARTITION
Norm of a partition P, denoted by $|P|$, is defined by $|P| = \max_{1 \leq i \leq n} \Delta x_i$. Namely, the norm of P is the length of largest subinterval of [a,b] induced by P. Norm of P is also denoted by $\mu(P)$.

DEFINITION 3: REFINEMENT OF A PARTITION
Let P_1 and P_2 be two partitions of [a,b]. We say that P_2 is finer than P_1 or P_2 refines P_1 or P_2 is a refinement of P_1 if $P_1 \subset P_2$, that is, every point of P_1 is a point of P_2.

You may note that, if P_1 and P_2 are any two partitions of [a,b], then $P_1 \cup P_2$ is a common refinement of P_1 and P_2.

For example, if $P_1 = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ and $P_2 = \left\{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\right\}$ are partitions of [0,1] such that $P_1 \subset P_2$. Then P_2 is a refinement of P_1 and $P_1 \cup P_2 = \{0, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ is their common refinement.

We now introduce the notions of upper sums and lower sums of a bounded function f on an interval [a,b], as given by Darboux. These are sometimes referred to as **Darboux Sums.**

DEFINITION 4: UPPER AND LOWER SUMS

Let f: [a,b] $\rightarrow$ R be a bounded function, and let P = $\{x_0, x_1 \ldots x_n\}$ be a partition of [a,b]. For i = 1, 2 ....., n, let M_i and m_i be defined by

$$M_i = \text{lub}\ \{f(x) : x_{i-1} \leq x \leq x_i\}$$
$$m_i = \text{glb}\ \{f(x) : x_{i-1} \leq x \leq x_i\}$$

i.e. M_i and m_i be the supremum and infimum of f in the sub-interval $[x_{i-1}, x_i]$.

Then, the upper (Riemann) sum of f corresponding to the partition P, denoted by U (P,f), is defined by

$$U(P,f) = \sum_{i=1}^{n} M_i\ \Delta x_i$$

The lower (Riemann) sum of f corresponding to the partition P, denoted by L(P, f), is defined by

$$L(P,f) = \sum_{i=1}^{n} m_i\ \Delta x_i.$$

Before we pass on to the definition of upper and lower integrals, it is good for you to have the geometrical meaning of the upper and lower sums and to visualize the above definitions pictorially. You would, then, have a feeling for what is going on, and why such definitions are made. Refer to figures 4(i), 4(ii), 4(iii).

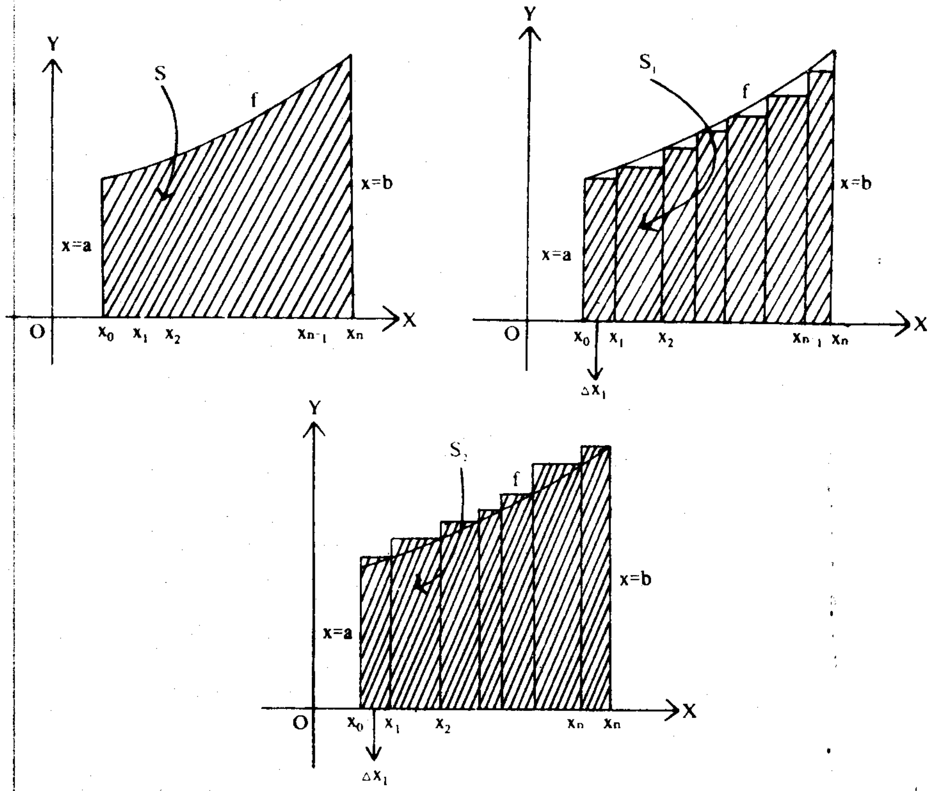


Fig. 4

In figure 4(i), the graph of f: [a,b] $\rightarrow$ R is drawn. The partition P = $\{x_0, x_1, \ldots, x_n\}$ divides the interval [a,b] into sub-intervals $[x_0, x_1]$, $[x_1, x_2]$, ... $[x_{n-1}, x_n]$. Consider the area S under the graph of f. In the first sub-interval $[x_0, x_1]$, m_1 is the g.l.b. of the set of values f(x) for x in $[x_0, x_1]$. Thus $m_1\ \Delta x_1$ is the area of the small rectangle with sides m_1 and Δx_1 as shown in the figure 4(ii). Similarly $m_2\ \Delta\ x_2 \ldots m_n\ \Delta\ x_n$ are areas of such small rectangles and $\sum_{i=1}^{n} m_i\ \Delta x_i$ i.e. lower sum L (P,f) is the area S_1 which is the sum of areas of such small rectangles. The area S_1 is less than the area S under the graph of f. In the same way $M_1 \Delta x_1$ is the area of the large rectangle with sides M_1 and Δx_1 and $\sum_{i=1}^{n} M_i \Delta x_i$ i.e. the upper sum U (P,f) is the area S_2 which is the sum of areas of such large rectangles as shown in figure 4(iii). The area S_2 is more than the area S under the graph of f. If the points in the partition P are increased, the areas S_1 and S_2 approach the area S.

We claim that the sets of upper and lower sums corresponding to different partitions of [a,b] are bounded. Indeed, let m and M be the infimum and supremum of f in [a,b]. Then $m \leq m_i \leq M_i \leq M$ and so

$$m \, \Delta x_1 \leq m_i \, \Delta x_i \leq M_i \, \Delta x_i \leq M \, \Delta x_i$$

Putting i = 1, 2, ...n and adding, we get

$$m \sum_{i=1}^{n} \Delta x_i \leq L(P,f) \leq U(P,f) \leq M \sum_{i=1}^{n} \Delta x_i.$$

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - x_{i-1}) = x_n - x_0 = b - a$$

Thus $m\,(b-a) \leq L(P,f) \leq U(P,f) \leq M\,(b-a)$

For every partition P, there is a lower sum and there is an upper sum. The above inequalities show that the set of lower sums and the set of upper sums are bounded, so that their supremum and infimum exist. In particular, the set of upper sums have an infimum and the set of lower sums have a supremum. This leads us to concepts of upper and lower integrals as given by Riemann and popularly known as Upper and Lower Riemann Integrals.

DEFINITION 5: UPPER AND LOWER RIEMANN INTEGRAL

Let f: [a,b] –> R be a bounded function. The infimum or the greatest lower bound of the set of all upper sums is called the upper (Riemann) integral of f on [a,b] and is denoted by

$$\overline{\int_a^b} f(x)\, dx.$$

i.e.

$$\overline{\int_a^b} f(x)\, dx. = \text{g.l.b.} \{U(P,f): P \text{ is a partition of } [a,b]\}.$$

The supremum or the least upper bound of the set of all lower sums is called the lower (Riemann) integral of f on [a,b] and is denoted by

$$\underline{\int_a^b} f(x)\, dx$$

i.e.

$$\underline{\int_a^b} f(x)\, dx = \text{l.u.b} \{L(P,f): P \text{ is a partition of } [a,b]\}.$$

Now we consider some examples where we calculate upper and lower integrals.

EXAMPLE 1: Calculate the upper and lower integrals of the function f defined in [a.b] as follows:

$$f(x) = \begin{cases} 1 \text{ when x is rational} \\ 0 \text{ when x is irrational} \end{cases}$$

SOLUTION: Let $P = \{x_0, x_1 \ldots\ldots x_n\}$ be any partition of [a,b]. Let M_i and m_i be respectively the sup. f and inf. f in $[x_{i-1}, x_i]$. You know that every interval contains infinitely many rational as well as irrational numbers. Therefore, $m_i = 0$ and $M_i = 1$ for i = 1, 2 ... n. Let us find U(P,f) and L(P,f).

$$U(P,f) = \sum_{i=1}^{n} M_i \, \Delta x_i = \sum_{i=1}^{n} \Delta x_i = b-a$$

$$L(P,f) = \sum_{i=1}^{n} m_i \, \Delta x_i = 0$$

Therefore U(P,f) = b−a and L(P,f) = 0 for every partition P of [a,b]. Hence

$\overline{\int_a^b} f(x)\,dx$ = g.l.b. {U(P,f): P is a partition of [a,b]}

= g.l.b. {b−a} = b−a.

$\underline{\int_a^b} f(x)\,dx$ = l.u.b. {L(P,f): P is a partition of [a,b]}

= l.u.b. {0} = 0.

EXAMPLE 2: Let f be a constant function defined in [a,b]. Let f(x) = k ∀ x ∈ [a,b]. Find the upper and lower integrals of f.

SOLUTION: With the same notation as in example 1, $M_i = k$ and $m_i = k \ \forall\, i$.

$$\text{So } U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k\,(b-a)$$

$$\text{and } L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i = k \sum_{i=1}^{n} \Delta x_i = k\,(b-a)$$

Therefore U(P,f) = k(b−a) and L(P,f) = k (b−a) for every partition P of [a,b].

Consequently $\overline{\int_a^b} f(x)\,dx = k\,(b-a)$ and $\underline{\int_a^b} f(x)\,dx = k\,(b-a)$

Now try the following exercise.

EXERCISE 1

Find the upper and lower Riemann integrals of the function f defined in [a,b] as follows

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

You have seen that sometimes the upper and lower integrals are equal (as in Example 2) and sometimes they are not equal (as in Example 1). Whenever they are equal, the function is said to be integrable. So integrability is defined as follows:

DEFINITION 6: RIEMANN INTEGRAL

Let f: [a,b] → R be a bounded function. The function f is said to be Riemann integrable or simply integrable or R-integrable over [a,b] if $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ and if f is Riemann integrable, we denote the common value by $\int_a^b f(x)\,dx$. This is called the Riemann integral or simply the integral of f on [a,b].

EXAMPLE 3: Show that the function f considered in Example 1 is not Riemann integrable.

SOLUTION: As shown in Example 1, $\overline{\int_a^b} f(x)\,dx = b-a$ and $\underline{\int_a^b} f(x)\,dx = 0$ and so $\overline{\int_a^b} f(x)\,dx \neq \underline{\int_a^b} f(x)\,dx$ and consequently f is not Riemann integrable.

EXAMPLE 4: Show that a constant function is Riemann integrable over [a,b] and find $\int_a^b f(x)\,dx$.

SOLUTION: As proved in Example 2, $\overline{\int_a^b} f(x)\,dx = k(b-a) = \underline{\int_a^b} f(x)\,dx$

Therefore, f is Riemann integrable on [a,b] and $\int_a^b f(x)\,dx = k\,(b-a)$.

EXERCISE 2

Show that the function f defined in Exercise 1 is not integrable.

It is not always easy to find the upper and lower Riemann integrals of a given bounded function f and thereby decide whether the function is integrable over the given interval or not. For this, we discuss some conditions of integrability with the help of which we can decide integrability of a function without finding upper and lower integrals. For proving these conditions of integrability we require some results which we give below in the form of theorems. Some are proved while others are given without proof.

THEOREM 1: If the partition P_2 is a refinement of the partition P_1 of [a,b], then $L(P_1,f) \leq L(P_2, f)$ and $U(P_2,f) \leq U(P_1,f)$.

PROOF: Suppose P_2 contains one point more than P_1. Let this extra point be c. Let $P_1 = \{x_0, x_1 \dots\dots x_n\}$ and $x_{i-1} < c < x_i$. Let M_i and m_i be respectively the sup. f and inf. f in $[x_{i-1}, x_i]$. Suppose sup. f and inf. f in $[x_{i-1}, c]$ are p_1 and q_1 and those in $[c,x_i]$ are p_2 and q_2.

Then $L(P_2,f) - L(P_1,f) = q_1 (c-x_{i-1}) + q_2 (x_i-c) - m_i \Delta x_i$

$= (q_1-m_i) (c-x_{i-1}) + (q_2-m_i) (x_i-c)$

{since $\Delta x_i = (x_i-c) + (c-x_{i-1})$).

Similarly $U (P_2,f) - U(P_1,f) = (p_1 - M_i) (c-x_{i-1}) + (p_2-M_i) (x_i-c)$

Now $m_i \leq q_1 \leq p_1 \leq M_i$

$m_i \leq q_2 \leq p_2 \leq M_i$

Therefore

$L(P_2, f) - L (P_1,f) \geq 0$ and $U (P_2, f) - U (P_1, f) \leq 0$

Therefore

$L(P_1,f) \leq L(P_2, f)$ and $U (P_2, f) \leq U (P_1, f)$.

If P_2 contains p points more than P_1, then adding these extra points one by one to P_1 and using the above results, the theorem is proved. We can also write the theorem as

$$L(P_1,f) \leq L(P_2,f) \leq U(P_2,f) \leq U(P_1,f)$$

from which it follows that $U(P_2,f) - L(P_2,f) \leq U(P_1,f) - L(P_1,f)$. As an illustration of theorem 1, we consider the following example.

EXAMPLE 5: Verify Theorem 1 for the function f(x) = x + 1 defined over [0,1] and the partition $P_1 = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{4}, 1\right\}$ and $P_2 = \left\{0, \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, 1\right\}$.

SOLUTION: For partition P_1, $n = 5$, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$, $x_4 = \frac{3}{4}$, $x_5 = 1$

and so $\Delta x_1 = \frac{1}{4}$, $\Delta x_2 = \frac{1}{12}$; $\Delta x_3 = \frac{1}{6}$, $\Delta x_4 = \frac{1}{4}$, $\Delta x_5 = \frac{1}{4}$.

Further $M_i = f(x_i)$ & $m_i = f(x_{i-1})$ for i = 1, 2, 3, 4, 5 and therefore $M_1 = \frac{5}{4}$, $M_2 = \frac{4}{3}$,

$M_3 = \frac{3}{2}$, $M_4 = \frac{7}{4}$, $M_5 = 2$, $m_1 = 1$, $m_2 = \frac{5}{4}$, $m_3 = \frac{4}{3}$, $m_4 = \frac{3}{2}$, $m_5 = \frac{7}{4}$. We have

$L(P,f) = \sum_{i=1}^{5} m_i \Delta x_i = \frac{25}{18}$ and $U(P_1,f) = \sum_{i=1}^{5} M_i \Delta x_i = \frac{29}{18}$. Similarly $L(P_2,f) = \frac{17}{12}$,

$U(P_2,f) = \frac{19}{12}$.

Hence $L(P_1,f) \leq L(P_2,f)$ and $U(P_2,f) \leq U(P_1,f)$.

Do the following exercise yourself.

EXERCISE 3

Verify Theorem 1 for the function f(x) = sin x defined over the interval $\left[0, \frac{\pi}{2}\right]$ and the partitions $P_1 = \left\{0, \frac{\pi}{4}, \frac{\pi}{2}\right\}$

and

$$P_2 = \left\{0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$$

By applying Theorem 1, it is easily proved that lower integral of a function is less than or equal to upper integral of the function. It is proved in the next theorem. From Examples 3 and 4, you can see the truth of this result.

THEOREM 2: $\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$

PROOF: If P_1 & P_2 be two partitions of $[a,b]$ and $P = P_1 \cup P_2$ be their common refinement, then using Theorem 1, we have $L(P_1,f) \le L(P,f) \le U(P,f) \le U(P_1,f)$ and

$$L(P_2,f) \le L(P,f) \le U(P,f) \le U(P_2,f).$$

Therefore, $L(P_1,f) \le U(P_2,f)$

Keeping P_2 fixed and taking l.u.b. over all P_1, we get

$$\underline{\int_a^b} f(x)\,dx \le U(P_2, f)$$

Now taking g.l.b. over all P_2, we obtain

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

This proves the result.

In Theorem 1, we have compared the lower and upper sums for a partition P_1 with those for a finer partition P_2. Next theorem, which we state without proof, gives the estimate of the difference of these sums.

THEOREM 3: If a refinement P_2 of P_1 contains p more points and

$$|f(x)| \le k \;\forall\, x \in [a,b] \text{ then}$$
$$L(P_1,f) \le L(P_2,f) \le L(P_1,f) + 2p\,k\,\delta$$

and $U(P_1,f) \ge U(P_2,f) \ge U(P,f) - 2p\,k\,\delta$

where δ is the norm of P_1.

This theorem helps us in proving Darboux's theorem which will enable us to derive conditions of integrability. Firstly we give Darboux's Theorem.

THEOREM 4: (DARBOUX'S THEOREM)

If $f: [a,b] \to R$ is a bounded function, then to every $\epsilon > 0$, there corresponds $\delta > 0$ such that

(i) $U(P,f) < \overline{\int_a^b} f(x)\,dx + \epsilon$

(ii) $L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon$

for every partition P of $[a,b]$ with $|P| < \delta$.

PROOF: We consider (i). As f is bounded, there exists a positive number k such that $|f(x)| \le k \;\forall\, x \in [a,b]$. As $\overline{\int_a^b} f(x)\,dx$ is the infimum of the set of upper sums, therefore to each $\epsilon > 0$, there is a partition P_1 of $[a,b]$ such that

$$U(P_1,f) < \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} \qquad (1)$$

Let $P_1 = \{x_0, x_1, \ldots\ldots.. x_p\}$ and δ be a positive number such that $2\,k\,(p-1)\,\delta = \epsilon/2$. Let P be a partition of $[a,b]$ with $|P| < \delta$. Consider the common refinement $P_2 = P \cup P_1$ of P and P_1. Each partition has the same end points 'a' and 'b'. So P_2 is a refinement of P having at the most $(p-1)$ more points than P. Consequently by Theorem 3,

$$\begin{aligned} U(P,f) - 2\,(p-1)\,k\,\delta &\le U(P_2,f) \\ &\le U(P_1,f) \\ &< \overline{\int_a^b} f(x)\,dx + \epsilon/2 \text{ (Using (1))} \end{aligned}$$

Thus

$$\begin{aligned} U(P,f) &< \overline{\int_a^b} f(x)\,dx + \frac{\epsilon}{2} + 2\,(p-1)\,k\,\delta \\ &= \overline{\int_a^b} f(x)\,dx + \epsilon \text{ with } |P| < \delta \end{aligned}$$

EXERCISE 4

Write down the proof of part (ii) of Darboux's Theorem.

As mentioned earlier, Darboux's Theorem immediatley leads us to the conditions of integrability. We discuss this in the form of the following theorem:

THEOREM 5: (CONDITION OF INTEGRABILITY)

FIRST FORM: The necessary and sufficient condition for a bounded function f to be integrable over [a.b] is that to every number $\delta > 0$ there corresponds $\delta > 0$ such that

$$U(P,f) - L(P,f) < \delta \ \forall\ P \text{ with } |P| < \delta$$

PROOF: (i) Condition is necessary:

Since the bounded function f is integrable on [a,b],

$$\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx = \int_a^b f(x)dx$$

Let $\epsilon > 0$ be any number. By Darboux Theorem, there is a number $\delta > 0$ such that

$$U(P,f) < \overline{\int_a^b} f(x)dx + \epsilon/2$$

$$= \int_a^b f(x)\,dx + \epsilon/2 \ \forall\ P \text{ with } |P| < \delta \qquad (2)$$

Also,

$$L(P,f) > \underline{\int_a^b} f(x)\,dx - \epsilon/2 = \int_a^b f(x)\,dx - \epsilon/2$$

$$\text{i.e.} - L(P,f) < -\int_a^b f(x)\,dx + \epsilon/2 \ \forall\ P \text{ with } |P| < \delta \qquad (3)$$

Adding (2) and (3), we get

$U(P,f) - L(P,f) < \epsilon \ \forall\ P$ with $|P| < \delta$.

(ii) Condition is sufficient:

It is given that for each number $\epsilon > 0$, there is a number $\delta > 0$ such that

$$U(P,f) - L(P,f) < \epsilon \ \forall\ P \text{ with } |P| < \delta.$$

Let P be a fixed partition with $|P| < \delta$. Then

$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f).$$

Therefore $\overline{\int_a^b} f(x)\,dx \le \underline{\int_a^b} f(x)\,dx \le U(P,f) - L(P,f) < \epsilon.$

Since ϵ is arbitrary, therefore the non-negative number

$$\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx$$

is less than every positive number. Hence it must be equal to zero that is $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$ and consequently f is integrable over [a,b].

Second Form: The necessary and sufficient condition for a bounded function f to be integrable over [a,b] is that to every number $\epsilon > 0$, there corresponds a partition P of [a,b] such that

$$U(P,f) - L(P,f) < \epsilon.$$

The proof is left as an exercise.

EXERCISE 5

Estal ish the condition of integrability in the second form.

14.3 RIEMANN INTEGRABLE FUNCTIONS

Having derived the necessary and sufficient conditions for the integrability of a function, we can now decide whether a function is Riemann integrable without finding the upper and lower integrals of the function. By using the sufficient part of the conditions, we test the integrability of the functions. In this section we discuss functions which are always integrable. We will show that a **continuous function is always Riemann integrable.** The integrability is not affected even when there are finites number of points of discontinuity or the set of points of discontinuity of the function has a finite number of limit points. It will also be shown that a **monotonic function is also always Riemann integrable.**

We shall denote by R [a,b], the family of all Riemann integrable functions on [a,b]. First we discuss results pertaining to continuous functions in the form of the following theorems:

THEOREM 6: If f: (a,b) $\to$ R is a continuous function, then f is integrable over [a,b], that is $f \in R[a,b]$.

PROOF: Recall from unit 10 that if f is a continuous function on [a,b] then f is bounded and is also uniformly continuous.

To show that $f \in R[a,b]$ you have to show that to each number $\epsilon > 0$, there is a partition P for which

$$U(P,f) - L(P,f) < \epsilon$$

Let $\epsilon > 0$ be given. Since f is uniformly continuous on [a,b], there is a number $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ whenever $|x-y| < \delta$. Let P be any partition of [a,b] with $|P| < \delta$. We show that, for such a partition P, $U(p,f) - L(p,f) < \epsilon$.

Now, $U(P,f) - L(P,f) = \sum_{i=1}^{n} M_i \Delta x_i - \sum_{i=1}^{n} m_i \Delta_i$

$$= \sum_{i=1}^{n} (M_i - m_I) \Delta x_i \qquad (4)$$

where

$$\Delta x_i = x_i - x_{i-1},$$

$M_i = \sup_{x_{i-1} \le x \le x_i} \{f(x)\} = f(\xi_i)$ for some $\xi_i \in [x_{i-1}, x_i]$

(Remember that a continuous function f attains its bounds on $[x_{i-1}, x_i]$.

Similarly $m_i = \inf_{x_{i-1} \le x \le x_i} \{f(x)\} = f(n_i)$ for some $n_i \in [x_{i-1}, x_i]$. Hence

$M_i - m_i = f(\xi_i) - f(n_i) \le |f(\xi_i) - f(n_i)| < \epsilon / b-a$,

Since $|\xi_i - n_i) \le \Delta x_i < \delta$.

Substituting in (4) we obtain

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_I) \Delta x$$

$$< \frac{\epsilon}{b-a} \left(\sum_{i=1}^{n} \Delta x_i \right)$$

$$= \frac{\epsilon}{b-a} (b-a) = \epsilon.$$

This proves the theorem.

Thus, **every continuous function is Riemann integrable.**

But as remarked earlier, even when there are discontinuities of the function, it is integrable. This is given in the next two theorems which we state without proof.

THEOREM 7: Let the bounded function f: [a,b] —> R have a finite number of discontinuities. Then $f \in R(a,b)$.

THEOREM 8: Let the set of points of discontinuity of a bounded function f: [a,b] —> R has a finite number of limit points. Then $f \in R(a,b)$.

We illustrate these theorems with the help of examples.

EXAMPLE 6: Show that the function f where $f(x) = x^2$ is integrable in every interval [a,b].

SOLUTION: You know that the function $f(x) = x^2$ is continuous. Therefore it is integrable in every interval [a,b].

EXAMPLE 7: Show that the function f where f(x) = [x] is integrable in [0,2] where [x] denotes the greatest integer not greater than x.

SOLUTION: $[x] = \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } 1 \leq x < 2 \\ 2 \text{ if } x = 2 \end{cases}$

The points of discontinuity of f in [0,2] are 1 and 2 which are finite in number and so it is integrable in [0,2].

EXAMPLE 8: Show that the function F defined in the interval [0,1] by

$$F(x) = \begin{cases} 2rx \text{ when } \frac{1}{r+1} < x < \frac{1}{r} \text{ where } r \text{ is a positive integer} \\ o \text{ elsewhere} \end{cases}$$

is Riemann integrable.

SOLUTION: The function F is discontinuous at the points $0, 1, \frac{1}{2}, \frac{1}{3}, \ldots$. The set of points of discontinuity has 0 as the only limit point. So the limit points are finite in number and hence the function F is integrable in [0,1].

You should now try the following exercises.

EXERCISE 6
Show that the function f where f(x) = x [x] is integrable in [0,2].

EXERCISE 7
Show that the function f defined in [0,2] such that f(x) = 0 when

$x = \frac{n}{n+1}, \frac{n+1}{n}$ (n = 1, 2, 3 ....) and f(x) = 1 elsewhere, is integrable.

EXERCISE 8
Prove that the function f defined in [0,1] by the condition that if r is a positive integer,

$f(x) = (-1)^{r-1}$ when $\frac{1}{r+1} < x < \frac{1}{r}$,

= o elsewhere
is integrable

There is one more class of integrable functions and this class is that of monotonic functions. This we prove in the following theorem.

THEOREM 9: Every monotonic function is integrable.

PROOF: We shall prove the theorem for the case where f: [a,b] —> R is a monotonically increasing function. The function is bounded, f(a) and f(b) being g.l.b. and l.u.b. Let $\epsilon > 0$ be given number. Let n be a positive integer such that

$$n > \frac{(b-a)[f(b) - f(a)]}{\epsilon}$$

Divide the interval [a,b] into n equal subintervals by the partition $P = \{x_0, x_1 \ldots, x_n\}$ of [a,b]. Then $U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)(\Delta x_i)$

$$= \frac{b-a}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]$$

$$= \frac{(b-a)}{n} [f(b)-f(a)] < \epsilon.$$

This proves that f is integrable. Discuss the case of monotonically decreasing function as an exercise.

EXERCISE 9

Show that a monotonically decreasing function is integrable.

Now we give example to illustrate the theorem.

EXAMPLE 9: Show that the function f defined by the condition $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$, n = 0,1,2 ....

$f(0) = 0,$

is integrable in [0,1]

SOLUTION: Here we have f (0) = 0,

$$f(x) = 1 \text{ when } \frac{1}{2} < x \leq 1$$

$$f(x) = \frac{1}{2} \text{ when } \left(\frac{1}{2}\right)^2 < x \leq \frac{1}{2}$$

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Clearly f is monotonically increasing in [0,1]. Hence it is integrable.

EXERCISE 10

Show that the function f defined in [0,1] by $f(x) = \frac{1}{a^{r-1}}$ when $\frac{1}{a^r} < x \leq \frac{1}{a^{r-1}}$ for r = 1,2,3, ..., f(0) = 0,
where a is an integer greater than 2 is integrable.

14.4 ALGEBRA OF INTEGRABLE FUNCTIONS

In unit 11,we discussed the algebra of the derivable functions. Likewise, we shall now study the algebra of the integrable functions. In the previous sections, you have seen that there are integrable as well as non-integrable functions. In this section you will see that the set of all integrable functions on [a,b] is closed under addition and multiplication by real numbers, and that the integral of a sum equals the sum of the integrals. You will also see that difference, product and quotient of two integrable functions is also integrable. All these results are given in the following theorems.

THEOREM 10: If $f \in R[a,b]$, and λ is any real number, then

$$\lambda f \in R[a,b] \text{ and } \int_a^b \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx.$$

PROOF: Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of [a,b]. Let M_i and m_i be the respective l.u.b. and g.l.b. of f in $[x_{i-1}, x_i]$. Then λM_i and λ_{mi} are the respective l.u.b. and g.l.b. of λf in $[x_{i-1}, x_i]$ if $\lambda \geq 0$ and λm_i and λM_i are the respective l.u.b. and g.l.b. of λf in $[x_{i-1}, x_i]$ if $\lambda < 0$.

When $\lambda \geq 0$, then $U(P,\lambda f) = \sum_{i=1}^{n} \lambda M_i \Delta x_i = \sum_{i=1}^{n} M_i \Delta x_i = \lambda U(P,f)$

$$\Rightarrow \overline{\int_a^b} \lambda f(x)\, dx = \lambda \overline{\int_a^b} f(x)\, dx$$

Similarly $L(P,\lambda f) = \lambda\, L(P,f)$.

$$\Rightarrow \underline{\int_a^b} \lambda f(x)\, dx = \lambda \underline{\int_a^b} f(x)\, dx$$

If $\lambda < 0$, $U(P,\lambda f) = \sum_{i=1}^{n} \lambda\, m_i\, \Delta x_i = \lambda\, L(P,f)$.

$$\Rightarrow \overline{\int_a^b} \lambda f(x)\, dx = \lambda \underline{\int_a^b} f(x)\, dx$$

Similarly $L(P,\lambda f) = \lambda U\, L(P,f)$.

$$\Rightarrow \underline{\int_a^b} \lambda f(x)\, dx = \lambda \overline{\int_a^b} f(x)\, dx$$

Since f is integrable in [a,b], therefore

$$\overline{\int_a^b} f(x)\, dx = \underline{\int_a^b} f(x)\, dx = \int_a^b f(x)\, dx$$

Hence $\overline{\int_a^b} \lambda f(x)\, dx = \underline{\int_a^b} \lambda f(x)\, dx = \lambda \int_a^b f(x)\, dx$

whether $\lambda \geq 0$ or $\lambda > 0$.

Hence $\lambda f \in R\,[a,b]$ and $\int_a^b \lambda\, f(x)\, dx = \lambda \int_a^b f(x)\, dx$.

Now suppose that $\lambda = -1$. In this case the theorem says that if $f \in R\,[a,b]$, then $(-f) \in R\,[a,b]$ and $\int_a^b [-f(x)]\, dx = -\int_a^b f(x)\, dx$.

THEOREM 11 : If $f \in R\,[a,b]$, $g \in R\,[a,b]$,. then $f + g \in R\,[a,b]$ and $\int_a^b (f+g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$.

PROOF : We first show that $f+g \in R\,[a,b]$. Let $\epsilon > 0$ be a given number. Since $f \in R\,[a,b]$, $g \in R\,[a,b]$, there exist partitions P and Q of [a,b] such that

$U(P,f) - L(P,f) < \epsilon/2$ and $U(Q,g) - L(Q,g) < \epsilon/2$

If T is a partition of [a,b] which refines both P and Q, then

$U(T,f) - L(T,f) < \epsilon/2\ [U(T,f) - L(T,f) \leq U(P,f) - L(P,f)]$.

Similarly,

$U(T,g) - L(T,g) < \epsilon/2$ (5)

Also note that, if $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$

and

$N_i = \sup\{g(x): x_{i-1} \leq x \leq x_i\}$

then,

$\sup\{f(x) + g(x): x_{i-1} \leq x \leq x_i\} \leq M_i + N_i$.

Using this, it readily follows that

$U(T, f+g) \leq U(T,f) + U(T,g)$

for every partition T of [a,b]. Similarly

$L(T,f+g) \geq L(T,f) + L(T,g)$

for every partition T of [a,b].

Thus $U(T,f+g) - L(T,f+g) \leq [U(T,f) + U(T,g) - L[(T,f) + L(T,g)]$

$$= [U(T,f) - L(T,f)] + [U(T,g) - L(T,g)] < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \text{ for}$$

T occurring in (5). This shows that $f + g \in R(a,b)$

It remains to show that $\int_a^b [f(x) + g(x)]\, dx = \int_a^b [f(x)\, dx + \int_a^b g(x)$

Now

$$\int_a^b (f + g)(x)\, dx = \overline{\int_a^b} (f + g)(x)\, dx \leq U(P, f + g) \leq U(P,f) + U(P,g) \quad \ldots (6)$$

for any partition P of [a,b]. Given any $\epsilon > 0$ we can find a partition P of [a,b] such that

$$U(P,f) < \int_a^b f(x)\,(x)\,dx + \epsilon/2$$

$$U(P,g) < \int_a^b g(x)\,dx + \epsilon/2 \qquad \dots (7)$$

Substituting (7) in (6), we obtain

$$\int_a^b (f+g)(x)\,dx < \int_a^b f(x)\,dx + \int_a^b g(x)\,dx + \epsilon \qquad \dots (8)$$

Since (8) holds for arbitrary $\epsilon > 0$, we obtain

$$\int_a^b (f+g)(x)\,dx \leq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \dots (9)$$

Replacing f and g by $-f$ and $-g$ in (9) we obtain

$$\int_a^b (-f-g)(x)\,dx \leq \int_a^b \{-f(x)\}\,dx + \int_a^b \{-g(x)\}\,dx$$

or

$$-\int_a^b (f+g)(x)\,dx \leq -\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

This is equivalent to

$$\int_a^b (f+g)(x)\,dx \geq \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \qquad \dots (10)$$

Combining (9) and (10), we get

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx \int_a^b g(x)\,dx$$

Which proves the theorem.

THEOREM 12 : If $f \in R[a,b]$ and $g \in R[a,b]$ then $f - g \in R[a,b]$ and

$$\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx \quad g(x)\,dx$$

PROOF: Since $g \in R[a,b]$, therefore $-g \in R[a,b]$ and

$$\int_a^b -[g(x)]\,dx = -\int_a^b g(x)\,dx$$

Now $f \in R[a,b]$ and $-g \in R[a,b]$ implies that $f + (-g) \in R[a,b]$ and

therefore,

$$\int_a^b [f+(-g)](x)\,dx = \int_a^b f(x)\,dx + \int_a^b [-g(x)]\,dx$$

that is $(f-g) \in R[a,b]$ and

$$\int_a^b (f-g)(x)\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$$

For the product and quotient of two functions, we state the theorems without proof.

THEOREM 13: If $f \in R[a,b]$ and $g \in R[a,b]$ then $fg \in R[a,b]$.

THEOREM 14: If $f \in R[a,b]$, $g \in R[a,b]$ and there exists a number $t > 0$ such that $|g(x)| \geq t \ \forall\ x \in [a,b]$, then $f/g \in R[a,b]$. Now we give some examples.

EXAMPLE 10: Show that the function f where $f(x) = x + [x]$ is integrable is [0,2].

SOLUTION: The function $F(x) = x$, being continuous is integrable in [0,2] and the function $G(x) = [x]$ is integrable as it has only two points namely 1 and 2 as points of discontinuity. So their sum i.e. f(x) is integrable in [0,2].

EXAMPLE 11: Give an example of functions f and g such that $f + g$ is integrable but f and g are not integrable in [a,b].

SOLUTION: Let f and g be defined in [a,b] such that

$$f(x) = \begin{cases} 0 \text{ when x is rational} \\ 1 \text{ when x is irrational} \end{cases}$$

$$g(x) = \begin{cases} 1 \text{ when } x \text{ is rational} \\ 0 \text{ when } x \text{ is irrational} \end{cases}$$

f and g are not integrable but $(f + g) = 1 \; \forall \; x \in [a,b]$, being a constant function, is integrable.

EXERCISE 11

Give example of functions f and g such that f−g, fg, f/g are integrable but f and g may not be integrable over [a,b].

Example 11 and Exercise 11 show that converse of each of Theorems 11 to 14 may not be true.

14.5 COMPUTING AN INTEGRAL

So far, we have discussed several theorems for testing whether a given function is integrable on a closed interval [a,b]. For example, we can see that a function $f(x) = x^4$, $\forall x \in [0,2]$ is continuous as well as monotonic on the given interval and hence it is integrable over [0,2]. But this information does not give us a method for finding the value of the integral of this function. In practice, this is not so easy as we might think of. The reason is that there are some functions which are integrable by conditions of integrability but it is difficult to find the values of their integrals. For example, suppose a function is given by $f(x) = e^{x^2}$. This is continuous over every closed interval and hence it is integrable. But we cannot find its integral by our usual method of antiderivative since there is no function for which e^{x^2} is the derivative. If possible, try to find the antiderivative for this function!

In such situations, to find the integral of a given function, we use the basic definition of the integral to evaluate its integral. Indeed, the definition of integral as a limit of sum helps us in such situations.

In this section, we demonstrate this method by means of certain examples. We have found the integral $\int_a^b f(x)\,dx$ via the sums U(P,f) and L(P,f). The numbers M_i and m_i which appear in these sums are not necessarily the values of f(x), if f is not continuous. In fact, we shall now show that $\int f(x)\,dx$ can be considered as limit of sums in which M_i and m_i are replaced by values of f. This approach gives us a lot of latitude in evaluating $\int_a^b f(x)\,dx$, as we shall see in several examples.

Let f: [a,b] −> R be a bounded function. Let
$\{a = x_0 < x_1 < \ldots\ldots x_n = b\}$

be a partition P of [a,b]. Let us choose points $t_1, \ldots. t_n$, such that

$x_{i-1} \leq t_i \leq x_i$ (i = 1, ... n). Consider the sum

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} f(t_i)\,(x_i - x_{i-1}).$$

Notice that, instead of M_i in U(P,f) and m_i in L(P,f), we have $f(t_i)$ in S(P,f). Since t_i's are arbitrary points in $[x_{i-1}, x_i]$, S(P,f) is not quite well-defined. However, this will not cause any trouble in case of integrable functions.

S(P,f) is called **Riemann Sum** corresponding to the partition P.
We say that $\lim_{|P| \to 0} S(P,f) = A$

or S(P,f) −> A as |P| −> 0 if for every number $\epsilon < 0 \; \exists \; \delta > 0$ such that

$$|S(P,f) - A| < \epsilon \text{ for P with } |P| < \delta.$$

We give a theorem which expresses the integral as the limit of S(P,f).

THEOREM 15: If $\lim_{|P| \to 0} S(P,f)$ exists, then $f \subset R[a,b]$

and

$$\lim_{|P| \to 0} S(P,f) = \int_a^b f(x)\,dx$$

PROOF: Let $\lim_{P \to 0} S(P,f) = A$. Then given a number $\epsilon > 0$, there exists

a number $\delta > 0$ such that

$$|S(P,f) - A| < \epsilon/4 \text{ for P with } |P| < \delta.$$
$$\text{i.e. } A - \epsilon/4 < S(P,f) < A + \epsilon/4 \text{ for P with } |P| < \delta. \text{ (11)} \quad \text{...(11)}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$. Suppose the points $t_1, \ldots, t_n$ vary in $[x_0, x_1], \ldots\ldots [x_{n-1}, x_n]$, The l.u.b. of the numbers S(P, f) are given by

$$\text{l.u.b. } S(P,f) = \text{l.u.b.} \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} M_i\,\Delta x_i = U(P,f)$$

Similarly g.l.b. $S(P,f) = L(P,f)$.
Then from (11), we get

$$A - \epsilon/4 \le L(P,f) \le U(P,f) \le A + \epsilon/4 \quad \text{...(12)}$$

Therefore

$$U(P,f) - L(P,f) \le (A + \epsilon/4) - (a - \epsilon/4)$$
$$= \epsilon/2$$
$$< \epsilon.$$

In other words,

$$\therefore\ f \in R[a,b]$$

Thus

$$\overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$

Since $L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f)$, therefore

$$L(P,f) \le \int_a^b f(x)\,dx \le U(P,f) \quad \text{...(13)}$$

From (12) and (13), we get

$$A - \epsilon/4 \le \int_a^b f(x)\,dx \le A + \epsilon/4.$$

or

$$\left|\int_a^b f(x)\,dx - A\right| \le \epsilon/2 < \epsilon.$$

Since ϵ is arbitrary, therefore $\int_a^b f(x)\,dx = A = 0$, that is,

$$\int_a^b f(x)\,dx = A = \lim_{|P| \to 0} S(P,f)$$

To illustrate this theorem, we give two examples.

EXAMPLE 12: Show that $\int_a^b dx = \int_a^b 1\,dx = b - a$.

SOLUTION: Here, the function f: {a,b} –> R is the constant function $f(x) = 1$. Clearly, for any partition $P = (x_0, x_1, \ldots\ldots, x_n)$ of [a,b] $S(P,f) = (x_1 - x_0) f(t_1) + (x_2 - x_1) f(t_2) + \ldots\ldots (x_n - x_{n-1}) f(t_n) = (x_1, x_0)\,1 + (x_2 - x_1)\,1 + \ldots\ldots x\,(x_n - x_{n-1})\,1 = b - a$

Since $S(P,f) = b - a$ for all partitions, $\int_a^b 1\,dx = \lim_{|P| \to 0} S(P,f) = b - a$.

EXAMPLE 13: Show that $\int_a^b x\,dx = \dfrac{b^2 - a^2}{2}$

SOLUTION: The function f: [a,b] $\rightarrow$ R in this example is the identity function f(x) = x

Let P = ($a = x_0, x_1, \ldots\ldots, x_n = b$) be any partition of [a,b]. Then
$S(P,f) = (x_1 - x_0) f(t_1) + (x_2 - x_1) f(t_2) + \ldots\ldots x_n - x_{n-1}) f(t_n)$ where $t_1 \in [x_1 - x_0]$, $t_2 \in (x_1 - x_2) \ldots\ldots t_n \in [x_{n-1} - x_n]$ are arbitrary.

Let us choose

$$t_i = \frac{x_0 + x_1}{2}\ .\ t_2 = \frac{x_1 + x_2}{2} \ldots\ldots t_n = \frac{x_{n-1} + x_n}{2}$$

$$\text{Then } S(P,f) = (x_1 - x_0)\frac{x_1 + x_0}{2} + (x_2 - x_1)\frac{x_2 + x_1}{2} + \ldots\ldots + (x_n - x_{n-1})\frac{x_n + x_{n-1}}{2}$$

$$= \frac{1}{2}[(x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \ldots + (x_n^2 - x_{n-1}^2)]$$

$$= \frac{1}{2}(x_n^2 - x_0^2)$$

$$= \frac{1}{2}(b^2 - a^2).$$

Here again $S(P,f) = 1/2\,(b^2 - a^2)$, no matter what the partition P is.

Hence $\int_a^b f(x)\,dx = \lim_{|P| \to 0} S(P,f) = 1.2\,(b^2 - a^2)$

The converse of theorem 15 is also true which we state as the next theorem without proof.

THEOREM 16: If a function f is Riemann integrable on a closed interval [a,b], then $\lim_{|P| \to 0} S(P,f)$ exists and

$$\lim_{|P| \to 0} S(P,f) = \int (fx)\,dx.$$

Theorem 16 is used for computing the limit of sums of some series. For that, let us consider a partition P of [a,b] having n sub-intervals, each of length h so that nh = b − a. Then P can be written as

$$P = (a, a + h, a + 2h, \ldots\ldots a + nh = b)$$

Let $t_i = a + ih$, $i = 1,2, \ldots\ldots n$. Then

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = h\,[f(a + h) + f(a + 2h) + \ldots\ldots + f(a + nh)]$$

when $\lim_{|P| \to 0} S(P,f)$ exists, then

$$\lim_{\substack{n \to \infty \\ h \to 0}} h\,[f(a + h) + f(a + 2h) + \ldots\ldots + f(a + nh)] = \int_a^b f(x)\,dx$$

We can change the limits of integration from a, b to 0,1. Changing h to $\frac{b-a}{n}$ you see that

$$(b - a)\lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n} f\left[a + (b - a)\frac{r}{n}\right] = \int_a^b f(x)\,dx$$

$$\text{But } \lim_{n \to \infty} \frac{1}{n}\sum_{r=1}^{n} f\left[a + (b - a)\frac{r}{n}\right] = \int_0^1 f[a + (b - a)\,x]\,dx \qquad \ldots(14)$$

$$\text{Therefore} \int_a^b f(x)\,dx = (b - a)\int_0^1 f[a + (b - a)\,x]\,dx \qquad \ldots(15)$$

In (14), put a = 0, b = 1. We get the following result:
If f is intrgrable in [0,1], then

$$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} f\left(\frac{1}{n}\right) = \int_0^1 f(x)\,dx$$

This gives you the following method for finding the limit of sum of n terms of a series:

I Write the general rth term of the series.

II Express it as $\frac{1}{n} f(\frac{r}{n})$, the product of $\frac{1}{n}$ and a function of $\frac{r}{n}$.

III Change $\frac{r}{n}$ to x and $\frac{1}{n}$ to dx and integrate between the limits 0 and 1. The value of the integral gives the limit of the sum of n terms of the series.

Since each term of the series tends to 0, the addition or deletion of a finite number of terms of the series does not affect the value of the limit.
Similarly you can verify that

$$\lim_{n \to \infty} \sum_{r=1}^{2n} \left[\frac{1}{n} \phi\left(\frac{r}{n}\right)\right] = \int_0^2 \phi(x)\,dx.$$

$$\text{and } \lim_{n \to \infty} \sum_{r=1}^{3n} \left[\frac{1}{n} \phi\left(\frac{r}{n}\right)\right] = \int_0^3 \phi(x)\,dx.$$

As an illustration of these results, consider the following examples:

EXAMPLE 14: Find the limit, when n tends to infinity of the series

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots\dots + \frac{1}{n+n}$$

SOLUTION: General (rth) term of the series is $\sum_{r=1}^{n} \frac{1}{n} \frac{1}{1+\frac{r}{n}}$

$$\text{Hence } \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{1+\frac{r}{n}} = \int_0^1 \frac{1}{1+x}\,dx$$

which can be easily evaluated.

EXAMPLE 15: Find the limit, when n tends to infinity, of the series.

$$\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \dots\dots\dots + \frac{1}{\sqrt{n^2-(n-1)^2}}$$

SOLUTION: Here the rth term = $\frac{1}{\sqrt{n^2-(r-1)^2}}$,

Since it contains (r−1), some consider (r+1) th term i.e.

$$(r+1) \text{ th term} = \frac{1}{\sqrt{n^2-r^2}} = \frac{1}{n} \frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

$$\text{Therefore } \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n\sqrt{1-\left(\frac{r}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$$

Which can be evaluated and its value is $\pi/2$.

EXAMPLE 16: Find $\lim_{n \to \infty} \sum_{r=1}^{3n} \frac{n^2}{(3n+r)^3}$

SOLUTION: We have

$$\frac{n^2}{(3n+r)^3} = \frac{1}{n} \frac{1}{\left(3+\frac{r}{n}\right)^3}$$

Since the number of terms in the summation is 3n, the resulting definite integral will have the limits 0 and 3.

$$\text{Therefore } \lim_{n \to \infty} \sum_{r=1}^{n} \frac{n^2}{(3n+r)^3} = \lim_{n \to \infty} \sum_{r=1}^{3n} \frac{1}{n} \frac{1}{\left(3+\frac{r}{n}\right)^3}$$

$$\int_0^3 \frac{dx}{(3+x)^3}$$

This you can evaluate easily.
Now try the following exercises yourself:

EXERCISE 12.
Find the limit, when n tends to infinity, of the series

$$\frac{\sqrt{n}}{\sqrt{n^3}} + \frac{\sqrt{n}}{\sqrt{(n+4)^3}} + \frac{\sqrt{n}}{\sqrt{(n+8)^3}} + \ldots + \frac{\sqrt{n}}{\sqrt{[n+4(n-1)]^3}}$$

EXERCISE 14
Find the limit, when n tends to infinity, of the series

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{3n}.$$

14.6 SUMMARY

In this unit, you have been introduced to the concept of integration without bringing in the idea of differentiation. In section 14.2, upper and lower sums and integrals of a bounded function f over closed interval [a,b] have been defined. You have seen that upper and lower Riemann integrals of a bounded function always exist. Only when the upper and lower Riemann integrals are equal, the function f is said to be Riemann integrable or simply integrable over [a,b] and we write it as $f \in R[a,b]$ and the value of the integral of f over [a,b] is denoted by $\int_a^b f(x)\,dx$. The definition of Riemann integral is given in this section. Also in this section, it has been shown that in passing from a partition P_1 to a finer partition P_2, the upper sum does not increase and the lower sum does not decrease. Further you have seen that the lower integrable of a function is less than or equal to the upper integral. Further condition of integrability has been derived with the help of which the integrability of a function can be decided without finding the upper and lower integrals. Using the condition of integrability, it has been shown in section 14.3 that a function f is integrable on [a,b] if it is continuous or it has a finite number of points of discontinuities or the set of points of discontinuities have finite number of limit points. Also in this section you have seen that a monotonic function is integrable. As in the case of continuous and derivable functions, the sum, difference, product and quotient of integrable functions is integrable. This has been discussed in section 14.4. Finally in section 14.5, Riemann sum S(P,f) of a function f for a partition P has been defined and you have been shown that $\lim_{|P| \to 0} S(P,f)$ exists if and only if $f \in R[a,b]$ and $\int_a^b f(x)\,dx = \lim_{|P| \to 0} S(P,f)$. Using this idea a number of problems can be solved.

14.7 ANSWERS/HINTS/SOLUTIONS

E1) If $P = \{x_0, x_1 \ldots x_n\}$ be any partition of [a,b] and M_i, m_i be l.u.b. and g.l.b. of f in $[x_{i-1}, x_i]$, then $M_i = 1$, $m_i = -1$ $\forall i = 1, 2, \ldots n$

$U(P,f) = b-a$ and $L(P,f) = a-b$. Therefore $\int_a^{\bar{b}} f(x)\,dx = b-a$ and $\int_{\underline{a}}^{b} f(x)\,dx = a-b$.

E 2) In Exercise 1, you see that $\int_a^{\bar{b}} f(x)\,dx \neq \int_{\underline{a}}^{b} f(x)\,dx$ and therefore f is not integrable in [a,b].

E 3) For P_1, $n = 3$, $x_0 = 0$, $x_1 = \pi/4$, $x_2 = \pi/2$.

So $\Delta x_1 = \pi/4$, $\Delta x_2 = \pi/4$. $M_1 = \frac{1}{\sqrt{2}}$, $M_2 = 1$, $m_1 = 0$, $m_2 = \frac{1}{\sqrt{2}}$

$$U(P_1,f) = \sum_{i=1}^{2} M_i\,\Delta x_i = \frac{\pi}{4}\left(\frac{1}{\sqrt{2}} + 1\right) \text{ and}$$

$$L(P_1,f) = \sum_{i=1}^{n} m_i\,\Delta x_i = \frac{1}{\sqrt{2}}\,\pi/4$$

For P_2, $n = 4$, $x_0 = 0$, $x_1 = \pi/6$, $x_2 = \pi/4$, $x_3 = \pi/2$

$\Delta x_1 = \pi/6$, $\Delta x_2 = \frac{\pi}{12}$, $\Delta x_3 = \pi/4$, $M_1 = \frac{1}{2}$, $M_2 = \frac{1}{\sqrt{2}}$, $M_3 = 1$

$m_1 = 0$, $m_2 = \frac{1}{2}$, $m_3 = \frac{1}{\sqrt{2}}$. So $U(P_2,f) = \frac{\pi}{12}\left(4 + \frac{1}{\sqrt{2}}\right)$ and

$L(P_2,f) = \frac{\pi}{24}\left(1 + \frac{6}{\sqrt{2}}\right)$. So $L(P_1, f) \leq L(P_2, f)$ and $U(P_2,f) \leq U(P_1, f)$

E 4) As f is bounded, there exists a positive number k such that

$|f(x)| \leq k \;\forall\; x \in [a,b]$. As $\int_{\underline{a}}^{b} f(x)\,dx$ is the supremum of the set of lower sums, for each $\epsilon > 0$, there exists a partition P_1 of [a,b] such that

$$L(P_1,f) > \int_{\underline{a}}^{b} f(x)\,dx - \epsilon/2 \qquad (16)$$

Taking $P_1 = \{x_0, x_1, \ldots, x_p\}$ and a positive number δ defined by $2k(p-1)\,\delta = \epsilon/2$ and proceeding as in (i) part of Darboux Theorem with $P_2 = P_1 \cup P$,
$L(P,f) \geq L(P_2,f) - 2(p-1)k\delta$ (Using Theorem 3).

$$\geq L(P,f) - 2(p-1)k\delta$$

$$> \int_{\underline{a}}^{b} f(x)\,dx - \epsilon/2 - \epsilon/2 \text{ (Using (16))}$$

$$= \int_{\underline{a}}^{b} f(x)\,dx - \epsilon \;\forall\; P \text{ with } |P| < \delta.$$

E 5) Condition is necessary:
Proceeding as in First Form, you will get
$U(P,f) - L(P,f) < \epsilon \;\forall\; P$ with $|P| < \delta$.
Now fix a partition P having $|P| < \delta$. So far this partition P,

$$U(P,f) - L(P,f) < \epsilon.$$

Condition is sufficient: For $\epsilon > 0$, $\exists$ a partition P of [a,b] such that $U(P,f) - L(P,f) < \epsilon$. Then proceeding as in first form, you will get that f is integrable in [a,b,].

E 6)
$$f(x) = \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ x \text{ if } 1 \leq x < 2 \\ 4 \text{ if } x = 2 \end{cases}$$

This function has 1,2, as the only points of discontinuity and so it is integrable in [0,2].

E 7) The function f has the following points of discontinuity:

$1, \frac{1}{2}, \frac{2}{1}, \frac{2}{3}, \frac{3}{2}, \frac{3}{4}, \frac{4}{3}, \ldots$ and this set of points of discontinuity has 1 as the only limit point and so the function f is integrable by Theorem 8.

E 8) The function f has the following points of discontinuity: $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots$

This set has 0 as the only limit point and hence f is integrable on $[0,1]$.

E 9) In this case $f(a)$ is the l.u.b. and $f(b)$ is the g.l.b. of f. If $\epsilon > 0$ be any number, you have to choose a positive integer $n > \frac{(b-a)[f(a)-f(b)]}{\epsilon}$. Proceed as in theorem 9 by taking $M_i = f(x_{i-1})$ and $m_i = f(x_i)$

E 10) Here $f(0) = 0$, $f(x) = 1$ when $\frac{1}{a} < x \leq 1$.

$f(x) = \frac{1}{a}$, when $\frac{1}{a^2}, < x \leq \frac{1}{a}, \ldots\ldots$

f is monotonically increasing in $[0,1]$ and so by Theorem 9, it is integrable in $[0,1]$

E 11) Consider the functions f and g defined in $[a,b]$ as follows:

$$f(x) = g(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

f and g are not integrable (Proved in Ex. 1).
$f(x) - g(x) = 0 \ \forall x \in [a,b]$ and so it is a constant function and hence integrable in $[a,b]$.

$f(x)\, g(x) = 1 \ \forall \ x \in [a,b]$ and also

$$\frac{f(x)}{g(x)} = 1 \ \forall x \in [a,b].$$

Both fg and f/g are constant functions and so they are integrable.

E 12) $(r+1)$ th term $= \dfrac{\sqrt{n}}{\sqrt{(n+4r)^3}} = \dfrac{1}{n\sqrt{\left(1+\frac{4r}{n}\right)^3}}$

So the required limit $= \int_0^1 \dfrac{1}{(1+4x)^{3/2}}$, which is easy to evaluate.

E 13) $(r+1)$th term $= \dfrac{1}{n+r} = \dfrac{1}{n}\dfrac{1}{1+\frac{r}{n}}$

Given series $= \sum_{r=1}^{2n} \dfrac{1}{n} \dfrac{1}{1+\frac{r}{n}}$

Since the number of terms in the series is 2n, the required limit is $\int_0^2 \dfrac{1}{1+x}\, dx$.

UNIT 15 INTEGRABILITY AND DIFFERENTIABILITY

Structure

15.1 INTRODUCTION

In unit 14, the notion of the integral of a function was developed as a limit of sums of the series. Nowhere, the concept of differentiation was used. Apparently, you may conclude as if there is no relation between the integration and differentiation. But this is not true in all cases. No doubt, the notion of integral as a limit of sums allows us to compute the integrals in some simple cases which you have seen in section 14.5. Nevertheless, it is not convenient for a large number of cases. We do require the process of differentiation to compute the integrals for a certain class of functions. This means there must be some relationship between differentiability and integrability of a function. What is that relationship between the two notions? We shall bring forth this intimate connection between the notions of differentiation and integration for a certain class of functions. In the case of continuous functions, this relationship is expressed in the form of an important theorem called the Fundamental Theorem of Calculus, which we discuss in section 15.3. Prior to this, we need a few important properties of the definite integral which you have studied in your previous course on Calculus. We shall review these properties in section 15.2. In section 15.3, we shall study two additional theorems of integrability which use the process of differentiation. These theorems are known as the Mean-value Theorems of integrability analogous to the mean-value theorems of differentiability.

Objectives

After the study of this unit you should, therefore, be able to

- know some important properties of the Riemann Integral
- establish the inverse relationship between integration and differentiation
- apply Fundamental theorems of calculus to evaluate large number of integrals
- learn the mean value theorems of integrability and their applications.

15.2 PROPERTIES OF RIEMANN INTEGRAL

In section 14.4, you were introduced to some methods which enabled you to associate with each integrable function f defined on [a,b], a unique real number called the integral $\int_a^b f(x)\,dx$ in the sense of Riemann. In section 14.5, a method of computing this integral as a limit of a sum was explained. All this leads us to consider some nice properties which are presented as follows:

PROPERTY I: If f and g are integrable on [a,b] and ii

$$f(x) \le g(x) \quad \forall\, x \in [a,b],$$

then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

PROOF: Define a function $h: [a,b] \to R$ as
$h = g - f$.

Since f and g are a integrable on [a,b], therefore the difference h is integrable on [a,b]. Since

$$f(x) \leq g(x) \Rightarrow g(x) - f(x) \geq 0,$$

therefore $h(x) \geq 0$ for all $x \in [a,b]$.

Consequently If $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of [a,b] and m_i be the inf. of h in $[x_{i-1}, x_i]$, then

$$m_i \geq 0 \;\forall\; i = 1,2, \ldots n$$

$$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \geq 0$$

$$\Rightarrow L(P,h) \geq 0$$

Thus for every partition P, the lower sum $L(P,h) \geq 0$.
In other words, Sup. $\{L(P,h)$: P is a partition of $[a,b]\} \geq 0$

or

$$\underline{\int_a^b} f(x)\, dx \geq 0$$

Since h is integrable in [a,b], therefore

$$\overline{\int_a^b} h(x)\, dx = \underline{\int_a^b} h(x)\, dx = \int_a^b h(x)\, dx.$$

Thus

$$\int_a^b h(x)\, dx \geq 0$$

or

$$\int_a^b (g-f)(x)\, dx \geq 0$$

$$\Rightarrow \quad \int_a^b g(x)\, dx \geq \int_a^b f(x)\, dx$$

which proves the property

PROPERTY II: If f is integrable on [a,b] then $|f|$ is also integrable on [a,b] and $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|\, dx$

PROOF : The inequality follows at once from Property I provided it is known that $|f|$ is integrable on [a,b]. Indeed, you know that $-|f| \leq f \leq |f|$.
Therefore,

$$-\int_a^b |f(x)|\, dx \leq \int_a^b f(x)\, dx \leq \int_a^b |f(x)|\, dx$$

which proves the required result. Thus, it remains to show that $|f|$ is integrable.

Let $\epsilon > 0$ be any number. There exists a partition P of [a,b] such that

$$U(P,f) - L(P,f) \leq \epsilon$$

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$.
Let M'_i and m'_i denote the supremum and infimum of $|f|$ and M_i and m_i denote the supremum and infimum of f in $[x_{i-1}, x_i]$.
You can easily check that

$$M_i - m_i \geq M'_i - m_i.$$

This implies that $\sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i \geq \sum_{i=1}^{n} (M'_i - m'_i)\, \Delta x_i$

i.e. $U(P,|f|) - L(P,|f|) \leq U(P,f) - L(P,f) < \epsilon$.
This shows that $|f|$ is integrable on [a,b].

Note that the inequality established in Property II may be thought of as a generalization of the well-known triangle inequality

$$|a + b| \leq |a| + |b|$$

discussed in Unit 3. In other words. the absolute value of the limit of a sum never exceeds the limit of the sum of the absolute values.

You know that in the integral $\int_a^b f(x)\,dx$, the lower limit a is less than the upper limit b. It is not always necessary. In fact the next property deals with the integral in which the lower limit a may be less than or equal to or greater than the upper limit b.

For that, we have the following definition:

DEFINITION 1: Let f be integrable on [a,b], that is, $\int_a f(x)\,dx$ exists when $b > a$. Then

$$\int_a^b f(x)\,dx = 0, \text{ if } a = b$$

$$= -\int_b^a f(x)\,dx, \text{ if } a > b.$$

Now have the following property.

PROPERTY III: If a function f is integrable in [a,b]and $|f(x)| \leq k \ \forall x \in [a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \leq k\,|b-a|.$$

PROOF: There are only three possibilities namely either $a < b$ or $a > b$ or $a = b$. We discuss the cases as follows:

Case (i) $a < b$:

Since $|f(x)| \leq k \quad \forall x \in [a,b]$, therefore

$-k \leq f(x) \leq k \quad \forall x \in [a,b]$

$$\Rightarrow \int_a^b -k\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b k\,dx \text{ (why?)}$$

$$\Rightarrow -k(b-a) \leq \int_a^b f(x)\,dx \leq k(b-a)$$

$$\therefore \left|\int_a^b f(x)dx\right| \leq k(b-a) = k\,|b-a|$$

which completes the proof of the theorem.

Case (ii) $a > b$:

In this case, interchanging a and b in the Case (i), you will get

$$\left|\int_b^a f(x)\,dx\right| \leq k(a-b)$$

i.e. $\left|-\int_a^b f(x)\,dx\right| \leq k(a-b)$

i.e. $\left|\int_a^b f(x)\,dx\right| \leq k(a-b) = k\,|b-a|.$

Case (iii) $a = b$:

In this case also, the result holds,

since $\int_a^b f(x)\,dx = 0$ for $a = b$ and $k|b-a| = 0$ for $a = b$.

Let [a,b] be a fixed interval. Let R [a,b] denote the set of all Riemann integrable functions on this interval. We have shown in Unit 14 that if $f,g \subset R[a,b]$, then $f + g$ f.g and λf for $\lambda \in R$ belong to R [a,b]. Combining these with Property II, we can say that the set R [a,b] of Riemann integrable functions is closed under addition, multiplication, scalar multiplication and the formation of the absolute value.

If we consider the integral as a function Int: $R[a,b] \rightarrow R$ defined by

$$\text{Int}(f) = \int_a^b f(x)\,dx$$

with domain R [a,b] and range contained in R, then this function has the following properties:

$$\text{Int}(f+g) = \text{Int}(f) + \text{Int}(g),\ \text{Int}(\lambda f) = \lambda\, \text{Int}(f)$$

In other words, the function Int preserves 'Vector sums' and the scalar products. In the language of Linear Algebra, the function Int acts as a linear transformation. This function also has an additional interesting property such as

$$\text{Int}(f) \leq \text{Int}(g)$$

whenever

$$f \leq g.$$

We state yet another interesting property (without proof) which shows that the Riemann Integral is additive on an interval.

PROPERTY IV: If f is integrable on [a,b] and $c \in [a,b]$, then f is integrable on [a,c] and [c,b] and conversely. Further in either case

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

According to this property, if we split the interval over which we are integrating into two parts, the value of the integral over the whole will be the sum of the two integrals over the subintervals. This amounts to dividing the region whose area must be found into two separate parts while the total area is the sum of the areas of the separate portions.

We now state a few more properties of the definite integral $\int_a^b f(x)\,dx$ which you ought to have studied in the course on Calculus **(MTE-01)**.

(i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$

(ii) $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx. + \int_0^a f(2a-x)\,dx.$

(iii) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx \text{ if f is an even function} \\ 0 \text{ if f is an odd function.} \end{cases}$

(iv) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if f is periodic with period 'a' and n is a positive integer provided the integrals exist.

15.3 FUNDAMENTAL THEOREM OF CALCULUS

In section 15.1, we raised a question, "What is the relationship between the two notions of differentiation and integration? Now we shall try to find an answer to this question. In fact, we shall show that differentiation and Integration are intimately related in the sense that they are inverse operations of each other.

Let us begin by asking ourselves the following question: **"When is a function f: [a,b] –> R, the derivative of some function F: [a,b] –> R?"**
For example consider the function f: [−1,1] –> R defined by

$$f(x) = \begin{cases} 0 \text{ if } -1 \leq x < 0 \\ 1 \text{ if } 0 \leq x < 1 \end{cases}$$

This function is not the derivative of any function F: [−1, 1] –> R. Indeed if f is the derivative of a function F: [−1, 1] –> R then (Refer to Unit 12 for the intermediate value property of derivatives) f must have the intermediate value property. But clearly, the function f given above does not have the intermediate value property. Hence f cannot be the derivative of any function F: [−1, 1] –> R.

However if f: [−1,1] –> R is continuous, then f is the derivative of a function F: [−1, 1] –> R. This leads us to the following general theorem:

THEOREM 1: Let f be integrable on [a,b]. Define a function F on [a,b] as

$F(x) = \int_a^x f(t)\,dt, \forall\, x \in [a,b].$

Then F is continuous on [a,b]. Furthermore, if f is continuous at a point x_0 of [a,b], then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

PROOF: Since f is integrable on [a,b] it is bounded. In other words, there exists a positive number M such that

$$|f(x)| \leq M \ \forall x \in [a,b].$$

Let $\epsilon > 0$ be any number.
Choose $x,y \in [a,b]$ where $x < y$ such that

$$|x-y| < \frac{\epsilon}{M}.$$

Then

$$\begin{aligned} |F(y) - F(x)| &= \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_a^x f(t)\,dt + \int_x^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| \\ &\leq \int_x^y |f(t)|\,dt \\ &\leq \int_x^y M\,dt = M(y-x) < \epsilon \end{aligned}$$

Similarly you can discuss the case when $y < x$. This shows that F is continuous on [a,b]. In fact this proves the uniform continuity of F.
Now, suppose f is continuous at a point x_0 of [a,b]

We can choose some suitable $h \neq 0$ such that $x_0 + h \in [a,b]$.
Then

$$F(x_0+h) - F(x_0) = \int_a^{x_0+h} f(t)\,dt - \int_a^{x_0} f(t)\,dt$$

$$= \int_a^{x_0} f(t)\,dt + \int_{x_0}^{x_0+h} f(t)\,d(t) - \int_a^{x_0} f(t)\,dt = \int_{x_0}^{x_0+h} f(t)\,dt$$

Thus

$$F(x_0+h) - F(x_0) = \int_{x_0}^{x_0+h} f(t)\,dt \ \ldots\ldots\ldots.. (1)$$

Now

$$\left|\frac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| = \left|\frac{1}{h}\int_{x_0}^{x_0+h} f(t)\,dt - \frac{1}{h}\int_{x_0}^{x_0+h} f(x_0)\,dt\right|$$

$$= \frac{1}{|h|}\int_{x_0}^{x_0+h} [f(t) - f(x_0)]\,dt.$$

Since f is continuous at x_0, given a number $\epsilon > 0$. $\exists$ a number $\delta > 0$ such that

$|f(x) - f(x_0)| < \in/2$ whenever $|x-x_0) < \delta$, $x \in [a,b]$.
So if $|h| < \delta$, then $|f(t) - f(x_0)| < \epsilon/2$ for $t \in [x_0, x_0 + h]$ and consequently

$$\left|\int_{x_0}^{x_0+h} [f(t) - f(x_0)]\,dt\right| \leq \in 2\,|h|. \text{ Therefore}$$

$$\left|\frac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| \leq \in 2 < \text{if } |h| < \delta.$$

$$\therefore \lim_{h \to 0} \frac{F(x_0+h) - F(x_0)}{h} = f(x_0)$$

i.e. $F'(x_0) = f(x_0)$.

Which shows that F is differentiable at x_0 and $F'(x_0) = f(x_0)$ From Theorem 1, you can easily deduce the following theorem:

THEOREM 2: Let f: [a,b] → R be a continuous function. Let F: [a,b] → R be a function defined by

$$F(x) = \int_a^x f(t)\, dt, \; x \in [a,b].$$

Then $F'(x) = f(x),\ a \le x \le b$.

This is the first result which links the concepts of integral and derivative. It says that, if f is continuous on [a,b] then there is a function F on [a,b] such that $F'(x) = f(x),\ \forall x \in [a,b]$.

You have seen that if f: [a,b] —> R is continuous, then there is a function F: [a,b] → R such that $F'(x) = f(x)$ on [a,b]. Is such a function F unique? Clearly the answer is 'no'. For, if you add a constant to the function F, the derivative is not altered. In other words, if $G(x) = c + \int_a^x f(t)\, dt$ for $a \le x \le b$ then also $G'(x) = f(x)$ on [a,b].

Such a function F or G is called primitive of f. We have the formal definition as follows:


Isaac Barrow

DEFINITION 2: PRIMITIVE OF A FUNCTION
If f and F are functions defined on [a,b] such that $F'(x) = f(x)$ for $x \in [a,b]$ then F is called a 'primitive' or an 'antiderivative' of f on [a,b].

Thus from Theorem 1, you can see that every continuous function on [a,b] has a primitive. Also there are infinitely many primitives, in the sense that adding a constant to a primitive gives another primitive.

"Is it true that any two primitives differ by a constant?"

The answer to this question is yes. Indeed if F and G are two primitives of f in [a,b], then $F'(x) = G'(x) = f(x)\ \forall x \in [a,b]$ and therefore $[F(x) - G(x)]' = 0$. Thus $F(x) - G(x) = k$ (constant), for $x \in [a,b]$.

Let us consider an example.

EXAMPLE 1 What is the primitive of $f(x) = \log x$ in [1,2]

SOLUTION: Since $\frac{d}{dx}(x \log x - x) = \log x\ \forall x \in [1,2]$, therefore $F(x) = x \log x - x$ is a primitive of f in [1,2].
Also $G(x) = x \log x - x + k$, k being a constant, is a primitive of f.
Try the following exercise yourself.

EXERCISE 1
Find the primitive of the function f defined in [0,2] by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1[\\ 1 & \text{if } x \in [1,2] \end{cases}$$

According to this theorem, differentiation and integration are inverse operations.

We now discuss a theorem which establishes the required relationship between differentiation and integration. This is called the **Fundamental Theroem of Calculus.**

It states that the integral of the derivative of a function is given by the function itself.

The Fundamental Theorem of Calculus was given by an English mathematician Isaac Barrow [1630-1677], the teacher of great Isaac Newton.

THEOREM 3: (FUNDAMENTAL THEOREM OF CALCULUS)

If f is integrable on [a,b] and F is a primitive of f on [a,b], then $\int_a^b f(x)\, dx = F(b) - F(a)$.

PROOF: Since $f \in R[a,b]$, therefore $\lim_{|P| \to 0} S(P,f) = \int_a^b f(x)\,dx$

where $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a,b]$. The Riemann sum $S(P,f)$ is given by

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1});\ x_{i-1} \le t_i \le x_i.$$

Since F is the primitive of f on $[a,b]$, therefore $F'(x) = f(x),\ x \in [a,b]$.

Hence $S(P,f) = \sum_{i=1}^{n} F'(t_i)(x_i - x_{i-1})$. We choose the points t_i as follows:

By Lagrange's Mean Value theorem of Differentiability (Unit 12), there is a point t_i in $]x_{i-1}, x_i[$ such that

$$F(x_i) - F(x_{i-1}) = F'(t_i)(x_i - x_{i-1})$$

Therefore $S(P,f) = \sum_{i=1}^{n} [F(x_i) - F(x_i - 1)] = F(x_n) - F(x_0) = F(b) - F(a)$.

Take the limit as $|P| \to 0$. Then $\int_a^b f(x)\,dx = F(b) - F(a)$. This completes the proof.

Alternatively, the Fundamental Theorem of Calculus is also interpreted by stating that the derivative of the integral of a continuous function is the function itself.

If the derivative f' of a function f is integrable on $[a,b]$,

then $\int_a^b b(x)\,dx = f(b) - f(a)$.

Applying this theorem, we can find the integral of various functions very easily. Consider the following example:

EXAMPLE 2: Show that $\int_o^t \sin x\,dx = 1 - \cos t$.

SOLUTION: Since $g(x) = -\cos x$ is the primitive of $f(x) = \sin x$ in the interval $[0,t]$, therefore

$\int_o^t \sin x\,dx = g(t) - g(o) = 1 - \cos t$.

Try the following exercises.

EXERCISE 2

Find $\int_o^2 f(x)\,dx$ where f is the function given in Exercise 1.

EXERCISE 3

Evaluate $\int_a^b x^n\,dx$ where n is a positive integer.

We have, thus, reduced the problem of evaluating $\int_a^b f(x)\,dx$ to that of finding primitive of f on $[a,b]$. Once the primitive is known, the value of $\int_a^b f(x)\,dx$ is easily given by the Fundamental Theorem of Calculus.

You may note that any suitable primitive will serve the purpose because when the primitive is known, then the process described by the Fundamental Theorem is much simpler than other methods. However, it is just possible that the primitive may not exist while you may keep on trying to find it. It is, therefore, essential to formulate some conditions which can ensure the existence of a primitive. Thus now the next step is to find the conditions on the integrand (function to be integrated) which will ensure the existence of a primitive. One such condition is provided by the theorem 2.

According to theorem 2 if f is continuous in $[a,b]$, then the function F given by

$F(x) = \int_a^x f(t)\,dt$, $x \in [a,b]$ is differentiable in [a,b] and $F'(x) = f(x)\ \forall x \in [a,b]$

i.e. F is the primitive of f in [a,b]

The following observations are obvious from the theorems 1 and 2.

(i) If f is integrable on [a,b], then there is a function F which is associated with f through the process of integration and the domain of F is the same as the interval [a,b] over which f is integrated.

(ii) F is continuous. In other words, **the process of integration generates continuous function.**

(iii) If the function f is continuous on [a,b] then F is differentiable on [a,b]. Thus, **the process of integration generates differentiable functions.**

(iv) At any point of continuity of f, we will have $f'(c) = f(c)$ for $c \in [a,b[$.
This means that **if f is continuous on the whole of [a,b], then F will be a member of the family of primitives of f on [a,b].**

In the case of continuous functions, this leads us to the notion

$$\int f(x)\,dx$$

for the family of primitives of f. Such an integral, as you know, is called an Indefinite integral of f. It does not simply denote one function, but it denotes a family of functions. Thus, **a member of the indefinite integral of f will always be an anti-derivative for f.**

Theorem 3 gives us a condition on the function to be integrated which ensures the existence of a primitive. But how to obtain the primitives once this condition is satisfied. In the next section, we look for the two most important techniques for finding the primitives. Before we do so, we need to study two important mean-values theorems of integrability.

15.4 MEAN VALUE THEOREMS

In Unit 12, we discussed some mean-value theorems concerning the differentiability of a function. Quite analogous, we have two mean value theorems of integrability which we intend to discuss in this section. You are quite familiar with the two well-known techniques of integration namely the **integration by parts** and **integration by substitution** which you must have studied in your study of Calculus. How these methods were devised? We shall discuss this question also in this section.

THEOREM 3 : FIRST MEAN VALUE THEOREM
Let f: [a,b] → R be a continuous function. Then there exists c ∈ [a,b] such that

$$\int_a^b f(x)\,dx = (b-a)\,f(c).$$

PROOF: Let $m = \text{g.l.b.}\{f(x): x \in [a,b]\}$ and
$M = \text{l.u.b}\{f(x): x \in [a,b]\}$.
If P is any partition of [a,b], then

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

Also we know that

$$L(P,f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P,f).$$

Since f, being a continuous function, is integrable in [a,b],

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$$

and hence

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Thus there is a number $\mu \in [m,M]$ such that $\int_a^b f(x)\,dx = \mu\,(b-a)$

Since f is continuous in [a,b], it attains its bounds and it also attains every value between the bounds. Consequently, there is a point $c \in [a,b]$ such that $f(c) = \mu$. Hence

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

or equivalently can be written as

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

This theorem is usually referred to as the **Mean Value theorem for integrals.** The Geometrical interpretation of the theorem is that for a non-negative continuous function f, the area between f, the lines $x = a$, $x = b$ and the X-Axis can be taken as the area of a rectangle having one side of length $(b-a)$ and the other $f(c)$ for some **$c \in [a,b]$ as shown in the figure 1.**

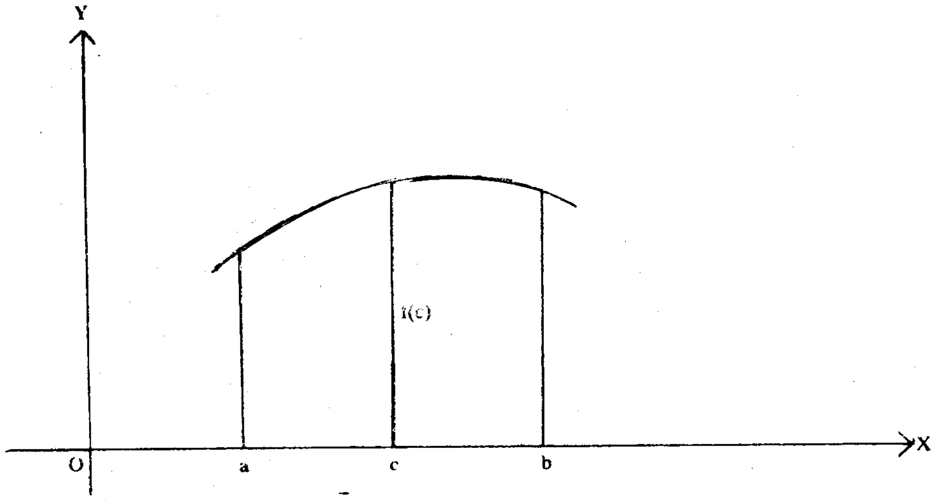


Fig. 1

We now discuss the generalised form of the first mean value theorem.

THEOREM 4: THE GENERALISED FIRST MEAN VALUE THEOREM

Let f and g be any two functions integrable in [a,b]. Suppose g(x) keeps the same sign for all $x \in [a,b]$. Then there exists a number μ lying between the bounds of f such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

PROOF: Let us assume that g(x) is positive over [a,b]. Since f and g are both integrable in [a,b], therefore both are bounded. Suppose that m and M are the g.l.b. and l.u.b. of f in [a,b].
Then

$$m \leq f(x) \leq M \;\forall\; x \in [a,b].$$

Consequently

$$mg(x) \leq f(x)\,g(x) \leq Mg(x) \;\forall\; x \in [a,b].$$

Therefore

$$m \int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M \int_a^b g(x)\,dx$$

It follows that there is a number $\mu \in [m,M]$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.$$

Now suppose further that f is continuous on [a,b]. Then there exists a point $c \in [a,b]$ such that

$$\int_a^b f(x)\,g(x)\,dx = f(c) \int_a^b f(x)\,dx$$

We use the first Fundamental Theorem of Calculus for integration by parts. We discuss it in the form of the following theorem:

THEOREM 5: If f and g are differentiable functions on [a,b] such that the derivatives f′ and g′ are both integrable on [a,b], then

$$\int_a^b fg' = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'g$$

PROOF: Since f and g are given to be differentiable on [a,b], therefore both f and g are continuous on [a,b]. Consequently both f and g are Riemann integrable on [a,b]. Hence both fg′ as well as f′ g are integrable.

$$fg' + f'g = (fg)'.$$

Therefore (fg)′ is also integrable and consequently, we have

$$\int_a^b (fg)' = \int_a^b fg' + \int_a^b f'g.$$

By Fundamental Theorem of Calculus, we can write

$$\int_a^b (fg)' = |fg|_a^b = f(b)\,g(b) - f(a)\,g(a)$$

Hence, we have

$$\int_a^b fg' = f(b)\,g(b) - f(a)\,g(a) - \int_a^b f'g.$$

i.e

$$\int_a^b f(x)\,g'(x)\,dx = [\,f(x)\,g(x)\,]_a^b - \int_a^b f'(x)\,g(x)\,dx.$$

This theorem is a formula for writing the integral of the product of two functions. What we need to know is that the primitive of one of the two functions should be expressible in a simple form and that the derivative of the other should also be simple so that the product of these two is easily integrable. You may note here that the source of the theorem is the well-known product rule for differentiation.

The Fundamental Theorem of Calculus gives yet another useful technique of integration. This is known as method by **Substitution** also named as the **change of variable** method. In fact this is the reverse of the well-known **chain** Rule for differentiation. In this method, we use the law of composition of functions which you have studied in Unit 1. In other words, we compose the given function f with another function g so that the composite $f_o g$ admits an easy integral. We deduce this method in the form of the following theorem:

THEOREM 6: Let f be a function defined and continuous on the range of a function g. If g′ is integrable on [c,d], then

$$\int_a^b f(x)\,dx = \int_c^d (f_o g)\,(x)\,g'(x)\,dx$$

Where a = g(c) and b = g(d).

PROOF: Let $F(x) = \int_a^b f(t)\,dt$ be a primitive of the function f.

Note that the function F is defined on the range of g.

Since f is continuous, therefore by theorem 2 it follows that F is differentiable and F′(t) = f(t) for any t. Denote $G(x) = (F_o g)\,(x)$.

Then clearly G is defined on [c,d] and it is differentiable because both F and g are differentiable. By the Chain Rule for differentiation, it follows that

$$G'(x) = (F_o g)\;(x)\,g'(x) = (f_o g)\;(x)\,g'(x).$$

Also $f_o g$ is continuous since both f and g are continuous. Therefore fog is integrable. Since g′ is integrable, therefore $(f_o g)\;g'$ is also integrable. Hence

$$\int_c^d (f_o g)\;(x)\,g'(x)\,dx = \int_c^d G'(x)\,dx$$

$$= G(d) - G(c) \qquad \text{(Why?)}$$
$$= F(g(d)) - F(g(c))$$
$$= F(b) - F(a)$$
$$= \int_a^b f(x)\,dx.$$

You have seen that the proof of the theorem is based on the Chain Rule for differentiation. In fact, this theorem is sometimes treated as a **Chain Rule for Integration** except that it is used exactly the opposite way from the Chain Rule for differentiation. The Chain Rule for differentiation tells us how to differentiate a composite function while the Chain Rule for Integration or the change of variable method tells us how to simplify an integral by rewriting it as a composite function. Thus, we are using the equalities in the opposite directions.

We conclude this section by a theorem (without Proof) known as the Second Mean Value Theorem for Integrals. Only the outlines of the proof are given.

THEOREM 7: SECOND MEAN VALUE THEOREM

Let f and g be any two functions integrable in [a,b] and g be monotonic in [a,b] then here exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,g(x)\,dx = g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx$$

PROOF: The proof is based on the following result (without proof) known as Bonnet's Mean Value Theorem, given by a French methematician 0. Bonnet [1819–1892].

Let f and g be integrable functions in [a,b]. If g is any monotonically decreasing function and positive in [a,b], then there exists a point $c \in [a,b]$ such that

$$\int_a^b f(x)\,\phi(x)\,dx = \phi(a)\int_a^c g(x)\,dx$$

Let g be monotonically decreasing so that ϕ where $\phi(x) = g(x) - g(b)$, is non-negative and monotonically decreasing in [a,b]. Then there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)\,[g(x) - g(b)]\,dx = [g(a) - g(b)]\int_a^c f(x)\,dx$$

i.e.

$$\int_a^b f(x)\,g(x)\,dx = g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx.$$

Now let g be monotonically increasing so that $-g$ is monotonically decreasing. Then there exists a number $c \in [a,b]$ such that

$$\int_a^b f(x)\,[-g(x)]\,dx = -g(a)\int_a^c f(x)\,dx - g(b)\int_c^b f(x)\,dx$$

i.e.

$$\int_a^b f(x)\,g(x)\,dx = g(a)\int_a^c f(x)\,dx + g(b)\int_c^b f(x)\,dx.$$

This completes the proof of the theorem.

There are several applications of the Second Mean Value Theorem. It is sometimes used to develop the trigonometric functions and their inverses which you may find in higher Mathematics. Here, we consider a few examples concerning the verification and application of the two Mean Value theorems.

EXAMPLE 3: Verify the two Mean Value Theorems for the functions

$$f(x) = x,\ g(x) = e^x \text{ in the interval } [-1, 1].$$

SOLUTION : VERIFICATION OF FIRST MEAN VALUE THEOREM:

Since f and g are continuous in $[-1, 1]$, so they are integrable in $[-1,1]$. Also g(x) is positive in $[-1, 1]$. By first Mean Value Theorem, there is a number μ between the bounds of f such that

$$\int_{-1}^{1} f(x)\, g(x)\, dx = \mu \int_{-1}^{1} g(x)\, dx \text{ i.e. } \int_{-1}^{1} x\, e^x\, dx = \mu \int_{-1}^{1} e^x\, dx.$$

$$\int_{-1}^{1} x\, e^x\, dx = |x\, e^x|_{-1}^{1} - \int_{-1}^{1} e^x\, dx = \frac{2}{e} \text{ and } \int_{-1}^{1} e^x\, dx = e\, 2 - \frac{1}{e}.$$

$$\therefore \frac{2}{e} = \mu\left(e - \frac{1}{e}\right) \text{ i.e. } \mu = \frac{2}{e^2-1} \sim \frac{2}{(2.7)^2-1} = \frac{2}{6.29}$$

g.l.b.f $= -1$ & l.u.b.f $= 1$ and so $\mu \in [-1,1]$. First Mean Value Theorem is verified.

VERIFICATION OF SECOND MEAN VALUE THEOREM:

As shown above, f and g are integrable in $[-1,1]$. Also g is monotonically increasing in $[-1,1]$. By second mean value theorem there is a points $c \in [-1,1]$ such that

$$\int_{-1}^{1} f(x)\, g(x)\, dx = g(-1) \int_{-1}^{c} f(x)\, dx + g(1) \int_{c}^{1} f(x)\, dx$$

$$\Rightarrow \quad \int_{-1}^{1} x\, e^x\, dx = \frac{1}{e} \int_{-1}^{c} x\, dx + e \int_{c}^{1} x\, dx$$

$$\Rightarrow \quad \frac{2}{e} = \frac{1}{e}\left(c^2/2 - 1/2\right) + e\left(1/2 - c^2/2\right)$$

Therefore $c^2 = \dfrac{e^2-5}{e^2-1} = \dfrac{2.29}{6.29}$ i.e. $c = \pm\sqrt{\dfrac{2.29}{6.29}} \in [-1, 1]$

Thus second mean value theorem is verified.

EXERCISE 4

Show that the second mean value theorem does not hold good in the interval $[-1,1]$ for $f(x) = g(x) = x^2$.

What do you say about the validity of the first mean value theorem.

Now we show the use of mean value theorems to prove some inequalities.

EXAMPLE 4: By applying the first mean value theorem of Integral calculus, prove that

$$\pi/6 \leq \int_0^{1/2} \frac{1}{\sqrt{[(1-x^2)(1-k^2x^2)]}}\, dx \leq \frac{\pi}{6} \frac{1}{\sqrt{(1-\frac{1}{4}k^2)}}$$

SOLUTION: In the first mean value theorem, take $f(x) = \dfrac{1}{\sqrt{(1-k^2x^2)}}$

$g(x) = \dfrac{1}{\sqrt{1-x^2}}$, $x \in [0, \frac{1}{2}]$. Being continuous functions, f and g are integrable in $[0, \frac{1}{2}]$.

By the first mean value theorem, there is a number. $\mu \in [m,M]$ such that

$$\int_0^{1/2} \frac{1}{\sqrt{[(1-x^2)(1-k^2x^2)]}}\, dx = \mu \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \mu\, \pi/6$$

where m = g.l.b. f and M = l.u.b. f. Now $m = 1$ & $M = \dfrac{1}{\sqrt{1-1/4\, k^2}}$

$$\therefore 1 \leq \mu \leq \frac{1}{\sqrt{1-1/4\, k^2}} \text{ i.e. } \frac{\pi}{6} \leq \mu\, \pi/6 \leq \frac{\pi}{6} \frac{1}{\sqrt{1-1/4\, k^2}}$$

and so $\dfrac{\pi}{6} \displaystyle\int_0^{1/2} \frac{1}{\sqrt{[(1-x^2)(1-k^2x^2)]}}\, dx \leq \frac{\pi}{6} \frac{1}{\sqrt{1-1\ 4\, k^2}}$

EXAMPLE 5: Prove that $\left| \int_p^q \frac{\sin x}{x} dx \right| \leq \frac{2}{p}$ if $q > p > 0$

SOLUTION: Let $f(x) = \sin x$, $\phi(x) = \frac{1}{x}$, $x \subset [p,q]$. Being continuous, these functions are integrable in $[p,q]$. By Bonnet form of **second mean value theorem**, there is a point $\zeta \in [p,q]$ such that

$$\int_p^q f(x)\, \phi(x)\, dx = \phi(p) \int_a^\zeta f(x)\, dx$$

$$\text{i.e} \int_p^q \frac{\sin x}{x} dx = \frac{1}{p} \int_p^q \sin x\, dx = \frac{1}{p} (\cos p - \cos \zeta)$$

$$\text{Hence } \left| \int_p^q \frac{\sin x}{x} dx \right| \leq \frac{1}{p} [|\cos p| + |\cos \zeta|] \leq \frac{2}{p}$$

EXERCISE 5

Show that $\left| \int_a^b \sin x^2\, dx \right| \leq \frac{1}{a}$ if $b > a > 0$.

EXERCISE 6 Prove the Bonnet's Mean-Value Theorem.

15.5 SUMMARY

The main thrust of this unit has been to establish the relationship between differentiation and integration with the help of the Fundamental Theorem of Calculus.

In section 15.2. we have discussed some important properties of the Riemann Integral. We have shown that the inequality between any two functions is preserved by their corresponding Riemann integrals; the modulus of the limit of a sum never exceeds the limit of the sum of their modulie and if we split the interval over which we are integrating a function into two parts, then the value of the integral over the whole will be the sum of the two integrals over the subintervals.

In section 15.3, primitive of a function has been defined. It has been proved that a continuous function has a primitive. Using the idea of a primitive, Fundamental Theorem of Calculus has been proved which shows that differentiation and integration are inverse process.

In section 15.4, indefinite integral also called the integral function of an integrable function is defined and you have seen that this function is continuous. This function is differentiable whenever the integrable function is continuous. Finally,in this section the First and Second Mean Value theorem have been discussed. The First Mean Value theorem states that if f is a continuous function in [a,b], then the value of the integral $\int_a^b f(x)\, dx$ is (b−a) times f(c) where $c \in [a,b]$.According to Generalised First Mean Value Theorem, if f and g are integrable in [a,b] and g(x) keeps the same sign, then the value of $\int_a^b f(x)\, g(x)\, dx$ is $\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$ where μ lies between the bounds of f. But in the second mean value theorem, if out of the integrable functions f and g, g is monotonic in [a,b], then the value of $\int_a^b f(x)\, g(x)\, dx$ is $g(a) \int_a^c f(x)\, dx + g(b) \int_c^b f(x)\, dx$ where c is point of [a,b].

15.6 ANSWERS/HINTS/SOLUTIONS

E 1) If we consider the function F defined in [0,2] by

$$f(x) = \begin{cases} \dfrac{x^2}{2} \text{ if } x \in [0,1[\\ x - \dfrac{1}{2} \text{ if } x \in [1,2] \end{cases}$$

then $F'(x) = f(x)\ \forall\, x \in [0,2]$. It is obviously true for all $x \neq 1$. At $x = 1$, calculate $R f'(1)$ & $L f'(1)$ and show that both are equal to 1.

E 2) By Fundamental Theorem of Calculus, $\int_0^2 f(x)\,dx = F(2) - F(0)$ where F is given in E (1). Put the values of F (2) & F(0) and you get the values of the integral.

E 3) Since $\dfrac{d}{dx}\left[\dfrac{n^{n+1}}{n+1}\right] = x^n$ and therefore $\dfrac{x^{n+1}}{n+1}$ is a primitive of f in [a,b] and

by the Fundamental Theorem of Calculus,

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$$

E 4) Functions f and g being continuous are integrable in [−1, 1]. The function g is not monotonic in [−1, 1], since in [−1,0], it is monotonically decreasing and in [0,1] it is monotonically increasing. So second mean value theorem does not hold good.

Now the function g is +ve in [−1, 1]. So first mean theorem holds good and by the theorem, there is a number $\mu \in [m,M]$ such that $\int_{-1}^{1} f(x)\,g(x) = \mu \int_{-1}^{1} g(x)\,dx$ where m = g.l.b. f & M = l.u.b. f i.e. $\int_{-1}^{1} x^4\,dx = \mu \int_{-1}^{1} x^2\,dx$ and so $\mu = \dfrac{3}{5}$.

Now m = 0, M = 1. So $\mu \in [0,1]$ and first mean value theorem is verified.

E 5) $\sin x^2 = \dfrac{1}{2x}(2x \sin x^2)$ for $x \neq 0$. Take $f(x) = 2x \sin x^2$ and $g(x) = \dfrac{1}{2x}$ and apply Bonnet Form of Second Mean Value Theorem as in Example 6.

E 6) To prove it, consider any partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b]. Let M_i and m_i be the bounds of f in $[x_{i-1}, x_i]$. Let $t_1 = a$ and t_i ($i \neq 1$) be any point of $[x_{i-1}, x_i]$

we have $m_i\,\Delta x_i \leq \int_{x_{i-1}}^{x_i} f(x)\,dx \leq M_i\,\Delta x_i$ and

$m_i\,\Delta x_i \leq f(t_i)\,\Delta x_i \leq M_i\,\Delta x_i$

Putting i = 1, 2 ...... p where $p \leq n$ and adding we get

$$\sum_{i=1}^{p} m_i\,\Delta x_i \leq \int_a^{x_p} f(x)\,dx \leq \sum_{i=1}^{p} M_i\,\Delta x_i.$$

$$\sum_{i=1}^{p} m_i\,\Delta x_i \leq \sum_{i=1}^{p} f(t_i)\,\Delta x_i \leq \sum_{i=1}^{p} (M_i - m_i)\,\Delta x_i.$$

Thus $\left|\int_a^{x_p} f(x)\,dx - \sum_{i=1}^{p} f(t_i)\,\Delta x_i\right| \leq \sum_{i=1}^{p} (M_i - m_i)\,\Delta x_i \leq \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$

$$\therefore \int_a^{x_p} f(x)\,dx - \sum_{i=1}^{n} o_i\,\Delta x_i \leq \sum_{i=1}^{p} f(t_i)\,\Delta x_i \leq \int_a^{x_p} f(x)\,dx + \sum_{i=1}^{n} M_i\,\Delta x_i$$

where $o_i = M_i - m_i$.

Now the indefinite integral $\int_a^t f(x)\,dx$ is continuous and so it is bounded. Let C &

D be its g.l.b. and l.u.b.

Then $C - \sum_{i=1}^{n} o_i \Delta x_i \leq \sum_{i=1}^{n} f(t_i) \Delta x_i \leq D + \sum_{i=1}^{n} o_i \Delta x_i$.

Now we use Abel's Lemma which states that if (i) $\{a_i, \ldots\ldots a_n\}$ is a monotonically decreasing set of any numbers (iii) k, K are two numbers such that

$$k \leq v_i + v_2 + \ldots\ldots + v_p \leq K \text{ for } p = 1, 2, \ldots., n,$$

then

$$a_1 k \leq \sum_{i=1}^{n} a_i V_i \leq a_1 k.$$

In this lemma, take $k = C - \sum_{i=1}^{n} o_i \Delta x_i$, $K = D + \sum_{i=1}^{n} o_i \Delta x_i$

$$a_i = g(t_i),\ v_i = f(t_i) \Delta x_i.$$

$$g(a)\,[C - \sum_{i=1}^{n} o_i \Delta x_i] \leq \sum_{i=1}^{n} f(t_i)\, g(ti) \Delta x_i \leq g(a)\,[D + \sum_{i=1}^{n} o_i \Delta x_i]$$

Let $|P| \to 0$. Then, it follows that

$$C\, g(a) \leq \int_a^b f(x)\, g(x)\, dx \leq D\, g(a)$$

Thus $\int_a^b f(x)\, g(x)\, dx = \mu\, g(a)$ where $\mu \in [C,D]$.

Now $\int_a^t f(x)\, dx$ being continuous, there exists $c \in [a,b]$ such that

$\mu = \int_a^c f(x)\, dx$. Therefore

$\int_a^b f(x)\, g(x)\, dx = g(a) \int_a^c f(x)\, dx$ which proves the result.

UNIT 16 SEQUENCES AND SERIES OF FUNCTIONS

Structure

16.1 INTRODUCTION

In unit 5, you were introduced to the notion of sequences of real numbers and their convergence. In units 6 and 7, convergence of the infinite series of real numbers was considered. In this unit, we want to discuss sequences and series whose members are functions defined on a subset of the set of real numbers. Such sequences or series are known as sequences or series of real functions. You will be introduced to the concepts of pointwise and uniform convergence of sequences and series of functions. Whenever they are convergent, their limit is a function called limit function. The question arises whether the properties of continuity, differentiability, integrability of the members of a sequence or series of functions are preserved by the limit function. We shall discuss this question also in this unit and show that these properties are preserved by the Uniform convergence and not by the pointwise convergence.

Objectives

After the study of this unit, you should be able to

- define sequence and series of functions
- distinguish between the pointwise and uniform convergence of sequences and series of functions
- know the relationship of uniform convergence with the notions of continuity, differentiability, and integrability.

16.2 SEQUENCES OF FUNCTIONS

In unit 5, you have studied that a sequence is a function from the set N of natural numbers to a set B. In that unit, sequences of real numbers have been considered in detail. You may recall that for sequences of real numbers, the set B is a sub-set of real numbers. If the set B is the set of real functions defined on a sub-set A of R, we get a sequence called sequence of functions. We define it in the following way:

DEFINITION 1: SEQUENCE OF FUNCTIONS

Let A be a non-empty sub-set of R and let B be the set of all real functions each defined on A. A mapping from the set N of natural numbers to the set B of real functions is called a sequence of functions.

The sequences of functions are denoted by (f_n), (g_n) etc. It (f_n) is a sequence of functions defined on A, then its members $f_1, f_2, f_3, \ldots\ldots$ are real functions with domain as the set A. These are also called the terms of the sequence (f_n).
For example, let

$f_n(x) = x^n$, $n = 1, 2, 3 \ldots\ldots$, where $x \in A \{x: 0 \leq x \leq 1\}$. Then (f_n) is a sequence of functions defined on the closed interval [0,1].

Similarly consider (f_n) where $f_n(x) = \sin nx$, $n = 1, 2, 3, \ldots\ldots$, $x \in R$.Then $\{f_n\}$, is a sequence of functions defined on the set R of real numbers.

Suppose (f_n) is a sequence of functions defined on a set A and we fix a point x of A, then the sequence $(f_n(x))$, formed by the values of the members of (f_n), is a sequence of real numbers. This sequence of real numbers may be convergent or divergent. For example suppose that $f_n(x) = x^n$, $x \in [-1, 1]$. If we consider the point $x = \frac{1}{2}$, then the sequence $(f_n(x))$ is $((\frac{1}{2})^n)$ which converges to 0. If we take the point $x = 1$, the sequence $(f_n(x))$ is the constant sequence (1,1,1.....) which converges to 1. If $x = -1$, the sequence $(f_n(x))$ is $(-1, 1, -1, 1, ...)$ which is divergent.

Thus, you have seen that the sequence $(f_n(x))$ may or may not be convergent. If for a sequence (f_n) of functions defined on a set A, the sequence of numbers $(f_n(x))$ converges for each x in A, we get a function f with domain A whose value f(x) at any point x of A is $\lim_{n \to \infty} f_n(x)$. In this case (f_n) is said to be pointwise convergent to f. We define it in the following way:

DEFINITION 2: POINTWISE CONVERGENCE

A sequence of functions (f_n) defined on a set A is said to be convergent pointwise to f if for each x in A, we have $\lim_{n \to \infty} f_n(x) = f(x)$. Generally, we write $f_n \to f$ (pointwise) on A or $\lim_{n \to \infty} f_n(x) = f(x)$ pointwise on A. Also f is called pointwise limit or limit function of (f_n) on A.

Equivalently, we say that a sequence $\{f_n\}$ converges to f pointwise on the set A if for each $\epsilon > 0$ and each $x \in A$, there exists a positive integer depending both ϵ and x such that

$$|f_n(x) - f(x)| < \epsilon \text{ whenever } n \geq m.$$

Now we consider some examples.

EXAMPLE 1: Show that the sequence (f_n) where $f_n(x) = x^n$, $x \in [0,1]$ is pointwise convergent. Also find the limit.

SOLUTION: If $0 \leq x < 1$, then $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} x^n = 0$. (Recall unit 5).

$$\text{If } x = 1, \text{ then } \lim_{n \to \infty} f(x) = \lim_{n \to \infty} 1 = 1.$$

Thus (f_n) is pointwise convergent to the limit function f where $f(x) = 0$ for $0 \leq x < 1$ and $f(x) = 1$ for $x = 1$.

EXERCISE 1

Show that the sequence of functions (f_n) where $f_n(x) = x^n$, for $x \in [-1, 1]$ is not pointwise convergent.

EXAMPLE 2: Define the functions f_n, $n = 1, 2$ .... as follows

$$f_n(x) = \begin{cases} 0 \text{ if } x = 0 \\ 2n^2 x \text{ if } 0 < x < 1/2n \\ 2n - 2n^2 x \text{ if } 1/2n \leq x \leq \frac{1}{n} \\ 0 \text{ if } 1/n < x \leq 1 \end{cases}$$

Show that the sequence (f_n) is pointwise convergent.

SOLUTION: The graph of function f_n looks as shown in the figure 1.

When $x = 0$, $f_n(x) = 0$ for $n = 1,2, ...$ Therefore, the sequence $(f_n(0))$ tends to 0.

If x is fixed such that $0 < x \leq 1$; then choose m large enough so that $\frac{1}{m} < x$ or $m > \frac{1}{x}$. Then $f_m(x) = f_{m+1}(x) = ... = 0$. Consequently the sequence $(f_n(x)) \to 0$ as $n \to \infty$.

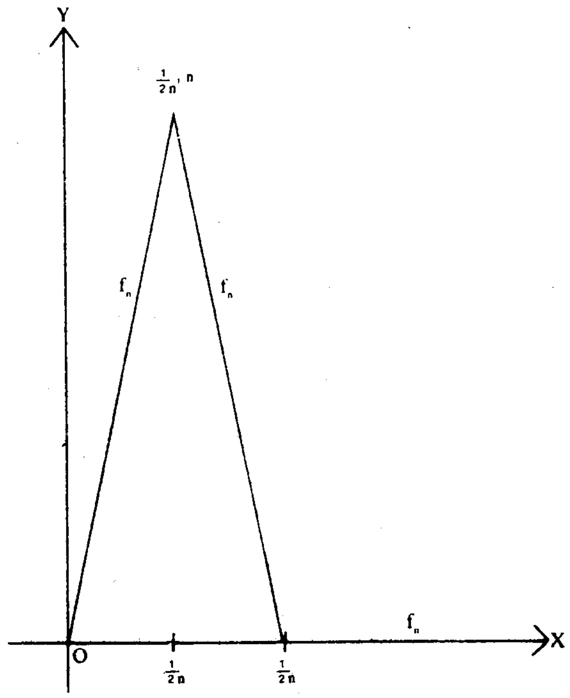


Fig. 1

Thus we see that $f_n(x)$ tends to 0 for every x in $0 \leq x \leq 1$ and consequently (f_n) tends pointwise to f where $f(x) = 0 \ \forall\, x \in [0,1]$.

EXAMPLE 3: Consider the sequence of functions f_n defined by $f_n(x) = \cos nx$ for $-\infty < x < \infty$ i.e. $x \in R$. Show that the sequence is not convergent pointwise for every real x.

SOLUTION: If $x = \pi/4$ then $(f_n(x))$ is the sequence

$(1/\sqrt{2}, 0, -1/\sqrt{2}, -1, -1/\sqrt{2}, 0, \ldots\ldots)$ which is not convergent.

You should be able to solve the following exercises.

EXERCISE 2

Show that the sequence (f_n) where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$, $x \in R$, is pointwise convergent. Also find the pointwise limit.

EXERCISE 3

Examine which of the following sequences of functions converge pointwise

(i) $f_n(x) = \sin nx, -\infty < x < +\infty$.

(ii) $f_n(x) = \dfrac{nx}{1+n^2x^2}, -\infty < x < +\infty$

If the sequence of functions (f_n) converges pointwise to a function f on a subset A of R, then the following question arises? **"If each member of (f_n) is continuous, differentiable or integrable, is the limit function f also continuous, differentiable or integrable?"**. The answer is no if the convergence is only pointwise. For instance in example 1, each of the functions f_n is continuous (in fact uniformly continuous) but the sequence of these functions converges to a limit function f(x)

$$f(x) = \begin{cases} 0, & \text{for } 0 \leq x < 1 \\ 1, & \text{for } x = 1 \end{cases}$$

which is not continuous. Thus, the pointwise convergence does not preserve the property of continuity. To ensure the passage of the properties of continuity, differentiability or integrability to the limit function, we need the notion of uniform convergence which we introduce in the next section.

16.3 UNIFORM CONVERGENCE

From the definition of the convergence of the sequence or real numbers, it follows that the sequences (f_n) of functions converges pointwise to the function f on A if and only if for each $x \in A$ and for every number $\epsilon > 0$, there exists a positive integer m such that

$$| f_n (x) - f(x)| < \epsilon \text{ whenever } n \geq m.$$

Clearly for a given sequence (f_n) of functions, this m will, in general, depend on the given ϵ and the point x under consideration. Therefore it is, sometimes, written as m (ϵ,x).. The following example illustrates this point.

EXAMPLE 5: Define $f_n (x) = \frac{x}{n}$ for $-\infty < x < \infty$.

For each fixed x the sequence $(f_n(x))$ clearly converges to zero. For a given $\epsilon > 0$, we must show the existence of an m, such that for all $n \geq m$,

$$| f_n(x) - f(x) | = | \frac{x}{n} - 0 | = \frac{|x|}{n} < \epsilon.$$

This can be achieved by choosing $m = [\frac{|x|}{\epsilon}] + 1$ where $[\frac{|x|}{\epsilon}]$ denotes the

integral part of $\frac{|x|}{\epsilon}$ (i.e. the integer m is next to $\frac{|x|}{\epsilon}$ in the real line). Clearly this choice of m depends both on ϵ and x.

For example let $\epsilon = \frac{1}{10^3}$. If $x = \frac{1}{10^3}$ then $\frac{|x|}{\epsilon} = 1$ and so m can be chosen

to be 2. If x = 1 then $\frac{|x|}{\epsilon} = 10^3$ and therefore m should be larger than 10^3.

If $x = 10^3$, then $\frac{|x|}{\epsilon} = 10^6$ and so m should be larger than 10^6. Note that it is

impossible to find an m that serve for all x. For, if it were, then $|x| m < \epsilon$ for all x. Consequently $|x|$ is smaller than (ϵm), which is not possible. Geometrically the f_n's can be described as shown in the figure 2:

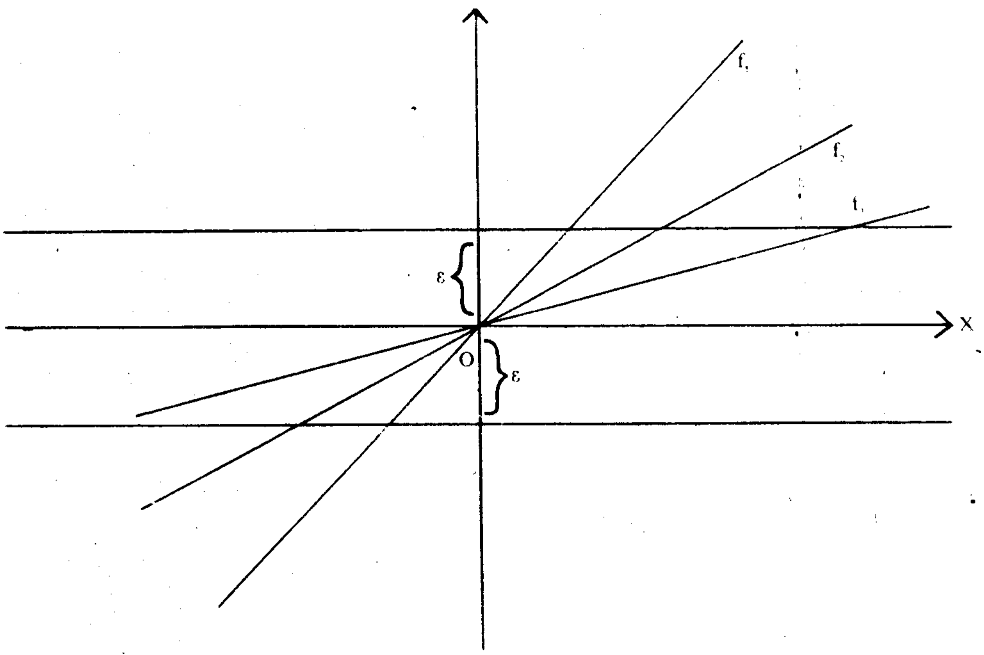


Fig. 2

By putting $y = f_n (x)$, we see that $y = \frac{1}{n} x$ is the line with slope $\frac{1}{n}$. f_1 is the line

$y = x$ with slope 1, f_2 is the line with slope $\frac{1}{2}$ and so on. As n tends to ∞, the lines approach the X-axis. But if we take any strip of breadth 2ϵ around X- axis, parallel to the X- axis as shown in the figure 2, it is impossible to find a stage m such that all the lines after the stage m, i.e. f_m, f_{m+1} ...... lie entirely in this strip.

If it is possible to find m which depends only on ϵ but is independent of the point x under consideration, we say that (f_n) is uniformly convergent to f. We define uniform convergence as follows:

DEFINITION 3: UNIFORM CONVERGENCE

A sequence of functions (f_n) defined on a set A is said to be uniformly convergent to a function f on A if given a number $\epsilon > 0$, there exists a positive integer m depending only on ϵ such that

We write it as $f_n \to f$ uniformly on A or $\lim f_n(x) = f(x)$ uniformly on A. Also f is called the uniform limit of f on A.

Note that if $f_n \to f$ uniformly on the set A, for a given $\epsilon > 0$, there exists m such that

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon$$

for all $x \in A$ and $n \geq m$. In other words, for $n \geq m$, the graph of f_n lies in the strip between the graphs of $f - \epsilon$ and $f + \epsilon$. As shown in the figure 3, the graphs of f_n for $n \geq m$ will all lie between the dotted lines.

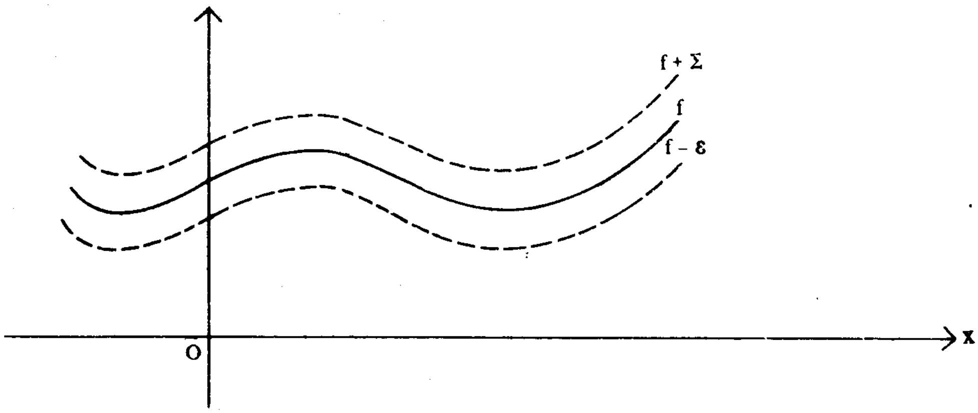


Fig. 3

From the definition of uniform convergence, it follows that uniform convergence of a sequence of functions implies its pointwise convergence and uniform limit is equal to the pointwise limit. We will show below by suitable examples that the converse is not true.

EXAMPLE 6: Show that the sequence (f_n) where $f_n(x) = \frac{x}{n}$, $x \in R$ is pointwise but not uniformly convergent in R.

SOLUTION: In example 5, you have seen that (f_n) is pointwise convergent to f where $f(x) = 0 \; \forall \; x \in R$. In the same example, at the end, it is remarked that given $\epsilon > 0$, it is not possible to find a positive integer m such that $\frac{|x|}{n} < \epsilon$ for $n \geq m$ & $\forall \; x \in R$ i.e. $|f_n(x) - f(x)| < \epsilon$ for $n \geq m$ & $\forall \; x \in R$. Consequently (f_n) is not uniformly convergent in R.

EXAMPLE 7: Show that the sequence (f_n) where $f_n(x) = x^n$ is convergent pointwise but not uniformly on [0,1].

SOLUTION: In example 2, you have been shown that (f_n) is pointwise convergent to f on [0,1] where

$f(x) = 0 \ \forall x \in [0,1[\ \& \ f(1) = 1$

Let $\epsilon > 0$ be any number. For $x = 0$ or $x = 1$, $|f_n(x) - f(x)| < \epsilon$ for $n \geq 1$.

For $0 < x < 1$, $|f_n(x) - f(x)| < \epsilon$ if $x^n < \epsilon$ i.e. $n \log x < \log \epsilon$ i.e $n > \dfrac{\log \epsilon}{\log x}$,

since log x is negative for $0 < x < 1$. If we choose $m = \left[\dfrac{\log \epsilon}{\log x}\right] + 1$,

then $|f_n(x) - f(x)| < \epsilon$ for $n \geq m$. Clearly m depends upon ϵ and x.

We will now prove that the convergence is not uniform by showing that it is not possible to find an m independent of x.

Let us suppose that $0 < \epsilon < 1$. If there exists m independent of x in [0,1] so that

$$|f_n(x) - f(x)| < \epsilon \text{ for all } n \geq m,$$

then $x^n < \epsilon$ for all $n \geq m$, whatever may be x in $0 < x < 1$.
If the same m serves for all x for a given $\epsilon > 0$ then $x^m < \epsilon$ for all x, $0 < x < 1$. This implies that $m > \dfrac{\log \epsilon}{\log x}$ (since log x is negative). This is not possible since log x decreases to zero as x tends to 1 and so $\log \epsilon / \log x$ is unbounded.

Thus we have shown that the sequence (f_n) does not converge to the function f uniformly in [0,1] even though it converges pointwise.

EXAMPLE 8: Show that the sequence (g_n) where $g_n(x) = \dfrac{x}{1 + nx}$, $x \in [0, \infty[$ is uniformly convergent in $[0, \infty[$.

SOLUTION: $\lim\limits_{n \to \infty} g_n(x) = 0$ for all x in the interval $[0, \infty[$. Thus (g_n) is pointwise convergent to f where $f(x) = 0 \ \forall x \in [0, \infty[$.

Now $|g_n(x) - f(x)| = \dfrac{x}{1 + nx} < \dfrac{1}{n}$ for all x in $[0, \infty[$.

Since $\lim\limits_{n \to \infty} \dfrac{1}{n} = 0$, therefore given $\epsilon > 0$, there exists a positive integer m such that

$\dfrac{1}{n} < \epsilon$ for $n \geq m$.

Thus m depends only on ϵ. Therefore,

$$|g_n(x) - f(x)| < \epsilon \text{ for } n \geq m \ \& \ \forall x \in [0, \infty[.$$

Therefore $(g_n) \to f$ uniformly in $[0, \infty[$.

EXERCISE 4

Test the uniform convergence of the following sequence of functions in the specified domains

(i) $f_n(x) = \dfrac{1}{nx}$ in $0 < x < \infty$

(ii) $f_n(x) = \dfrac{nx}{1 + n^2 x^2}$, $-\infty < x < \infty$

(iii) $f_n(x) = \dfrac{x^n}{1 + x^n}$, $0 \leq x \leq 1$

(iv) $f_n(x) = \dfrac{1}{n} e^{-nx}$, $0 \leq x < \infty$

Just as you have studied Cauchy's Criterion for convergence of sequence of real numbers, we have Cauchy's Criterion for uniform convergence of sequence of functions which we now state and prove.

THEOREM 1: CAUCHY'S PRINCIPLE OF UNIFORM CONVERGENCE

The necessary and sufficient condition for a sequence of functions (f_n) defined on A to converge uniformly on A is that for every $\epsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f_k(x)| < \epsilon \text{ for } n > k \geq m \text{ \& } \forall x \in A$$

PROOF: Condition is Necessary. It is given that (f_n) is uniformly convergent on A. Let $f_n \to f$ uniformly on A. Then given $\epsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \epsilon/2 \text{ for } n \geq m \text{ \& } \forall x \in A.$$

$$\therefore |f_n(x) - f_k(x)| = |f_n(x) - f(x)| + f(x) - f_k(x)|$$

$$\leq |f_n(x) - f(x)| + |f(x) - f_k(x)| \text{ (By triangular inequality)}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ for } n > k \geq m \text{ and } \forall x \in A = \epsilon$$

This proves the necessary part. Now we prove the sufficient part.

Condition is sufficient: It is given that for every $\epsilon > 0$, there exists a positive integer m such that $|f_n(x) - f_k(x)| < \epsilon$ for $n > k \geq m$ and for all x in A. But by Cauchy's principle of convergence of sequence of real numbers, for each fixed point x of A, the sequence of numbers $(f_n(x))$ converges. In other words, (f_n) is pointwise convergent say to f on A. Now for eacn $\epsilon > 0$, there exists a positive integer m such that

$$|f_n(x) - f(x)| < \frac{\epsilon}{2} \text{ for } n > k \geq m.$$

Fix h and let $n \to \infty$. Then $f_n(x) \to f(x)$ and we get

$$|f(x) - f_k(x)| \leq \frac{\epsilon}{2} \text{ i.e. } |f_k(x) - f(x)| < \epsilon.$$

This is true for $h \geq m$ and for all x in A. This shows that (f_n) is uniformly convergent to f on A which proves the sufficient part.

As remarked in the introduction, uniform convergence is the form of convergence of the sequence of function (f_n) which preserves the continuity, differentiability and integrability of each term f_n of the sequence when passing to the limit function f. In other words if each member of the sequence of functions (f_n) defined on a set A is continuous on A, then the limit function f is also continuous provided the convergence is uniform. The result may not be true if the convergence is only pointwise. Similar results hold for the differentiability and integrability of the limit function f. Before giving the theorems in which these results are proved, we discuss some examples to illustrate the results.

EXAMPLE 9: Discuss for continuity the convergence of a sequence of functions (f_n). where $f_n(x) = 1 - |1-x^2|^n$ $x \in \{x: |1-x^2| \leq 1\} = [-\sqrt{2}, \sqrt{2}]$.

SOLUTION: Here $\lim_{n \to \infty} f(x) = \begin{cases} 1 \text{ when } |1-x^2| < 1 \\ 0 \text{ when } |1-x^2| = 1 \text{ i.e } x = 0, \pm\sqrt{2}. \end{cases}$

Therefore the sequence (f_n) is pointwise convergent to f where

$$f(x) = \begin{cases} 1 \text{ when } |1-x^2| < 1 \\ 0 \text{ when } |1-x^2| = 1 \end{cases}$$

Now each member of the sequence (f_n) is continuous at 0 but f is discontinuous at 0. Here (f_n) is not uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$ as shown below.

Suppose (f_n) is uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$, so that f is its uniform limit.

Taking $\epsilon = \frac{1}{2}$, there exists an integer m such that

$$|f_n(x) - f(x)| < \frac{1}{2} \text{ for } n \geq m \text{ \& } \forall x \in [-\sqrt{2}, \sqrt{2}].$$

In particular $|f_m(x) - f(x)| < \frac{1}{2}$ for $x \in [-\sqrt{2}, \sqrt{2}]$.

Now $|f_m(x) - f(x)| = \begin{cases} |1-x^2|^m & \text{when } |1-x^2| < 1 \\ 0 & \text{when } |1-x^2| = 1 \end{cases}$

Since $\lim_{x \to 0} |1-x^2|^m = 1$, $\exists$ a + v no. δ such that

$$||1-x^2|^m - 1| < 1/4 \text{ for } 0 < |x| < \delta$$
$$\text{i.e. } 3/4 < |1-x^2|^m < 5/4 \text{ for } |x| < \delta.$$

So $|1-x^2|^m > \frac{1}{2}$ for $|x| < \delta$, which is a contradiction.

Consequently (f_n) is not uniformly convergent in $[-\sqrt{2}, \sqrt{2}]$.

EXAMPLE 10: Discuss, for continuity, the convergence of the sequence (f_n) where

$f_n(x) = \frac{x}{1 + nx}$ $x \in [0, \infty[$.

SOLUTION: In example 8, you have seen that $(f_n) \to f$ uniformly in $[0, \infty[$ where $f(x) = 0$, $x \in [0, \infty[$.

Here each f_n is continuous in $[0, \infty[$ and also the uniform limit is continuous in $[0, \infty[$.

EXAMPLE 11: Discuss for differentiability the sequence (f_n) where

$(f_n)(x) = \frac{\sin nx}{\sqrt{n}}$, $\forall x \in R$.

SOLUTION: Here $(f_n) \to f$ uniformly where $f(x) = 0 \; \forall x \in R$. You can see that each f_n and f are differentiable in R and

$$f_n'(x) = \sqrt{n}\cos nx \;\&\; f'(x) = 0 \;\forall x \in R.$$
$$f_n'(0) = \sqrt{n} \to \infty \text{ whereas } f'(0) = 0$$
$$\lim_{n \to \infty} f_n'(0) \neq f'(0)$$

i.e. limit of the derivatives is not equal to the derivative of the limit.

As you will see in the theorem for the differentiability of f and the equality of the limit of the derivatives and the derivative of the limit, we require the uniform convergence of the sequence (f_n').

EXAMPLE 12: Discuss for integrability the sequence (f_n) where

$$f_n(x) = nx\,e^{-nx^2},\; x \in [0,1].$$

SOLUTION: If $x = 0$, then $f_n(0) = 0$ and $\lim f_n(0) = 0$. If $x \neq 0$, $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{e^{nx^2}}$ which is of the form $\frac{\infty}{\infty}$.

Applying L Hopital's Rule, we have

$$\lim_{n \to \infty} \frac{nx}{e^{nx^2}} = \lim_{n \to \infty} \frac{x}{e^{nx^2} \cdot x^2} = 0.$$

So $(f_n) \to f$ pointwise where $f(x) = 0 \;\forall x \in [0, 1]$.

You may find that $\int_0^1 f_n(x)\,dx = \frac{1}{2}(1-e^{-n})$ & $\int_0^1 f(x)\,dx = 0$

Therefore $\lim_{n \to \infty} \int_0^1 f_n(x) = \frac{1}{2} \neq \int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 \lim_{n \to \infty} f_n(x)\,dx$

that is, the integral of the limit is not equal to the limit of the sequence of integrals. In fact, (f_n) is not uniformly convergent to f in [0,1]. This we prove by the contradiction method. If possible, let the sequence be uniformly convergent in [0,1]. Then

for $\epsilon = \frac{1}{4}$, there exists a positive integer m such that

$$|f_n(x) - f(x)|\; 1/4 \text{ for } n \geq m \;\&\; \forall x \in [0,1]$$

i.e. $\frac{nx}{e^{nx^2}} < \frac{1}{4}$ for $n \geq m$ & $\forall x \in [0,1]$.

Choose a + ve integer $M \geq m$ such that $\frac{1}{M} \in [0,1]$.

Take $n = M$ and $x = \frac{1}{\sqrt{M}}$ We get

$$\frac{\sqrt{M}}{e} < \frac{1}{4} \text{ i.e. } M < \frac{e^2}{16}$$

which is a contradiction. Hence (f_n) is not uniformly convergent in $[0,1]$.
Now try the following exercises.

EXERCISE 5
Show that the limit function of the sequence (f_n) where $(f_n)(x) = \frac{x}{n}$, $x \in R$, is continuous in R while (f_n) is not uniformly convergent.

EXERCISE 6.
Show that for the sequence (f_n) where $(f_n)(x) = n\,x\,(1-x^2)^n$, $x \in [0,1]$, the integral of the limit is not equal to the limit of the sequence of integrals.

Now we give the theorems without proof which relate uniform convergence with continuity, differentiability and integrability of the limit function of a sequence of functions.

THEOREM 2: (UNIFORM CONVERGENCE AND CONTINUITY)
If (f_n) be a sequence of continuous functions defined on $[a,b]$ and $(f_n) \rightarrow f$ uniformly on $[a,b]$, then f is continuous on $[a,b]$.

THEOREM 3: (UNIFORM CONVERGENCE AND DIFFERENTIATION)

Let (f_n) be a sequence of functions, each differentiable on $[a,b]$ such that $(f_n(x_o))$ converges for some point x_o of $[a,b]$. If (f_n') converges uniformly on $[a,b]$ then (f_n) converges uniformly on $[a,b]$ to a function f such that

$$f'(x) = \lim_{n \to \infty} f_n'(x);\ x \in [a,b].$$

THEOREM 4: (UNIFORM CONVERGENCE AND INTEGRATION)
If a sequence (f_n) converges uniformly to f on $[a,b]$ and each function f_n is integrable on $[a,b]$, then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \int_a^b f_n(x)\,dx$$

16.4 SERIES OF FUNCTIONS

Just as we have studied series of real numbers, we can study series formed by a sequence of functions defined on a given set A. The ideas of pointwise convergence and uniform convergence of sequence of functions can be extended to series of functions.

DEFINITION 4: (SERIES OF FUNCTIONS)

A series of the form $f_1 + f_2 + f_3 + \ldots\ldots + f_n + \ldots.$ where the f_n's are real functions defined on a given set $A \subset R$ is called a series of functions and is denoted by $\sum_{n=1}^{\infty} f_n$. The function f_n is called nth term of the series.

For each x in A, $f_1(x) + f_2(x) + f_3(x) + \ldots\ldots +$ is a series of real numbers. We put

$S_n(x) = \sum_{k=1}^{n} f_k(x)$. Then we get a sequence (S_n) of real functions defined on A. We

say that the given series $f_1 + f_1 + \ldots\ldots + f_n + \ldots..$ of functions converges to a function pointwise if the sequence (S_n) associated to the given series of functions converges pointwise to the function f. i.e. $(S_n(x))$ converges to $f(x)$ for every x in A.

We also say that f is the pointwise sum of the series Σf_n on A.
If the sequence (S_n) of functions converges uniformly to the function f, then we say that the given series $f_1 + f_2 + \ldots + f_n + \ldots$ of functions converges uniformly to the function f on A and f is called uniform sum of $\sum_{i=1}^{n} f_n$ on A. The function S_n is called the sum of n terms of the given series or the n^{th} partial sum of the series and the sequence (S_n) is called the sequence of partial sums of the series $\sum_{i=1}^{n} f_n$. To make the ideas clear, we consider some examples.

EXAMPLE 13: Let $f_n(x) = x^{n-1}$ where $x^0 = 1$ and $-r \leq x \leq r$ where $0 < r < 1$. Then the associated series is $1 + x + x^2 + \ldots$

In this case, $S_n(x) = 1 + x + x^2 + \ldots + x^{n-1}$. It is clear that $S_n(x) = \frac{1 - x^n}{1 - x}$

This sequence $(S_n(x))$ of functions is easily seen to converge pointwise to the function

$f(x) = \frac{1}{1-x}$, since $x^n \to 0$ as $n \to \infty$, since $|x| < r < 1$ but the convergence is not uniform as shown below:

Let $\epsilon > 0$ be given.

$$| S_n(x) - f(x) | = \frac{|x|^n}{|1-x|} \leq \frac{r^n}{1-r} < \epsilon \text{ if } r^n < \epsilon(1-r)$$

i.e. $\quad n > \dfrac{\log(\epsilon(1-r))}{\log r}$

If $m = \left[\dfrac{\log(\epsilon(1-r))}{\log r}\right] + 1$, then

$| s_n(x) - f(x) | < \epsilon$ if $n \geq m$ and for $-r \leq x \leq r$.
Therefore (S_n) converges uniformly in $[-r, r]$. Thus the geometric series $1 + x + x^2 + \ldots$ converges uniformly in $[-r, r]$ to the sum function $f(x) = \frac{1}{1-x}$

EXAMPLE 14 : Let $f_n(x) = n x e^{-nx^2} - (n-1) x e^{-(n-1)x^2}$, $x \in [0,1]$.

Consider the series $\sum_{n=1}^{\infty} f_n(x)$.

In this case $S_n(x) = \sum_{k=1}^{n} (k x^{-kx^2} - (k-1) x e^{-(k-1)x^2}) = n x e^{-nx^2}$

In example 12, you have seen that this sequence (S_n) is pointwise but not uniformly convergent to the function f where $f(x) = 0, x \in (0, 1)$. Thus the series $\Sigma f_n(x)$ is pointwise convergent but not uniformly to the function f where $f(x) = 0$, $x \in [0, 1]$.
Try the following exercises.

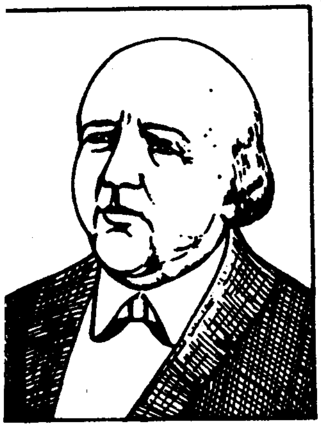
EXERCISE 7
Show that the series

$$\frac{x}{x+1} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \ldots \text{ is uniformly}$$

convergent in $]k, \infty[$ where k is a positive number.

EXERCISE 8

Show that the series $\Sigma \frac{x}{n(n+1)}$ is uniformly convergent in $[0,k]$ where k is any positive number but it does not converge uniformly in $[0,\infty]$.



There is a very useful method to test the uniform convergence of a series of functions. In this method, we relate the terms of the series with those a series with constant terms. This method is popularly called Weierstrass's M-test given by the German mathematician K.W.T. Weierstrass (1815-1897). We state this test in the form of the following theorem (without proof) and illustrate the method by an example.

THEOREM 5: WEIERSTRASS M-TEST

Let Σf_n be a series of functions defined on a set A⊂R and let (M_n) be a sequence of real numbers such that ΣM_n is convergent and $| f_n (x)| \leq M_n$ ∀ n and ∀ x ∈ A. Then Σf_n is uniformly and absolutely convergent on A.

Consider the following example and the exercise.

EXAMPLE 14 : Test the uniform convergence of the series

$$\sum_{n=1}^{\infty} \frac{x}{x^2 (n+1)}$$

$$| f_n (x) | = \frac{x}{x^2 (n+1)}, \leq - \frac{k}{n^3} \ \forall n \text{ and } \forall x \in [0,k]$$

Now the series $\Sigma M_n = K \Sigma \frac{1}{n^3}$ is convergent

Therefore by Weierstrass M-test, the given series is uniformly convergent.

EXERCISE 9

Show that $\sum_{n=1}^{\infty} \frac{x}{n^4 + x^2}$ converges uniformly ∀ x ∈ R.

16.5 SUMMARY

In this unit you have learnt how to discuss the pointwise and uniform convergence of sequences and series of functions. In section 16.2, sequence of functions is defined and pointwise convergence of the sequence of functions has been discussed. We say that a sequence of functions (f_n) is pointwise convergent to f on a set A if given a number $\epsilon > 0$, there is a positive integer m such that

$$|f_n (x) - f(x)| < \epsilon \text{ for } n \geq m,\ x \in A.$$

m in general depends on ϵ and the point x under consideration. If it is possible to find m which depends only on ϵ and not the point x under consideration, then (f_n) is said to be uniformly convergent to f on A. Uniform convergence has been defined in section 16.3. Further in this section, Cauchy's criteria for uniform convergence is discussed. Also in this section you have seen that if the sequence of functions (f_n) is uniformly convergent to a function f on [a,b] and each f_n is continuous or integrable, then f is also continuous or integrable on [a,b]. Further it has been discussed that if (f_n) is a sequence of functions, differentiable on [a,b] such that $(f_n (x_0))$ converges for some point x_0 of [a,b] and if (f_n') converges uniformly on [a,b], then (f_n) converges uniformly to a differentiable function f such that $f'(x) = \lim_{n \to \infty} f_n' (x)$; x ∈ [a,b].

Finally in section 16.4, pointwise and uniform convergence of series of functions is given. The series of functions is said to be pointwise or uniformly convergent on a set A according as the sequence of partial sums (s_n) of the series is pointwise or uniformly convergent on A.

16.6 ANSWERS/HINTS/SOLUTIONS TO EXERCISES

E 1) When $x = -1$, $\lim x^n$ does not exist. When $x= 1$, $\lim x^n = 1$.

When $|x| < 1$, $\lim x^n = 0$. Therefore $(f_n(x))$ converges at all points of $[-1, 1]$ except $x = -1$. Since the sequence $(f_n(x))$ does not converges at each point of $[-1,1]$, it is not pointwise convergent in $[-1,1]$. However it is pointwise convergent in $]-1,1]$.

E 2) Since sin n x is bounded, and $\frac{1}{\sqrt{n}}$ converges to 0. therefore $\left(\frac{\sin nx}{\sqrt{n}}\right)$ converges to 0 for every fixed value of x in R and so $\lim_{n\to\infty} f_n(x) = 0$, $x \in R$.

Therefore (f_n) is pointwsie convergent to the limit function f where $f(x) = 0 \ \forall x \in R$.

E 3) (i) For any value of $x \neq 0$, the sequence of numbers (sin n x) is not convergent.

For example for $x = \pi/2$, the sequence is $(\sin \frac{n\pi}{2})$ i.e $(1, 0, -1, 0, ...)$ which is not convergent. Thus sequence (f_n) is not pointwise convergent.

(ii) For $x = 0$, $(f_n(x)) = (0, 0, 0, ...)$ which is convergent. For $x \neq 0$,

$f_n(x) = \frac{x/n}{1/n^2 + x^2} = 0$ as $n \to \infty$. So (f_n) is pointwise convergent to f where $f(x) = 0$ for $-\infty < x < \infty$.

E 4) (i) $\lim_{n\to\infty} f_n(x) = 0$, $0 < x < \infty$. So (f_n) is pointwise convergent to f where $f(x) = 0$, $0 < x < \infty$. Let $\epsilon > 0$ be given.

$|f_n(x) - f(x)| = \frac{1}{nx} \quad (0 < x < \infty)$

$< \epsilon$ if $nx > \frac{1}{\epsilon}$ i.e. $n > \frac{1}{\epsilon x}$

If $m = [\frac{1}{\epsilon x}] + 1$, then for $n \geq m$, $|f_n(x) - f(x)| < \epsilon$.

m depends upon ϵ & x. $\frac{1}{\epsilon x}$ is unbounded for $0 < x < \infty$, so it is not possible to find m which serves for all x. If such a m exists, then we have

$\frac{1}{nx} < \epsilon$ for $n \geq m$ and $\forall x$ in $]0, \infty[$.

$\therefore \frac{1}{mx} < \epsilon$ i.e. $m > \frac{1}{\epsilon x}$ which is not possible,

since $\frac{1}{\Sigma x} \to \infty$ as $x \to 0+$.

so (f_n) is not uniformly convergent in $]0,\infty[$.

(ii) (f_n) is pointwise convergent to f where $f(x) = 0$ for $-\infty < x < \infty$. It is not uniformly convergent. If it is, then for $\Sigma = \frac{1}{4}$, there is a positive integer m such that $|f_n(x) - f(x)| < \frac{1}{4}$ i.e. $\left|\frac{nx}{1 + n^2x^2}\right| < \frac{1}{4}$ for $n \geq m$ & for $-\infty < x < \infty$. Take $n = m$ and $x = \frac{1}{m}$.

$\therefore \left|\frac{nx}{1 + n^2x^2}\right| = \frac{1}{2} < \frac{1}{4}$

which is a contradiction. So (f_n) is not uniformly convergent in $-\infty < x < \infty$.

(iii) For $0 \leq x < 1$, $\frac{x^n}{1 + x^n} \to 0$ as $n \to \infty$, since $x^n \to 0$.

For $x = 1$, $\frac{x^n}{1 + x^n} = \frac{1}{2} \to \frac{1}{2}$ as $n \to \infty$.

$\therefore$ $(f_n) \to f$ pointwise where $f(x) = 0$ for $0 \le x < 1$ and $f(x) = \frac{1}{2}$ for $x = 1$.

let $\epsilon > 0$ be given.
For $x = 0$ or 1, $|f_n(x) - f(x)| = 0 < \epsilon$ for $n \ge 1$.

For $0 < x < 1$, $|f_n(x) - f(x)| = \dfrac{x^n}{1 + x^n} < \epsilon$ if
$x^n < \epsilon\,(1 + x^n)$
i.e. $x^n(1 - \epsilon) < \epsilon$

i.e. $x^n < \dfrac{\epsilon}{1-\epsilon}$

i.e. $n \log x < \log \dfrac{\epsilon}{1-\epsilon}$

i.e. $n < \dfrac{\log \frac{\epsilon}{1-\epsilon}}{\log x}$ (log x is negative)

Take $m = \left[\dfrac{\log \frac{\epsilon}{1-\epsilon}}{\log x}\right] + 1$. Then
$|f_n(x) - f(x)| < \epsilon$ for $n \ge m$.
m depends upon ϵ and x. Since $\log x \to 0$ as x tends to 1, it is not possible to find same m for all x.

Therefore, the convergence is not uniform.

(iv) $(f_n) \to f$ pointwise where $f(x) = 0$ for $0 \le x < \infty$, since $\lim\limits_{n \to \infty} \frac{1}{n} e^{-nx} = 0$.

Let $\epsilon > 0$ be given.

$|f_n(x) - f(x)| = \dfrac{1}{ne^{nx}}$

$\le \dfrac{1}{n}$ for $0 \le x < \infty$

Since $\frac{1}{n} \to 0$, so there exists a positive integer m such that $\frac{1}{n} < \epsilon$ for $n \ge m$.

Therefore $|f_n(x) - f(x)| < \epsilon$ for $n \ge m$ and for $0 \le x < \infty$ and consequently $f_n \to f$ uniformly for $0 \le x < \infty$.

E 5) $\lim\limits_{n \to \infty} \frac{x}{n} = 0$, so $(f_n) \to f$ pointwise in R where $f(x) = 0\ \forall x \in R$. Obviously each f_n, being a polynomial function, is continuous and also f being a constant function is continuous in R. (f_n) is not uniformly convergent, since if it were, for a given $\epsilon > 0$, there is a positive integer m such that
$|f_n(x) - f(x)| < \epsilon$ for $n \ge m$ & $\forall x \in R$.

i.e. $\frac{x}{n} < \epsilon$ for $n \ge m$ and $\forall x \in R$.

Take $\epsilon = \frac{1}{2}$. Then $\frac{x}{n} < \frac{1}{2}$ for $n \ge m$ & $\forall x \in R$.

Take $n = m$ & $x = m$, then $1 < \frac{1}{2}$ which is a contradiction. So (f_n) is not uniformly convergent in R.

E 6) $f_n(0) = 0 = f_n(1)$.

For $0 < x < 1$, $\lim\limits_{n \to \infty} f_n(x) = x \lim\limits_{n \to \infty} \dfrac{n}{(1-x^2)^{-n}}$ $\left(\dfrac{\infty}{\infty}\right)$

$= x \lim\limits_{n \to \infty} \dfrac{1}{(1-x^2)^{-n} \log(1-x^2)}$ (L' Hôpital Rule)

$= 0$

Therefore, $(f_n) \to f$ pointwise where $f(x) = 0\ \forall x \in [0, 1]$.

$$\int_0^1 f(x)\,dx = 0 \ \& \int_0^1 f_n(x)\,dx = \frac{n}{2(n+1)} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty.$$

Thus $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx.$

E 7) If $S_n(x)$ denotes the sum of n terms of the series,

$$S_n(x) = \frac{x}{x+1} + \frac{}{(x+1\,(2x+1)} + \ldots\ldots \text{ upto n terms}$$

$$= \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \ldots\ldots + \left(\frac{1}{n-1\,x+1} - \frac{1}{nx+1}\right)$$

$$= 1 - \frac{1}{nx+1}$$

(s_n) is pointwise convergent to f where
$f(x) = 1 \ \forall\ x \in\ ]k, \infty[$.

$$|S_n(x) - f(x)| = \frac{1}{nx+1} < \frac{1}{nk+1} < \frac{1}{nk} \ \forall\ x \in\ ]k, \infty[$$

$\frac{1}{nk} \rightarrow 0$ an so for given $\epsilon > 0$, there is an integer m such that $\frac{1}{nk} < \epsilon$ for $n \geq m$.

Consequently

$|S_n(x) - f(x)| < \epsilon$ for $n \geq m \ \& \ \forall\ x \in\ ]k, \infty[$.

Therefore (S_n) is uniformly convergent and so is the given series in $]k, \infty[$.

E 8) If $S_n(x)$ is the sum of n terms of the series, then

$$S_n(x) = \frac{x}{1.2} + \frac{x}{2.3} + \ldots + \frac{x}{n(n+1)}$$

$$= x\left(\frac{1}{1} + \frac{1}{2}\right) + x\left(\frac{1}{2} - \frac{1}{3}\right) + \ldots\ldots + x\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= x\left(1 - \frac{1}{n+1}\right)$$

$S_n(x) \rightarrow x$ as $n \rightarrow \infty$.

Therefore $(S_n) \rightarrow f$ pointwise where

$f(x) = x$ for $0 \leq x < \infty$

$$|S_n(x) - f(x)| = \frac{x}{n+1}$$

In $[0,k]$, $|S_n(x) - f(x)| \leq \frac{k}{n+1}$ and $\frac{k}{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

For given $\epsilon > 0$, there is a positive integer m such that

$|S_n(x) - f(x)| < \epsilon$ for $n \geq m \ \& \ \forall\ x \in [0, k]$.

Therei· re (S_n) is uniformly convergent in $[0, k]$.

If (s_n) is uniformly convergent in $[0, \infty[$, then for $\epsilon = 1/2$ there is a positive integer m such that

$$|S_n(x) - f(x)| = \frac{x}{n+1} < \frac{1}{2} \text{ for } n \geq m \ \& \ x \in [0, \infty[$$

Take $n = m \ \& \ x = m + 1$ and then there is a contradiction. So (S_n) is not uniformly convergent in $[0, \infty[$.

Hence the series is uniformly convergent in $[0, k]$ but is not uniformly convergent in $[0, \infty[$.

E 9) $f_n(x) = \dfrac{1}{n^4 + x^2}$

$$= < \frac{1}{n^4} \text{ for every } n \geq 1 \text{ and } \forall\, x \in R.$$

But $\Sigma \dfrac{1}{n^4}$ is a convergent series. Therefore, by Weierstrass M-test the series

$$\Sigma\, f_n(x) = \Sigma \;\frac{1}{n^4 + x^2}$$

is uniformly convergent in R.

REVIEW

This is the fifth and the last block in the course on Real Analysis.

It consists of three units namely the Units 14, 15 and 16. In Unit 14, you have been introduced to the fundamental notion of the integral of a function, popularly known as the **Riemann Integral.** This is introduced through the supremum principle and is defined as the limit of a set of suitable sums thus ruling out the common misconception that integration is merely the reverse of differentiation. Also, the criteria for the integrability of a function have been given.

Some important properties of integrable functions have been discussed in Unit 15. The Fundamental Theorem of Calculus is, then, established which brings forth the relationship between differential and Integral Calculus. It turns out that for a certain class of functions, integration is indeed a reverse process of differentiation. Two mean-value theorems and the well-known basic techniques of integration namely **integration by parts** and **integration by substitution** (change of variables) have been deduced with the help of the Fundamental Theorem of Calculus. Finally, Unit 16 deals with sequences and series of functions which is a generalization of the notions of the sequences and series of real numbers discussed in block 2. The notion of pointwise and uniform convergence has been introduced as well as their relevance to the notions of continuity, differentiability and integrability of functions has been discussed.

You will do well if you try to attempt the following questions as a self-test to know whether or not you have actually grasped the material given in this block. You may check your answers with the ones given at the end.

1. If $f(y,x) = 1 + 2x$ for y rational and $F(y,x) = 0$ for y irrational. Calculate

   $F(y) = \int_0^1 f(y,x)\,dx.$

2. By applying the generalised first mean theorem of integral calculus, show that

   $$\frac{\pi}{2} \leq \int_0^1 \frac{dx}{\sqrt{(1-x)^2\left(1-\frac{x}{4}\right)^2}} \leq \pi - \frac{2}{\sqrt{5}}$$

3. Show by means of an example that the product of two non-integrable functions may be integrable.

4. If $f(x) = 0 \ \forall\ x \in [a,b]$, then show that

   $$\int_a^b f(x)\,dx = 0.$$

5. Give an example to show that f is integrable in [a,b] but has (i) finite number of points of discontinuity (ii) infinite number of points of discontinuity.

6. Show that $\int_0^4 [x]\,dx = 6$

7. Does the First Mean Value Theorem of Integral Calculus apply to the function
   (i) $f(x) = |x|$ if $0 \leq x \leq 2$
   (ii) $f(x) = 2$ if $2 \leq x \leq 3$.

8. Let (f_n) be a sequence such that

   $$f_n(x) = \begin{cases} 1 \text{ if } x \neq 1 \\ 2 \text{ if } x = 1 \end{cases}$$

   Show that the limit function f is discontinuous in [0, 1]

9. Using Weierstrass M-Test, show that the following series converge uniformly:

   (i) $\sum_{n=1}^{\infty} n^2 x^n,\ x \in [-\frac{1}{2}, \frac{1}{2}]$

   (ii) $\sum_{n=1}^{\infty} e^{nx} \cos n x,\ x \in [\pi, 2\pi]$

10. Test the convergence of the following series:

(i) Σr^n converges and $\Sigma r^n \sin n\theta$ v $\forall\ \theta \in R$ and $r \in [0, \frac{1}{2}]$

(ii) Σn^{-x}, $x \in [1+\alpha\ \infty[\ (\alpha > 0)$

ANSWERS/HINTS

1. $F(y) = 2$ if y is rational and $F(y) = 0$ if y is irrational

2. Apply generalised first mean value theorem by taking

$$f(x) = \frac{1}{\sqrt{1-4^2x^2}} \quad \text{and } g(x) \quad \frac{1}{\sqrt{1-x^2}}$$

3. Take $f(x) = g(x) = \begin{cases} 1 \text{ when x is rational} \\ -1 \text{ when x is irrational, } x \in [a,b]. \end{cases}$

Then $(fg)(x) = 1\ \forall\ x \in [a,b]$.

4. A constant function $f(x) = k\ \forall x \in [a,b]$ is integrable in $[a,b]$ and $\int_a^b f(x)\,dx =$

$k(b-a)$ (Recall unit 14)

Take $k = 0$.

5. (i) Take $f(x) = \begin{cases} 1 \text{ when } 1 \leq x < 2 \\ 3 \text{ when } 2 \leq x \leq 3. \end{cases}$

(ii) $f(x) = \frac{1}{4^n}$ when $\frac{1}{4^{n+1}} < x \leq \frac{1}{4^n}$, $n = 0, 1, 2 \ldots.$

$f(0) = 0$

$a = 0,\ b = 1$

6. $$\int_0^4 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx \int_2^3 [x]\,dx + \int_3^4 [x]\,dx$$

$$= \int_0^1 dx + \int_1^2 !\ dx + \int_2^3 2\,dx + \int_3^4 3\,dx$$

7. Yes, since the functions are continuous

8. $f(x) = \begin{cases} 1 \text{ if } x \neq 1 \\ 2 \text{ if } x = 1 \end{cases}$

It is discontinuous at 1.

9. (i) $|n^2x^n| = n^2|x|^n \leq n^2 (\frac{1}{2})^n\ \forall\ x \in [-\frac{1}{2}, \frac{1}{2}]$

$\Sigma \frac{n^2}{2^n}$ is convergent.

(ii) $|e^{-nx} \cos nx| \leq e^{-nx} \leq e^{-n\pi} \forall x \in [\pi, 2n]$

$\Sigma e^{-n\pi}$ is a G.P. with common ration less than unity and so it is convergent.

10. (i) $|r^n \sin n\theta| \leq r^n$. Σr^n is a G.P. with common ratio r which is less than 1 and so it is convergent.

(ii) $|x^{-x}| = \frac{1}{n^x} \leq \frac{1}{n^{1+\alpha}}\ \forall\ x \in [1+\alpha, \infty[$ and $\Sigma \frac{1}{n^{1+\alpha}}$ is convergent.