स्वाध्याय स्वमन्थन स्वावलम्बन

UTTAR PRADESH RAJARSHI TANDON OPEN UNIVERSITY

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UGMM-07
Advanced Calculus

FIRST BLOCK
R∞ AND R^n

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Block

1

R_∞ And R^n

ADVANCED CALCULUS

This course takes you a step further in the study of calculus. Here we shall first tackle infinite limits of functions of one variable, discuss L'Hopital's rule, and then go on to study functions of several variables.

You are already familiar with some aspects of the calculus of functions of one variable. As we have mentioned in our calculus course (MTE-01), Newton and Leibniz are considered to be the founding fathers of calculus. You have also read that the major developments in calculus took place in the seventeenth century. Later, in the eighteenth century, the basic concepts of calculus, for example, limit, continuity and differentiability were extended to functions of more than one variable. The need for studying functions of several variables arose when some mathematicians like Euler, Daniel Bernoulli, Fourier and d' Alembert were investigating some physical problems.

Augustin-Louis Cauchy (1789-1857) was a dominant mathematical figure in Paris, the centre of the mathematical world in those days. Cauchy too has contributed to the development of several variables to a great extent.

The concepts which you are going to study in this course are bound to be a little more complex than the corresponding concepts for functions of one variable. But you will see that one variable case shows us the way in which these concepts could be generalised. So, each time we introduce a new concept we'll recall its parallel in the one variable case, and then see how it is extended to the several variables' case. In this course we'll restrict ourselves mainly to functions of two or three variables.

We have interspersed the text with a lot of solved examples. These examples will help you understand the theory better. We have given the answers of all the exercises in each unit at the end of the unit. As you will see, we often have to refer back to results or definitions from earlier units. For this we'll refer to sub-section y.z of Unit x as Sec. x.y.z, or to section y of Unit x as Sec. x.y. We'll also recall some results from our earlier course MTE-01 on calculus. We shall refer to a unit in this course as "Unit x of Calculus". What we had said in our calculus course remains true for this course too—to master the various techniques presented here, you will need to put in a lot of practice.

In case you want to seek some additional information about the concepts discussed here, or to solve some more exercises, you can consult the following book:

Calculus III by Jerrold Marsden and Alan Weinstein

This will be available in your study centre library.

We hope you find the techniques developed in this course useful in your further studies.

Notations and Symbols

$\in$	belongs to
$\notin$	does not belong to
$A \cup B$	union of the sets A and B
$A \cap B$	intersection of the sets A and B
$A \setminus B$	the set of elements of A that are not in B
$A \times B$	the Cartesian product of A and B
N	the set of natural numbers
Z(Z*)	the set of integers (non-zero integers)
Q(Q*)	the set of rational numbers (non-zero rational numbers)
R(R*)	the set of real numbers (non-zero real numbers)
∞	infinity
$\mathbf{R}_\infty$	$\mathbf{R} \cup \{\pm\infty\}$
$\mathbf{R}^n$	the Cartesian product of n copies of **R**
$\Rightarrow$	implies
$\Leftrightarrow$	implies and is implied by
iff	if and only if
$\exists$	there exists
$\forall$	for all
$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + \ldots\ldots + a_n$
w.r.t	with respect to
$x \to a$	x tends to a
$f : X \to Y$ $x \to f(x)$	f is a function from X to Y taking x to f(x)
$\lim_{x \to a} f(x)$	limit of f(x) as x tends to a
$\frac{dy}{dx}, y_1, f'(x)$	derivative of y = f(x) w.r.t. x
$f^{(k)}(x)$	kth derivative of f(x) w.r.t x
n!	factorial $n = n(n-1) \ldots. 3.2.1$
max {x,y}	the maximum of x and y
min {x,y}	the minimum of x and y
$(x_1, \ldots\ldots, x_n)$	n-tuple of $x_1, x_2, \ldots\ldots, x_n$
$f \circ g$	composite of f and g
$\lvert x \rvert$	the absolute value of x
[x]	the greatest integer $\leq$ x

BLOCK 1 INTRODUCTION

In this course we are going to generalise the concepts learnt in the calculus course. For instance, in the calculus course, we had restricted our discussion to finite limits only. You may remember that we said that the (finite) limit of $f(x) = \frac{1}{x}$, as x tends to zero, does not exist. In this block we shall enlarge the notion of limit to include infinite limits too. For this we'll first extend the real number system to R_- by adding two symbols ∞ and $-\infty$ to it. We'll also refresh your knowledge of the limit of a function as the independent variable tends to infinity. This discussion will then help us to study L'Hopital's rule. This rule is a simple technique which helps us to find the limits of various functions which are in indeterminate forms. In the first two units of this block we shall be dealing with functions of a single variable.

Unit 3, which is the last unit in this block, introduces you to functions of several variables. In future blocks you will study the concepts of limit, continuity, differentiability and integrability of these functions. To enable you to appreciate these concepts, in Unit 3, we'll give a detailed description of the algebraic structure of R^n. We shall also discuss the distance function in R^n there. So, Unit 3 forms the basis of the rest of this course.

Lastly, we remind you once again to carefully go through the solved examples, and to attempt all the exercises in each unit. This will help you grasp the theory better.

Greek Alphabets

α	Alpha
β	Beta
γ	Gamma
δ, Δ	Delta (capital delta)
ε	Epsilon
ζ	Zeta
η	Eta
θ	Theta
ι	Iota
κ	Kapa
λ	Lambda
μ	Mu
ν	Nu
ξ	Xi
ο	Omicron
π, Π	Pi (capital Pi)
ρ	Rho
σ, Σ	Sigma (capital Sigma)
τ	Tau
υ	Upsilon
φ	Phi
χ	Chi
ψ	Psi
ω	Omega
∂	del

UNIT 1 INFINITE LIMITS

Structure

1.1 INTRODUCTION

You are already familiar with the notion of the limit of a real-valued function f(x) as x tends to a real number a. You have also come across functions like sinx, cosx, e^x, lnx which are defined for arbitrary large values of x. In our earlier course on calculus, we said that $\lim_{x\to 0} \frac{1}{x^2}$ or $\lim_{x\to 0} \frac{-1}{x^2}$ does not exist. But in both these cases the functions under consideration have a definite behaviour in a neighbourhood of zero. It is clear that $\frac{1}{x^2}$ can be made as large as we like while $\frac{-1}{x^2}$ can be made as small as we like by choosing x sufficiently close to 0.

In order to study the behaviour of functions like e^x and lnx when x is large, or to study the behaviour of $\frac{1}{x^2}$ or $\frac{-1}{x^2}$ when x approaches 0, in Sec. 1.2 we extend the real number system by adding two new symbols $+\infty$ (simply written as ∞), called plus infinity or infinity and $-\infty$, called minus infinity.

In Sec. 1.3 we extend the notion of limit to include ∞ and $-\infty$ as limits. We also define the limit of f(x) when x approaches ∞ or $-\infty$. You have already studied the limits of some functions as x approaches ∞ or $-\infty$ in the earlier course on calculus.

Objectives

After reading this unit you should be able to:

- define $\lim_{x\to a} f(x) = L$, where $a \in R$ and L may be a real number or ∞ or $-\infty$
- define $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to -\infty} f(x) = L$, when L may be a real number or ∞ or $-\infty$
- evaluate $\lim_{x\to a} f(x) = L$, where a and L are elements of the extended real number system.

1.2 THE EXTENDED REAL NUMBER SYSTEM R_∞

The extended real number system is the set consisting of the set of real numbers and two new symbols $+\infty$ (plus infinity) and $-\infty$ (minus infinity).

Henceforth, we shall denote the extended real number system by R_∞ and write ∞ for $+\infty$. Thus,

$R_\infty = R \cup \{\infty\} \cup \{-\infty\}$,

where **R** denotes the set of real numbers.

You are already familiar with the arithmetic operations in R. Let us see if we can define similar operations on R_∞.

1.2.1 Arithmetic Operations in R_∞

The basic operations of addition, subtraction, multiplication and division in R are extended to R_∞ by the following formulas:

1. If x, y are in R, then $x \pm y$, xy, $\frac{x}{y}$ ($y \neq 0$) have their usual meaning.

2. For any real number x, we define

   (i) $x + \infty = \infty + x = \infty$

   (ii) $x + (-\infty) = (-\infty) + x = -\infty$

   (iii) $x . \infty = \infty . x = \infty$ if $x > 0$

   (iv) $x . \infty = \infty . x = -\infty$ if $x < 0$

   (v) $x . (-\infty) = (-\infty) . x = -\infty$ if $x > 0$

   (vi) $x(-\infty) = (-\infty) x = \infty$ if $x < 0$

   (vii) $\frac{\pm\infty}{x} = \pm\infty$ if $x > 0$

   (viii) $\frac{\pm\infty}{x} = \mp\infty$ if $x < 0$

   (ix) $\frac{x}{\pm\infty} = 0$

3. We define

   (i) $\infty + \infty = \infty$

   (ii) $-\infty + (-\infty) = -\infty$

   (iii) $\infty . \infty = \infty$

   (iv) $(-\infty)(-\infty) = \infty$

   (v) $\infty . (-\infty) = (-\infty) . \infty = -\infty$

Note that if x and y are two real numbers, then $x + y$, $x . y$, $x - y$ and $\frac{x}{y}$ ($y \neq 0$), have the same value, whether x and y are considered as elements of R or of R_∞.

Remark 1 : You would notice that the formulas 1, 2 and 3 do not cover cases like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 . \infty$, $\infty - \infty$. These symbols are not defined because it is not possible to assign a unique value to these expressions consistent with the above formulas.

For example, if we define, $\infty - \infty = \alpha$, where α is a real number, then

$\infty = \infty + \alpha = \infty + (\infty - \infty) = (\infty + \infty) - \infty = \infty - \infty = \alpha$,

which is a contradiction. If $\infty - \infty$ is defined to be equal to ∞, then, for any real $a < 0$,

$-\infty = a.(\infty) = a(\infty - \infty) = a . \infty - a . \infty = -\infty + \infty = \infty$.

A similar contradiction would arise if we defined $\infty - \infty$ to be equal to $-\infty$.

Proceeding exactly as above, you should have no difficulty in checking that a unique value cannot be assigned to the other symbols mentioned above. That is why expressions like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 . \infty$, $\infty - \infty$ are called **indeterminate forms.**

E 1) Prove that a unique value cannot be assigned to

a) $\frac{0}{0}$; b) $\frac{\infty}{\infty}$; c) $0 . \infty$

Now after defining arithmetic operations, let's see if we can define an order relation in $\mathbf{R}_{\infty}$.

Order relation in $\mathbf{R}_{\infty}$: We extend the order relation 'less than or equal to' ($\leq$) to $\mathbf{R}_{\infty}$ as follows:

(i) If x and y are two real numbers, then $x \leq y$ in $\mathbf{R}_{\infty}$ if and only if $x \leq y$ in $\mathbf{R}$.

(ii) $x < \infty$ for any real number x.

(iii) $-\infty < x$ for any real number x.

Clearly ∞ is larger than any real number and $-\infty$ is smaller than any real number. Intuitively speaking, $\mathbf{R}_{\infty}$ has been obtained by adding two more points to the real line, namely, ∞ to the extreme right and $-\infty$ to the extreme left.

Remark 2 : In the first course on calculus, we used the notations

$]-\infty, \infty[\;=\; \mathbf{R} = \{ x \mid x \in \mathbf{R}, -\infty < x < \infty \}$

$]a, \infty[\;=\; \{ x \mid x \in \mathbf{R}, x > a \}$

$]-\infty, a[\;=\; \{ x \mid x \in \mathbf{R}, x < a \}$

$[a, \infty[\;=\; \{ x \mid x \in \mathbf{R}, x \geq a \}$

$]-\infty, a] \;=\; \{ x \mid x \in \mathbf{R}, x \leq a \}$

for different infinite intervals on the real line. You will notice that these notations are consistent with our definition of the order relation in $\mathbf{R}_{\infty}$.

Now that we have defined an order relation in $\mathbf{R}_{\infty}$, we can discuss upper and lower bounds of subsets of $\mathbf{R}_{\infty}$.

1.2.2 Bounds in $\mathbf{R}_{\infty}$

You are already familiar with the notions of upper and lower bounds of subsets of real numbers (Definition 1 in Unit 1 of Calculus). We now introduce these concepts for subsets of $\mathbf{R}_{\infty}$. You would observe that the definitions given here are exactly similar to the ones given for subsets of $\mathbf{R}$.

Definition 1: Let S be a non-empty subset of $\mathbf{R}_{\infty}$. An element $a \in \mathbf{R}_{\infty}$ is said to be an **upper bound** of S, if $s \leq a$ (or $a \geq s$) for every $s \in S$.

An element $u \in \mathbf{R}_{\infty}$ is said to be a **least upper bound** of S (denoted by lub S) if

- u is an upper bound of S, and
- any number $u' \in \mathbf{R}_{\infty}$, $u' < u$, is not an upper bound of S.

lub S is also called **supremum** of S and is also denoted by sup S.

Definition 2 : Let S be a non-empty subset of $\mathbf{R}_{\infty}$. An element $x \in \mathbf{R}_{\infty}$ is said to be a **lower bound** of S, if $x \leq s$ for every $s \in S$.

An element $x_0 \in \mathbf{R}_{\infty}$ is said to be a **greatest lower bound** of S (denoted by glb S) if

- x_0 is a lower bound of S, and
- any number $x_1 \in \mathbf{R}_{\infty}$, $x_1 > x_0$ (i.e. $x_0 < x_1$) is not a lower bound of S.

glb S is also called **infimum** of S and is also denoted by inf S.

The least upper bound and greatest lower bound of any non-empty set S are unique.

It is obvious from Definition 1 that if S is a non-empty subset of $\mathbf{R}$, which is bounded above, then the least upper bound of S in $\mathbf{R}$ is the same as the least upper bound of S in $\mathbf{R}_{\infty}$.

If S is a non-empty subset of $\mathbf{R}$, which is not bounded above, then ∞ is the only upper bound of S, when S is considered as a subset of $\mathbf{R}_{\infty}$, and therefore, l.u.b. $S = \infty$.

Similarly, for a non-empty set $S \subseteq \mathbf{R}$, which is bounded below, glb S in $\mathbf{R}$ is the same as glb S in $\mathbf{R}_\infty$. And for a non-empty subset S of $\mathbf{R}$, which is not bounded below, glb S is $-\infty$.

Thus, every non-empty subset of $\mathbf{R}$ is bounded in $\mathbf{R}_\infty$, and has a unique lub and a unique glb.

1.2.3 Extension of Exponential and Logarithmic Functions to $\mathbf{R}_\infty$

You know that the natural exponential function is defined on the set $\mathbf{R}$. Let us see if we can extend this function to $\mathbf{R}_\infty$. To do so, we will have to define e^∞ and $e^{-\infty}$. You know that when $x > 0$ and large, then so is e^x (see Fig.1). We, therefore, define $e^\infty = \infty$ and $e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$.

Next we extend the definition of $\ln x$ by setting $\ln 0 = -\infty$ and $\ln \infty = \infty$.

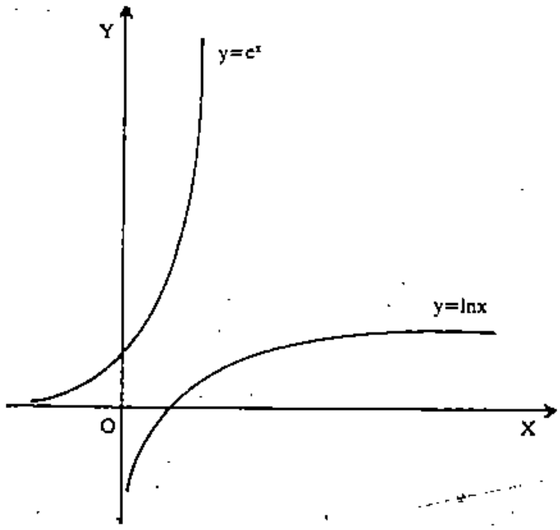


Fig.1

The power function a^x is extended to $\mathbf{R}_\infty$ as follows:

$$a^\infty = \begin{cases} \infty, & a > 1 \\ 0, & 0 < a < 1 \end{cases}$$

$$a^{-\infty} = \begin{cases} 0, & a > 1 \\ \infty, & 0 < a < 1 \end{cases}$$

The expressions $1^\infty, 0^0, \infty^0, 0^\infty$ have not been defined, as no unique value, which is consistent with the various definitions given above, can be assigned to any of these expressions. Thus, $1^\infty, 0^0, \infty^0$ and 0^∞ are also indeterminate forms.

Try to solve these exercises now.

E 2) a) Let S be an unbounded subset of $\mathbf{R}$. Prove that, if S is considered as a subset of $\mathbf{R}_\infty$, then either lub $S = \infty$ or glb $S = -\infty$.

b) Give an example of a subset S of $\mathbf{R}$ which is unbounded and for which lub $S = \infty$ but glb $S \in \mathbf{R}$.

c) Give an example of an unbounded subset of $\mathbf{R}$ for which glb $S = -\infty$ and lub $S \in \mathbf{R}$.

d) Give an example of an unbounded subset S of $\mathbf{R}$ such that lub $S = \infty$, glb $S = -\infty$.

E 3) Let $S = \left\{ x + \frac{1}{x} \;\middle|\; 0 < x < 1 \right\} \subset \mathbf{R}_\infty$.

a) Give two lower bounds and one upper bound of S.

b) Find lub S and glb S.

Before we conclude this section we would like to remind you once again that $\infty, -\infty$ are just symbols and not real numbers.

In what follows in this unit and the subsequent unit, unless specifically stated, all sets will be subsets of the set of real numbers, and a reference to their bounds will be to their bounds as subsets of **R**.

1.3 INFINITE LIMITS

In this section we shall extend the notion of limits. Now there are four ways in which the notion of limits can be extended. One way is to consider the behaviour of f(x) as x approaches ∞. Another way would be to examine the behaviour of f(x) as x tends to $-\infty$. We'll cover these in Sec. 1.3.3. Two other ways would be to consider the cases where f(x) becomes arbitrarily large (tends to ∞), and where f(x) becomes arbitrarily small (tends to $-\infty$), as x approaches $a \in \mathbf{R}$.

We'll take up these in Sec. 1.3.1 now.

1.3.1 Infinite Limits as the Independent Variable $x \to a \in \mathbf{R}$

You are already familiar with the concept of finite limits. Let us quickly recall what we mean when we say that

$$\lim_{x\to 0} f(x) = a$$

This means that given $\varepsilon > 0, \exists\ \delta > 0$, s.t.

$$x \in\ ]-\delta, \delta[\ \setminus \{0\} \Rightarrow f(x) \in\ ]a-\varepsilon, a+\varepsilon[$$

We can interpret this as:

$$\{f(x) \mid x \in\ ]-\delta, \delta[\ \setminus \{0\}\} \subseteq\ ]a-\varepsilon, a+\varepsilon[$$

This means that the set $\{f(x) \mid x \in\ ]-\delta, \delta[, x \neq 0\}$ is a bounded set.

What we can gather from this discussion is that if $\lim_{x\to 0} f(x)$ exists (and is finite) then for some neighbourhood $]-\delta, \delta[$ of zero, the set $\{f(x) \mid x \in\ ]-\delta, \delta[, x \neq 0\}$ is a bounded set.

By a neighbourhood of a, we mean a set of the type $]a-\delta, a+\delta[, \delta \in \mathbf{R}$; $\{x \mid x > r\}$ or $\{x \mid x < r\}$, $r \in \mathbf{R}$, according as a is equal to a finite real number, ∞ or $-\infty$.

Now suppose we want to prove that a function, say $f(x) = \frac{1}{x^2}$, does not tend to a **finite** limit as $x \to 0$. Then it would be enough to prove that f is not bounded in any neighbourhood of 0, i.e., in any interval of the type $]-\delta, \delta[$. Since $f(x) = \frac{1}{x^2}$ is positive for all x, to show the unboundedness of f(x) we need to show that f(x) is very large.

Let us consider an interval $]-\delta, \delta[$, for some $\delta > 0$. We want to prove that given any real number $M > 0$, however large M may be, we can find an $x \in\ ]-\delta, \delta[$ such that $f(x) > M$.

Let $M > 0$ be chosen. Then $f(x) > M \Rightarrow \frac{1}{x^2} > M$

$$\Rightarrow x^2 < \frac{1}{M}$$

$$\Rightarrow |x| < \frac{1}{\sqrt{M}}$$

Now given M, either $\delta \le \frac{1}{\sqrt{M}}$ or $\delta > \frac{1}{\sqrt{M}}$. If $\delta \le \frac{1}{\sqrt{M}}$, then for any $x \in\ ]-\delta, \delta[$, we have $|x| < \delta \Rightarrow x^2 < \delta^2 \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} > M$. That is, $f(x) > M$.

If $\delta > \frac{1}{\sqrt{M}}$, then the interval $\left]-\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right[\subseteq\ ]-\delta, \delta[$.

Choose $x \in \left]-\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right[$ Then $|x| < \frac{1}{\sqrt{M}} \Rightarrow \frac{1}{x^2} > M$. This means, $f(x) > M$.

Thus, in both the cases we could prove that given $M > 0$, $\exists\, x \in\ ]-\delta, \delta[$ such that $f(x) > M$. However, in the case of $f(x) = \frac{1}{x^2}$ we can prove a stronger statement. That is, we can prove that given a real number $M > 0$, there exists a real number $\delta > 0$, (δ depending on M), such that $f(x) > M$ for all x in $]-\delta, \delta[$, $x \neq 0$. In fact, the above discussion shows that given M, any δ such that $\delta < \frac{1}{\sqrt{M}}$ would suffice. We express the above facts by saying that the function $\frac{1}{x^2}$ tends to ∞ as x tends to 0.

Similarly, if we consider the function $f(x) = \frac{-1}{(x-a)^2}$, $x \neq a$, then we can show that f(x) can be made to assume values less than any given number for all x which are sufficiently close to a, but are not equal to a. In fact, for a given $m < 0$, any positive $\delta < \sqrt{\frac{-1}{m}}$ would suffice. We express this by saying that f(x) tends to $-\infty$ as x tends to a.

We are giving the precise definitions below.

Definition 3 : Let f be a real-valued function defined in an open interval $]a-h, a+h[$ except possibly at a. f(x) is said to tend to the limit ∞ as x tends to a, if given any real number M, there exists a positive real number δ (depending on M), $\delta < h$, such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M.$$

Definition 4 : Let f be a real-valued function defined in an open interval $]a-h, a+h[$ except possibly at a. Then f(x) is said to approach the limit $-\infty$ as x approaches a, if given any real number m, there exists a positive number $\delta < h$, δ depending on m, such that

$$0 < |x - a| < \delta \Rightarrow f(x) < m.$$

We'll use the symbols $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) or $f(x) \to \infty$ (or $-\infty$) as $x \to a$ to express that f(x) tends to ∞ (or $-\infty$) as x tends to a.

Remark 3 : (i) In Definition 3, if we have found a real number $\delta > 0$ for one M, then the same δ would suffice for all real numbers smaller than M. Therefore, there is no loss of generality, if we take $M > 0$ in Definition 3.

(ii) Similarly, in Definition 4, if we have found a real number $\delta > 0$ for one m, then the same δ would suffice for all real numbers greater than m. Therefore, there is no loss of generality, if we take $m < 0$ in Definition 4.

We now illustrate the above definitions with the help of a few examples:

Example 1: Let us prove that

(i) $\lim_{x \to 1} \frac{x}{(x-1)^2} = \infty$

(ii) $\lim_{x \to 0} \frac{1}{1 - \cos 2x} = \infty$

We'll prove them one by one.

(i) Let $M > 0$ be given. Our aim is to find a real number $\delta > 0$ such that

$$0 < |x - 1| < \delta \Rightarrow \frac{x}{(x-1)^2} > M$$

Now, $0 < |x - 1| < \delta \Rightarrow 1 - \delta < x < 1 + \delta$, $x \neq 1$. Thus, $x > 1 - \delta$ and $(x-1)^2 < \delta^2$. This means that

$$0 < |x - 1| < \delta \Rightarrow \frac{x}{(x-1)^2} > \frac{1-\delta}{\delta^2}.$$

And if we choose a positive δ less than a fixed real number, say $\frac{1}{2}$, then

$$0 < |x - 1| < \delta < \frac{1}{2} \Rightarrow \frac{x}{(x-1)^2} > \frac{1}{2\delta^2}.$$

Now when will $\frac{x}{(x-1)^2}$ be greater than M ?

This will happen if $\frac{1}{2\delta^2} > M$, i.e., if

$$\delta < \frac{1}{\sqrt{2M}}$$

So we have put two restrictions on $\delta : 0 < \delta < \frac{1}{2}$ and $\delta < \frac{1}{\sqrt{2M}}$ to achieve our aim. Thus, if we choose any δ such that

$$0 < \delta < \min\left\{\frac{1}{2}, \frac{1}{\sqrt{2M}}\right\}$$

then,

$$0 < |x-1| < \delta \Rightarrow \frac{x}{(x-1)^2} > M,$$

showing that $\lim_{x \to 1} \frac{x}{(x-1)^2} = \infty$.

(ii) Let $f(x) = \frac{1}{1 - \cos 2x}$, $x \neq 0$

Then $f(x) = \frac{1}{2\sin^2 x}$. If $|x| < \frac{\pi}{2}$, then $|\sin x| \leq |x|$, and therefore,

$$f(x) > \frac{1}{2x^2}.$$

Thus, for $0 < |x| < \delta < \frac{\pi}{2}$, we have

$$f(x) > \frac{1}{2x^2} > \frac{1}{2\delta^2}$$

Consequently, $0 < |x| < \delta$, and $f(x) > \frac{1}{2\delta^2} > M$ if $\delta < \frac{1}{\sqrt{2M}}$.

Hence, we have proved that for any $M > 0$, if we choose a real number δ such that $0 < \delta < \min\left\{\frac{\pi}{2}, \frac{1}{\sqrt{2M}}\right\}$, then

$$0 < |x| < \delta \Rightarrow \frac{1}{1 - \cos 2x} > M,$$

showing that $\lim_{x \to 0} \frac{1}{1 - \cos 2x} = \infty$.

Note that in (i) there was nothing special about our choice of $\delta < \frac{1}{2}$. We could have chosen any other δ, say $\delta < \frac{2}{3}$. Then we would have obtained

$$\frac{x}{(x-1)^2} > \frac{1}{3\delta^2},$$

and we could choose any $\delta < \min\left\{\frac{2}{3}, \frac{1}{\sqrt{3M}}\right\}$ to reach our goal.

Example 2 : Suppose we want to prove that

i) $\lim_{x \to -2} \frac{4x}{(x+2)^2} = -\infty$

ii) $\lim_{x \to 0} \frac{-1}{\sin^2 x} = -\infty$

Let us take these one by one.

(i) Let $M > 0$ be given. If $0 < |x + 2| < \delta < 1$, then

$$\frac{1}{(x+2)^2} > \frac{1}{\delta^2}$$

Now $|x + 2| < 1 \Rightarrow x < -1$, and therefore

$$\frac{1}{(x+2)^2} > \frac{1}{\delta^2} \Rightarrow \frac{4x}{(x+2)^2} < \frac{4x}{\delta^2} < \frac{-4}{\delta^2}.$$

Now, $\frac{-4}{\delta^2}$ can be made smaller than $-M$, if we take $\delta < \frac{2}{\sqrt{M}}$.

Thus, if $0 < \delta < \min\left\{1, \frac{2}{\sqrt{M}}\right\}$, then

$$0 < |x + 2| < \delta \Rightarrow \frac{4x}{(x+2)^2} < -M.$$

Hence, $\lim_{x \to -2} \frac{4x}{(x+2)^2} = -\infty$.

(ii) Let $M > 0$ be given. If $0 < \delta < \frac{\pi}{2}$, then for $x \in]-\delta, \delta[$, $x \neq 0$, we have $\sin^2 x < x^2$

or $\frac{-1}{\sin^2 x} < \frac{-1}{x^2} < -\frac{1}{\delta^2}$.

Thus, if $0 < \delta < \min\left\{\frac{\pi}{2}, \frac{1}{\sqrt{M}}\right\}$, then

$0 < |x| < \delta \Rightarrow \frac{-1}{\sin^2 x} < \frac{-1}{\delta^2} < -M$, and therefore

$$\lim_{x \to 0} \frac{-1}{\sin^2 x} = -\infty.$$

While studying the earlier course on calculus, you would have observed that it is not easy to evaluate the limits of most of the functions by directly applying the definition. The same holds here too.

The following theorem, which is very easy to prove, is very useful for applications, as you will see later. It also provides a connection between the notions of finite and infinite limits.

Theorem 1 : (i) $\lim_{x \to a} f(x) = \infty$ if and only if $f(x)$ is positive in $]a - \delta, a + \delta[$ except possibly at a for some $\delta > 0$, and $\lim_{x \to a} \frac{1}{f(x)} = 0$.

(ii) $\lim_{x \to a} f(x) = -\infty$ if and only if $f(x)$ is negative in $]a - \delta, a + \delta[$ except possibly at a for some $\delta > 0$, and $\lim_{x \to a} \frac{1}{f(x)} = 0$.

Proof : (i) Suppose that $\lim_{x \to a} f(x) = \infty$.

We have to show that $\lim_{x \to a} \frac{1}{f(x)} = 0$. Let $\varepsilon > 0$ be given. Then for this $\varepsilon > 0$, choose M s.t. $M > \frac{1}{\varepsilon}$. For this M, $\exists\, \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow f(x) > M > \frac{1}{\varepsilon}$$

i.e., $0 < |x - a| < \delta \Rightarrow \frac{1}{f(x)} < \frac{1}{M} < \varepsilon$,

i.e., $0 < |x - a| < \delta \Rightarrow \left|\frac{1}{f(x)} - 0\right| < \varepsilon$.

Thus, $\lim_{x\to a} \frac{1}{f(x)} = 0$.

Also, $f(x) > M$, for all x such that $0 < |x - a| < \delta \Rightarrow f(x)$ is positive in $]a - \delta, a + \delta[$, except possibly at a.

To prove the converse, suppose f(x) is positive in $]a - \delta, a + \delta[$, for some $\delta > 0$ (except possibly at a), and $\lim_{x\to a} \frac{1}{f(x)} = 0$. Then given $M > 0$, choose a $\delta_1 > 0$, $\delta_1 < \delta$ such that

$$0 < |x - a| < \delta_1 \Rightarrow \left|\frac{1}{f(x)}\right| < \frac{1}{M}$$

$$\Rightarrow \frac{1}{f(x)} < \frac{1}{M}, \text{ since } f(x) > 0.$$

$$\Rightarrow f(x) > M.$$

This means that $\lim_{x\to a} f(x) = \infty$.

The proof of (ii) is easy and we are leaving it to you as an exercise (see E 4).

The following example will illustrate the usefulness of this theorem.

Example 3 : Let us show that

(i) $\lim_{x\to \pi/2} \frac{1}{1-\sin x} = \infty$

(ii) $\lim_{x\to 0} \frac{-1}{(e^x + e^{-x} - 2)^2} = -\infty$

(iii) $\lim_{x\to \pi/2} \frac{x}{\sin 2x \cos x} = \infty$

We will take up (i) first.

(i) Clearly $\frac{1}{1-\sin x} > 0$ for all x and $\lim_{x\to\pi/2} (1 - \sin x) = 0$.

Therefore, by Theorem 1,

$$\lim_{x\to \pi/2} \frac{1}{1-\sin x} = \infty.$$

(ii) $\lim_{x\to 0} (e^x + e^{-x} - 2) = 0 \Rightarrow \lim_{x\to 0} (e^x + e^{-x} - 2)^2 = 0$.

Also, $\frac{-1}{(e^x + e^{-x} - 2)^2}$ is negative for all x. Therefore, it follows from Theorem 1 that

$$\lim_{x\to 0} \frac{-1}{(e^x + e^{-x} - 2)^2} = -\infty.$$

(iii) Since $\frac{x}{\sin 2x \cos x} = \frac{x}{2\sin x\cos^2 x}$, and $\sin x > 0$ and $\cos^2 x > 0$ for all x such that $0 < x < \pi$, $x \neq \frac{\pi}{2}$, we have

$\frac{x}{\sin 2x \cos x} > 0$ for all x such that $0 < x < \pi$, except for $x = \frac{\pi}{2}$.

Further, $\lim_{x\to \pi/2} \frac{\sin 2x \cos x}{x} = 0$.

Thus, $\lim_{x\to \pi/2} \frac{x}{\sin 2x \cos x} = \infty$.

We have seen that the existence of $\lim_{x\to a} f(x) = L$ implies that the function f(x) is bounded in a neighbourhood of a (except possibly at a). $\lim_{x\to a} f(x) = \infty$ (or $-\infty$) on the other hand, implies that f(x) is not bounded above (resp., below) in any neighbourhood of a (except possibly at a). From this we can infer that these three cases are mutually exclusive, i.e., as

$x \to a$, f(x) cannot tend to a finite number (L) as well as to ∞ or $-\infty$. In other words, $\lim_{x\to a} f(x) = L$, where L is any real number, ∞ or $-\infty$, is unique.

Try the following exercises now.

E 4) Prove part (ii) of Theorem 1.

E 5) Evaluate the following limits

a) $\lim_{x\to 2} \frac{1}{|x-2|}$

b) $\lim_{x\to 0} \left(2x^2 - \frac{5}{x^2}\right)$

c) $\lim_{x\to 0} \frac{1}{x^2(x-5)}$

d) $\lim_{x\to 0} \frac{1}{2x(e^x - 1)}$

e) $\lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{h}}\right)$

f) $\lim_{x\to 0} e^{-1/x^2}$

g) $\lim_{x\to 0} \frac{1}{x^{2n+1}(e^x - 1)}$, n is a non-negative integer.

h) $\lim_{x\to \pi/2} \frac{1}{(\frac{\pi}{2} - x)\cos^3 x}$

i) $\lim_{x\to \pi/2} x \tan^2 x$

j) $\lim_{x\to \pi} \frac{1}{(\sin x)(x-\pi)}$

1.3.2 One-sided Infinite Limits

We now generalise the concept of one-sided limits (Sec. 2.2 of Calculus) to include ∞ or $-\infty$ as a limit.

Definition 5 : Let f be a real-valued function defined in the open interval] a, a + h [, h > 0.

The condition $\delta < h$ is necessary since f is defined in] a, a + h[.

i) f(x) is said to approach the limit ∞ as x approaches a from the right if given a real number M there exists a positive real number δ (depending on M), $\delta < h$, such that

$a < x < a + \delta \Rightarrow f(x) > M.$

ii) f(x) is said to tend to the limit $-\infty$ as x tends to a from the right if given a real number m there exists a positive real number δ (depending on m), $\delta < h$, such that

$a < x < a + \delta \Rightarrow f(x) < m.$

Definition 6 : Let f be a real-valued function defined in the open interval]a – h, a[.

i) We say that f(x) approaches the limit $+\infty$ as x approaches a from the left if given a real number M there exists a positive real number δ (depending on M), $\delta < h$, such that

$a - \delta < x < a \Rightarrow f(x) > M$

ii) We say that f(x) approaches the limit $-\infty$ as x approaches a from the left if given a real number m there exists a positive real number δ (depending on m), $\delta < h$, such that

$a - \delta < x < a \Rightarrow f(x) < m.$

The symbols $\lim_{x\to a^+} f(x) = \pm\infty$, or

$f(x) \to \pm\infty$ as $x \to a^+$, or

$\lim_{x\to a+0} f(x) = \pm\infty$, or

$f(x) \to \pm\infty$ as $x \to a + 0$

will be used to express the fact that f(x) approaches ∞ or $-\infty$ as x approaches a from the right. For the left-sided limit, we shall use the minus sign in place of the plus sign. For example, $\lim_{x\to a-0} f(x) = \infty$ means that f(x) approaches ∞ as x approaches a from the left.

Remark 4 : (i) There is no loss of generality if we take $M > 0$ in Definitions 5(i) and 6(i), and $m < 0$ in Definitions 5(ii) and 6(ii). (Compare with Remark 3).

(ii) With suitable modifications Theorem 1 holds for one-sided limits. More precisely, we can easily prove that

$\lim\limits_{x \to a^+} f(x) = \infty$ (or $-\infty$) if and only if $f(x) > 0$ (or < 0) in some open interval $]a, a + \delta[$ and $\frac{1}{f(x)} \to 0$ as $x \to a^+$.

$\lim\limits_{x \to a^-} f(x) = \infty$ (or $-\infty$) if and only if $f(x) > 0$ (or < 0) in some open interval $]a, a + \delta, a[$ and $\frac{1}{f(x)} \to 0$ as $x \to a^-$.

In Unit 2 of Calculus you have studied a result (Theorem 4) which brings out the connection between limits and one-sided limits. A similar result holds for infinite limits too. Thus we have

$\lim\limits_{x \to a} f(x) = \infty$ (or $-\infty$) if only if

$\lim\limits_{x \to a^+} f(x) = \infty$ (or $-\infty$) and $\lim\limits_{x \to a^-} f(x) = \infty$ (or $-\infty$).

The following examples will give you some practice in dealing with one-sided limits.

Example 4 : Let us check the following one-sided limits.

(i) $\lim\limits_{x \to 5^+} \frac{1}{\sqrt{x^2 - 25}} = \infty$

(ii) $\lim\limits_{x \to 0^+} \frac{1}{e^x - 1} = \infty$

(iii) $\lim\limits_{x \to 2^-} \frac{x^2}{(x-2)^3} = -\infty$

(iv) $\lim\limits_{x \to 0^-} \frac{\cos x}{x^2 \sin x} = -\infty$

We shall take them one by one.

(i) Clearly $\sqrt{x^2 - 25} > 0$ for $x > 5$ and $\sqrt{x^2 - 25} \to 0$ as $x \to 5^+$.

Therefore, in view of Remark 4(ii),

$$\lim_{x \to 5^-} \frac{1}{\sqrt{x^2 - 25}} = \infty$$

(ii) Since $e^x - 1 > 0$ for $x > 0$ and $e^x - 1 \to 0$ as $x \to 0$, it follows that $\frac{1}{e^x - 1} \to \infty$ as $x \to 0^+$.

(iii) Since, $\frac{(x-2)^3}{x^2} \to 0$ as $x \to 2$ and $\frac{x^2}{(x-2)^3} < 0$ for $x < 2$, we have,

$$\lim_{x \to 2^-} \frac{x^2}{(x-2)^3} = -\infty.$$

(iv) For $-\frac{\pi}{2} < x < 0$, $\frac{\cos x}{x^2 \sin x} < 0$ and $\frac{x^2 \sin x}{\cos x} \to 0$ as $x \to 0$. It follows that

$$\lim_{x \to 0^-} \frac{\cos x}{x^2 \sin x} = -\infty$$

Example 5 : We shall now use the observation made after Remark 4 to show that the limits of the following functions as $x \to 0$, do not exist.

(i) $f(x) = \frac{1}{e^x - 1}$,

(ii) $f(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ 2x + 3, & x < 0 \end{cases}$

Let us start with (i).

(i) We have already seen in Example 4 that

$$\lim_{x \to 0^+} \frac{1}{e^x - 1} = \infty.$$

Since, $\frac{1}{e^x - 1} < 0$ for $x < 0$, and $e^x - 1 \to 0$ as $x \to 0$, we have

$$\lim_{x \to 0^-} \frac{1}{e^x - 1} = -\infty, \text{ and therefore } \lim_{x \to 0} \frac{1}{e^x - 1} \text{ does not exist.}$$

ii) It can be seen easily that

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \text{ and } \lim_{x \to 0^-} 2x + 3 = 3,$$

which implies that the given limit does not exist.

See if you can do these exercises now.

E 6) Evaluate the following limits :

a) $\lim_{x \to \pi/2^+} x \tan x$

b) $\lim_{x \to 0^+} \frac{x + 2}{2x(e^x - 1)}$

c) $\lim_{x \to 1^-} \frac{1}{(1 - x)\ln x}$

d) $\lim_{x \to 3^+} \frac{1 + 2x + x^2}{3 - x}$

e) $\lim_{x \to 1^+} \frac{x^2[x] + 2}{x^2 + x - 2}$, where $[x]$ denotes the greatest integer $\leq x$.

E 7) Discuss whether the following limits exist or not. Evaluate the limit, if it exists.

a) $\lim_{x \to \pi/2} \frac{x^2 + 2}{\sin x \cos x}$

b) $\lim_{x \to 1} \frac{x^2[x] + 2}{x^2 + x - 2}$

c) $\lim_{x \to \pi/2} x \tan x$

d) $\lim_{x \to 0} f(x)$, where

$$f(x) = \begin{cases} \frac{1}{\sin x}, & x > 0 \\ x^2 + 2x + 3, & x < 0 \end{cases}$$

e) $\lim_{x \to 0} \sin \frac{1}{x}$

E 8) Let $f(x)$ and $g(x)$ be two polynomials with real coefficients having α as a root with multiplicity m and n, respectively. That is, $f(x) = (x - \alpha)^m f_1(x)$ and $g(x) = (x - \alpha)^n g_1(x)$ where α is not a root of $f_1(x)$ or $g_1(x)$. Then prove the following :

a) $\lim_{x \to \alpha} \frac{f(x)}{g(x)} = 0$ if $m > n$

b) $\lim_{x \to \alpha} \frac{f(x)}{g(x)}$ is finite and different from zero if $n = m$.

c) $\lim_{x \to \alpha} \frac{f(x)}{g(x)}$ does not exist if $m - n$ is odd and $m < n$.

d) $\lim_{x \to \alpha} \frac{f(x)}{g(x)} = \infty$ or $-\infty$

according as $\frac{f(x)}{(x - \alpha)^m} \cdot \frac{g(x)}{(x - \alpha)^n} > 0$ or < 0 at $x = \alpha$,

if $m - n$ is even and $m < n$.

1.3.3 Limit as the Independent Variable Tends to ∞ or $-\infty$

Until now we have considered the limit of a function f as the independent variable approaches a finite real number a. The existence of $\lim_{x \to a} f(x)$ gives us some information about the behaviour of the function near a only. But functions like e^x, $\sin x$, $\sqrt{x}$, $\sqrt{1-x}$, $\frac{1}{x^2-1}$, which are defined for large values of x or for small values of x, arise naturally in many contexts. We would like to know the behaviour of such functions for large or small values of x. For this, we extend the notion of limit to include the cases when the independent variable x "approaches ∞" or "approaches −∞". You have already studied such cases in Calculus. Here we'll recall these definitions and also extend them to include infinite limits.

Definition 7 : Let f be a real-valued function defined for all $x > r$, where r is some real number.

(i) f(x) is said to approach a real number L as x approaches ∞, i.e., $\lim_{x \to \infty} f(x) = L$, if given any real number $\varepsilon > 0$ there exists a real number G (depending on ε), $G > r$, such that

$x > G \Rightarrow |f(x) - L| < \varepsilon.$

ii) f(x) is said to approach ∞ as x approaches ∞,

i.e., $\lim_{x \to \infty} f(x) = \infty$, if given any real number M there exists a real number G (depending on M), $G > r$ such that

$x > G \Rightarrow f(x) > M.$

iii) The function f(x) is said to approach $-\infty$ as x approaches ∞ i.e., $\lim_{x \to \infty} f(x) = -\infty$, if given any real number m there exists a real number G (depending upon m), $G > r$, such that

$x > G \Rightarrow f(x) < m.$

Definition 8 : Let f(x) be a real-valued function defined for all $x < r$, where r is some real number.

i) The function f(x) is said to approach a real number L as x approaches $-\infty$, if given any real number $\varepsilon > 0$, there exists a real number g (depending on ε), $g < r$, such that

$x < g \Rightarrow |f(x) - L| < \varepsilon.$

ii) The function f(x) is said to approach ∞ as x approaches $-\infty$ if given any real number M there exists a real number g (depending on M), $g < r$, such that

$x < g \Rightarrow f(x) > M.$

iii) The function f(x) is said to approach $-\infty$ as x approaches $-\infty$ if given any real number m there exists a real number g (depending on m), $g < r$, such that

$x < g \Rightarrow f(x) < m.$

Remark 5 : (i) As remarked earlier, (See Remarks 3 and 4) we can assume without loss of generality that $M > 0$ and $m < 0$ in the above definitions.

(ii) Clearly $f(x) \to \infty$ (or $-\infty$) as $x \to \infty$, (i.e., $\lim_{x \to \infty} f(x) = \infty$ or $-\infty$) if and only if $f(x) > 0$ (or $f(x) < 0$) for all large values of x and $\frac{1}{f(x)} \to 0$ as $x \to \infty$.

(iii) $\lim_{x \to -\infty} f(x) = \infty$ (or $-\infty$) if and only if $f(x) > 0$ (or $f(x) < 0$) for all small values of x and $\frac{1}{f(x)} \to 0$ as $x \to -\infty$.

Thus, in this section we have defined $\lim_{x \to a} f(x) = L$, where a is any real number or $a = \infty$ or $-\infty$ and L is any real number or $L = \infty$ or $-\infty$.

At the end of Sec. 1.3.1 we have noted that $\lim_{x \to a} f(x)$, $a \in \mathbf{R}$, if it exists, is unique. Similar reasoning shows that $\lim_{x \to a} f(x)$, $a \in \mathbf{R}_\infty$, if it exists, whether finite or not, is unique.

Note that the value of f(x) at a is immaterial for the existence of $\lim_{x \to a} f(x)$. In fact, f(x) may not even be defined at $x = a$. You know that $\lim_{x \to 0} \frac{\sin x}{x} = 1$. But $\frac{\sin x}{x}$ is not defined at $x = 0$.

Further, if $\lim_{x \to a} f(x)$ exists, it will continue to exist and its value will remain unaltered, if the function f(x) is changed arbitrarily outside a neighbourhood of a. For example, consider the function given by $f(x) = x, x \in \mathbf{R}$. Here $\lim_{x \to 0} f(x) = 0$. Now if we define a function g by $g(x) = f(x) = x$ for $x \in]-\delta, \delta[$ for some $\delta > 0$, and $g(x) = 1$ otherwise, then we get $\lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 0$.

Thus, changing f(x) outside a neighbourhood of 0 does not have any effect on the value of the limit.

In the example below we show how to evaluate certain limits using only the definition.

Example 6 : Let us show that

i) $\lim_{x \to \infty} \frac{3x^2}{x^2 + 2} = 3$

ii) $\lim_{x \to \infty} e^{-ax} = 0$, where a is a positive real number

iii) $\lim_{x \to \infty} \frac{e^x}{e^x + 1} = 1$

We'll take these one by one.

i) Let $\varepsilon > 0$ be given. Then

$$\left|\frac{3x^2}{x^2 + 2} - 3\right| = \left|\frac{6}{x^2 + 2}\right| < \frac{6}{x^2} < \varepsilon \text{ if } x > \sqrt{\frac{6}{\varepsilon}}.$$

Thus, given $\varepsilon > 0$, we have found $G = \sqrt{\frac{6}{\varepsilon}}$ such that

$$x > G \Rightarrow \left|\frac{3x^2}{x^2 + 2} - 3\right| < \varepsilon$$

This shows that $\lim_{x \to \infty} \frac{3x^2}{x^2 + 2} = 3$.

ii) Let ε be such that $0 < \varepsilon < 1$. Then

$\left|e^{-ax}\right| < \varepsilon$ if and only if $\frac{1}{\varepsilon} < e^{ax}$

if and only if $x > \frac{1}{a} \ln \frac{1}{\varepsilon}$

Thus, $\lim_{x \to \infty} e^{-ax} = 0$ when $a > 0$.

(iii) Let ε be such that $0 < \varepsilon < 1$. Then

$$\left|\frac{e^x}{e^x + 1} - 1\right| = \frac{1}{e^x + 1} < \frac{1}{e^x} < \varepsilon, \text{ if } x > \ln \frac{1}{\varepsilon}$$

Thus, $x > G = \ln \frac{1}{\varepsilon} \Rightarrow \left|\frac{e^x}{e^x + 1} - 1\right| < \varepsilon$

This shows that $\lim_{x \to \infty} \frac{e^x}{e^x + 1} = 1$.

Now let us turn our attention to algebra of limits.

1.3.4 Algebra of Limits

The algebra of limits, which was stated for finite limits in Section 2, Unit 2 of Calculus holds good for infinite limits also. We now state it (without proof) in the following theorem. You will realise that it greatly reduces our labour for evaluating certain limits.

Theorem 2 (Algebra of Limits) : Let $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$ where a, L, $M \in \mathbf{R}_{\infty}$. Then

i) $\lim_{x \to a} c\, f(x) = c.L$, where c is a constant

ii) $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} (f(x) \pm g(x)) = L \pm M$

iii) $\lim_{x \to a} (fg(x)) = \lim_{x \to a} (f(x)\, g(x)) = LM$

iv) $\lim_{x \to a} \left(\frac{f}{g}(x) \right) = \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \cdot M \neq 0,$

provided the right-hand side makes sense, i.e. cL, $L \pm M$, LM, $\frac{L}{M}$ do not become indeterminate forms.

The algebra of limits states that if $\lim_{x \to a} f(x) * \lim_{x \to a} g(x)$, where $*$ denotes any one of the symbols +, –, ×, ÷, does not become an indeterminate form, then it is equal to $\lim_{x \to a} (f * g(x))$. However, if $\lim_{x \to a} f(x) * \lim_{x \to a} g(x)$ becomes an indeterminate form, then it does not mean that $\lim_{x \to a} (f*g)(x)$ does not exist. It simply means that the algebra of limits does not apply in these cases. For such cases, where the algebra of limits fails, new methods have to be devised to calculate $\lim_{x \to a} (f \pm g)(x)$, $\lim_{x \to a} (fg)(x)$ or $\lim_{x \to a} \frac{f(x)}{g(x)}$.

We'il study these in the next unit.

Our next example illustrates the usefulness of algebra of limits.

Example 7 : Suppose we want to show that

(i) $\lim_{x \to \infty} \frac{e^x - 1}{e^x + 1} = 1$

(ii) $\lim_{x \to -\infty} \frac{e^x + 1}{e^x - 1} = -1$

(iii) $\lim_{x \to \infty} \frac{x^5 + 4x^3 + 3x^2 + 7}{2x^5 + x^4 + 3x + 6} = \frac{1}{2}$

(iv) $\lim_{x \to \infty} \frac{x^4 + 7x^3 + 3x^2 + 2}{x^3 + 6x + 5} = \infty$

Let us start with the first limit.

(i) $\lim_{x \to \infty} \frac{e^x - 1}{e^x + 1} = \lim_{x \to \infty} \frac{1 - e^{-x}}{1 + e^{-x}}$ (dividing numerator and denominator by e^x)

But, $\lim_{x \to \infty} (1 - e^{-x}) = \lim_{x \to \infty} (1 + e^{-x}) = 1.$

Therefore by using algebra of limits we obtain

$$\lim_{x \to \infty} \frac{e^x - 1}{e^x + 1} = 1.$$

(ii) Again, by using algebra of limits we can easily show that

$\lim_{x \to -\infty} \frac{e^x - 1}{e^x + 1} = -1$, because

$\lim_{x \to -\infty} e^x - 1 = -1$ and $\lim_{x \to -\infty} e^x + 1 = 1$

(iii) For $x \neq 0$, we can write

$$\frac{x^5 + 4x^3 + 3x^2 + 7}{2x^5 + x^4 + 3x + 6} = \frac{1 + \frac{4}{x^2} + \frac{3}{x^3} + \frac{7}{x^5}}{2 + \frac{1}{x} + \frac{3}{x^4} + \frac{6}{x^5}}$$

Let $f(x) = 1 + \frac{4}{x^2} + \frac{3}{x^3} + \frac{7}{x^5}$ and $g(x) = 2 + \frac{1}{x} + \frac{3}{x^4} + \frac{6}{x^5}$

Then $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(1 + \frac{4}{x^2} + \frac{3}{x^3} + \frac{7}{x^5}\right) = 1$ and

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} \left(2 + \frac{1}{x} + \frac{3}{x^4} + \frac{6}{x^5}\right) = 2$$

as $\frac{1}{x^n} \to 0$ for all integral $n \geq 1$ when $x \to \infty$.

Thus, $\lim\limits_{x \to \infty} \dfrac{x^5 + 4x^3 + 3x^2 + 7}{2x^5 + x^4 + 3x + 6} = \lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to \infty} f(x)}{\lim\limits_{x \to \infty} g(x)} = \dfrac{1}{2}$

(iv) For $x \neq 0$, we can write

$$\frac{x^4 + 7x^3 + 3x^2 + 2}{x^3 + 6x + 5} = \frac{x + 7 + \frac{3}{x} + \frac{2}{x^3}}{1 + \frac{6}{x^2} + \frac{5}{x^3}} = \frac{f(x)}{g(x)},$$

where $f(x) = x + 7 + \frac{3}{x} + \frac{2}{x^3}$ and $g(x) = 1 + \frac{6}{x^2} + \frac{5}{x^3}$.

It is obvious that

$\lim\limits_{x \to \infty} f(x) = \infty$ and $\lim\limits_{x \to \infty} g(x) = 1$

Therefore, we obtain

$$\lim_{x \to \infty} \frac{x^4 + 7x^3 + 3x^2 + 2}{x^3 + 6x + 5} = \lim_{x \to \infty} \frac{x + 7 + \frac{3}{x} + \frac{2}{x^3}}{1 + \frac{6}{x^2} + \frac{5}{x^3}}$$

$$\frac{\lim\limits_{x \to \infty}\left(x + 7 + \frac{3}{x} + \frac{2}{x^3}\right)}{\lim\limits_{x \to \infty}\left(1 + \frac{6}{x^2} + \frac{5}{x^3}\right)} = \frac{\infty}{1} = \infty.$$

We now give two results (Theorem 3 and Theorem 4) which you would find very useful in evaluating $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to \infty} f(x)$.

Theorem 3 : (i) $\lim\limits_{x \to \infty} f(x) = L$ if and only if

$$\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L$$

(ii) $\lim\limits_{x \to -\infty} f(x) = L$ if and only if

$$\lim_{x \to 0^-} f\left(\frac{1}{x}\right) = L.$$

where L is any real number, ∞ or $-\infty$.

Proof : (i) Suppose $\lim\limits_{x \to \infty} f(x) = L$. We have to prove that

$\lim\limits_{x \to 0^+} f\left(\frac{1}{x}\right) = L$. Now for a given $\varepsilon > 0$, choose $M > 0$ s.t.

$x > M \Rightarrow f(x) \in \,]L - \varepsilon, L + \varepsilon[$, that is,

$\frac{1}{x} < \frac{1}{M} \Rightarrow f(x) \in \,]L - \varepsilon, L + \varepsilon[$.

If we take $\delta = \frac{1}{M}$ and write $y = \frac{1}{x}$, then

$$0 < y < \delta \Rightarrow f\left(\frac{1}{y}\right) \in \,]L - \varepsilon, L + \varepsilon[.$$

This means that

$$\lim_{y \to 0^+} f\left(\frac{1}{y}\right) = \lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L.$$

By reversing the arguments we can prove that if

$\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = L$, then $\lim_{x \to \infty} f(x) = L$.

The proof of (ii) follows on similar lines, and we are leaving it to you as an exercise (see E 9).

E 9) Prove Part (ii) of Theorem 3.

We now illustrate the utility of Theorem 3 with the help of an example.

Example 8 : Suppose we want to prove that $\lim_{x \to \infty} \sin \frac{1}{x} = 0$.

Here $f(x) = \sin \frac{1}{x}$. Therefore, $f\left(\frac{1}{x}\right) = \sin x$.

Now, $\lim_{x \to 0^+} f\left(\frac{1}{x}\right) = \lim_{x \to 0^+} \sin x = 0$.

Therefore, by Theorem 3, we can conclude that $\lim_{x \to \infty} \sin \frac{1}{x} = 0$.

Now here is an example which illustrates yet another useful result about limits.

Example 9 : Let us show that

(i) $\lim_{x \to \infty} \frac{1}{x^2} \sin x = 0$ and ii) $\lim_{x \to -\infty} e^{ax} \sin \frac{1}{x} = 0$, $a > 0$.

We start with (i).

(i) Let ε be such that $0 < \varepsilon < 1$. Then

$$\left|\frac{1}{x^2} \sin x\right| \le \frac{1}{x^2} < \varepsilon$$

for all $x > \sqrt{\frac{1}{\varepsilon}}$, showing that $\lim_{x \to \infty} \frac{1}{x^2} \sin x = 0$.

(ii) Let ε be such that $0 < \varepsilon < 1$. It is obvious that

$$\left|e^{ax} \sin \frac{1}{x}\right| \le e^{ax} < \varepsilon \text{ if } x < \frac{1}{a} \ln \varepsilon.$$

This proves that $\lim_{x \to -\infty} e^{ax} \sin \frac{1}{x} = 0$.

Both the results mentioned in the above example are particular cases of the following simple result.

"Let $\lim_{x \to a} f(x) = 0$, where $a \in R_{\infty}$. If $g(x)$ is a bounded function defined in a neighbourhood of a, then $\lim_{x \to a} f(x)\, g(x) = 0$".

Example 10 : To show that $\lim_{x \to \infty} \cos x$ does not exist, we'll start by assuming that this limit does exist, and then arrive at a contradiction.

Since $\cos x$ is a bounded function on the whole real line, it follows that $\lim_{x \to \infty} \cos x$, if it exists, has to be finite.

Let $\lim_{x \to \infty} \cos x = L$. Then for $\varepsilon = \frac{1}{2}$ in particular, there exists a real number $G > 0$ such that $x > G \Rightarrow |\cos x - L| < \varepsilon$.

If $x_1 > G$, $x_2 > G$, then

$$|\cos x_1 - \cos x_2| = |\cos x_1 - L + L - \cos x_2|$$
$$\leq |\cos x_1 - L| + |\cos x_2 - L|$$
$$< 2\varepsilon = 1 \;......(*)$$

Let n be any natural number such that $n\pi > G$. If we take $x_1 = n\pi$ and $x_2 = \frac{(2n+1)\pi}{2}$, then x_1 and x_2 both are greater than G, but

$$|\cos x_1 - \cos x_2| = 1$$

This contradicts (*). This proves that $\lim_{x \to \infty} \cos x$ does not exist.

Similarly we can prove that $\lim_{x \to -\infty} \cos x$, $\lim_{x \to -\infty} \sin x$ and $\lim_{x \to \infty} \sin x$ do not exist.

Now we will state and prove a theorem about the limit of a composite function.

Theorem 4 (Composite Function Rule) : Let f and g be two real-valued functions such that $g \circ f$ is defined for all $x > r$, where r is some real number. If $\lim_{x \to \infty} f(x)$ is finite and g is continuous at $\lim_{x \to \infty} f(x)$, then

$$\lim_{x \to \infty} g(f(x)) = g(\lim_{x \to \infty} f(x))$$

Proof: Suppose that $\lim_{x \to \infty} f(x) = y_0$. Let $\varepsilon > 0$ be given. Then since g is continuous at y_0, there exists a real number $\delta > 0$ such that

$$|y - y_0| < \delta \Rightarrow |g(y) - g(y_0)| < \varepsilon \qquad ...(*).$$

Since $\lim_{x \to \infty} f(x) = y_0$, given $\delta > 0$, there exists a real number $G > r$ such that

$$x > G \Rightarrow |f(x) - y_0| < \delta \qquad ...(**).$$

Combining (*) and (* *) we obtain that

$$x > G \Rightarrow |g(f(x)) - g(y_0)| < \varepsilon$$

i.e. $\lim_{x \to \infty} g(f(x)) = g(y_0) = g(\lim_{x \to \infty} f(x))$

This result remains true when ∞ is replaced by $-\infty$ or any finite real number with suitable modification. More precisely, we have the following theorem.

Theorem 5 (Composite Function Rule) : Let f and g be two real-valued functions such that $g \circ f$ is defined for all x in a neighbourhood of a, except possibly at a, if a is a finite real number. If $\lim_{x \to a} f(x)$ is finite and g is continuous at $\lim_{x \to a} f(x)$, then

$$\lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x)).$$

You should modify the proof of Theorem 4 to obtain a proof of the above theorem in case a is a finite real number or a is $-\infty$.

We now give an example to illustrate the usefulness of these theorems.

Example 11 : Let us evaluate the following limits :

i) $\lim_{x \to 7} \sqrt[3]{5x - 8}$

ii) $\lim_{x \to -\infty} \sqrt{\frac{x^2}{2x^2 - 5}}$

Let us consider these one by one.

(i) Let $h(x) = \sqrt[3]{5x-8}$. Then h(x) is the composite of two functions f(x) and g(x) given by

$f(x) = 5x - 8$ and $g(x) = \sqrt[3]{x}$.

That is. $h(x) = g \circ f(x)$. Also

$\lim\limits_{x \to 7} f(x) = \lim\limits_{x \to 7} 5x - 8 = 27.$

Now, since the function g is continuous for all x, g is continuous at 27. Therefore, by Theorem 5 we get

$$\lim_{x \to 7} \sqrt[3]{5x-8} = \sqrt[3]{\lim_{x \to 7} (5x-8)} = \sqrt[3]{27} = 3.$$

(ii) Let $h(x) = \sqrt{\dfrac{x^2}{2x^2-5}}$. Then $h(x) = g \circ f(x)$ where $f(x) = \dfrac{x^2}{2x^2-5}$ and $g(x) = \sqrt{x}$. Also

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{x^2}{2x^2-5} = \lim_{x \to -\infty} \frac{1}{2 - \frac{5}{x^2}} = \frac{1}{2}$$ and g(x) is continuous at $\frac{1}{2}$.

Therefore by Theorem 5,

$$\lim_{x \to -\infty} \sqrt{\frac{x^2}{2x^2-5}} = \sqrt{\lim_{x \to -\infty} \frac{x^2}{2x^2-5}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

We are sure you can do these exercises now.

E 10) Evaluate the following limits:

a) $\lim\limits_{x \to \infty} \dfrac{x^3 - 2x + 5}{3x^3 + 6x^2 + 7}$ b) $\lim\limits_{x \to -\infty} \dfrac{x^9 + 5x^4 + 6x + 7}{x^7 + 6x^3 + 3x + 5}$

c) $\lim\limits_{t \to \infty} \dfrac{3t^2 + 4t}{4t + 5}$ d) $\lim\limits_{x \to -\infty} \dfrac{x + \cos x}{x + \sin x}$

e) $\lim\limits_{x \to \infty} 2\cos\dfrac{1}{x} + e^{-x} + 5$

f) $\lim\limits_{x \to \infty} ([x] + 1)$, where [x] denotes the greatest integer $\leq x$.

E 11) Using only the definition prove the following.

a) $\lim\limits_{x \to \infty} \dfrac{3x^2}{x^4 + 8} = 0$ b) $\lim\limits_{x \to \infty} (2 + e^{-5x}) = 2$

c) $\lim\limits_{x \to \infty} \dfrac{1}{\ln x} = 0$ d) $\lim\limits_{x \to -\infty} \dfrac{2 + x^5}{x^5} = 1.$

e) $\lim\limits_{x \to \infty} \dfrac{1}{1 + \ln(x-2)} = 0$

Let us quickly recall what we have covered in this unit.

1.4 SUMMARY

In this unit, we have

1. extended the real number system by adding two new symbols ∞ and $-\infty$.
2. defined $\lim\limits_{x \to a} f(x) = L$, where a is any real number, ∞ or $-\infty$ and L is any real number, ∞ or $-\infty$.
3. used the algebra of limits to calculate the limits of some functions.
4. developed some techniques to calculate these limits.

1.5 SOLUTIONS AND ANSWERS

E 1) a) Suppose $\frac{0}{0} = k \in \mathbb{R}$. Then

$2k = 2.\frac{0}{0} = \frac{0}{0} = k.$

This can happen only if k = 0.

Now, if $\frac{0}{0} = 0$, then

$x + 0 = x + \frac{0}{0} = \frac{x.0 + 0}{0} = \frac{0}{0} = 0$, or

$x = 0 \ \forall \ x \in \mathbb{R}$.

This is contradiction.

If $\frac{0}{0} = \infty$, then for $x < 0$, $\frac{0}{0} = \frac{0}{0} \cdot x$, or $\infty = -\infty$, which is again a contradiction.

Similarly we can not say that $\frac{0}{0} = -\infty$.

Hence we cannot assign any value to $\frac{0}{0}$.

b) Suppose $\frac{\infty}{\infty} = k$.

Then $x.k = x . \frac{\infty}{\infty} = \frac{x . \infty}{\infty} = \frac{\infty}{\infty} = k$.

$\Rightarrow k = 0$.

If $\frac{\infty}{\infty} = 0$, then as in a) above, we arrive at a contradiction. Similarly prove that it is not possible to assign the value ∞ or $-\infty$ to $\frac{\infty}{\infty}$.

c) Suppose $0 . \infty = k$.

$x . k = x (0 . \infty) = (x . 0)\infty = 0.\infty = k$.

$\Rightarrow k = 0$.

Now if $0 . \infty = 0$, then $\frac{\infty}{\infty} = \infty . \frac{1}{\infty} = \infty . 0 = 0$. But we have seen that $\frac{\infty}{\infty}$ is undefined.

Also prove that $0 . \infty$ cannot be equal to ∞ or $-\infty$.

E 2) a) If S is an unbounded subset of $\mathbb{R}$, then either S is not bounded below, or it is not bounded above. Suppose S is not bounded below. Thus, S does not have a lower bound in $\mathbb{R}$, and we have seen that in such a case glbS $= -\infty$.

Similarly, if S is not bounded above, then lubS $= \infty$.

b) Take $\mathbb{N} = \{1, 2, 3, ...\}$. $\mathbb{N}$ is not bounded above in $\mathbb{R}$. Therefore, lub$\mathbb{N} = \infty$. But glb$\mathbb{N} = 1 \in \mathbb{R}$.

c) Take $S = \{x \in \mathbb{R} \mid x < 0\} \subset \mathbb{R}$. Then lubS $= 0 \in \mathbb{R}$, But glbS $= -\infty$.

d) Consider $\mathbb{Z} = \{ ...., -2, -1, 0, 1, 2, ...\} \subset \mathbb{R}$.

lub$\mathbb{Z} = \infty$ and glb$\mathbb{Z} = -\infty$.

E 3) a) 0 and 1 are two lower bounds of S and the only upper bound of S $= \infty$.

b) lubS $= \infty$ and glbS = 2.

E 4) Suppose $\lim_{x \to a} f(x) = -\infty$. Then $\forall$ m < 0, $\exists$ $\delta > 0$, s.t.

$0 < |x - a| < \delta \Rightarrow f(x) < m$.

$\Rightarrow f(x)$ is negative.

We have to prove that $\lim_{x \to a} \frac{1}{f(x)} = 0$.

Now given $\varepsilon > 0$, choose $m < -\frac{1}{\varepsilon}$. For this m, $\exists$ $\delta > 0$, s.t.

$$0 < |x - a| < \delta \Rightarrow f(x) < m < -\frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{f(x)} > -\varepsilon.$$

$$\Rightarrow -\frac{1}{f(x)} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{f(x)} \right| < \varepsilon, \text{ since } f(x) \text{ is negative.}$$

$$\Rightarrow \lim_{x \to a} \frac{1}{f(x)} = 0.$$

Now, if $f(x) < 0$ for x such that $0 < |x - a| < \delta_1$ and $\lim_{x \to a} \frac{1}{f(x)} = 0$, then given $m < 0$, choose $\varepsilon < -\frac{1}{m}$. For this ε, $\exists$ a $\delta > 0$, $\delta < \delta_1$ s.t.

$$0 < |x - a| < \delta \Rightarrow \left|\frac{1}{f(x)}\right| < \varepsilon$$

$$\Rightarrow -\frac{1}{f(x)} < -\frac{1}{m}$$

$$\Rightarrow f(x) < m.$$

$$\Rightarrow \lim_{x \to a} f(x) = -\infty.$$

E 5) a) $\frac{1}{|x - 2|} > 0$ and $|x - 2| \to 0$ as $x \to 2$.

$$\Rightarrow \lim_{x \to 2} \frac{1}{|x - 2|} = \infty.$$

b) $2x^2 - \frac{5}{x^2} = \frac{2x^4 - 5}{x^2} < 0$ if $0 < |x| < \sqrt[4]{\frac{5}{2}}$

and $\lim_{x \to 0} \frac{x^2}{2x^4 - 5} = 0$.

$$\therefore \lim_{x \to 0} \left(2x^2 - \frac{5}{x^2} \right) = -\infty.$$

c) $-\infty$.

d) For $x > 0$, $e^x - 1 > 0$ and for $x < 0$, $e^x - 1 < 0$, which shows that $x^{2n+1}(e^x - 1) > 0$ for all $x \neq 0$, where n is any non-negative integer. Since $2x(e^x - 1) \to 0$ as $x \to 0$, it follows that $\frac{1}{2x(e^x - 1)} \to \infty$ as $x \to 0$.

$$\therefore \lim_{x \to 0} \frac{1}{2x(e^x - 1)} = \infty.$$

e) $\frac{-1}{2h^{3/2}}$. (Hint : Rationalise the numerator).

f) 0.

Let ε be such that $0 < \varepsilon < 1$. Then $e^{-1/x^2} < \varepsilon$.

$$\Leftrightarrow \frac{1}{\varepsilon} < e^{1/x^2}$$

$$\Leftrightarrow \ln\frac{1}{\varepsilon} < \frac{1}{x^2}$$

$$\Leftrightarrow x^2 < \frac{1}{\ln\frac{1}{\varepsilon}}$$

Take any δ such that $0 < \delta < \sqrt{\frac{1}{\ln\frac{1}{\varepsilon}}}$. Then $0 < |x| < \delta \Rightarrow e^{-1/x^2} < \varepsilon$.

g) ∞ (see d) above)

h) ∞

i) ∞

j) − ∞.

E 6) a) $x \tan x < 0$ if $\frac{\pi}{2} < x < \pi$, and

$$\frac{1}{x \tan x} = \frac{\cos x}{x \sin x} \to 0 \text{ as } x \to \frac{\pi^+}{2}.$$

$$\therefore \lim_{x \to \pi/2^+} x \tan x = -\infty.$$

b) $\frac{x+2}{2x\cdot(e^x - 1)} > 0$ for $x > 0$, and

$$\frac{2x\ (e^x - 1)}{x + 2} \to 0 \text{ as } x \to 0.$$

Hence, $\lim_{x \to 0^+} \frac{x+2}{2x(e^x - 1)} = \infty.$

c) − ∞,

d) − ∞

e) ∞. If $1 < x < 2$, then $[x] = 1$ and $x^2 + x - 2 = (x + 2)(x - 1) > 0$.

Consequently, $\frac{x^2 [x] + 2}{x^2 + x - 2} > 0$ for all x such that $1 < x < 2$.

Since $\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 [x] + 2} = 0$, the required limit is equal to ∞.

E 7) a) Does not exist.

$$\lim_{x \to \pi/2^+} \frac{x^2 + 2}{\sin x \cos x} = -\infty, \lim_{x \to \pi/2^-} \frac{x^2 + 2}{\sin x \cos x} = \infty.$$

b) Does not exist.

c) Does not exist.

d) Does not exist.

Note that $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to 0^-} f(x) = 3$.

e) Does not exist. To prove this it will be enough to prove that $\lim_{x \to 0^+} \sin\frac{1}{x}$ does not exist.

Now, since $\left|\sin\frac{1}{x}\right| \le 1$, $\lim_{x \to 0^+} \sin\frac{1}{x}$, if it exists, has to be finite. Let, if possible, $\lim_{x \to 0^+} \sin\frac{1}{x} = L$. Then, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < x < \delta \Rightarrow \left|\sin\frac{1}{x} - L\right| < \varepsilon$$

Let $x_1 = \frac{2}{(4n+1)\pi}$, $x_2 = \frac{1}{2n\pi}$.

Then for sufficiently large n, $0 < x_1 < \delta$, $0 < x_2 < \delta$ and

$$1 = \left|\sin\frac{1}{x_1} - \sin\frac{1}{x_2}\right| = \left|\sin\frac{1}{x_1} - L + L - \sin\frac{1}{x_2}\right|$$

$$\leq \left|\sin\frac{1}{x_1} - L\right| + \left|\sin\frac{1}{x_2} - L\right| < 2\varepsilon < 1,$$

if $\varepsilon < \frac{1}{2}$, which is a contradiction.

Therefore, $\lim_{x \to 0^+} \sin\frac{1}{x}$ does not exist.

E 8) In view of the given data we can write

$f(x) = (x-\alpha)^m f_1(x)$ with $f_1(\alpha) \neq 0$

$g(x) = (x-\alpha)^n g_1(x)$ with $g_1(\alpha) \neq 0$

Thus,

a) $\frac{f(x)}{g(x)} = \frac{(x-\alpha)^{m-n} f_1(x)}{g_1(x)}$ for $m > n$,

and therefore,

$$\lim_{x \to \alpha} \frac{f(x)}{g(x)} = \lim_{x \to \alpha} (x-\alpha)^{m-n} \frac{f_1(x)}{g_1(x)}$$

$$= 0 . \frac{f_1(\alpha)}{g_1(\alpha)} = 0.$$

b) Since $\frac{f(x)}{g(x)} = \frac{f_1(x)}{g_1(x)}$ for $m = n$ and $g_1(\alpha) \neq 0$,

it follows that $\lim_{x \to \alpha} \frac{f(x)}{g(x)} = \frac{f_1(\alpha)}{g_1(\alpha)} \neq 0$ if $m = n$,

c) If $m < n$, then

$$\frac{f(x)}{g(x)} = (x-\alpha)^{m-n} \frac{f_1(x)}{g_1(x)}$$

Since $\frac{f_1(x)}{g_1(x)} \neq 0$ at $x = \alpha$ and is continuous there, it has the same sign as $\frac{f_1(\alpha)}{g_1(\alpha)}$ in a neighbourhood of α. But $(x-\alpha)^{m-n}$ is positive for $x > \alpha$ and $(x-\alpha)^{m-n} < 0$ for $x < \alpha$ as m–n is odd. Now $\lim_{x \to \alpha^+} \frac{g(x)}{f(x)} = \infty$, $\lim_{x \to \alpha^-} \frac{g(x)}{f(x)} = -\infty$, where m–n is odd and $m < n$.

$\therefore \lim_{x \to \alpha} \frac{g(x)}{f(x)}$ does not exist and hence $\lim_{x \to \alpha} \frac{f(x)}{g(x)}$ does not exist.

d) For m–n even, $m < n$, $(x-\alpha)^{m-n}$ is always positive. It follows that

$$(x-\alpha)^{m-n} \frac{f_1(x)}{g_1(x)}$$

is positive or negative according as

$\frac{f_1(\alpha)}{g_1(\alpha)} > 0$ or $\frac{f_1(\alpha)}{g_1(\alpha)} < 0$ in a neighbourhood of α.

Since $(x-\alpha)^{n-m} \frac{g_1(x)}{f_1(x)} \to 0$ as $x \to \alpha$, the required result

E 9) Suppose $\lim_{x \to -\infty} f(x) = L$. We have to prove that $\lim_{x \to 0^-} f\left(\frac{1}{x}\right) = L$.

For $\varepsilon > 0$, choose $m < 0$ s.t. $x < m \Rightarrow f(x) \in \,]L-\varepsilon, L+\varepsilon[$, or,

$\frac{1}{x} < -\frac{1}{m} \Rightarrow f(x) \in \,]L-\varepsilon, L+\varepsilon[$.

If $\delta = -\frac{1}{m}$, then $\delta > 0$. Take $y = \frac{1}{x}$.

Then $0 < y < \delta \Rightarrow f\left(\frac{1}{y}\right) \in \,]L-\varepsilon, L+\varepsilon[$.

$\Rightarrow \lim_{y \to 0^-} f\left(\frac{1}{y}\right) = \lim_{x \to 0^-} f\left(\frac{1}{x}\right) = L.$

Similarly prove the reverse.

E 10) a) $\lim_{x \to \infty} \frac{x^3-2x+5}{3x^3+6x^2+7} = \lim_{x \to \infty} \frac{1 - \frac{2}{x^2} + \frac{5}{x^3}}{3 + \frac{6}{x} + \frac{7}{x^3}} = \frac{1}{3}$.

b) ∞

c) ∞

d) $\lim_{x \to -\infty} \frac{x + \cos x}{x + \sin x} = \lim_{x \to -\infty} \frac{1 + \frac{\cos x}{x}}{1 + \frac{\sin x}{x}}$

Since cos x and sin x are bounded functions for all x and

$\lim_{x \to -\infty} \frac{1}{x} = 0$, it follows that

$\lim_{x \to -\infty} \frac{\cos x}{x} = \lim_{x \to -\infty} \frac{\sin x}{x} = 0.$

Thus, $\lim_{x \to -\infty} \frac{x + \cos x}{x + \sin x} = \frac{1+0}{1+0} = 1$.

e) 7.

Since $\lim_{x \to \infty} \frac{1}{x} = 0$ and cos x is continuous everywhere, we obtain

(Theorem 5)

$\lim_{x \to \infty} \frac{1}{x} = \cos\left(\lim_{x \to \infty} \frac{1}{x}\right) = 1.$

The rest is obvious.

f) ∞.

Let M be given. Let N be a natural number so chosen that

N > M. Then

$x > N-1 \Rightarrow [x] + 1 \geq N - 1 + 1 > M$

showing that $\lim_{x \to \infty} ([x] + 1) = \infty$.

E 11) a) Given $\varepsilon > 0$, $\left|\frac{3x^2}{x^4+8}\right| = \frac{3x^2}{x^4+8} < \varepsilon$ iff

$3x^2 - \varepsilon x^4 < 8\varepsilon \Leftrightarrow x^2(\varepsilon x^2 - 3) > -8\varepsilon$

if $(\varepsilon x^2 - 3) > -8\varepsilon$ and $x > 1$.

i.e., if $\varepsilon x^2 > 3 - 8\varepsilon$

i.e., if $x > \sqrt{\frac{3-8\varepsilon}{\varepsilon}}$

$\therefore$ given $\varepsilon > 0$, if $M = \max\left\{1, \sqrt{\frac{3-8\varepsilon}{\varepsilon}}\right\}$, then

$x > M \Rightarrow \left|\frac{3x^2}{x^4+8}\right| < \varepsilon.$

$\Rightarrow \lim_{x\to\infty} \frac{3x^2}{x^4+8} = 0.$

b) Given $\varepsilon > 0$, $\left|2 + e^{-5x} - 2\right| < \varepsilon$ iff

$\left|e^{-5x}\right| = e^{-5x} < \varepsilon$

$\Leftrightarrow -5x < \ln \varepsilon$

$\Leftrightarrow x > \frac{-\ln\varepsilon}{5}$

$\therefore$ If $M = \frac{-\ln\varepsilon}{5}$, then $x > M \Rightarrow \left|2 + e^{-5x} - 2\right| < \varepsilon$

Therefore, $\lim_{x\to\infty} (2 + e^{-5x}) = 2.$

c) Given $\varepsilon > 0$, let $x > 1$. Then $\left|\frac{1}{\ln x}\right| > \varepsilon \Leftrightarrow \frac{1}{\ln x} < \varepsilon$

$\Leftrightarrow x > e^{\varepsilon}$

$\therefore$ Take $M = \max.\{1, e^{\varepsilon}\}$

d) $\left|\frac{2+x^5}{x^5} - 1\right| = \left|\frac{2}{x^5}\right|$

Given $\varepsilon > 0$, let $x < 0$. Then $\left|\frac{2}{x^5}\right| < \varepsilon \Leftrightarrow \frac{2}{x^5} > -\varepsilon$

$\Leftrightarrow x < \sqrt[5]{\frac{-2}{\varepsilon}}$

$\therefore$ Take $m = \min\left\{0, \sqrt[5]{\frac{-2}{\varepsilon}}\right\}$

Then $x < m \Rightarrow \left|\frac{2+x^5}{x^5} - 1\right| < \varepsilon.$

Therefore, $\lim_{x\to-\infty} \frac{2+x^5}{x^5} = 1.$

e) Let $x > 3$. Then since $\ln x > 0$ for $x > 1$,

$\left|\frac{1}{1+\ln(x-2)}\right| = \frac{1}{1+\ln(x-2)} < \varepsilon$

$\Leftrightarrow 1 + \ln(x-2) > \frac{1}{\varepsilon}$

$\Leftrightarrow \ln(x-2) > \frac{1}{\varepsilon} - 1.$

$\Leftrightarrow x - 2 > e^{(1/\varepsilon - 1)}$

$\Leftrightarrow x > 2 + e^{(1/\varepsilon - 1)}$

$\therefore$ Take $M = \max.\{3, 2 + e^{(1/\varepsilon - 1)}\}.$

UNIT 2 L'HOPITAL'S RULE

Structure

2.1 INTRODUCTION

In the last unit after stating the theorem about algebra of limits (Theorem 2, Unit 1), we had remarked that infinite limits do not always obey the four fundamental laws of arithmetic. This means that the sum or product of the limits need not be equal to the limit of the sum or product function. In this unit we describe such situations in some detail, and develop methods to cope with them.

A method which enables us to evaluate most of the limits in such exceptional cases is known as L' Hopital's rule, after the French mathematician L'Hopital. In this rule we use the derivative for evaluating limits. This is in contrast with what we have been doing so far, i.e., evaluating derivatives of functions by calculating certain limits.

Objectives

After reading this unit you should be able to

- identify the types of indeterminate forms,
- evaluate the following limits:

  i) $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ when $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$

  ii) $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ when $\lim\limits_{x \to a} f(x) = \infty = \lim\limits_{x \to a} g(x)$

  iii) $\lim\limits_{x \to a} (f(x) - g(x))$ when $\lim\limits_{x \to a} f(x) = \infty = \lim\limits_{x \to a} g(x)$

  iv) $\lim\limits_{x \to a} f(x)\, g(x)$ when $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = \infty$

  v) $\lim\limits_{x \to a} (f(x))^{g(x)}$ when $\lim\limits_{x \to a} f(x) = 0 = \lim\limits_{x \to a} g(x)$, or

  $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = 0$, or

  $\lim\limits_{x \to a} f(x) = 1$ and $\lim\limits_{x \to a} g(x) = \infty$, where $a \in \mathbb{R}$.

- obtain all the above limits when a is $+\infty$ or $-\infty$.

2.2 INDETERMINATE FORMS

In Unit 1 we have seen that we cannot assign any particular value to symbols like $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$. This is why these symbols are called indeterminate forms. We have also seen that the algebra of limits: $\lim_{x \to a} (f * g)(x) = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$ cannot be applied if the right-hand side is in an indeterminate form. In such a situation we say that $f(x) * g(x)$ is in an indeterminate form as $x \to a$. Now let us get familiar with the various types of such forms.

Suppose f(x) and g(x) are two real-valued functions defined in a neighbourhood V of a, except possibly at a when a is a finite real number.

(i) $\frac{0}{0}$ **Form**: Suppose that $g(x) \neq 0$ for any x in V.

If $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, then $\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ is an expression of the form $\frac{0}{0}$. In this case we say that $\frac{f(x)}{g(x)}$ is in an indeterminate form of the type $\frac{0}{0}$ at $x = a$ or as $x \to a$. For example,

$\frac{e^x - \cos x}{x}$ is in the $\frac{0}{0}$ form at $x = 0$,

$\frac{e^{-x} + \frac{1}{x}}{\frac{1}{x^2} + \frac{2}{x}}$ is in the $\frac{0}{0}$ form at $x = \infty$, and

$\frac{e^x}{1/x^2}$ is in the $\frac{0}{0}$ form at $x = -\infty$.

(ii) $\frac{\infty}{\infty}$ **Form** : If $\lim_{x \to a} f(x) = \pm\infty = \lim_{x \to a} g(x)$, then we say that $\frac{f(x)}{g(x)}$ is in an indeterminate form of the type $\frac{\infty}{\infty}$ at $x = a$.

For example,

$\frac{e^x}{\ln x}$ is in the $\frac{\infty}{\infty}$ form at ∞,

$\frac{x^2}{e^{-x}}$ is in the $\frac{\infty}{\infty}$ form at $-\infty$, and

$\frac{\ln x}{x^{-2n}}$ $(n \in N)$ is in the $\frac{\infty}{\infty}$ form at $x = 0$.

(iii) $\infty - \infty$ **Form** : If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then we say that $f(x) - g(x)$ is in an indeterminate form of the type $\infty - \infty$ at $x = a$. For instance,

$\tan^2 x - (\pi/2 - x)^{-2}$ is in the $\infty - \infty$ form at $x = \pi/2$, and $x^2 - e^x$ is in the $\infty - \infty$ form at ∞.

In general, let $*$ denote any of the four algebraic operations of addition, subtraction, multiplication and division.

If $\lim_{x \to a} f(x) * \lim_{x \to a} g(x)$ is in an indeterminate form, i.e. an expression of the type $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \cdot \infty$, $\infty - \infty$, then we say that $f(x) * g(x)$ is in an indeterminate form at a or as x approaches a.

Other Indeterminate Forms: If $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$, so that $\left[\lim_{x \to a} f(x)\right]^{\lim_{x \to a} g(x)}$ is an expression of the type 1^∞, we say that $(f(x))^{g(x)}$ is in an indeterminate form of the type 1^∞ at $x = a$. For example,

$\left[\frac{\sin x}{x}\right]^{1/x^2}$ is in an indeterminate form of the type 1^∞ at $x = 0$, while

$(1 + 1/x)^x$ is in an indeterminate form of the type 1^∞ at ∞.

Indeterminate forms of the type 0^0, 0^∞, ∞^0 are similarly defined. For example,

$(e^x - 1)^{1-\cos x}$ is in an indeterminate form of type 0^0 at $x = 0$,

$(\sin x)^{1/x^2}$ is in an indeterminate form of the type 0^∞ at $x = 0$, and

$(e^x)^{1/x^2}$ is in an indeterminate form of the type ∞^0 at ∞.

We have said it before, and we repeat it once again that the methods developed by us so far do not enable us to calculate the limits in the situations mentioned above. In what follows we describe methods which would enable us to deal with almost all these situations. But first, see if you can do this exercise.

E 1) Identify the types of indeterminate forms in the following cases.

a) $\frac{e^x}{x^n}$ as $x \to \infty$

b) $\frac{\sin 2x}{x \cos x^2}$ as $x \to 0$

c) $\operatorname{cosec} x - \frac{1}{x}$ as $x \to 0$.

d) $\frac{\sin x}{x}$ as $x \to 0$.

In the next section we will give a simple method for calculating the limits of functions which are in the $\frac{0}{0}$ form.

2.3 L'HOPITAL'S RULE FOR $\frac{0}{0}$ FORM

Guillaume Francois Antoine De L'Hopital, a French mathematician, was a student of Johann Bernoulli. He published the first book on calculus in 1696. This book, based on Bernoulli's lectures, contains a method for evaluating $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ when $\frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ at $x = a$. This result is now universally known as L'Hopital's rule, even though it was proved by Bernoulli. Here we first state and prove the simplest form of this rule. In the subsequent sub-sections, we shall state and sometimes prove other versions of this rule.

2.3.1 Simplest Form of L'Hopital's Rule

We now state the simplest form of L'Hopital's rule as a theorem.

Theorem 1 : If f(x) and g(x) are two real-valued functions such that

i) f(x) and g(x) are differentiable at $x = a$, $a \in \mathbf{R}$,

ii) $f(a) = 0 = g(a)$, and

iii) $g'(a) \neq 0$,

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

Proof : Observe that the existence of $f'(a)$ and $g'(a)$ implies that both the functions are defined in a neighbourhood of a. Moreover, the hypothesis $g'(a) \neq 0$, $g(a) = 0$ implies that g(x) is different from 0 in a deleted neighbourhood of a so that the quotient $\frac{f(x)}{g(x)}$ is defined in a deleted neighbourhood of a. Clearly

A deleted neighbourhood of a means a neighbourhood of a from which the point a has been removed. See the figure below.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)}, \text{ since } f(a) = g(a) = 0.$$

$$= \frac{(f(x) - f(a))/(x - a)}{(g(x) - g(a))/(x - a)}$$

Now since $\lim\limits_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$, and

$\lim\limits_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a) \neq 0$, we can use the algebra of limits to obtain

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}}$$

$$= \frac{\lim_{x \to a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \to a} \frac{g(x) - g(a)}{x - a}}$$

$$= \frac{f'(a)}{g'(a)}.$$

Note that the right hand side of this equation is not in an indeterminate form, since $g'(a) \neq 0$ and is finite.

Here is an example which illustrates the utility of Theorem 1.

Example 1 : Let us find the following limits.

i) $\lim_{x \to 0} \frac{1 - \cos x}{\sin x}$

ii) $\lim_{x \to \pi/2} \frac{(x - \pi/2)^2}{\cos x}$

We'll start with i)

i) Let $f(x) = 1 - \cos x$, $g(x) = \sin x$. Then the hypotheses of Theorem 1 are satisfied and we obtain

$$\lim_{x \to 0} \frac{1 - \cos x}{\sin x} = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0.$$

ii) Since $\frac{d}{dx}(\cos x)$ at $x = \pi/2$ is -1, and

$\frac{d}{dx}(x - \pi/2)^2 = 0$ at $x = \pi/2$, applying Theorem 1, we get

$$\lim_{x \to \pi/2} \frac{(x - \pi/2)^2}{\cos x} = 0$$

In this example we could not have obtained $\lim_{x \to a} \frac{f(x)}{g(x)}$ by applying the algebra of limits directly, because the functions $f(x)$ and $g(x)$, being differentiable at $x = a$, are continuous there, and therefore $\lim_{x \to a} f(x) = f(a) = 0$ and $\lim_{x \to a} g(x) = g(a) = 0$.

Theorem 1 has a very restrictive use. According to the hypotheses of the theorem, the functions $f(x)$ and $g(x)$ have to be defined at a. Whereas for the existence of the limit of the quotient $\frac{f(x)}{g(x)}$, it is not essential that $f(x)$ and $g(x)$ be defined at a. In the next sub-section we'll present another theorem (Theorem 3), in which the conditions of Theorem 1 are replaced by another set of conditions.

Now try and evaluate the limits in the following exercise by applying Theorem 1.

E 2) Find the following limits

a) $\lim_{x \to 0} \frac{x - \sin x}{\sin x}$ b) $\lim_{x \to 1} \frac{x^4 - 1}{x^2 - 6x + 5}$

2.3.2 Another Form of L'Hopital's Rule for $\frac{0}{0}$ Form

In this sub-section we'll state and prove L'Hopital's rule for $\frac{0}{0}$ form (Theorem 3). But to prove it, we'll have to take the help of Cauchy's mean value theorem.

You have already studied Rolle's theorem and Lagrange's mean value theorem (Theorems 2 and 3 in Unit 7, Calculus). Here we shall only state Cauchy's mean value theorem. It is an

Augustin Louis Cauchy (1789–1857) of France is one of the all time great mathematicians of the nineteenth century. His interests spread over a broad area in mathematics. He paid great attention to the logical foundations of mathematical analysis.

easy consequence of Rolle's theorem. You can find the proofs of all these mean value theorems in the course **Real Analysis.**

Theorem 2 (Cauchy's Mean Value Theorem) :

Let f and g be two real-valued functions defined on the closed interval [a, b] such that

i) f and g are continuous on [a, b],

ii) f and g are differentiable on]a, b[, and

iii) $g'(x) \neq 0$ for any x in]a, b[.

Then there exists a real number c in]a, b[such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

We are now ready to state and prove L'Hopital's Rule for the $\frac{0}{0}$ form at a, where a is any real number.

Theorem 3 : Let f(x) and g(x) be two real-valued functions such that

i) $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$, where a is a real number, and

ii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists (and may be $+\infty$ or $-\infty$).

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Proof : According to the hypothesis $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists. This means that f'(x) and g'(x) exist in a deleted neighbourhood of a, i.e., both f and g are differentiable in an interval $]a - \delta, a + \delta[$ except possibly at x = a for some $\delta > 0$. Further, this also means that $g'(x) \neq 0$ for $0 < |x - a| < \delta$, so that $\frac{f'(x)}{g'(x)}$ is defined for $0 < |x - a| < \delta$.

The hypothesis does not say anything about the values of f and g at x = a. In fact, f(x) and g(x) need not be defined at x = a. But let us assume that f(a) = 0, and g(a) = 0. Note that this assumption does not affect the existence or the value of the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$. However, our assumption makes the functions continuous at x = a.

In Sec.1.3.3 we have noted that the value of f(x) at a is immaterial for the existence or the value of $\lim_{x \to a} f(x)$.

Now for any x such that $0 < |x - a| < \delta$, the functions f(x) and g(x) satisfy the requirements of Cauchy's mean value theorem (Theorem 2) in the interval [a, x] or [x, a] according as a < x or x < a. That is,

- f(x) and g(x) are continuous in the closed interval [a, x] or [x, a].
- differentiable in the open interval]a, x[or]x, a[, and
- $g'(y) \neq 0$ for any y in]a, x[or]x, a[.

Thus, applying Theorem 2 we can say that there exists a real number c between a and x such that

Note that f (a) = g (a) = 0

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

Obviously $c \to a$ when $x \to a$, and therefore,

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, and the proof is complete.

Remark 1 : The condition (ii) stated in Theorem 3 is only sufficient and is not necessary for the existence of $\lim_{x \to a} \frac{f(x)}{g(x)}$. Thus, if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ does not exist, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist, and we have to use different techniques to establish its existence and to evaluate it (See Example 3).

If we modify the proof of Theorem 3 a bit, we get the following result, which is referred to as L'Hopital's rule for one-sided limits.

Theorem 4 : Let f(x) and g(x) be two real-valued functions.

i) If $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$, and

$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists (and may be ∞ or $-\infty$),

then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

ii) If $\lim_{x \to a^-} f(x) = 0 = \lim_{x \to a^-} g(x)$, and

$\lim_{x \to a^-} \frac{f'(x)}{g'(x)}$ exists (and may be ∞ or $-\infty$),

then

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}.$$

We, now, illustrate the above discussion with the help of a few examples.

Example 2 : Let us find

i) $\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x}$, and

ii) $\lim_{x \to 1} \frac{\ln x}{x - \sqrt{x}}$

We'll take these one by one.

i) It is obvious that $\frac{1 - \sin x}{\cos x}$ is in the $\frac{0}{0}$ form at $x = \pi/2$.

The functions $1 - \sin x$ and $\cos x$ satisfy the hypotheses of Theorem 3.

Therefore we can apply L'Hopital's rule here. Thus,

$$\lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} = \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = \frac{\cos \pi/2}{\sin \pi/2} = 0$$

ii) Set $f(x) = \ln x$, $g(x) = x - \sqrt{x}$. Clearly, f(x) and g(x) are differentiable in $]1 - \delta, 1 + \delta[$ for a suitable $\delta > 0$ say $\delta = \frac{1}{2}$. Moreover,

$\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} g(x)$, and

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{1/x}{1 - \frac{1}{2\sqrt{x}}} = 2.$$

Thus, L'Hopital's rule gives us

$$\lim_{x \to 1} \frac{\ln x}{x - \sqrt{x}} = 2.$$

Now we give an example of a situation in which L'Hopital's rule cannot be applied. You will see that we can still find the limit of $\frac{f(x)}{g(x)}$.

Example 3 : To find $\lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{\sin x}$, we first note that $\frac{x^2 \sin \frac{1}{x}}{\sin x}$ is in the $\frac{0}{0}$ form at $x = 0$. But L'Hopital's rule is not applicable, because,

$$\lim_{x \to 0} \frac{\frac{d}{dx} x^2 \sin\frac{1}{x}}{\frac{d}{dx}\sin x} = \lim_{x \to 0} \frac{-\cos\frac{1}{x} + 2x\sin\frac{1}{x}}{\cos x} \text{ does not exist.}$$

How can we be sure that this limit does not exist?

Note that if $\lim_{x \to 0} \frac{2x \sin 1/x - \cos 1/x}{\cos x}$ exists, then

As $x \to 0$, $\cos\frac{1}{x}$ oscillates wildly between -1 and 1, and does not tend to any limit.

$\lim_{x \to 0}\left[2x\sin\frac{1}{x} - \cos\frac{1}{x}\right]$ would exist and consequently $\lim_{x \to 0} \cos\frac{1}{x}$ would exist, which is not true.

However, we can still evaluate the limit of $\frac{x^2 \sin 1/x}{\sin x}$ as $x \to 0$.

We have $\left|\frac{x^2 \sin 1/x}{\sin x}\right| < \left|\frac{x^2}{\sin x}\right| = \left|\frac{x}{\sin x}\right| |x|$.

Since, $\frac{x}{\sin x} \to 1$ as $x \to 0$, it follows that

$\frac{x^2}{\sin x} \to 0$ as $x \to 0$, and therefore,

$$\lim_{x \to 0} \frac{x^2 \sin 1/x}{\sin x} = 0.$$

The next example shows that L'Hopital's rule $\left(\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}\right)$ is applicable only to those quotients which are in indeterminate forms of the type $\frac{0}{0}$ at the given point. In other words, the first condition in Theorem 3 is very important.

Example 4 : Suppose we want to find $\lim_{x \to 0} \frac{x + \sin x}{x + \cos x}$. Can we argue that

$$\lim_{x \to 0} \frac{x + \sin x}{x + \cos x} = \lim_{x \to 0} \frac{1 + \cos x}{1 + \sin x} = \frac{2}{1} = 2?$$

It is true that the functions $x + \sin x$ and $x + \cos x$ are differentiable functions, but this argument is not correct, because L'Hopital's rule cannot be applied as $\frac{x + \sin x}{x + \cos x}$ is not in an indeterminate form at $x = 0$. Actually,

$$\lim_{x \to 0} \frac{x + \sin x}{x + \cos x} = \frac{\lim_{x \to 0} (x + \sin x)}{\lim_{x \to 0} (x + \cos x)} = \frac{0}{1} = 0.$$

Now we give another example to show that the conditions stated in Theorem 3 are sufficient and not necessary to evaluate the limit of functions of the form $\frac{f(x)}{g(x)}$.

Example 5 : Let

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $g(x) = x$. Let us evaluate $\lim_{x \to 0} \frac{f(x)}{g(x)}$, and show that Theorem 3 cannot be applied.

Clearly,

(i) $f(x)$ and $g(x)$ are differentiable at 0,

(ii) $f(0) = 0 = g(0)$, and

(iii) $g'(0) \neq 0$.

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

since $\left|h \sin\frac{1}{h}\right| \leq |h| \to 0$

Therefore, by Theorem 1,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{x^2 \sin 1/x}{x} = \frac{f'(0)}{g'(0)} = 0$$

But, Theorem 3 is not applicable because

$\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$ does not exist. That is, condition (ii) in Theorem 3 is violated. And yet, $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$ exists.

You must have noticed that here we have a situation where $\lim\limits_{x \to 0} \frac{f'(x)}{g'(x)}$ does not exist, but $\frac{f'(0)}{g'(0)}$ exists. This means that $\frac{f'(x)}{g'(x)}$ is not continuous at $x = 0$. We suggest that you go over this example again, and try to understand all these points.

The conditions in Theorem 1 are also sufficient and not necessary for the evaluation of $\lim\limits_{x \to 0} \frac{f(x)}{g(x)}$. In our next example we give a situation where Theorem 3 can be applied but Theorem 1 cannot be applied. So Examples 5 and 6 together tell us that Theorem 1 and 3 are independent of each other.

Example 6 : Suppose we want to find $\lim\limits_{x \to 0} \frac{x^2}{\sin^2 x}$. Let $f(x) = x^2$ and $g(x) = \sin^2 x$. It is obvious that $\frac{x^2}{\sin^2 x}$ is in the $\frac{0}{0}$ form at $x = 0$. Now,

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{2x}{2\sin x \cdot \cos x}$$

$$= \lim_{x \to 0} \frac{1}{\frac{\sin x}{x} \cos x}$$

$$= \frac{1}{\lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \lim\limits_{x \to 0} \cos x}$$

$$= 1$$

Therefore by Theorem 3, $\lim\limits_{x \to 0} \frac{x^2}{\sin^2 x} = 1$.

Note that here we can't apply Theorem 1 as $g'(0) = 0$.

Now you can try this exercise.

E 3) Evaluate the following limits:

a) $\lim\limits_{x \to 0} \frac{1 - \cos^2 x}{\sin x^2}$

b) $\lim\limits_{x \to 1/2} \frac{2x^2 - 9x + 4}{\cos \pi x}$

c) $\lim\limits_{x \to \alpha} \frac{x^m - \alpha^m}{x^n - \alpha^n}$, $\alpha > 0$

d) $\lim\limits_{x \to 0} \frac{\ln(1 + 4x)}{4x}$

e) $\lim\limits_{x \to \pi/2} \frac{\ln \sin x}{1 - \sin x}$

f) $\lim\limits_{x \to 0} \frac{3^x - 2^x}{x}$

g) $\lim\limits_{x \to \pi/4} \frac{\sin x - \cos x}{x - \pi/4}$

h) $\lim\limits_{x \to 2} \frac{\sqrt{x^2 + 12} - 4}{x^2 - 4}$

Consider the functions $f(x) = 1 - \cos x$ and $g(x) = x^2$, which are differentiable everywhere. Let us try to evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$. Here $f'(x) = \sin x$, and $g'(x) = 2x$, and $\frac{f'(x)}{g'(x)}$ is again in an indeterminate form of the type $\frac{0}{0}$ at $x = 0$. But let us now turn our attention to the functions $f'(x)$ and $g'(x)$. We find that the functions $f'(x)$ and $g'(x)$ are also differentiable functions, and $f''(x) = \cos x$, $g''(x) = 2$. Clearly, $\lim_{x \to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}$. This means we can apply L'Hopital's rule to the quotient of $f'(x)$ and $g'(x)$ at $x = 0$, and get

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \lim_{x \to 0} \frac{f''(x)}{g''(x)} = \frac{1}{2}.$$

Now since $\lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{2}$, applying L'Hopital's rule to $\frac{f(x)}{g(x)}$ we get

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{1}{2}.$$

Thus, we can write,

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

We often come across similar situations where repeated use of L'Hopital's rule enables us to evaluate the required limits. We now state the general result in the following theorem.

Theorem 5 : Let $f(x)$ and $g(x)$ be two real-valued functions such that

$$\lim_{x \to a} f^{(k)}(x) = 0 = \lim_{x \to a} g^{(k)}(x),\ 0 \le k \le n-1, \text{ for some } n \in \mathbf{N}.$$

(Here, $f^{(0)} = f$, $g^{(0)} = g$, and $f^{(k)}$ denotes the k-th order derivative of f for $1 \le k \le n-1$.)

If $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists (may be equal to ∞ or $-\infty$), then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

The proof of this theorem follows on the lines of the proof of Theorem 3. But let us not worry about it here. We will be interested only in the application of this theorem.

Now we give some general observations in the form of remarks.

Remark 2 : Note that if for some n, $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ does not exist, and $\lim_{x \to a} f^{(k)}(x) = 0 = \lim_{x \to a} g^{(k)}(x)$, $0 \le k \le n-1$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ cannot be evaluted using L'Hopital's rule.

Remark 3 : We can now state the general L'Hopital's rule for one sided limits.

Let $f(x)$ and $g(x)$ be two real-valued functions such that

$$\lim_{x \to a^+} f^{(k)}(x) = 0 = \lim_{x \to a^+} g^{(k)}(x),\ 0 \le k \le n-1 \text{ for some } n \in \mathbf{N}.$$

If $\lim_{x \to a^+} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists (may be equal to $+\infty$ or $-\infty$), then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

If we replace a^+ wherever it occurs by a^-, we get the statement for the left hand limit.

We now give an example to illustrate the above discussion.

Example 7 : Let us evaluate

i) $\lim_{x \to 1} \frac{x^5 - 5x + 4}{x^3 - x^2 - x + 1}$, and

ii) $\lim_{x \to 0} \frac{e^{3x} - 3x - 1}{1 - \cos x}$.

We start with i).

i) If we take $f(x) = x^5 - 5x + 4$ and $g(x) = x^3 - x^2 - x + 1$, then

$$\lim_{x \to 1} f(x) = 0 = \lim_{x \to 1} g(x)$$

$$\lim_{x \to 1} f'(x) = \lim_{x \to 1} (5x^4 - 5) = 0$$

$$\lim_{x \to 1} g'(x) = \lim_{x \to 1} (3x^2 - 2x - 1) = 0$$

and $\lim_{x \to 1} \frac{f''(x)}{g''(x)} = \lim_{x \to 1} \frac{20x^3}{6x - 2} = 5.$

Therefore, by Theorem 5, i.e., by repeated use of L'Hopital's rule, we obtain,

$$\lim_{x \to 1} \frac{x^5 - 5x + 4}{x^3 - x^2 - x + 1} = \lim_{x \to 1} \frac{5x^4 - 5}{3x^2 - 2x - 1} = \lim_{x \to 1} \frac{20x^3}{6x - 2} = 5.$$

ii) If $f(x) = e^{3x} - 3x - 1$ and $g(x) = 1 - \cos x$, then

$f'(x) = 3e^{3x} - 3$ and $g'(x) = \sin x$. Also,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} f'(x) = 0, \text{ and}$$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} g'(x) = 0, \text{ and}$$

$\lim_{x \to 0} \frac{f''(x)}{g''(x)} = \lim_{x \to 0} \frac{9e^{3x}}{\cos x} = 9$, which shows that

$$\lim_{x \to 0} \frac{e^{3x} - 3x - 1}{1 - \cos x} = 9.$$

See if you can solve these exercises now.

E 4) Evaluate the following limits:

a) $\lim_{x \to 0} \frac{(\tan^{-1} x)^2}{\ln(1 + x^2)}$

b) $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x - \sin x}$

c) $\lim_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin^2 x}$

d) $\lim_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$

e) $\lim_{x \to 0} \frac{\sin 3x - 3x}{x^3}$

f) $\lim_{x \to \pi/2} \frac{1 - \sin x}{1 + \cos 2x}$

g) $\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

E 5) Find the value of t for which

$$\lim_{x \to 0} \frac{\sinh 2x + t \sin 2x}{x^3}$$

is finite. Evaluate the limit.

$\sinh x = \frac{e^x - e^{-x}}{2}$

E 6) Find the value of t for which

$$\lim_{x \to 0} \frac{e^x + te^{-x} - 2x}{1 - \cos x}$$

is finite. Find the value of the limit.

E 7) Show that

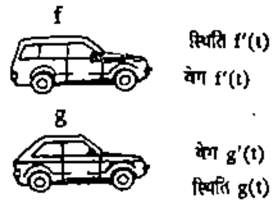
$$\lim_{x \to 0} \frac{x^2 \sin 1/x}{5 \tan x} = 0.$$

Also show that this limit cannot be evaluated by using L'Hopital's rule.

Till now we have considered $\lim_{x \to a} \frac{f(x)}{g(x)}$, where $a \in \mathbf{R}$. What happens if a is equal to ∞ or $-\infty$? Let's see.

L'Hopital's Rule for $\frac{0}{0}$ Form at ∞ or $-\infty$

Theorem 3 and Theorem 5 about L'Hopital's rule at $x = a$ hold good even if $a = \infty$ or $-\infty$.



We shall not try to prove this statement. But intuitively we can say that it has to be true. Imagine that two cars, the f-car and the g-car are on endless journeys. Suppose f (t) and g (t) represent the positions of these cars at time t. The respective velocities of these cars will be $f'(t)$ and g (t). Now, suppose we are given that

$$\lim_{t \to \infty} \frac{f'(t)}{g'(t)} = L.$$

This means that in the long run the f-car travels L times as fast as the g-car. Therefore, is it not reasonable to say that in the long run the f-car will travel L times as far as the g-car, that is,

$$\lim_{t \to \infty} \frac{f(t)}{g(t)} = L?$$

We now solve an example using this fact.

Example 8 : Suppose we want to evaluate

i) $\lim_{x \to \infty} \frac{\tan (3/x)}{\sin (1/x)}$, and

ii) $\lim_{x \to -\infty} x \sin \frac{5}{x}$.

Let's tackle these limits one by one.

i) Let $f(x) = \tan \frac{3}{x}$ and $g(x) = \sin \frac{1}{x}$. Then $\frac{f(x)}{g(x)}$ is in an indeterminate form of the type $\frac{0}{0}$ as $x \to \infty$. Clearly, both f(x) and g(x) are differentiable for all $x \neq 0$, and

$$\lim_{x \to \infty} \frac{(-3/x^2) \sec^2 (3/x)}{(-1/x^2) \cos (1/x)} = \lim_{x \to \infty} \frac{3 \sec^2 (3/x)}{\cos (1/x)} = 3.$$

Thus, L'Hopital's rule for $\frac{0}{0}$ form at ∞ is applicable, and therefore,

$$\lim_{x \to \infty} \frac{\tan (3/x)}{\sin (1/x)} = 3.$$

ii) Let $f(x) = \sin \frac{5}{x}$, $g(x) = \frac{1}{x}$. Then $\frac{f(x)}{g(x)}$ is $\frac{0}{0}$ form at $-\infty$, and

$$\lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = \lim_{x \to -\infty} \frac{(-5/x)^2 \cos (5/x)}{-1/x^2} = 5.$$

Therefore,

$$\lim_{x \to -\infty} \frac{\sin (5/x)}{1/x} = \lim_{x \to -\infty} \frac{f'(x)}{g'(x)} = 5.$$

Now you should be able to do this exercise.

E 8) Evaluate the following limits:

a) $\lim_{x \to \infty} x \tan^{-1} (1/x)$

b) $\lim_{x \to -\infty} \frac{\sin (1/x)}{e^{1/x} - 1}$

c) $\lim_{x \to \infty} x(e^{1/x} - 1)$

d) $\lim_{x \to \infty} 2x(\ln (x + 1) - \ln x)$

e) $\lim_{x \to -\infty} x^2 \ln \left[\frac{x^2 + 1}{x^2}\right]^3$

In the next section we will consider functions in the $\frac{\infty}{\infty}$ form.

2.4 L'HOPITAL'S RULE FOR $\frac{\infty}{\infty}$ FORM

In the last section we have seen how to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$ by L'Hopital's rule when $\frac{f(x)}{g(x)}$ is in the $\frac{0}{0}$ form at x = a. Now we shall study the rule for evaluating $\lim_{x \to a} \frac{f(x)}{g(x)}$ when $\frac{f(x)}{g(x)}$ is in the $\frac{\infty}{\infty}$ form at x = a.

In order to evaluate $\lim_{x \to a} \frac{f(x)}{g(x)}$ when $\lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x)$, we have results similar to those proved in the last section, but their proofs are more complicated. Therefore, we state these results without proofs, and then illustrate them with the help of examples.

Theorem 6 : Let f(x) and g(x) be two real-valued functions such that

i) $\lim_{x \to a} f(x) = \pm \infty = \lim_{x \to a} g(x)$,

where a is any real number, ∞ or $-\infty$,

ii) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, which may even be infinite.

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

We shall not prove this theorem here.

Now as in the last section (see Theorem 4) we can modify the statement of Theorem 6 to cover the evaluation of one-sided limits.

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} \text{ and } \lim_{x \to a^-} \frac{f(x)}{g(x)}.$$

We had also seen, in Theorem 5, how repeated use of L'Hopital's rule sometimes helps us in evaluating the required limit. We now state an analogous result for indeterminate forms of the type $\frac{\infty}{\infty}$.

Theorem 7 : Let f (x) and g (x) be two real-valued functions such that

i) $\lim_{x \to a} f^{(k)}(x) = \pm \infty = \lim_{x \to a} g^{(k)}(x)$,

where $0 \le k \le n-1$, n is a natural number and a is any real number, ∞ or $-\infty$, and

ii) $\lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}$ exists, and may even be infinite.

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

We remind you once again that, as in the last section, the conditions stated in Theorem 6 and Theorem 7 are sufficient and not necessary.

Here is another point that you should note.

If $\frac{f^{(k)}(x)}{g^{(k)}(x)}$ is in an indeterminate form for $0 \le k < n$ and $\frac{f^{(n)}(x)}{g^{(n)}(x)}$ fails to tend to a limit as $x \to a$, then this does not mean that $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist. It only means that we cannot apply L'Hopital's rule, and that we have to adopt a different procedure to establish the existence or non-existence of the limit under consideration.

We shall bring out this and various other points with the help of a number of examples. Go through these carefully. They will help you to get a better understanding of the concepts involved.

Example 9 : Let us try to show that

i) $\lim_{x \to \infty} \frac{e^x}{x^n} = \infty, n \geq 1,$

ii) $\lim_{x \to \infty} \frac{\ln x}{x^n} = 0, n > 0,$ and

iii) $\lim_{x \to 0^+} \frac{\ln \tan 2x}{\ln \tan x} = 1.$

Let us evaluate these limits one by one.

i) Let $f(x) = e^x, g(x) = x^n, n \geq 1$. Then

$$\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x).$$

If $n = 1$, then $\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{e^x}{1} = \infty$

and therefore by L'Hopital's rule

$$\lim_{x \to \infty} \frac{e^x}{x} = \infty.$$

If $n > 1$, then it is clear that,

$$\lim_{x \to \infty} f^{(k)}(x) = \infty = \lim_{x \to \infty} g^{(k)}(x), 0 \leq k < n \text{ and}$$

$$\lim_{x \to \infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} = \lim_{x \to \infty} \frac{e^x}{n!} = \infty.$$

Consequently,

$$\lim_{x \to \infty} \frac{e^x}{x^n} = \infty \text{ for all } n \geq 1,$$

ii) Let $f(x) = \ln x, g(x) = x^n, n > 0.$

The functions f (x) and g (x) satisfy the requirements of Theorem 6. Therefore,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{1/x}{n\, x^{n-1}} = \lim_{x \to \infty} \frac{1}{nx^n} = 0.$$

iii) Let $f(x) = \ln \tan 2x, g(x) = \ln \tan x$

Then $\lim_{x \to 0^+} f(x) = -\infty = \lim_{x \to 0^+} g(x)$ and

$$\lim_{x \to 0^+} \frac{f'(x)}{g'(x)} = \lim_{x \to 0^+} \frac{4 \operatorname{cosec} 4x}{2 \operatorname{cosec} 2x}$$

$$= \lim_{x \to 0^+} \frac{1}{\cos 2x} = 1.$$

Therefore,

$$\lim_{x \to 0^+} \frac{\ln \tan 2x}{\ln \tan x} = 1.$$

In the above example we cannot talk of $\lim_{x \to 0} \frac{\ln \tan 2x}{\ln \tan x}$ because tan2x and tanx are negative for $x < 0$ and therefore we cannot take their logarithms. You would also notice that $\lim_{x \to 0^+} f'(x) = \infty = \lim_{x \to 0^+} g'(x)$, but $\frac{f'(x)}{g'(x)}$ is not in an indeterminate form as $x \to 0^+$.

Example 10 : Suppose we want to prove that

i) $\lim_{x \to \infty} \frac{(\ln x)^n}{x} = 0$, where n is an integer, $n \geq 0$.

ii) $\lim_{x \to \infty} \frac{(\ln x)^m}{x^n} = 0$, $m > 0$, $n > 0$ and m is an integer.

Let's start with i).

i) For n = 0, the result is obvious. For n = 1, the result has been proved in Example 9, ii). Let n > 1.

Set $f(x) = (\ln x)^n$ and $g(x) = x$. Then the functions f(x) and g(x) are differentiable for $x > 0$, and $\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x)$.

Therefore,

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{n(\ln x)^{n-1}}{x},$$

provided the right-hand side limit exists. Considering the functions $(\ln x)^{n-1}$ and x instead of $(\ln x)^n$ and x, we get

$$\lim_{x \to \infty} \frac{(\ln x)^n}{x} = \lim_{x \to \infty} \frac{n(\ln x)^{n-1}}{x} = \lim_{x \to \infty} \frac{n(n-1)(\ln x)^{n-2}}{x},$$

provided the right-hand side limit exists. Repeating the above process, we obtain

$$\lim_{x \to \infty} \frac{(\ln x)^n}{x} = \lim_{x \to \infty} \frac{n!}{x} = 0.$$

ii) Let $f(x) = (\ln x)^m$, $g(x) = x^n$. Then f(x), g(x) are differentiable for $x > 0$, and

$\lim_{x \to \infty} f(x) = \infty = \lim_{x \to \infty} g(x)$.

Therefore,

$$\lim_{x \to \infty} \frac{(\ln x)^m}{x^n} = \lim_{x \to \infty} \frac{m(\ln x)^{m-1}}{n x^n},$$

provided the right hand side limit exists. Considering the functions $(\ln x)^{m-1}$ and x^n instead of $(\ln x)^m$ and x^n, we obtain

$$\lim_{x \to \infty} \frac{(\ln x)^m}{x^n} = \lim_{x \to \infty} \frac{m(m-1)(\ln x)^{m-2}}{n^2 x^n},$$

provided the right-hand side limit exists. Thus, repeating the above process, we obtain,

$$\lim_{x \to \infty} \frac{(\ln x)^m}{x^n} = \lim_{x \to \infty} \frac{m!}{n^m} \cdot \frac{1}{x^n} = 0.$$

You must have observed that in the above example we are using Theorem 6 again and not Theorem 7. In fact, Theorem 7 cannot be applied. because $\lim_{x \to \infty} f'(x) \neq \infty$ in both the cases, i) and ii).

Example 11 : Let $P(x) = a_m x^m + a_{m-1} x^{m-1} + \dots\dots\dots + a_0$ and

$Q(x) = b_n x^n + b_{n-1} x^{n-1} + \dots\dots\dots + b_0$ be two polynomials with real coefficients, $a_m \neq 0$, $b_n \neq 0$.

We'll show that

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)} = \begin{cases} \frac{a_m}{b_n} \text{ if } m = n \\ 0, \text{ if } m < n \\ \pm\infty, \text{ if } m > n, \text{ according to } \frac{a_m}{b_n} \text{ is positive or negative.} \end{cases}$$

Let us take the case when m = n.

If $0 \leq k < m$, $\lim_{x \to \infty} P^{(k)}(x)$ and $\lim_{x \to \infty} Q^{(k)}(x)$ are infinite, and

$$\lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = \lim_{x\to\infty} \frac{a_m\, m!}{b_m\, m!} = \frac{a_m}{b_m}$$

Therefore,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{f^{(m)}(x)}{g^{(m)}(x)} = \frac{a_m}{b_m} \text{ by Theorem 7.}$$

Now, suppose m < n,

Again $\lim_{x\to\infty} P^{(k)}(x)$, $\lim_{x\to\infty} Q^{(k)}(x)$ are infinite for $0 \le k < m$, and

$$\lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = \lim_{x\to\infty} \frac{m!\, a_m}{\sum_{r=m}^{n} b_r\, r(r-1)\ldots\ldots(r-m+1)x^{r-m}}$$

$$= 0$$

Thus,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{P^{(m)}(x)}{Q^{(m)}(x)} = 0$$

Now, if m > n, it is obvious that

$\lim_{x\to\infty} P^{(k)}(x)$, $\lim_{x\to\infty} Q^{(k)}(x)$ are infinite for $0 \le k < n$, and

$$\lim_{x\to\infty} \frac{P^{(n)}(x)}{Q^{(n)}(x)} = \lim_{x\to\infty} \frac{\sum_{r=n}^{m} a_r\, r(r-1)\ldots\ldots(r-n+1)x^{r-n}}{n!\, b_n}$$

$$= \pm\infty, \text{ according as } \frac{a_m}{b_n} > 0 \text{ or } < 0.$$

Thus,

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} = \lim_{x\to\infty} \frac{P^{(n)}(x)}{Q^{(n)}(x)} = \infty \text{ or } -\infty, \text{ according as } \frac{a_m}{b_n} > 0 \text{ or } < 0.$$

This example could also be solved by using the method described in Example 7 of Unit 1. We shall illustrate this through a specific function in the next example.

Example 12 : We'll show that $\lim_{x\to\infty} \frac{4x^3+3x^2+5x+6}{5x^4+6x+7}$ can be evaluated in two ways.

By applying L'Hopital's rule (as in Example 11), we get

$$\lim_{x\to\infty} \frac{4x^3+3x^2+5x+6}{5x^4+6x+7} = \lim_{x\to\infty} \frac{12x^2+6x+5}{20x^3+6}$$

$$= \lim_{x\to\infty} \frac{24x+6}{60x^2} = \lim_{x\to\infty} \frac{24}{120x} = 0.$$

Using algebra of limits, we can find the above limit in a very simple way as follows:

$$\lim_{x\to\infty} \frac{4x^3+3x^2+5x+6}{5x^4+6x+7} = \lim_{x\to\infty} \frac{4+3/x+5/x^2+6/x^3}{5x+6/x^2+7/x^3} = 0, \text{ as } 1/x \to 0$$

when $x \to \infty$.

In the next example you will find a situation where L'Hopital's rule is not applicable.

Example 13 : Consider,

$\lim_{x\to\infty} \frac{2x \sin x}{1+x^2}$. Can we apply L'Hopital's rule to evaluate this limit?

No, L'Hopital's rule is not applicable because $\lim_{x\to\infty} 2x \sin x$ does not exist.

However,

$$\lim_{x\to\infty} \frac{2x \sin x}{1+x^2} = 0,$$

because,

$$\left| \frac{2x \sin x}{1 + x^2} \right| \leq \left| \frac{2y}{1 + x^2} \right|, \text{ and}$$

$$\lim_{x \to \infty} \frac{2x}{1 + x^2} = 0.$$

We now give an example where L'Hopital's rule is applicable but it yields no result. But such situations are very rare.

Example 14 : Consider the function

$$\frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Let us see what happens if L'Hopital's rule is applied to evaluate its limit as $x \to \infty$. We get

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The right-hand side is again in the $\frac{\infty}{\infty}$ form, but if we apply L'Hopital's rule to evaluate it, we get back to where we started. Thus, it is useless to apply L'Hopital's rule in this case. But we can still evaluate the limit as follows.

$$\lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1,$$

because $\lim_{x \to \infty} e^{-x} = 0$.

After going through the above examples you should have no difficulty in solving these exercises.

E 9) Evaluate the following limits.

a) $\lim_{x \to \infty} \frac{a_m x^m + a_{m-1} x^{m-1} + \ldots\ldots\ldots + a_0}{e^x}$, where $a_i \in \mathbf{R}, \forall\ i = 0, 1, \ldots\ldots, m$.

b) $\lim_{x \to (\pi/2)^-} \frac{-\tan x}{\ln \cos x}$,

c) $\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x}$

d) $\lim_{x \to \infty} \frac{x^6 + \ln x}{2x^6 + 5x^4 + 1}$

e) $\lim_{x \to \infty} \frac{2x^8 + 5x^7 + 6x^3 + 1}{3x^8 + 5x^7 + 5x + 1}$

E 10) Show that

$$\lim_{x \to \infty} \frac{x + \sin x}{x + \cos x} = 1,$$

and that L'Hopital's rule cannot be used to evaluate it.

E 11) Evaluate the following limits and show that L'Hopital's rule is not applicable in each case.

a) $\lim_{x \to \infty} \frac{x^2 - \sin x^2}{x^2}$

b) $\lim_{x \to \infty} \frac{x - \cos x}{x}$

c) $\lim_{x \to \infty} \frac{|\sin x| + |\cos x|}{x}$

d) $\lim_{x \to \infty} \frac{x \sin x + \cos x}{x^2}$

2.5 OTHER TYPES OF INDETERMINATE FORMS

So far we have concentrated on two types of indeterminate forms : $\frac{0}{0}$ and $\frac{\infty}{\infty}$. In this section we shall take up the remaining indeterminate forms one by one, and study the evaluation of limits in each case. Given any indeterminate form, our standard procedure would be to first bring it in the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, and then apply L'Hopital's rule.

2.5.1 Indeterminate Forms of the Type $\infty - \infty$

By some algebraic or trigonometric process we can transform an indeterminate form of this type to one of the two standard forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then, we can use L'Hopital's rule to evaluate its limit.

We'll now illustrate this procedure with an example.

Example 15 : Let us evaluate the following limits.

i) $\lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x} \right)$

ii) $\lim_{x \to (\pi/2)^-} \left[\sec x - \frac{1}{(1 - \sin x)} \right]$

iii) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{\ln(1+x)}{x^2} \right]$

Clearly all the three functions are of the type $\infty - \infty$.

i) We can write

$$\operatorname{cosec} x - \frac{1}{x} = \frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

so that the right-hand side is in the $\frac{0}{0}$ form at $x = 0$, to which L'Hopital's rule is applicable.

Thus, $\lim_{x \to 0} \left(\operatorname{cosec} x - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x}$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x\cos x}$$

$$= \lim_{x \to 0} \frac{\sin x}{2\cos x - x\sin x}, \text{ by differentiating}$$

$$= \frac{\lim_{x \to 0} \sin x}{\lim_{x \to 0} (2\cos x - x\sin x)} = \frac{0}{2} = 0.$$

ii) Now,

$$\sec x - \frac{1}{1 - \sin x} = \frac{1}{\cos x} - \frac{1}{1 - \sin x} = \frac{1 - \sin x - \cos x}{\cos x\,(1 - \sin x)}$$

and the right-hand side is in the $\frac{0}{0}$ form as $x \to \frac{\pi}{2}^-$, to which L'Hopital's rule is applicable. Thus,

$$\lim_{x \to \pi/2^-} \left(\sec x - \frac{1}{1 - \sin x} \right) = \lim_{x \to \pi/2^-} \frac{1 - \sin x - \cos x}{\cos x - \sin x \cos x}$$

$$= \lim_{x \to \pi/2^-} \frac{-\cos x + \sin x}{-\sin x - \cos 2x}, \text{ by differentiating}$$

$$= \lim_{x \to \pi/2^-} (-\cos x + \sin x) . \lim_{x \to \pi/2^-} \left(\frac{1}{-\sin x - \cos 2x} \right)$$

$$= 1.\infty = \infty$$

$\lim_{x \to \pi/2^-} \frac{1}{-\sin x - \cos 2x} = \infty$
since
$-\sin x - \cos 2x \to 0$ as $x \to \pi/2^-$.

iii) It is obvious that

$$\lim_{x \to 0} \left[\frac{1}{x} - \frac{\ln(1+x)}{x^2}\right] = \lim_{x \to 0} \frac{x - \ln(1+x)}{x^2}$$

and L'Hopital's rule can be applied to evaluate the limit on the right-hand side.

Therefore,

$$\lim_{x \to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x \to 0} \frac{1 - \frac{1}{1+x}}{2x} = \lim_{x \to 0} \frac{1/(1+x)^2}{2}$$

$$= \frac{1}{2}.$$

Now, we'll look at another type of indeterminate form.

2.5.2 Indeterminate form of the type 0.∞

Let f(x) and g(x) be two real-valued functions, such that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$, where a is any real number, ∞ or – ∞. Then, as we have seen in Sec. 2.2, f(x).g(x) is in an indeterminate form of the type 0.∞ as $x \to a$. In order to evaluate $\lim_{x \to a} f(x)\,g(x)$, we can write

$$\lim_{x \to a} f(x)\,g(x) = \lim_{x \to a} \frac{f(x)}{1/g(x)}, \text{ or}$$

$$\lim_{x \to a} f(x)\,g(x) = \lim_{x \to a} \frac{g(x)}{1/f(x)},$$

so that the right-hand side is $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form to which L'Hopital's rule can be applied.

The conversion from 0.∞ form to $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form would depend upon the particular functions under consideration. You will understand this more clearly if you study the following example.

Example 16 : Let us evaluate (i) $\lim_{x \to 1} \tan \frac{\pi x}{2} \ln x$ and

(ii) $\lim_{x \to \infty} x^p e^{-qx}$ where p, q are positive integers.

We start with (i).

(i) Note that $\tan \frac{\pi x}{2} \ln x$ is a 0.∞ form at x = 1. Now we write

$$\tan \frac{\pi x}{2} \ln x = \frac{\sin(\pi x/2)}{\cos(\pi x/2)} \ln x$$

We know that $\lim_{x \to 1} \sin \frac{\pi x}{2} = 1$. So let us try to find $\lim_{x \to 1} \frac{\ln x}{\cos(\pi x/2)}$.

Now, $\frac{\ln x}{\cos(\pi x/2)}$ is $\frac{0}{0}$ form at x = 1.

Therefore, by L'Hopital's rule

$$\lim_{x \to 1} \frac{\ln x}{\cos(\pi x/2)} = \lim_{x \to 1} \frac{1/x}{-\sin(\pi x/2)\,.\,\pi/2} = -\frac{2}{\pi}.$$

Thus,

$$\lim_{x \to 1} \left(\tan \frac{\pi x}{2}\right) \ln x = \lim_{x \to 1} \frac{\sin(\pi x/2) \ln x}{\cos(\pi x/2)}$$

$$= \lim_{x \to 1} \sin \frac{\pi x}{2} \lim_{x \to 1} \frac{\ln x}{\cos(\pi x/2)}$$

$$= 1\,.\left(-\frac{2}{\pi}\right)$$

$$= -\frac{2}{\pi}.$$

ii) We can write

$$\lim_{x \to \infty} x^p e^{-qx} = \lim_{x \to \infty} \frac{x^p}{e^{qx}}$$

Now, $\frac{x^p}{e^{qx}}$ is in an indeterminate form of the type $\frac{\infty}{\infty}$ to which L'Hopital's rule is applicable. Thus, we get,

$$\lim_{x \to \infty} \frac{x^p}{e^{qx}} = \lim_{x \to \infty} \frac{p!}{q^p e^{qx}} = 0,$$

so that $\lim_{x \to \infty} x^p e^{-qx} = 0$.

We shall now end this section by discussing the remaining types of indeterminate forms.

2.5.3 Indeterminate Forms of the type 0^0, ∞^0, 1^∞

Let f(x) and g(x) be two real-valued functions defined in a neighbourhood V of a, except perhaps at a, if a is a finite real number. Suppose that f(x) > 0 for all x in V. Then

$$\ln (f(x))^{g(x)} = g(x) \ln f(x) \text{ and}$$

$$\lim_{x \to a} (f(x))^{g(x)} = \lim_{x \to a} e^{\ln(f(x))^{g(x)}}$$

$$= e^{\lim_{x \to a} (g(x) \ln f(x))}$$

because the exponential function e^x is a continuous function. Thus, in order to evaluate $\lim_{x \to a} (f(x))^{g(x)}$ it is enough to evaluate $\lim_{x \to a} g(x) \ln f(x)$. You will agree that if as $x \to a$, $(f(x))^{g(x)}$ is in any one of the forms, 0^∞, ∞^0, 1^∞, then g(x) ln f(x) is in the form $0.\infty$. And in Sec. 2.5.2, we have already seen how to evaluate the limit for this type.

Here is an example to illustrate this procedure.

Example 17 : Suppose we want to evaluate

i) $\lim_{x \to 1^+} x^{1/(x-1)}$

ii) $\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x$ and

iii) $\lim_{x \to \pi/2^-} (\cos x)^{\cos x}$

i) It is clear that $x^{1/(x-1)}$ is in an indeterminate form of the type 1^∞ as $x \to 1^+$. Let $y = x^{1/(x-1)}$. Then

$$\ln y = \frac{1}{x-1} \ln x$$

Now, $\frac{\ln x}{x-1}$ is in the $\frac{0}{0}$ form as $x \to 1^+$, and L'Hopital's rule is applicable to it. Therefore

$$\lim_{x \to 1^+} \frac{\ln x}{x-1} = \lim_{x \to 1^+} \frac{1/x}{1} = 1$$

Hence

$$\lim_{x \to 1^+} x^{1/(x-1)} = \lim_{x \to 1^+} e^{\ln y} = e^{\lim_{x \to 1^+} \ln y} = e.$$

ii) Let $y = \left(\ln \frac{1}{x}\right)^x$, so that y is in the form ∞^0 as $x \to 0^+$.

Then

$$\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^{\lim_{x \to 0^+} \ln y}$$

But

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln \left(\ln \frac{1}{x}\right)$$

$$= \lim_{x \to 0^+} \frac{\ln\left(\ln \frac{1}{x}\right)}{1/x}$$

Now $\frac{\ln\left(\ln \frac{1}{x}\right)}{1/x}$ is in the $\frac{\infty}{\infty}$ form as $x \to 0^+$. Therefore, by L'Hopital's rule

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\frac{x}{\ln(1/x)} \cdot \frac{-1}{x^2}}{-1/x^2} = 0.$$

Substituting this in (I) we get

$$\lim_{x \to 0^+} \left(\ln \frac{1}{x}\right)^x = e^0 = 1.$$

iii) Let $y = (\cos x)^{\cos x}$, $0 < x < \pi/2$. Then,

$\ln y = \cos x \ln \cos x = \frac{\ln \cos x}{\sec x}$, and therefore by applying L'Hopital's rule we obtain

$$\lim_{x \to \pi/2^-} \ln y = \lim_{x \to \pi/2^-} \frac{\ln \cos x}{\sec x} = \lim_{x \to \pi/2^-} \frac{-\tan x}{\sec x \tan x} = 0.$$

Thus.

$$\lim_{x \to \pi/2^-} (\cos x)^{\cos x} = e^{\lim_{x \to \pi/2^-} \ln y} = e^0 = 1.$$

Now you can try this exercise on your own.

E 12) **Evaluate the following limits. In each case you will have to first identify the type of indeterminate form, and then decide upon the procedure.**

a) $\lim_{x \to 0^+} (1 + x)^{1/x}$

b) $\lim_{x \to 0} (\cos x)^{1/x^2}$

c) $\lim_{x \to 0^+} x^x$

d) $\lim_{x \to 0^+} (\tan x)^{\sin 2x}$

e) $\lim_{x \to \pi/2^-} (\tan x)^{\sin 2x}$

f) $\lim_{x \to \infty} x \sin \frac{1}{x}$

That brings us to the end of this unit. Let us summarise all that we have done in it.

2.6 SUMMARY

In this unit we have

1) introduced indeterminate forms of functions at a point,

2) stated and proved L'Hopital's rule for the $\frac{0}{0}$ form at a, where a is a real number.

   Thus, if $\frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$ at $x = a$, then

   $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$, if the right-hand side limit exists.

3) extended L'Hopital's rule to $\frac{0}{0}$ forms at ∞ and $-\infty$.

4) stated (without proof) L'Hopital's rule for $\frac{\infty}{\infty}$ form, and demonstrated it with the help of examples.

5) described how to reduce indeterminate forms of the types $\infty - \infty$, 1^∞, 0^∞ and 0^0, to the forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

6) given a few examples where L'Hopital's rule fails.

2.7 SOLUTIONS AND ANSWERS

E 1) a) $\frac{\infty}{\infty}$, b) $\frac{0}{0}$, c) $\infty - \infty$, d) $\frac{0}{0}$.

E 2) a) $\lim\limits_{x \to 0} \frac{x - \sin x}{\sin x} = \lim\limits_{x \to 0} \frac{1 - \cos x}{\cos x} = \frac{1 - 1}{1} = 0$

b) $\lim\limits_{x \to 1} \frac{x^4 - 1}{x^2 - 6x + 5}$

$= \lim\limits_{x \to 1} \frac{x^3 + x^2 + x + 1}{x - 5} = \frac{4}{-4} = -1.$

E 3) a) By L'Hopital's rule

$$\lim_{x \to 0} \frac{1 - \cos^2 x}{\sin x^2} = \lim_{x \to 0} \frac{2 \sin x \cos x}{2x \cos x^2}$$

Now, $\lim\limits_{x \to 0} \frac{2 \sin x \cos x}{2x \cos x^2} = \lim\limits_{x \to 0} \frac{\sin x}{x} \lim\limits_{x \to 0} \frac{\cos x}{\cos x^2} = 1.$

Therefore, the required limit is 1.

b) $7/\pi$

c) $\left(\frac{m}{n}\right) \alpha^{m-n}$

d) 1

e) -1

f) $\ln 3/2$

g) $\sqrt{2}$

h) $1/6$

E 4) a) 1

b) 2

c) $\lim\limits_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin^2 x} = \lim\limits_{x \to 0} \left(\frac{1 - \cos x^2}{x^4} \frac{x^2}{\sin^2 x} \right)$

$\lim\limits_{x \to 0} \frac{1 - \cos x^2}{x^4} = \lim\limits_{x \to 0} \frac{2x \sin x^2}{4x^3} = \lim\limits_{x \to 0} \frac{1}{2} \left(\frac{\sin x^2}{x^2} \right) = \frac{1}{2}$

$\therefore \lim\limits_{x \to 0} \frac{1 - \cos x^2}{x^2 \sin^2 x} = \lim\limits_{x \to 0} \frac{1 - \cos x^2}{x^4} \cdot \lim\limits_{x \to 0} \left(\frac{x^2}{\sin^2 x} \right) = \frac{1}{2}$

A direct application of L'Hopital's rule will also yield the result.

d) $\lim\limits_{x \to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x} = \lim\limits_{x \to 0} \left(\frac{\tan^2 x - x^2}{x^4} \cdot \frac{x^2}{\tan^2 x} \right)$

$\lim\limits_{x \to 0} \frac{\tan^2 x - x^2}{x^4} = \lim\limits_{x \to 0} \frac{2 \tan x \sec^2 x - 2x}{4x^3}$

$= \lim\limits_{x \to 0} \frac{\sec^4 x - 2 \sec^2 x \tan^2 x - 1}{6x^2} = 0,$

For,

$\lim\limits_{x \to 0} \frac{\sec^4 x - 1}{6x^2} = \lim\limits_{x \to 0} \frac{4 \sec^4 x \tan x}{12x}$

$$= \lim_{x \to 0} \frac{1}{3} \sec^4 x . \lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{3},$$

and $\lim_{x \to 0} \frac{2 \sec^2 x \tan^2 x}{6x^2} = \frac{1}{3}$

e) $-\frac{9}{2}$

f) $\frac{1}{4}$

g) 2

E 5) $t = -1, \frac{8}{3}$

E 6) $t = -1, 0.$

E 7) $\lim_{x \to 0} \frac{x^2 \sin 1/x}{5 \tan x} = \lim_{x \to 0} \frac{x \sin \frac{1}{x}}{5 \frac{\tan x}{x}}$

$$= \frac{\lim_{x \to 0} x \sin \frac{1}{x}}{5 \lim_{x \to 0} \frac{\tan x}{x}}$$

$= \frac{0}{5}$, since $\sin \frac{1}{x} \le 1$, and since $\lim_{x \to 0} \frac{\tan x}{x} = 1.$

$= 0.$

L'Hopital's rule cannot be applied as

$\lim_{x \to 0} \frac{2x \sin \frac{1}{x} - \cos \frac{1}{x}}{5 \sec^2 x}$ does not exist, because $\lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$ does not exist as we have seen in Example 3.

E 8) a) 1, b) 1, c) 1

d) $\lim_{x \to \infty} 2x (\ln (x + 1) - \ln x)$

$$= \lim_{x \to \infty} \frac{2 (\ln(x + 1) - \ln x)}{1/x}$$

$$= \lim_{x \to \infty} \frac{2\left(\frac{1}{x + 1} - \frac{1}{x}\right)}{-1/x^2}$$

$$= \lim_{x \to \infty} \frac{2x}{x + 1}$$

$$= \lim_{x \to \infty} \frac{2}{1 + \frac{1}{x}} = 2.$$

e) $\lim_{x \to -\infty} \frac{\ln \left[\frac{x^2 + 1}{x^2}\right]^3}{\frac{1}{x^2}}$

$$= \lim_{x \to -\infty} \frac{\left[\frac{x^2}{1 + x^2}\right]^3 . 3 \left[\frac{x^2 + 1}{x^2}\right]^2 \left[\frac{-2}{x^3}\right]}{\frac{-2}{x^3}}$$

$$= \lim_{x \to -\infty} \frac{3x^2}{x^2 + 1} = 3.$$

E 9) a) $\lim\limits_{x \to \infty} \dfrac{a_m x^m + \dots + a_0}{e^x} = \lim\limits_{x \to \infty} \dfrac{m!\, a_m}{e^x} = 0$

b) ∞

c) By L'Hopital's rule

$$\lim_{x \to \pi/2} \frac{\tan 3x}{\tan x} = \lim_{x \to \pi/2} \frac{3\sec^2 3x}{\sec^2 x}$$

which is again $\frac{\infty}{\infty}$ form. It can be handled more easily by converting it into $\frac{0}{0}$ form.

$$\lim_{x \to \pi/2} \frac{3\sec^2 3x}{\sec^2 x} = \lim_{x \to \pi/2} \frac{3\cos^2 x}{\cos^2 3x}$$

$$= \lim_{x \to \pi/2} \frac{-6\cos x \sin x}{-6\cos 3x \sin 3x}$$

$$= \lim_{x \to \pi/2} \frac{\sin 2x}{\sin 6x}$$

$$= \lim_{x \to \pi/2} \frac{2\cos 2x}{6\cos 6x} = \frac{1}{3}$$

d) $\frac{1}{2}$

e) $\frac{2}{3}$

E 10) $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x + \cos x} = \lim\limits_{x \to \infty} \dfrac{1 + \frac{\sin x}{x}}{1 + \frac{\cos x}{x}} = \dfrac{1}{1} = 1.$

since $\sin x$ and $\cos x$ are bounded functions, and $\frac{1}{x} \to 0$ as $x \to \infty$.

L'Hopital's rule cannot be applied because $\lim\limits_{x \to \infty} (x + \sin x)$ and $\lim\limits_{x \to \infty} (x + \cos x)$ do not exist.

E 11) a) $\lim\limits_{x \to \infty} \dfrac{x^2 - \sin x^2}{x^2} = \lim\limits_{x \to \infty} \dfrac{1 - \frac{\sin x^2}{x^2}}{1}$

$= 1$, since $\sin x^2$ is bounded, and $\frac{1}{x^2} \to 0$ as $x \to \infty$.

L'Hopital's rule is not applicable because $\lim\limits_{x \to \infty} (x^2 - \sin^2 x)$ does not exist.

b) 1. L'Hopital's rule cannot be applied by the argument similar to the one in a).

c) 0. L'Hopital's rule is not applicable as $\frac{f(x)}{g(x)}$ is not in an indeterminate form as $x \to \infty$.

d) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} + \dfrac{\cos x}{x^2} = 0.$

L'Hopital's rule is not applicable since $\lim\limits_{x \to \infty} (x \sin x + \cos x)$ does not exist.

E 12) a) 1^∞. If $y = (1 + x)^{1/x}$, then $\ln y = \dfrac{\ln(1+x)}{x}$ is in the $\frac{0}{0}$ form

$\therefore \lim\limits_{x \to 0^+} \ln y = \lim\limits_{x \to 0^+} \dfrac{\ln(1+x)}{x} = \lim\limits_{x \to 0^+} \dfrac{1}{1+x} = 1.$

$\therefore \lim\limits_{x \to 0^+} (1 + x)^{1/x} = \lim\limits_{x \to 0^+} y = e^1 = e.$

b) 1^∞. If $y = \cos x^{1/x^2}$, $\ln y = \dfrac{\ln \cos x}{x^2}$, which is in the $\frac{0}{0}$ form as $x \to 0$.

$$\therefore \lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln\cos x}{x^2} = \lim_{x \to 0} \frac{-\tan x}{2x} = -\frac{1}{2}$$

$$\therefore \lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

c) 0^0. If $y = x^x$; then $\ln y = x \ln x$, which is $0(-\infty)$ form as $x \to 0^+$.

$$\therefore \lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x\ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \to 0^+} -x = 0.$$

$$\therefore \lim_{x \to 0^+} x^x = e^0 = 1.$$

d) 0^0. $\lim_{x \to 0^+} (\tan x)^{\sin 2x} = 1.$

e) ∞^0 . If $y = (\tan x)^{\sin 2x}$, $\ln y = \sin 2x \ln \tan x$

$$\therefore \lim_{x \to \pi/2^-} \ln y = \lim_{x \to \pi/2^-} \frac{\ln \tan x}{\operatorname{cosec} 2x}$$

$$= \lim_{x \to \pi/2^-} \frac{\frac{1}{\tan x}\sec^2 x}{2 \operatorname{cosec} 2x \cot 2x}$$

$$= \lim_{x \to \pi/2^-} \frac{\frac{2}{\sin 2x}}{\frac{-2\cos 2x}{\sin^2 2x}}$$

$$= \lim_{x \to \pi/2^-} \frac{-\sin 2x}{\cos 2x} = 0.$$

$$\therefore \lim_{x \to \pi/2^-} (\tan x)^{\sin 2x} = e^0 = 1.$$

f) $\infty.0$. $\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{x \to \infty} \frac{\sin 1/x}{1/x}$

$$= \lim_{x \to \infty} \cos \frac{1}{x} = 1.$$

UNIT 3 FUNCTIONS OF SEVERAL VARIABLES

Structure

3.1 INTRODUCTION

In the first course of calculus and the first two units of this block we have studied the concepts of limit, continuity and differentiability of real-valued functions of a real variable, i.e., functions whose domain and range are subsets of R, the set of real numbers. In Block 2 we shall study these concepts for functions of several variables, i.e., those functions whose domain is a subset of $\mathbf{R}^n$, the Cartesian product of n-copies of R. These functions arise naturally in various contexts. For instance, the insurance premium is a function of a large number of parameters like the sum insured, the age of the insured person and life expectancy. Similarly the price of a commodity is dependent on a number of factors like cost of production, permissible profit margin and state taxes.

You have seen that a knowledge of the algebraic structure of R and familiarity with the properties of the distance $|x - y|$ between two points x, y of R is necessary to study the notions of limit and continuity of functions of one variable. The same is true for functions of several variables. Therefore, in this unit we first define $\mathbf{R}^n$ and describe its algebraic structure. We then introduce the notion of a distance between two points of $\mathbf{R}^n$ and deduce its elementary properties. We end this unit by defining a function of several variables, and by giving various examples of such functions.

Objectives

After reading this unit, you should be able to :

- define a real Euclidean space of dimension n
- give examples of real-valued and vector-valued functions of several variables.

3.2 THE SPACE $\mathbf{R}^n$

We have mentioned in the introduction that in this unit we are going to study functions whose domain is a subset of $\mathbf{R}^n$. But what is $\mathbf{R}^n$? In this section we shall define $\mathbf{R}^n$, and study its algebraic structure. We'll also study the distance function in $\mathbf{R}^n$. But let us start with the definition of $\mathbf{R}^n$. For this we'll need to define Cartesian products of sets.

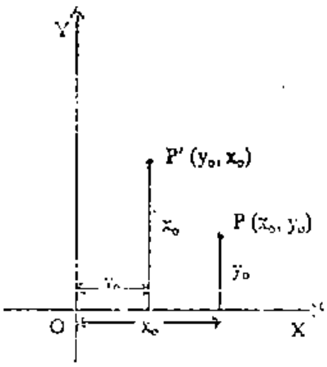


Fig. 1

The word Cartesian product comes from the name of the French mathematician Rene' Descartes (1596–1650), who first thought of representing points in a plane by ordered pairs of numbers.b

3.2.1 Cartesian Products

Let X and Y be two non-empty sets. By (x, y) where $x \in X$ and $y \in Y$, we denote the ordered pair whose first member or coordinate is x and whose second member or coordinate is y. Two ordered pairs (x_1, y_1), (x_2, y_2) are said to be equal, i.e. $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2$, $y_1 = y_2$. You are already familiar with this concept. While studying coordinate geometry you must have represented a point P in the Cartesian plane by (x_0, y_0), where x_0 is the abscissa and y_0 is the ordinate of P (see Fig. 1). Clearly the point (x_0, y_0) is different from the point (y_0, x_0) if $x_0 \neq y_0$. Thus you know that a point in the plane is represented by an ordered pair (x, y), where x and y are real numbers.

Note that the ordered pair (x, y) is different from the set {x, y}, because the ordered pair (x, y) is different from the ordered pair (y, x) if $x \neq y$, while the sets {x, y} and {y, x} are equal.

The set of all ordered pairs (x,y) where $x \in X$, $y \in Y$ is called the **Cartesian product** of the sets X and Y. We denote it by $X \times Y$.
Thus, $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$.

For example, if $X = \{0, 1, 2\}$ and $Y = \{0, 1\}$, then

$X \times Y = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$

If $X = \mathbf{R}$, $Y = \mathbf{R}$, then

$X \times Y = \mathbf{R} \times \mathbf{R} = \mathbf{R}^2 = \{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}\}$

is nothing but the Cartesian plane.

We now extend this idea to get a product of n sets.

Let $X_1, X_2, \ldots, X_n$ be any n non-empty sets. By $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$, $1 \leq i \leq n$, we shall denote an n-tuple. Two n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal, i.e.

$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$

if and only if $x_i = y_i$ for all i, $1 \leq i \leq n$.

The set of all n-tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in X_i$, is called the Cartesian product of n sets $X_1, X_2, \ldots, X_n$. It is denoted by $X_1 \times X_2 \times \ldots \times X_n$. Thus,

$X_1 \times X_2 \times \ldots \times X_n = \{(x_1, x_2, x_3, \ldots, x_n) \mid x_i \in X_i,\ 1 \leq i \leq n\}$

Now, if $X_1 = \{1, 2\}$, $X_2 = \{1, 2\}$, $X_3 = \{0\}$, then what will $X_1 \times X_2 \times X_3$ be? You can easily check that $X_1 \times X_2 \times X_3 = \{(1, 1, 0), (1, 2, 0), (2, 1, 0), (2, 2, 0)\}$.

If $X_i = \mathbf{R}$ for all i, $1 \leq i \leq n$, then

$X_1 \times \ldots \times X_n = \mathbf{R} \times \mathbf{R} \times \ldots \times \mathbf{R}$ (n times) $= \mathbf{R}^n$

$= \{(x_1, \ldots, x_n) \mid x_i \in \mathbf{R}, 1 \leq i \leq n\}$

is called the Cartesian product of n-copies of **R**.

Note that $\mathbf{R}^1 = \mathbf{R}$, $\mathbf{R}^2$ is the Cartesian plane, and $\mathbf{R}^3$ is nothing but the set of all points in 3-dimensional space.

Let V denote the set of all vectors $\overrightarrow{OP}$ in the Cartesian plane where O is the origin and P is any point with coordinates (x, y) in the plane. Then there is a one-to-one correspondence between V and $\mathbf{R}^2$ given by $\overrightarrow{OP} \rightarrow (x, y)$. Similarly there is a one-to-one correspondence between the vectors $\overrightarrow{OP}$ (where O is the origin and P is any point (x, y, z) in space) and the points in space given by $\overrightarrow{OP} \rightarrow (x, y, z)$. It is because of these correspondences that elements of $\mathbf{R}^2$ and $\mathbf{R}^3$ are called vectors. In what follows, we shall call elements of $\mathbf{R}^n$, vectors and the elements of **R** will be referred to as scalars.

If $x = (x_1, x_2, \ldots, x_n)$ is any point of $\mathbf{R}^n$, then x_i is called **the i-th coordinate** or the **i-th component** of x.

Now having defined the set $\mathbf{R}^n$ let us see if we can define any algebraic operations on the elements of $\mathbf{R}^n$.

3.2.2 Algebraic Structure of $\mathbf{R}^n$

In the last sub-section we have seen that for any integer $n \geq 1$

$\mathbf{R}^n = \{x = (x_1, \ldots, x_n) \mid x_i \in \mathbf{R},\ 1 \leq i \leq n\}$,

and two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are equal if and only if $x_i = y_i$, $1 \leq i \leq n$.

We'll now introduce an algebraic structure on $\mathbf{R}^n$:

The zero-vector : The vector, all of whose coordinates are 0, will be denoted by 0 and it will be clear from the context whether 0 represents an element of $\mathbf{R}^n$ or the real number 0.

Sum of two vectors: Let $x = (x_1, \ldots\ldots, x_n)$ and $y = (y_1, \ldots\ldots, y_n)$ be two elements of $\mathbf{R}^n$. Then $x + y$, the sum of the two vectors x and y, is defined to be the vector whose i-th coordinate is $x_i + y_i$, $1 \le i \le n$. That is,

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots\ldots, x_n + y_n)$$

Using the properties of the operation of addition in the set of real numbers, we can easily prove the following :

A1 If $x, y \in \mathbf{R}^n$, then $x + y \in \mathbf{R}^n$.

A2 $x + 0 = 0 + x = x$ for any vector x in $\mathbf{R}^n$.
(Here 0 denotes the zero-vector.)

A3 Given any vector x in $\mathbf{R}^n$, there exists a unique vector y in $\mathbf{R}^n$ such that
$x + y = y + x = 0$.

A4 $(x + y) + z = x + (y + z)$ for any three vectors x, y, z in $\mathbf{R}^n$.

A5 $x + y = y + x$ for any two vectors x, y in $\mathbf{R}^n$.

If $x = (x_1, \ldots\ldots, x_n)$, then the unique vector y mentioned in A3 above is clearly equal to $(-x_1, -x_2, \ldots\ldots, -x_n)$. We shall denote it by $-x$ and call it the **additive inverse or negative** of x.

If x and y are two vectors in $\mathbf{R}^n$, then $x - y$, the difference of x and y, will denote the vector $x + (-y)$, where $-y$ is the negative of y.

Scalar Multiplication: Let $x = (x_1, x_2, \ldots\ldots, x_n)$ be any vector, and let a be any element of **R**. We define a new vector ax by

$$ax = (ax_1, ax_2, \ldots\ldots, ax_n)$$

We say that the vector ax has been obtained by multiplying the vector x by the scalar a and this particular operation is called **scalar multiplication in $\mathbf{R}^n$**. Clearly $0.x = 0$ for every x in $\mathbf{R}^n$. Note that 0 on the left denotes the real number 0 and 0 on the right hand side denotes the vector 0 in $\mathbf{R}^n$.

The following properties of scalar multiplication are easy to prove, and we leave them to you as an exercise. See E1).

S1. For $a \in \mathbf{R}$, $x \in \mathbf{R}^n$, $ax \in \mathbf{R}^n$.

S2. $a(x+y) = ax + ay$ for every $x, y \in \mathbf{R}^n$, $a \in \mathbf{R}$.

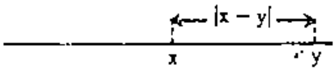
S3. $a(bx) = (ab)x$ for any $x \in \mathbf{R}^n$, $a, b \in \mathbf{R}$.

S4. $(a+b)x = ax + bx$ for $x \in \mathbf{R}^n$, $a, b \in \mathbf{R}$.

S5. $ax = 0$ for every $x \in \mathbf{R}^n$ if and only if $a = 0$.

E 1) Prove S1, S2, S3, S4 and S5 by using the corresponding properties of real numbers.

You must have noticed that the addition of vectors and scalar multiplication in $\mathbf{R}^n$ are identical with the usual operations of addition and multiplication of a vector by a scalar in the plane or space (when n = 2 or 3).



Further, you must be aware that multiplication or division of vectors is not defined in the plane or the space. Similarly, we do not define these operations in $\mathbf{R}^n$ for $n \ge 2$.

Those of you who have studied the course Linear Algebra, would have recognised that $\mathbf{R}^n$ is a vector space over **R** w.r.t. the operations of addition of vectors and scalar multiplication defined above.

Now after this discussion of the algebraic structure of $\mathbf{R}^n$, let us define a distance function in $\mathbf{R}^n$.

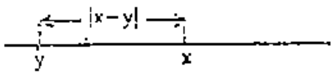


Fig. 2
Distance between x and y is $|x - y|$

3.2.3 Distance in $\mathbf{R}^n$

You know that for any two real numbers x, y, the absolute value $|x - y| = \sqrt{(x - y)^2}$ denotes the distance between the points represented by x and y on the real line. See Fig. 2. Similarly, from your study of coordinate geometry you know that the expression

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the distance between two points with coordinates (x_1, y_1) and (x_2, y_2) in the Cartesian plane. We define the distance between any two points of $\mathbf{R}^n$ in such a way that on taking n = 1 or n = 2, our distance formula reduces, respectively, to the two expressions mentioned above.

Definition 1: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two points of $\mathbf{R}^n$, we define $|x - y|$, the distance of x from y by

$$|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

You can see that for n = 1, $|x - y| = \sqrt{(x - y)^2}$ is nothing but the absolute value of x – y, which is the distance between the points x and y on the real line.

For n = 2, $|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, which is the distance between the points with coordinates (x_1, x_2) and (y_1, y_2) in the Cartesian plane.

Those of you who have studied coordinate geometry of 3-dimensions would recognise that, for n = 3

$|x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$ is the distance between two points with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) in space.

The distance between two points of $\mathbf{R}^n$, defined in this way, has the following properties, which are easily deducible from the definition.

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be any two points of $\mathbf{R}^n$. Then

D 1. $|x - y| = 0$ if and only if x = y.

D 2. $|x - y| = |y - x|$

You know that the sum of two sides of a triangle in the Cartesian plane or space is greater than the third. This means that if x, y, z are three points in $\mathbf{R}^n$, n = 1, 2, 3, then

$|x - y| \le |x - z| + |z - y|$ (Triangle inequality)

The same is true for $\mathbf{R}^n$ for n > 3, i.e.

$|x - y| \le |x - z| + |z - y|$

for any three points x, y, z of $\mathbf{R}^n$. But to prove this fact, we have to take the help of Cauchy's inequality which we shall now state.

Theorem 1 (Cauchy's Inequality) : If $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are any 2n real numbers, then

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Proof If all the real numbers a_i are equal to zero, then there is nothing to prove. Assume, therefore, that at least one $a_i \neq 0$. Consider the expression

$$\sum_{i=1}^{n} (a_i x + b_i)^2 = ax^2 + 2bx + c$$

with $a = \sum_{i=1}^{n} a_i^2$, $b = \sum_{i=1}^{n} a_i b_i$, $c = \sum_{i=1}^{n} b_i^2$

Clearly $ax^2 + 2bx + c \ge 0$ for all real x, and the identity

$a(ax^2 + 2bx + c) = (ax + b)^2 + ac - b^2$

implies that $ac - b^2 \ge 0$ as a > 0.

Thus $b^2 \le ac$, or

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2$$

This means

$$\left|\sum_{i=1}^{n} a_i b_i\right| \le \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

and the proof is complete.

Now for any point $x = (x_1, \ldots\ldots, x_n)$ in $\mathbf{R}^n$,

$$|x| = |x - 0| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

is called the **norm** or **modulus** of x. Recall that we use the same terms when $x \in \mathbf{R}$, or x is a vector in the plane or the space. Now let us get back to the triangle inequality.

Theorem 2 (Triangle inequality) : For any three points x, y, z in $\mathbf{R}^n$

$$|x - y| \le |x - z| + |z - y|$$

Proof : Let us first prove that

$$|x + y| \le |x| + |y|$$

for any two points x, y in $\mathbf{R}^n$.

Let $x = (x_1, \ldots\ldots, x_n)$, $y = (y_1, \ldots\ldots, y_n)$.

Then, $|x + y|^2 = \sum_{i=1}^{n} (x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$

$$\le \sum_{i=1}^{n} x_i^2 + 2\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2} + \sum_{i=1}^{n} y_i^2$$

in view of Cauchy's inequality.

Consequently,

$|x + y|^2 \le |x|^2 + 2|x|\,|y| + |y|^2$,

or $|x + y|^2 \le (|x| + |y|)^2$

or $|x + y| \le |x| + |y|$

Now, if x, y, z are any three points in $\mathbf{R}^n$, then

$|x - y| = |(x - z) + (z - y)| \le |x - z| + |y - z|$, and the proof is complete.

The set $\mathbf{R}^n$ together with the distance between two points of $\mathbf{R}^n$ defined above is called the **Euclidean space of dimension n.**

We would like to tell you that

$|x - y| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is not the only way in which we can define distance in $\mathbf{R}^n$. In fact, there are many other ways in which we can define distance between any two points of $\mathbf{R}^n$ which will also satisfy the triangle inequality. But $\mathbf{R}^n$ endowed with any distance function different from the one defined above is not called a Euclidean space. We shall not deal with any space other than the Euclidean space here.

You know that sets of the type $]a, b[= \{x \in \mathbf{R} \mid a < x < b\}$ where a and b are real numbers, ∞ or $-\infty$ are called open intervals in $\mathbf{R}$. We now, introduce analogues of open intervals in the Euclidean space $\mathbf{R}^n$.

Definition 2 : Let $x_0 \in \mathbf{R}^n$ and $r > 0$ be any real number. Then the set

$S(x_0, r) = \{ x \mid x \in R^n, \mid x - x_0 \mid < r \}$

is called an **open sphere** or **open ball** or **open disc** with centre x_0 and radius r.

Remark 1: i) If n = 1, then $S(x_0, r)$ is nothing but the open interval $]x_0 - r, x_0 + r[$. See Fig: 3(a).

ii) If n = 2 and x_0 is the point with coordinates (a, b), then $S(x_0, r)$ is the interior of the disc in the plane whose centre is (a, b), and whose radius is r. That is,

$S(x_0, r) = \{ (x, y) \mid \sqrt{(x-a)^2 + (y-b)^2} < r \}$.

See Fig. 3 (b).

iii) If n = 3 and x_0 is the point in space with coordinates (a, b, c), then

$S(x_0, r) = \{ (x, y, z) \mid \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r \}$, i.e., the interior of the sphere whose centre is (a, b, c) and radius is r. Also see Fig. 3 (c).

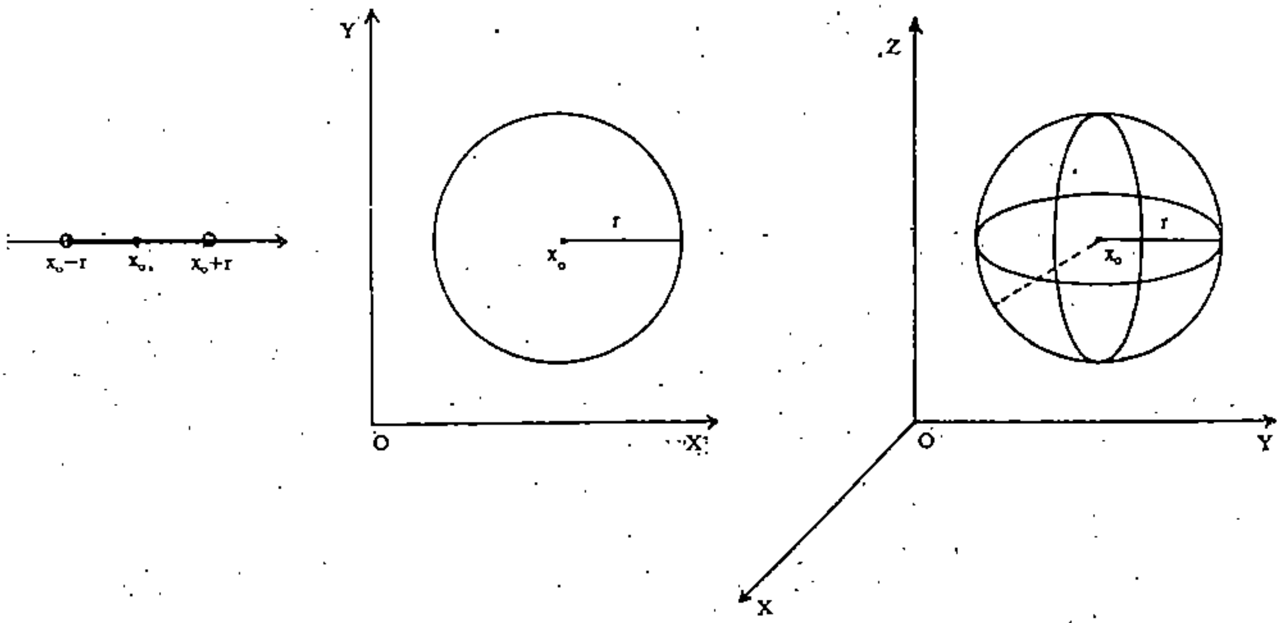


Fig. 3 : (a) $S(x_0, r)$ in R, (b) $S(x_0, r)$ in R^2, (c) $S(x_0, r)$ in R^3

Remark 2 : In an analogy with neighbourhoods on the real line, we shall call the open sphere $S(x_0, r)$ an **r-neighbourhood** of the point x_0 in R^n. By the **deleted r- neighbourhood** of x_0, we shall mean the set of points

$\{ x \mid x \in R^n, 0 < \mid x - x_0 \mid < r \} = S(x_0, r) \setminus \{x_0\}$.

We conclude this section with a few examples and exercises. Go through the examples carefully and try all the exercises. A thorough knowledge of the structure of R^n will help you while studying the limit and continuity of functions of several variables in Block 2.

Example 1 : Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. Then we can show that $x = (x_1, x_2, x_3)$ in R^3 can be uniquely written as

$x = x_1 e_1 + x_2 e_2 + x_3 e_3$.

By definition of scalar multiplication we get

$x_1 e_1 = (x_1, 0, 0)$

$x_2 e_2 = (0, x_2, 0)$

$x_3 e_3 = (0, 0, x_3)$

and therefore $x_1 e_1 + x_2 e_2 + x_3 e_3 = (x_1, x_2, x_3) = x$. Now let us prove that these x_1, x_2, x_3 are unique. Thus, if

$x = a_1e_1 + a_2e_2 + a_3e_3$, a_1, a_2, a_3 real, then
$x = (a_1, a_2, a_3) = (x_1, x_2, x_3)$, and therefore
$a_1 = x_1, a_2 = x_2, a_3 = x_3$.
The vectors e_1, e_2, e_3 are called **unit vectors** along the coordinate axes.

Example 2 : Let $x = e_1 + e_2 - 2e_3$, $y = 2e_1 - e_2 + e_3$ where e_1, e_2, e_3 are the unit vectors defined in Example 1. Let us find $|x + 2y|$, $|x + y|$.

Now, $x + 2y = e_1 + e_2 - 2e_3 + 4e_1 - 2e_2 + 2e_3$

$= 5e_1 - e_2 = (5, -1, 0)$,

and therefore $|x + 2y| = \sqrt{5^2 + (-1)^2 + 0^2} = \sqrt{26}$

Similarly,

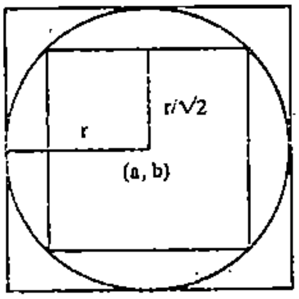
$x + y = 3e_1 - e_3 = (3, 0, -1)$ and $|x + y| = \sqrt{10}$.

Example 3 : The open disc S with centre (a, b) and radius r in $\mathbf{R}^2$ lies in the open square

$S_1 = \{ (x, y) \mid |x - a| < r, |y - b| < r \}$

and contains the open square

$S_2 = \{ (x, y) \mid |x - a| < r/\sqrt{2}, |y - b| < r/\sqrt{2} \}$. See Fig. 4.



Now let us try to prove this.

If $(x, y) \in S$, then we know that

$\sqrt{(x-a)^2 + (y-b)^2} < r$

and therefore,

$|x - a| = \sqrt{(x-a)^2} < r$

$|y - b| = \sqrt{(y-b)^2} < r$

That is, $(x, y) \in S_1$. This means $S \subset S_1$.

Now, if $(x, y) \in S_2$, then

$|x - a| < \frac{r}{\sqrt{2}}, |y - b| < \frac{r}{\sqrt{2}}$.

and therefore,

$(x - a)^2 + (y - b)^2 < \frac{r^2}{2} + \frac{r^2}{2} = r^2$

That is, $(x, y) \in S$. Thus, $S_2 \subset S$.

See if you can do these exercises now.

E 2) Let $e_i = (\delta_{i1}, \delta_{i2}, \ldots\ldots, \delta_{in})$, $1 \le i \le n$ where δ_{ij} is the Kronecker symbol, ($\delta_{ij} = 0$ if $i \neq j$, $\delta_{i,i} = 1$) be n vectors in $\mathbf{R}^n$. Prove that any $x = (x_1, \ldots\ldots, x_n)$ in $\mathbf{R}^n$ can be written uniquely as

$$x = \sum_{i=1}^{n} x_i e_i$$

$e_1, \ldots\ldots, e_n$ are called the unit vectors along the coordinate axes.

E 3) Let $e = (1, 0)$, $f = (1, 1)$ be in $\mathbf{R}^2$.

Find $|x - y|$, $|2x - y|$, $|x|$ where $x = e + f$, $y = 2e + 3f$.

E 4) Show that the open sphere S with centre at (a, b, c) and radius r in $\mathbf{R}^3$ is contained in the open cube

$P_1 = \{(x, y, z) \mid |x - a| < r, |y - b| < r, |z - c| < r \}$

and contains the open cube

$P_2 = \{(x, y, z) \mid |x - a| < r/\sqrt{3},\ |y - b| < r/\sqrt{3},\ |z - c| < r/\sqrt{3}\}.$

Now we shall turn our attention to functions defined on subsets of R^n.

3.3 FUNCTIONS FROM R^n TO R^m

You have already come across the definition of a function earlier (see Definition 4, Unit 1 of Calculus). Thus, if X and Y are two non-empty sets, then a function from X to Y is a rule or correspondence which associates to each member of X, a unique member of Y. Here, we shall introduce a special type of function for which X is a subset of R^n, and Y is a subset of R^m, both m, n ≥ 1. If m = 1, such functions are called **real-valued** functions of n variables. And if m > 1, these functions are called **vector-valued** functions of n variables. More precisely, we have the following definitions:

Definition 3 : i) Let D be a non-empty subset of R^n, the Euclidean space of dimension n, n ≥ 1. A function from D to R is called a **real-valued function of n variables** with domain D.

ii) Let D be a non-empty subset of R^n, n ≥ 1. A function from D to R^m (m > 1) is called a **vector-valued function of n variables** with domain D.

A function of n variables is also called a **function of several variables.**

If $f : D \to R^m$, where $D \subset R^n$, then we denote the value of the function f at a point $x = (x_1, x_2, ..., x_n) \in D$ by f(x) or by $f(x_1, x_2, ..., x_n)$.

Now here are a few examples of functions of several variables.

i) For $(x, y) \in R^2$, define f(x, y) = sinx + cosy.

Then f(x,y) is a real-valued function of two variables with domain R^2.

ii) Let $D = [-1, 1] \times [-1, 1]$. For $(x, y) \in D$ define $f(x, y) = \sin^{-1} x \cos^{-1} y$. Then f(x, y) is a real-valued function of two variables with domain D.

iii) For $(x, y, z) \in R^3$, set $f(x, y, z) = |x| + 2|y| + |z|^2$. Then f(x, y, z) is a real-valued function of 3 variables with domain R^3.

iv) Let $x = (x_1, x_2, ..., x_n)$ be any element of R^n. For any j, $1 \le j \le n$, define

$\pi_j(x) = x_j$ = j-th coordinate of x.

Clearly the n-functions $\pi_1, \pi_2, ..., \pi_n$ are real-valued functions of n-variables with domain R^n. The function π_j, $1 \le j \le n$, is called the j-th **projection** from R^n to **R**.

v) For any $x \in R$, define f(x) = (x, 0).

Then $f : R \to R^2$ is a vector-valued function of one-variable.

vi) Let D be the open sphere with centre at (0, 0, 0) and radius 1 in R^3. Then

$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

is a real-valued function of 3 variables with domain D.

vii) For any (x, y) in R^2 define g(x, y) = (x, y, 0)

Then. $g : R^2 \to R^3$ is a vector-valued function of two variables.

viii) For $(x, y) \in R^2$ define $f(x,y) = (e^x \cos y, e^x \sin y)$.

Then f(x,y) is a vector-valued function of two variables with domain R^2.

ix) A polynomial in n variables, $x_1, x_2, ..., x_n$ is an expression of the form

$\Sigma a_{k_{1i} k_{2i} ...., k_{ni}}\ x_1^{k_{1i}} x_2^{k_{2i}} ......x_n^{k_{ni}}$, where k_{ji}'s are non-negative integers, and $k_{1i} + k_{2i} + ...+ k_{ni} = i$.

For example, $x^3y^2z + 10x^2yz^3 + 8xyz + z^5$ is a polynomial in 3 variables, and $xy^2 + 2xy - y^4$ is a polynomial in 2 variables. If we define $f : R^n \to R$ such that $f(x_1, x_2, ..., x_n)$ = a polynomial in n variables, then f is a real-valued function of n variables...

Remark 3 : As in the case of functions of one variable, we often define functions of several variables with the help of a formula. When a function of several variables is defined with the help of a formula, then its domain is the set of all those points where the given formula is valid. For example, the domain of the function

$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$

of three variables is the closed sphere

$\{(x, y, z) \mid x^2 + y^2 + z^2 \leq 1\}$.

Similarly, the domain of a function

$f(x, y) = \frac{2xy}{x^2 + y^2}$

of two variables is the set $R^2 \setminus \{(0, 0)\}$.

Remark 4 : Let $f : D \to R^m$, $m > 1$ be a vector-valued function, where D is a subset of R^n. Then the function f gives rise to m real-valued functions defined on D, which in turn determine f uniquely. The functions are given by $(\pi_j \circ f)(x) = \pi_j(f(x))$, $x \in D$ for $1 \leq j \leq m$, where π_j denotes the j-th projection,

$\pi_j : R^m \to R$ defined in iv) earlier.

Clearly, $f(x) = (\pi_1(f(x)), \pi_2(f(x)), ... \pi_m(f(x)))$.

Conversely, if $g_1, g_2, ..., g_m$ are m real-valued functions defined on D, then these functions give rise to a unique vector-valued function g on D defined by

$g(x) = (g_1(x), g_2(x), ..., g_m(x))$.

This means we can break up any vector-valued function into a number of real-valued functions. As a result, you will soon see that many times the consideration of vector-valued functions can be reduced to the consideration of real-valued functions. The functions $g_1, ..., g_m$ are usually referred to as the components or **component functions** of g.

You are already familiar with the graphs of a number of real-valued functions of a real variable. Now let us see how we can geometrically represent a real-valued function of two variables.

Definition 4 : Let f(x, y) be a real-valued function of two variables with domain D. Then the graph of the function f is the set of points (x, y, z) in the Euclidean space of 3-dimensions such that z = f(x, y), i.e.,

Graph of f = G (f) = $\{(x, y, z) \mid z = f(x, y), (x, y) \in D\}$

We are giving the graphs of some simple functions here in Fig. 5.

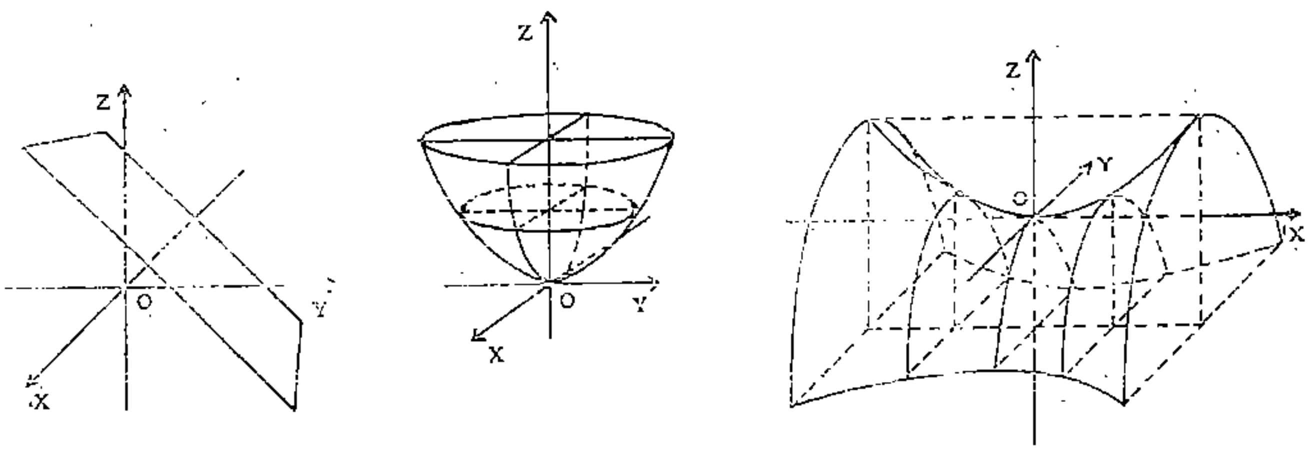


Fig. 5 : Graphs of (a) $f(x, y) = x - y + 2$, (b) $f(x, y) = x^2 + y^2$, (c) $f(x, y) = x^2 - y^2$

But in most cases you will find that it is not easy to plot the graph of a real-valued function of two variables. However, we can visualise the graph with the help of 'level curves' defined below.

Definition 5 : Let f(x, y) be a real-valued function of two variables and let c be a constant. Then the set of points (x, y) in the plane such that f (x, y) = c is called a **level curve** of the function with value c.

Clearly the level curve $f(x, y) = c$ is nothing but the intersection of the surface $z = f(x, y)$, i.e., the graph of f, with the plane $z = c$.

Roughly speaking, the graph of a real-valued function of two variables can be obtained by piling up the level curves $f(x, y) = c$, as c varies over the range, that is the set of values of f(x, y). Look at this example.

Example 4 : Let us find the domain and the range of

i) $f(x, y) = 100 - x^2 - y^2$,

ii) $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$, and examine their level curves.

i) The domain of the given function is the whole of $\mathbf{R}^2$. The range is the set of all real numbers ≤ 100. The level curves are the circles with centre at the origin. See Fig. 6(a)

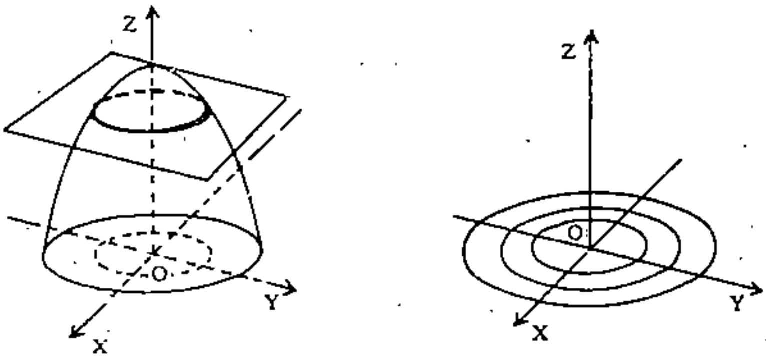


Fig. 6 : (a) Graph of $f(x, y) = 100 - x^2 - y^2$, (b) Level curves

ii) Note that here we have not expressed the function in the form $z = f(x, y)$, and we cannot write the value of z explicitly by substituting the values of x and y. Still we can find the level curves by putting $\frac{x^2}{9} + \frac{y^2}{16} = c$. These will be ellipses (see Fig. 7).

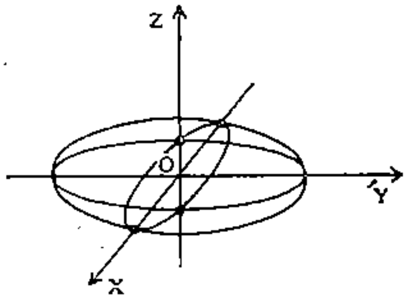


Fig. 7 Graph of $\frac{x^2}{9} + \frac{y^2}{16} + z^2 = 1$

Example 5 : We'll now draw the graph of the function $f(x, y) = x - y$.

The graph of this function is the plane $z = x - y$ in $\mathbf{R}^3$. See Fig. 8.

The level curves are the straight lines $x - y = c$.

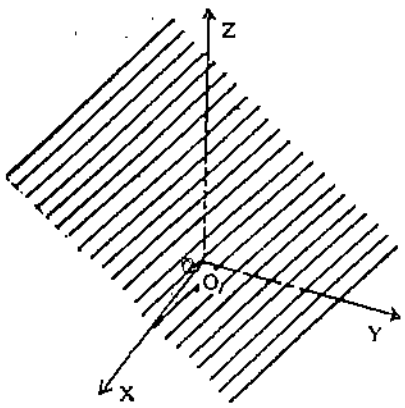


Fig. 8 : Graph of $z = x - y$

We can extend Definition 4 to the case of real-valued functions of three variables.

Definition 6 : Let $f(x, y, z)$ be a real-valued function of three variables with domain D. Then the graph of the function $f(x, y, z)$ is the set

Graph of $f = G(f) = \{(x, y, z, w) \mid w = f(x, y, z), (x, y, z) \in D\}$ in $\mathbf{R}^4$.

Since the graph of a real-valued function of three variables lies in the 4-dimensional Euclidean space, it is not possible to realise it geometrically. We can, however, visualise the level surfaces defined below.

Definition 7 : Let $f(x, y, z)$ be a real-valued function of three variables and let c be a constant. Then the set of points (x, y, z) in space such that $f(x, y, z) = c$ is called the **level surface** of the function f with value c.

What are the level surfaces of the function $f(x, y, z) = x + 2y + 3z$? These are the planes $x + 2y + 3z = c$, where c is a constant.

You will agree that the level surfaces of the function $f(x, y, z) = x^2 + y^2 + z^2 - a^2$ are spheres given by $x^2 + y^2 + z^2 - a^2 = c$, or $x^2 + y^2 + z^2 = c + a^2$, where $c > -\sqrt{a^2}$.

Try to do this exercise now.

E 5) Find the domain of the following functions :

a) $f(x, y) = \dfrac{xy}{x^4 + y^4}$

b) $f(x, y) = \dfrac{x + y}{x - y}$

c) $f(x, y) = x \sin \dfrac{1}{x} + y \sin \dfrac{1}{y}$

d) $f(x, y, z) = \dfrac{1}{\sqrt{4 - x^2 - y^2 - z^2}}$

e) $f(x, y, z) = \dfrac{z}{x^2 - y^2}$

Just as we can define the sum, product, quotient for functions from $\mathbf{R}$ to $\mathbf{R}$, we can define these algebraic operations on functions of several variables too. Let us consider these one by one.

Sum of Two Functions : Let $f : D_1 \to \mathbf{R}^m$ and $g : D_2 \to \mathbf{R}^m$, where D_1 and D_2 are subsets of $\mathbf{R}^n$. Let $D = D_1 \cap D_2 \neq \phi$. Then the function $f + g$ defined on D by

$(f + g)(x) = f(x) + g(x)$

is called the sum of the two vector-valued functions f and g.

Product of Two Functions : Let $f : D_1 \to \mathbf{R}$ and $g : D_2 \to \mathbf{R}$, where D_1 and D_2 are subsets of $\mathbf{R}^n$. Let $D = D_1 \cap D_2 \neq \phi$. Then the function fg defined on D by

$(fg)(x) = f(x)\, g(x)$

is called the **product** of the two real-valued functions f and g.

Quotient of Two Functions : Let f and g be the real–valued functions mentioned above. Suppose that the set

$D^* = \{x \mid x \in D, g(x) \neq 0\} \neq \phi$.

Then the function f/g defined on D^* by

$$(f/g)(x) = \frac{f(x)}{g(x)}$$

is called the quotient of the functions f and g.

You must have noticed that we have defined the sum f + g when f and g are two vector - valued functions. But we have defined the product fg and the quotient (f/g) only for real - valued functions f and g. This is because, as we have mentioned at the end of Sec. 3.2.2, the product and quotient of two vectors are not defined.

We shall now illustrate these operations with some examples.

Example 6 : i) Let $f(x, y) = y\sin\frac{1}{x}$ and $g(x, y) = x\sin\frac{1}{y}$. Then

D_1 = Domain of f = $\{(x, y) \mid x \neq 0\}$

D_2 = Domain of g = $\{(x, y) \mid y \neq 0\}$

Clearly $D_1 \cap D_2 = \{(x, y) \mid x \neq 0 \text{ and } y \neq 0\} \neq \phi$.

Thus, the sum function

$(f + g)(x, y) = f(x, y) + g(x, y) = y\sin\frac{1}{x} + x\sin\frac{1}{y}$ is defined on $D_1 \cap D_2$ i.e.,

$\mathbf{R}^2 \setminus \{\text{both the axes}\}$.

ii) Let $f(x, y) = (e^x \cos y, e^x \sin y)$ and $g(x, y) = (x^2, y^2)$. Then the sum function
$(f+g)(x, y) = f(x, y) + g(x,y) = (e^x \cos y + x^2, e^x \sin y + y^2)$
is defined on the whole of $\mathbf{R}^2$.

iii) Let $f(x, y, z) = |x|\ |y|^2$, $g(x, y, z) = \sin(x + y + z)$.

Then the product function fg is defined by

$(fg)(x, y, z) = f(x, y, z)\, g(x, y, z) = |x|\ |y|^2 \sin(x + y + z)$

and has its domain as $\mathbf{R}^3$.

iv) Let $f(x, y) = 2xy$, $g(x, y) = x^2 + y^2$.

Clearly

$D^* = \{(x, y) \mid g(x, y) \neq 0\} = \mathbf{R}^2 \setminus \{(0, 0)\}$.

Thus, the quotient of f and g is defined by

$\left(\frac{f}{g}\right)(x, y) = \frac{f(x, y)}{g(x, y)} = \frac{2xy}{x^2 + y^2}$ and its domain is $\mathbf{R}^2 \setminus \{(0, 0)\}$.

Here is an exercise which you can try.

E 6) Find the product and the quotient of the following pairs of function. State their domains in each case.

a) $f(x,y) = x^2 y$, $g(x,y) = x^2y^2$

b) $f(x,y) = \sin x + \sin y$, $g(x, y) = \frac{1}{x}\cos y$, $x \neq 0$.

You have already learnt how to define the composite of two real-valued functions of a real variable (Sec.6 of Unit 1 in Calculus). To recall the composite of two functions $f: \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$ and $g(x) = \sin x$, respectively, will be the function $g \circ f: \mathbf{R} \to \mathbf{R}$ such that $g \circ f(x) = g(f(x)) = \sin x^2$. Here, we extend this notion to functions of several variables.

Definition 8 : Let $g : D_1 \to \mathbf{R}^m$, where D_1 is a subset of $\mathbf{R}^n$ and $f : D_2 \to \mathbf{R}^p$, where D_2 is a subset of $\mathbf{R}^m$. Suppose that $g(D_1) \subset D_2$.

Then we can define a new function $\phi : D_1 \to \mathbf{R}^p$ by setting $\phi(x) = f(g(x))$

for all $x \in D_1$.

This new function $\phi : D_1 \to \mathbf{R}^p$ is called the composite of the functions f and g and is denoted by $f \circ g$. If $n = m = p = 1$, then this definition coincides with the definition given in the earlier course of Calculus.

Let us see some examples of composite functions.

Example 7 : Let $g(x, y) = x^2 + xy + y^2$ be a function from $\mathbf{R}^2 \to \mathbf{R}$ and $f(t) = \sin t$ be a function from $\mathbf{R} \to \mathbf{R}$. Then the composite function $f \circ g$ defined by

$$(f \circ g)(x, y) = f(g(x, y)) = f(x^2 + xy + y^2) = \sin(x^2 + xy + y^2)$$

is a function from $\mathbf{R}^2 \to \mathbf{R}$.

Note that $g \circ f$ does not make sense over here.

You may also come across functions f and g, for which $f \circ g$ and $g \circ f$ are both defined, but are not equal (see E 7a).

Example 8 : Let $f(x, y) = (x^2 + y^2, x+y, xy)$ be a function from $\mathbf{R}^2 \to \mathbf{R}^3$ and $g(x, y, z) = (e^{x+y}, \sin(y + z))$ be a function from $\mathbf{R}^3 \to \mathbf{R}^2$.

Then the composite function $g \circ f$ defined by

$$(g \circ f)(x, y) = g(f(x, y)) = g(x^2 + y^2, x+y, xy)$$

$$= (e^{x^2 + y^2 + x + y}, \sin(x + y + xy))$$

is a function from $\mathbf{R}^2 \to \mathbf{R}^2$.

You can now easily write the composite of the functions given in the following exercise.

E 7) Find $f \circ g$ and $g \circ f$, if they exist, for the functions given by

a) $f(x, y, z) = (e^x, \ln(x^2 + y^2 + 1), z^2)$, $g(x, y, z) = (x + y, 2y, 5z)$

b) $f(x, y) = \begin{cases} \dfrac{|2xy|}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ $\qquad g(t) = \sin^{-1} t$

Let us now summarise what we have covered in this unit

3.4 SUMMARY

In this unit we have

1. defined the Cartesian product of sets and discussed the algebraic structure of $\mathbf{R}^n$.
2. introduced a distance function on $\mathbf{R}^n$ and defined r-neighbourhood of points in $\mathbf{R}^n$.
3. defined real-valued and vector-valued functions of several variables.
4. introduced level curves and level surfaces, respectively, for functions of two and three variables.
5. defined the sum, product, quotient and composite of functions from $\mathbf{R}^n \to \mathbf{R}^m$.

3.5 SOLUTIONS AND ANSWERS

E 1) $S_1 : x \in \mathbf{R}^n \Rightarrow x = (x_1, x_2, ..., x_n)$, where each $x_i \in \mathbf{R}$

Now $ax = (ax_1, ax_2, ..., ax_n)$. Since $ax_i \in \mathbf{R}$ for $1 \le i \le n$, $ax \in \mathbf{R}^n$.

S_2 : Let $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$

Then $a(x + y) = a[(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n)]$

$a[x_1 + y_1, x_2 + y_2, ..., x_n + y_n]$.

$[a(x_1 + y_1), a(x_2 + y_2), ..., a(x_n + y_n)]$

$= [ax_1 + ay_1, ax_2 + ay_2, ..., ax_n + ay_n]$

$= (ax_1, ax_2, ..., ax_n) + (ay_1, ay_2, ..., ay_n)$

$= a(x_1, x_2, ..., x_n) + a(y_1, y_2, ..., y_n) = ax + ay$

S_3, S_4 follow similarly.

S_5 : Suppose $ax = 0 \ \forall\ x \in \mathbf{R}^n$. Choose $x \in \mathbf{R}^n$ such that $x = (1, 0, 0, ...,0)$.

Then, $ax = 0 \Rightarrow a(1, 0, 0, ...,0) = (a, 0, 0, ...,0) = 0$

$\Rightarrow a = 0$.

Conversely, suppose $a = 0$. Take $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$.

Then $ax = (ax_1, ax_2, ..., ax_n)$. But $ax_i = 0 \ \forall\ i = 1, 2, ..., n$.

Thus $ax = 0$.

This is true $\forall\ x \in \mathbf{R}^n$.

E 2) $e_1 = (1, 0, ...,0)$, $e_2 = (0, 1, 0, ...,0)$, ..., $e_n = (0, 0, ...,1)$

Let $x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$.

Then

$$x_1 e_1 + x_2 e_2 + ... + x_n e_n = x_1 (1, 0, ...,0) + x_2 (0, 1, ...,0) + ... + x_n (0, 0, ..., 1) = (x_1, x_2, ..., x_n)$$

This shows that every $x \in \mathbf{R}^n$ can be written as $x = \sum_{i=1}^{n} x_i e_i$.

To prove uniqueness, suppose x can be written as $x = \sum x_i e_i$ and $x = \sum y_i e_i$ where x_i and $y_i \in \mathbf{R} \ \forall\ i = 1, 2,......., n$.

Then $(x_1, x_2, ..., x_n) = (y_1, y_2, ...y_n)$. This shows that $x_i = y_i \ \forall\ i = 1, 2, ...n$.

Hence x can be written in only one way as $\sum_{i=1}^{n} x_i e_i$.

E 3) $x = e + f = (1, 0) + (1, 1) = (2, 1)$

$y = 2e + 3f = 2(1, 0) + 3(1, 1)$

$= (2, 0) + (3, 3)$

$= (5,3)$

$\therefore$, $x - y = (2, 1) - (5, 3)$

$= (-3, -2)$

$\therefore$, $|x - y| = \sqrt{(-3)^2 + (-2)^2}$

$= \sqrt{13}$

Similarly, $|2x - y| = |(1, -1)| = \sqrt{2}$ and $|x| = |(2,1)| = \sqrt{5}$.

E 4) $S = \{ (x, y, z) \in \mathbf{R}^3 \mid |(x - a, y - b, z - c)| < r \}$

Now, $(x, y, z) \in S \Rightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < r$

$$\Rightarrow (x-a)^2 + (y-b)^2 + (z-c)^2 < r^2$$

$$\Rightarrow (x-a)^2 < r^2,\ (y-b)^2 < r^2 \text{ and } (z-c)^2 < r^2$$

$$\Rightarrow |x - a| < r,\ |y - b| < r,\ |z - c| < r$$

$$\Rightarrow (x, y, z) \in P_1$$

$$\Rightarrow S \subset P_1$$

Now, $x \in P_2 \Rightarrow |x - a| < \frac{r}{\sqrt{3}},\ |y - b| < \frac{r}{\sqrt{3}},\ |z - c| < \frac{r}{\sqrt{3}}$

$$\Rightarrow \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \sqrt{\frac{r^2}{3} + \frac{r^2}{3} + \frac{r^2}{3}} < r$$

$$\Rightarrow |(x - a, y - b, z - c)| < r$$

$$\Rightarrow (x, y, z) \in S$$

$$\Rightarrow P_2 \subset S.$$

E 5) a) The domain consists of all the points in $\mathbf{R}^2$ except those for which $x^4 + y^4 = 0$.

Now, $x^4 + y^4 = 0 \Leftrightarrow x^4 = 0$ and $y^4 = 0$

$\Leftrightarrow x = 0$ and $y = 0$

Therefore, domain $= \mathbf{R}^2 \setminus \{(0, 0)\}$

b) $\{(x, y) \in \mathbf{R}^2 \mid x - y \neq 0\} = \{(x, y) \in \mathbf{R}^2 \mid x \neq y\}$

c) $\{(x, y) \in \mathbf{R}^2 \mid x \neq 0, y \neq 0\}$

d) The open sphere with radius 2 and centre (0, 0, 0), since $\sqrt{4 - x^2 - y^2 - z^2}$ has to be positive.

e) $\{(x, y, z) \in \mathbf{R}^3 \mid y \neq \pm x\}$

E 6) a) $(fg)(x, y) = f(x, y)\, g(x, y) = x^2y \cdot x^2y^2 = x^4y^3$

Domain $= \mathbf{R}^2$

$$\left(\frac{f}{g}\right)(x, y) = \frac{x^2y}{x^2y^2} = \frac{1}{y}$$

Domain $= \{(x,y) \in \mathbf{R}^2 \mid x \neq 0, y \neq 0\}$

b) $(fg)(x,y) = \frac{1}{x}(\sin x + \sin y)\cos y$

Domain $= \{(x, y) \in \mathbf{R}^2 \mid x \neq 0\}$

$$\left(\frac{f}{g}\right)(x,y) = \frac{x(\sin x + \sin y)}{\cos y}$$

Domain $= \{(x,y) \in \mathbf{R}^2 \mid y \neq (2n+1)\frac{\pi}{2},\ n \in \mathbf{Z}\}$

E 7) a) $(g \circ f)(x, y, z) = g(f(x, y, z))$

$= g(e^x, \ln(x^2 + y^2 + 1), z^2)$

$= (e^x + \ln(x^2 + y^2 + 1),\ 2\ln(x^2 + y^2 + 1),\ 5z^2)$.

and $(f \circ g)(x, y, z) = f(g(x, y, z))$

$= f(x + y, 2y, 5z)$

$= (e^{x+y}, \ln(x^2 + 5y^2 + 2xy + 1), 25z^2)$

Clearly, $f \circ g \neq g \circ f$, eventhough both are defined.

b) $g \circ f : \mathbf{R}^2 \to \mathbf{R}$

$$(g \circ f)(x, y) = \begin{cases} \sin^{-1}\left(\dfrac{2\,|xy|}{x^2 + y^2}\right), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$f \circ g$ does not exist.

NOTES

UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGMM-07
Advanced Calculus

Block

2

PARTIAL DERIVATIVES

BLOCK 2 PARTIAL DERIVATIVES

In Block 1 we introduced you to functions of several variabl.... In this block we shall first familiarise you with the notions of limit and continuity of functions of several variables. You will see that the distance function defined on $\mathbf{R}^n$, which you studied in Unit 3, comes in handy while studying these notions.

We will then turn our attention to derivatives of functions of several variables. In the remaining units of this block we shall show you some ways in which the concept of a derivative can be extended to functions of more than one variable. A major impetus to the work about derivatives of functions of two or three variables was provided by mathematicians studying partial differential equations arising out of physical situations.

In Unit 5 we will introduce you to partial derivatives. You will also study the notion of a differentiable function of several variables, which is a true generalisation of a differentiable function of one variable. We will also bring out the connection between the concepts of continuity, differentiability and partial derivatives. In all these considerations we will confine ourselves to functions of two or three variables.

The third unit in this block deals with higher order partial derivatives. In general, you will see that the order of the variables, with respect to which a function is differentiated, is important. We will also give sets of sufficient conditions for the equality of mixed partial derivatives of a function of two variables.

We shall then tackle the differentiation of composite functions in Unit 7. Here you will study the chain rule and Euler's theorem for homogeneous functions.

Finally, we will define the directional derivative of a function in a given direction at a given point. You will see that the partial derivatives of a function are nothing but its directional derivatives in the directions of the coordinate axes.

So, by the end of this block, you will be familiar with various ways of defining the derivatives of a function of several variables. We will also give geometrical interpretations of each of these concepts to help you understand them better. Our emphasis will be on the study of partial derivatives, since we will be using them throughout Block-3.

Notations and Symbols

$\lim_{x \to a} f(x)$	Limit of f(x) as x tends to a; where $x = (x_1, x_2, \ldots, x_n)$, $a = (a_1, a_2, \ldots, a_n)$
$\frac{\partial f}{\partial x}$, $D_1 f$, f_x	Partial derivative of f w.r.t. x.
$\frac{\partial^2 f}{\partial x^2}$, f_{xx}	second order partial derivative of f w.r.t. x.
$\frac{\partial^2 f}{\partial x \partial y}$, f_{yx}	second order partial derivative of f, first w.r.t. y and then w.r.t. x.
$D_\theta f(a)$, $f_\theta(a)$, $f_v(a)$	directional derivative of f at a in the direction $v = (\cos\theta, \sin\theta)$
∇f	$= (f_x, f_y)$, gradient of f.

UNIT 4 LIMIT AND CONTINUITY

Structure

4.1 INTRODUCTION

In Unit 3 you have seen some examples of functions of several variables. In this unit we introduce you to the notions of limit and continuity for functions of several variables. We shall define these concepts for functions of n variables where $n \geq 1$, but in the examples and exercises we shall confine our attention to functions of two or three variables only. We shall first deal with real-valued functions of several variables, and then with those functions of several variables, which are vector-valued, i.e., whose range is a subset of R^n, $n > 1$.

You will see that the definitions of limit and continuity for functions of several variables are similar to those for functions of a single variable.

Objectives

After reading this unit, you should be able to:

- define and evaluate the limits of functions of several variables,
- decide whether a function of several variables is continuous or not at a given point or a set of points

4.2 LIMITS OF REAL-VALUED FUNCTIONS

You are already familiar with the concept of limit for real-valued functions of one variable. We shall now study this concept for functions of several variables. We begin by discussing limits of real-valued functions of many variables. Go over the definitions of neighbourhood and distance in R^n again before you go further.

Definition 1: Let f(x) be a real-valued function defined in a neighbourhood

$$S(a, h) = \{x \in R^n : |x - a| < h\}$$

of a point a of R^n, except possibly at the point a. We say that the limit of f(x) as x tends to a is equal to a real number L, if given $\varepsilon > 0$, there exists a positive real number δ (depending on ε), $\delta < h$, such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Perhaps you have noted that this definition is exactly similar to the definition of the limit of a function of a single variable. The only difference is that the distance $|x - a|$ here is the distance of x from a in R^n. However, note that $|f(x) - L|$ is the absolute value of the real number $f(x) - L$.

If n = 1, then Definition 1 coincides with the definition of $\lim_{x \to a} f(x)$ for real-valued functions of a real variable. We shall use the notation $\lim_{x \to a} f(x) = L$ or $f(x) \to L$ as $x \to a$ to express the fact that the limit of the function f(x) as x tends to a is L. Proceeding exactly as in the case of real-valued functions of a real variable, we can prove that $\lim_{x \to a} f(x)$, if it exists, is unique. We are leaving the proof to you as an exercise (see E 1).

E1) If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M'$, where a is a point of $\mathbf{R}^n$, then prove that $L = M'$.

Here are some important facts related to limits.

Remark 1 : i) The following statement is equivalent to Definition 1 and therefore could have been adopted as the definition of $\lim_{x \to a} f(x)$.

"Let f(x) be a real-valued function defined in a neighbourhood S(a,h) of a point a of $\mathbf{R}^n$, except possibly at the point a. We say that the limit of f(x) as x tends to a is a real number L if, given as ε-neighbourhood of L, there exists a δ-neighbourhood of a (where $\delta < h$ and depends on the ε-neighbourhood of L) such that whenever x belongs to the δ-neighbourhood of a and $x \neq a$, then f(x) belongs to the ε-neighbourhood of L."

ii) If $\lim_{x \to a} f(x) = L$, then f(x) is bounded in a deleted neighbourhood of a. That is, there exist real numbers m and M such that

$m \leq f(x) \leq M$ for all x in a deleted neighbourhood of a.

How can we prove this? We'll use Definition 1. Thus, $\lim_{x \to a} f(x) = L$ means that given some $\varepsilon > 0$, $\exists\ \delta > 0$, such that

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$
$$\Rightarrow L - \varepsilon < f(x) < L + \varepsilon.$$

Now take $m = L - \epsilon$ and $M = L + \epsilon$. Then we have proved that if $\lim_{x \to a} f(x)$ exists, then f(x) is bounded in a deleted neighbourhood of a.

But we would like to tell you that the converse of this statement is not true. In other words, if a function is bounded in a deleted neighbourhood of some point a, it does not follow that the limit of the function exists at a. In Example 2 you will see a function which supports this statement.

We now state a theorem about the algebra of limits. It will be very useful to us in calculating the limits of some functions of several variables. You may recall that you had studied and used a similar theorem for the limits of functions of a single variable. We won't give the proof of this theorem here, as it is a bit technical.

Theorem 1 (Algebra of Limits): Let f and g be two real-valued functions defined in a deleted neighbourhood of a point a in $\mathbf{R}^n$. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then

i) $\lim_{x \to a} (\alpha f)(x) = \alpha \lim_{x \to a} f(x) = \alpha L$ for any $\alpha \in \mathbf{R}$.

ii) $\lim_{x \to a} (f \pm g)(x) = L \pm M$

iii) $\lim_{x \to a} (fg)(x) = LM$

iv) $\lim_{x \to a} (f/g)(x) = \frac{L}{M}$, provided $M \neq 0$.

In the Introduction we have mentioned that we would give examples and exercises only for functions of two or three variables. But before confining ourselves to functions of two or three variables, we will prove a simple result. With the help of this result we will be able to avoid the direct use of the distance formula in $\mathbf{R}^n$ in the definition of the limit.

Theorem 2: Let f(x) be a real-valued function defined in a deleted h-neighbourhood of a point a of $\mathbf{R}^n$. Than $\lim_{x \to a} f(x) = L$ if and only if given $\varepsilon > 0$, there exist positive real numbers $\delta_1, \delta_2, \ldots, \delta_n$ (which depend on ε), $\delta_i < h$, $1 \leq i \leq n$, such that whenever

$0 < |x_i - a_i| < \delta_i, \forall\ i = 1, 2, \ldots, n$, then $|f(x) - L| < \varepsilon$,

where $a = (a_1, \ldots, a_n)$ and $x = (x_1, \ldots, x_n)$.

Proof : Let $\lim_{x \to a} f(x) = L$. Then given $\varepsilon > 0$ there exists a real number $\delta > 0$, $\delta < h$, such that

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Let $\delta_i = \delta/\sqrt{n}$, $1 \le i \le n$.

Now, if for any point $x = (x_1, \ldots, x_n)$, we have $|x_i - a_i| < \delta_i = \delta/\sqrt{n}$, for all i such that $1 \le i \le n$, then $|x-a| < \delta$, and therefore whenever

$$0 < |x_i - a_i| < \delta_i, \forall \text{ i such that } 1 \le i \le n, \text{ then } |f(x) - L| < \varepsilon.$$

Conversely, suppose that the given condition is satisfied.

Let $\delta = \min \{\delta_1, \ldots, \delta_n\}$.

Then, $0 < |x-a| < \delta \Rightarrow 0 < |x_i - a_i| < \delta \le \delta_i, \forall$ i such that $1 \le i \le n$

$$\Rightarrow |f(x) - L| < \varepsilon,$$

which implies that $\lim_{x \to a} f(x) = L$.

We will now apply Theorems 1 and 2 to calculate the limits in the next example.

The limit of f(x,y) as $(x,y) \to (a,b)$ is also denoted by $\lim_{\substack{x \to a \\ y \to b}} f(x,y)$.

Example 1: Let us show that

i) $\lim_{\substack{x \to 0 \\ y \to 2}} (x^2 + y) = 2$

ii) $\lim_{\substack{x \to 0 \\ y \to 0}} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x}\right) = 0$

iii) $\lim_{\substack{x \to 2 \\ y \to 3}} (x^2 + xy + y^3) = 37$.

iv) $\lim_{(x, y, z) \to (a, b, c)} (xy + yz + zx) = ab + bc + ca$

We'll use Theorem 2 to check the limits in i) and ii).

Let $0 < \varepsilon < 1$ be given. Then

$$|f(x) - L| = |x^2 + y - 2| \le |x^2| + |y-2| < \varepsilon,$$

if $|x| < \varepsilon/2$, $|y-2| < \varepsilon/2$. Thus,

$$\lim_{\substack{x \to 0 \\ y \to 2}} (x^2 + y) = 2$$

in view of Theorem 2. Note that here we have taken $\delta_1 = \varepsilon/2 = \delta_2$.

ii) Clearly

$$\left| x \sin \frac{1}{y} + y \sin \frac{1}{x} \right| \le |x| + |y| < \varepsilon$$

if $|x| < \varepsilon/2$, $|y| < \varepsilon/2$. Thus, applying Theorem 2 again, we can say that

$$\lim_{\substack{x \to 0 \\ y \to 0}} \left(x \sin \frac{1}{y} + y \sin \frac{1}{x}\right) = 0.$$

We will use the algebra of limits (Theorem 1) to check the limits in iii) and iv).

iii) Since,

$$\lim_{\substack{x \to 2 \\ y \to 3}} x^2 = 4, \quad \lim_{\substack{x \to 2 \\ y \to 3}} xy = 6, \quad \lim_{\substack{x \to 2 \\ y \to 3}} y^3 = 27,$$

using algebra of limits, we get that the given limit is equal to 37.

iv) Using algebra of limits, we get

$$\lim_{(x, y, z) \to (a, b, c)} (xy + yz + zx) = \lim_{(x, y, z) \to (a, b, c)} xy + \lim_{(x, y, z) \to (a, b, c)} yz + \lim_{(x, y, z) \to (a, b, c)} zx$$

$$= ab + bc + ca.$$

We have mentioned in Remark 1 that a function may be bounded in a neighbourhood of a point, but may not have a limit at that point. In the next example we give two functions to illustrate this.

Example 2 : (i) If $f(x,y) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) \text{ does not exist.}$$

ii) If $f(x, y) = \begin{cases} 1, & x \text{ irrational} \\ 0, & x \text{ rational} \end{cases}$, then

$$\lim_{(x, y) \to (a, b)} f(x, y) \text{ does not exist for any point } (a, b).$$

Note that both these functions are bounded functions.
Let us prove i) first.

i) If possible let us suppose that $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = L$. Then, for a given ε s. t. $0 < \varepsilon < 1$, $\exists$ a real number $\delta > 0$ such that

$$0 < \sqrt{x^2+y^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon/2.$$

In particular, if (x_1, y_1), (x_2, y_2) are two points with $\sqrt{x_1^2 + y_1^2} < \delta$, $\sqrt{x_2^2 + y_2^2} < \delta$, then we get

$$\begin{aligned} |f(x_1, y_1) - f(x_2, y_2)| &= |f(x_1, y_1) - L + L - f(x_2, y_2)| \\ &\leq |f(x_1, y_1) - L| + |f(x_2, y_2) - L| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad \ldots\ldots\ldots(1) \end{aligned}$$

Now consider the points

$(x_1, y_1) = (\delta/2, 0)$ and $(x_2, y_2) = (0, \delta/2)$. For these points we have

$\sqrt{x_1^2 + y_1^2} = \dfrac{\delta}{2} < \delta$, and $\sqrt{x_2^2 + y_2^2} < \delta$. But
$|f(x_1, y_1) - f(x_2, y_2)| = 1 > \varepsilon$.

This contradicts (1). Therefore we can conclude that

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) \text{ does not exist.}$$

Let us consider the second function now.

ii) If possible, let us suppose that $\lim_{\substack{x \to a \\ y \to b}} f(x, y)$ exists. Then, proceeding as in i), we see that given any ε s.t. $0 < \varepsilon < 1$, there exists a real number $\delta > 0$ such that

$$|f(x_1, y_1) - f(x_2, y_2)| < \varepsilon$$

whenever (x_1, y_1), (x_2, y_2) belong to the open disc S with centre (a, b) and radius δ.

Now we can choose (x_1, y_1), (x_2, y_2) in the open disc S so that x_1 is irrational and x_2 is rational.

Then

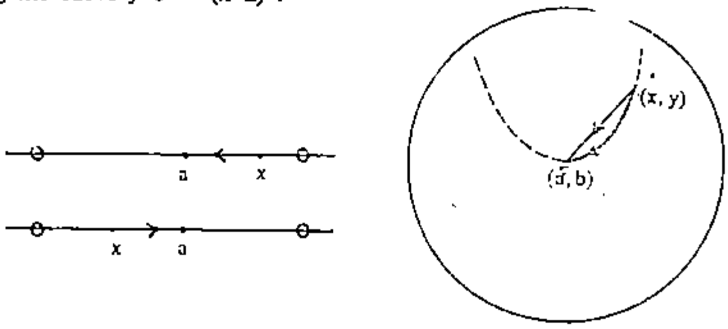
$$|f(x_1, y_1) - f(x_2, y_2)| = 1 > \varepsilon,$$

which shows that $\lim_{\substack{x \to a \\ y \to b}} f(x, y)$ does not exist.

We know that on the real line, a can be approached either from the left or from the right (see Fig. 1(a)). Accordingly, we have the left hand limit, $\lim_{x \to a^-} f(x)$, and the right hand limit, $\lim_{x \to a^+} f(x)$ of a real-valued function of a real variable. We also know that

$\lim_{x \to a} f(x) = L$, if and only if
$\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$.

Now let us consider the Cartesian plane $\mathbf{R}^2$ and a point (x, y) in a neighbourhood of a point (a, b). Then (x, y) can approach (a, b) in many different ways. For example, in Fig. 1(b) you can see that (x, y) can approach (a, b) along a straight line or along the curve $y-b = (x-a)^2$.



(a) (b)

Fig. 1 : (a) $x \in]a-\delta, a+\delta[$ approaches a either from the left or from the right (b) $x \in S(a, r)$ can approach a along a line or along a curve.

In the following theorem you will see that if $\lim\limits_{(x,y) \to (a,b)} f(x, y)$ exists and is equal to L, then f(x, y) approaches L, as (x, y) approaches (a, b) along any path.

Theorem 3 : Let f(x, y) be a real-valued function of two variables such that $\lim\limits_{(x,y) \to (a,b)} f(x, y) = L$.

If $\phi(x)$ is a real-valued function of a real variable such that $\lim\limits_{x \to a} \phi(x) = b$, then $\lim\limits_{x \to a} f(x, \phi(x)) = L$.

Proof : Let $\varepsilon > 0$. Then there exists a real number $\delta > 0$ such that f(x, y) is defined in the open disc with centre (a, b) and radius δ except possibly at (a, b) and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \Rightarrow |f(x, y) - L| < \varepsilon.$$

Since $\lim\limits_{x \to a} \phi(x) = b$, given $\delta > 0$ there exists a real number $\delta_1 > 0$, $\delta_1 < \delta/\sqrt{2}$, such that $\phi(x)$ is defined for all x, $0 < |x-a| < \delta_1$, and $0 < |x-a| < \delta_1 \Rightarrow |\phi(x) - b| < \delta/\sqrt{2}$

Thus,

$$0 < |x-a| < \delta_1 \Rightarrow \sqrt{(x-a)^2 + (\phi(x) - b)^2} < \delta, \text{ and therefore,}$$

$$|f(x, \phi(x)) - L| < \varepsilon$$

i.e., $\lim\limits_{x \to a} f(x, \phi(x)) = L$.

This theorem shows that if $\lim\limits_{\substack{x \to a \\ y \to b}} f(x) = L$ exists, then it is independent of the path along which the point (x, y) approaches the point (a, b). This result can also be interpreted as follows:

If f(x, y) tends to two different limits as (x, y) → (a, b) along two different paths, then $\lim\limits_{(x,y) \to (a,b)} f(x)$ does not exist.

You will find that this interpretation is very useful in proving the non-existence of certain limits. We'll now state this as a corollary to Theorem 3.

Corollary 1 : Suppose f(x,y) is a real-valued function defined in some deleted neighbourhood of the point (a,b). If there exist real valued functions $\phi_1(x)$ and $\phi_2(x)$ such that

$$\lim_{x \to a} \phi_1(x) = b = \lim_{x \to a} \phi_2(x)$$

and

$$\lim_{x \to a} f(x, \phi_1(x)) \neq \lim_{x \to a} f(x, \phi_2(x)),$$

then

$\lim\limits_{(x,y) \to (a,b)} f(x,y)$ does not exist.

We will illustrate the usefulness of this corollary with the help of an example now.

Example 3: We'll show that

i) the limit of $\frac{x^2-y^2}{x^2+y^2}$ does not exist as $(x,y) \to (0,0)$ and

ii) the limit of f(x,y) does not exist as $(x,y) \to (0,0)$,

$$\text{where } f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y}, & x \neq y \\ 0, & x = y \end{cases}$$

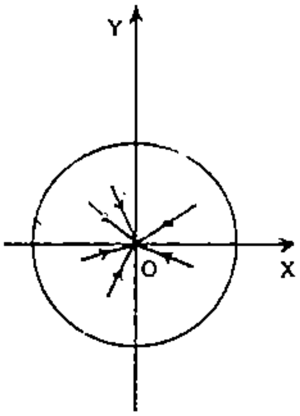


Fig 2: Radial vectors

Let us start with i).

i) Let $y = mx$. Then

$$\lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} = \frac{1-m^2}{1+m^2}$$

Now the value of $\frac{1-m^2}{1+m^2}$ is different for different m. This means that f(x,y) approaches different values as $(x, y) \to (0,0)$ along different radial vectors (see Fig. 2).

Thus, the limit of $\frac{x^2-y^2}{x^2+y^2}$ does not exist as $(x, y) \to (0,0)$ in view of Corollary 1. (Here we can take $\phi_1(x) = m_1x$, $\phi_2(x) = m_2x$, $m_1 \neq \pm m_2$).

ii) Let $\phi_1(x) = x - x^3$, $\phi_2(x) = x - x^2$.

Then

$$\lim_{x\to 0} f(x, \phi_1(x)) = \lim_{x\to 0} \frac{x^3+(x-x^3)^3}{x^3} = 2, \text{ and}$$

$$\lim_{x\to 0} f(x, \phi_2(x)) = \lim_{x\to 0} \frac{x^3+(x-x^2)^3}{x^2} = 0.$$

Therefore, the limit of f(x, y) does not exist as $(x,y) \to (0,0)$ by Corollary 1 again.

Many times you would find that conversion to polar coordinates, i.e., the use of as the substitution $x=r\cos\theta$, $y=r\sin\theta$ is very useful for evaluation of certain limits. Consider the following example.

Example 4: Let us prove that

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = 0.$$

For this we'll make the substitution $x=r\cos\theta$, $y=r\sin\theta$, so that $x^2+y^2=r^2$. Then

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right| \leq 2r = 2\sqrt{x^2+y^2}$$

because $|\cos^3\theta - \sin^3\theta| \leq |\cos^3\theta| + |\sin^3\theta| \leq 2$.

Now, if $|x| < \frac{\varepsilon}{\sqrt{8}}$ and $|y| < \frac{\varepsilon}{\sqrt{8}}$, then $2\sqrt{x^2+y^2} < \varepsilon$, and therefore

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| < \varepsilon$$

That is, $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = 0.$

Now see if you can solve these exercises.

E2) Show that

a) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$

b) $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2y^2}{\sqrt{x^2+y^2}} = 0$

c) $\lim\limits_{(x,\, y,\, z) \to (0,\, 1,\, 2)} \dfrac{x^2+3xyz - 5z^2}{xy^3+5z^2 - 3xy+x^3} = -1$

d) $\lim\limits_{(x,\, y) \to (0,\, 0)} \dfrac{x \sin y}{2x^2+1} = 0$

E3) Show that the limits of the following functions do not exist as $x \to 0$, $y \to 0$.

a) $\dfrac{xy}{x^2+y^2}$

b) $\dfrac{x^2}{x^2+y}$

c) $\dfrac{x^2 - y^2}{x^2+y^2} + \dfrac{2xy}{\sqrt{x^2+y^2}}$

E4) In the following problems, how close to the origin should we take the point (x,y) or (x, y, z) to make

$| f(x, y) - f(0, 0) | < \varepsilon$, or

$| f(x, y, z) - f(0, 0, 0) | < \varepsilon$

for the given ε?

a) $f(x, y, z) = x^2 + y^2 + z^2$, $\varepsilon = 0.01$

b) $f(x, y) = xy$, $\varepsilon = 0.0004$

By now you must have become familiar with the concept of the limit of a function of several variables. We shall now discuss the continuity of these functions.

4.3 CONTINUITY OF REAL-VALUED FUNCTIONS

In Unit 2 of the Calculus course you saw that the knowledge of the limit of a function of one variable is necessary for studying the continuity of these functions. To be precise, you know that a function $f: R \to R$ is continuous at a point $a \in R$ if $\lim\limits_{x \to a} f(x) = f(a)$.

In the last section we have studied the concept of limit for real-valued functions of several variables. Let us now see how we can use this knowledge to define continuous functions from $R^n \to R$.

Definition 2: Let f(x) be a real-valued function of n variables defined in a neighbourhood of the point $a \in R^n$. We say that the function f(x) is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a),$$

i.e., given $\varepsilon > 0$ there exists a real number $\delta > 0$ (depending upon ε such that

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

We say that a real-valued function of n variables is **continuous on a set A** contained in the domain of the function, if the function is continuous at each point of A

A real-valued function of n variables is said to be a **continuous function** if the function is continuous at every point of its domain of definition.

You have come across many examples of real valued functions in the last section. Let us check the continuity of some of these.

For example, consider $f(x, y) = x^2+y$. In Example 1 we have seen that

$$\lim_{(x,y)\to(0,2)} (x^2+y) = 2.$$

Now, $f(0, 2) = 2$.

Therefore, $\lim_{(x,y)\to(0,2)} (x^2+y) = f(0, 2)$.

This means that $f(x, y) = x^2+y$ is continuous at $(0,2)$.
The projections $\pi_j : \mathbf{R}^n \to \mathbf{R}$,

$\pi_j (x_1, x_2, \ldots\ldots, x_n) = x_j, 1 \leq j \leq n$

are continuous functions on $\mathbf{R}^n$. (We are leaving the proof to you. See E 6)).

You will agree that the function $f(x, y) = \dfrac{x^2 - y^2}{x^2+y^2}$ which we considered in Example 3 is not continuous at (0, 0). Remember, we have proved that it does not have a limit as $(x, y) \to (0, 0)$?

We now state a theorem which is an easy consequence of Theorem 1. You would find it very useful in establishing the continuity of many functions.

Theorem 4 (Algebra of Continuous Functions) : Let f and g be two real-valued functions of n variables, which are continuous at a point a. Then

i) $f \pm g$ are continuous at a

ii) αf is continuous at a for every $\alpha \in \mathbf{R}$.

iii) fg is continuous at a.

iv) f/g is continuous at a, provided $g(a) \neq 0$.

Proof : i) $\lim_{x\to a} (f \pm g)(x) = \lim_{x\to a} [f(x) \pm g(x)]$

$= \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$, by Theorem 1.

$= f(a) \pm g(a)$ since f and g are continuous at a.

ii) $\lim_{x\to a} (\alpha f)(x) = \lim_{x\to a} \alpha f(x) = \alpha \lim_{x\to a} f(x) = \alpha f(a)$

iii) $\lim_{x\to a} (fg)(x) = \lim_{x\to a} f(x) g(x) = \lim_{x\to a} f(x). \lim_{x\to a} g(x)$

$= f(a) g(a)$.

iv) $\lim_{x\to a} \left(\dfrac{f}{g}\right)(x) = \lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{f(a)}{g(a)}$.

This shows that $(f \pm g)$, αf, fg, f/g are all continuous at a.

From Theorem 4 we can conclude that if $f(x_1, x_2, \ldots\ldots, x_n)$ is any polynomial in the n variables $x_1, x_2, \ldots, x_n$, then $f : \mathbf{R}^n \to \mathbf{R}$ is a continuous function on $\mathbf{R}^n$.

Next we state and prove a result about the continuity of the composite of two continuous functions. You must have studied a similar result for functions of one variable (see Theorem 6, Unit 2, Calculus).

Theorem 5: Let f be a real-valued function of n variables, which is continuous at a point $a \in \mathbf{R}^n$ and let g be a real-valued function of a real variable, which is continuous at f(a). Then the composite function $g \circ f$ is continuous at a.

Proof : Let $\varepsilon > 0$. The continuity of the function g at f(a) implies that there exists a real number $\delta > 0$ such that

$|y-f(a)| < \delta \Rightarrow |g(y) - g(f(a))| < \varepsilon$ .... (*)

Now, the continuity of the function f at a implies that there exists a positive number $\eta > 0$ such that

$$|x-a| < \eta \Rightarrow |f(x) - f(a)| < \delta \text{ .... } (**)$$

Combining (*) and (**) we see that

$$|x-a| < \eta \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon,$$

i.e., $|x-a| < \eta \Rightarrow |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon,$

i.e., $g \circ f$ is continuous at a.

In the next example we will apply Theorem 5 to check the limit of a composite function.

Example 5: Let us show that

$$\lim_{(x,y) \to (0, \ln 5)} e^{x+y} = 5$$

Clearly, the functions $f(x,y) = x+y$ and $g(t) = e^t$ are continuous everywhere. Therefore, the composite function $g \circ f$ is continuous everywhere, in view of Theorem 5. Consequently,

$$\lim_{(x,y) \to (0, \ln 5)} e^{x+y} = e^{0+\ln 5} = 5.$$

We conclude this section with a simple result, which is found useful in many applications.

f(x) has the sign of f(a) means that either f(x) and f(a) are both positive or they are both negative.

Theorem 6: Let f(x) be a real-valued function of n variables which is continuous at a point a of $\mathbf{R}^n$. If $f(a) \neq 0$, then f(x) has the sign of f(a) in a neighbourhood of a.

Proof : Since f is given to be continuous at a, for $\varepsilon > 0$ there exists a real number $\delta > 0$ such that

$$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Or

$$f(a) - \varepsilon < f(x) < f(a) + \varepsilon \text{ for } |x-a| < \delta.$$

Let $\varepsilon = \dfrac{|f(a)|}{2}$. Then

$$\frac{f(a)}{2} < f(x) < \frac{3f(a)}{2} \text{ if } f(a) > 0,$$

and

$$\frac{3f(a)}{2} < f(x) < \frac{f(a)}{2} \text{ if } f(a) < 0$$

for all x in the δ-neighbourhood of a. That is, f(x) has the sign of f(a) for all x in $S(a,\delta)$. This completes the proof.

Note that in case $f(a) = 0$, nothing definite can be said about the sign of the function in a neighbourhood of a. For example

$f(x,y) = x^4+y^4$ is continuous at (0,0), $f(0,0) = 0$, and
$f(x,y) > 0$ for $(x,y) \neq (0,0)$.

On the other hand, for the function $g(x,y) = -(x^2+y^2)$, exactly the opposite happens.

Moreover, for the function $f(x,y) = x^3+y^3$, which is 0 at (0, 0), all the three possibilities occur:

$f(x, y) > 0$ for some (x, y),

$f(x, y) < 0$ for some (x, y) and

$f(x, y) = 0$ for some (x, y) in a neighbourhood of (0, 0).

Try to do these exercises now.

(5) Show that ...

a) f(x, y) = ...

b) f(x, ...

(6) Let ... of ...

(7) Show that the following ... continuous functions at ... their respective ...

a) x sin y ...

b) e^x cos ...

c) ln (1 + x^2 + y^2 ...

d) $|x_1|$ ...

In the next section we ... valued functions of several ...

4.4 LIMIT ... FROM ...

... vector ... continued ... separately

Definition 3: ...

S(a, h) ...

except possibly ... to have limit ... (depending upon ε) ...

... < |x−a| ...

lim f(x, ...

... that f(x) tends to ...

Note ... f(x) − ... the distance ...

... preceding ...

Now let us define ...

Definition 4: ... of a point a ∈ R^n ... defined in some neighbourhood ... continuous at ...

Let $D \subseteq \mathbf{R}^n$ and let $f : D \to \mathbf{R}^m$. **Then we say that f is continuous on a subset A** of D if f is continuous at every point a in A. The function f is said to be a **continuous function** if f is continuous at every point of its domain of definition.

If $f : D \to \mathbf{R}^m$, $D \subseteq \mathbf{R}^n$, then we have seen that there exist m real-valued functions $f_1, \ldots\ldots, f_m$ on D, which are determined by f, and determine f uniquely. In fact, $f_j = \pi_j \circ f$, where $\pi_j : \mathbf{R}^m \to \mathbf{R}$ is the j-th projection, $1 \leq j \leq m$. We can prove that f is continuous at a point a if and only if each component function f_j, $1 \leq j \leq m$, is continuous at a.

Now, if $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at $a \in \mathbf{R}^n$, then $f_j = \pi_j \circ f : \mathbf{R}^n \to \mathbf{R}$ being a composite of two functions, continuous respectively, at a and f(a), is also continuous at a. So it only remains to prove that if f_j, $1 \leq j \leq m$ are continuous at $a \in \mathbf{R}^n$, then f is also continuous at a. This proof is easy and we are leaving it to you as an exercise. (See E 8).

Thus the continuity of vector-valued functions really reduces to the consideration of the continuity of real-valued functions.

For example, the function $t \to (\cos t, \sin t)$ from $\mathbf{R}$ to $\mathbf{R}^2$ is continuous, as the component functions $t \to \cos t$ and $t \to \sin t$ are continuous everywhere.

Similarly, you can check that the function

$$(x, y) \to (\cos x, \sin x \sin y, e^x \sin y) \text{ from } \mathbf{R}^2 \text{ to } \mathbf{R}^3$$

is continuous on $\mathbf{R}^2$. You should be able to do this exercise now.

E8) Let $f : \mathbf{R}^n \to \mathbf{R}^m$ with component functions $f_1, \ldots\ldots, f_m$. If each f_j is continuous at a point $a \in \mathbf{R}^n$, prove that f is continuous at a.

We have seen how the concepts of limit and continuity are extended to functions of several variables. In the next section we will discuss one more way of defining the limit of a function of several variables.

4.5 REPEATED LIMITS

The definitions of limit which you have studied in Sec. 4.2 and Sec. 4.4 are generalisations of the definition of limit for functions from $\mathbf{R} \to \mathbf{R}$. We shall now consider another type of limit, which is peculiar to functions of several variables. For the sake of simplicity we confine ourselves to functions of two variables.

Let f(x, y) be a real-valued function of two variables defined on some deleted neighbourhood of the point (a,b). Then we can certainly talk about the limits.

$$\lim_{x \to a} \left(\lim_{y \to b} f(x, y) \right)$$

and

$$\lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right)$$

These two limits, which are called repeated limits are independent of the $\lim_{(x, y) \to (a, b)} f(x, y)$ which we have considered so far. We refer to $\lim_{(x, y) \to (a, b)} f(x, y)$ as the simultaneous limit as x and y are approaching a and b, respectively, at the same time. The following examples show that the existence of the simultaneous limit need not imply the existence of repeated limits and vice-versa.

Example 6: Let $f(x, y) = \dfrac{(x - y)^2}{x^2 + y^2}$. Then the two repeated limits

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right)$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right)$$

exist and are equal, but the simultaneous limit $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist.

Clearly $\lim_{x \to 0} f(x, y) = 1 = \lim_{y \to 0} f(x, y)$ and therefore

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right) = 1 = \lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right)$$

Now let $y = mx$. Then

$$f(x, y) = \frac{(1 - m)^2 x^2}{(1 + m^2) x^2}, \text{ and therefore}$$

$$\lim_{x \to 0} f(x, mx) = \frac{(1 - m)^2}{1 + m^2},$$

which is clearly different for different m. (Check for $m = 1, 2$).

Thus, $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist.

Example 7 : Let $f(x, y) = \frac{xy}{|y|}$. Then $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ exists, but

$\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y) \right)$ does not exist and

$\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y) \right) = 0.$

Since

$$| f(x, y) | = \frac{|xy|}{|y|} = |x|,$$

it follows that $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y) = 0.$

Further, $\lim_{y \to 0^+} \frac{xy}{|y|} = x$ and $\lim_{y \to 0^-} \frac{xy}{|y|} = -x.$

This shows that $\lim_{y \to 0} \frac{xy}{|y|}$ does not exist when $x \neq 0$, and therefore we cannot even think of $\lim_{x \to 0} \left(\lim_{y \to 0} \frac{xy}{|y|} \right)$. You can easily check that

$$\lim_{y \to 0} \left(\lim_{x \to 0} \frac{xy}{|y|} \right) = 0.$$

So, does it mean that the simultaneous limit and the repeated limits are totally unrelated? No. The situation is not so bad. In some cases we can relate the two.

Here is a theorem which gives some connection between simultaneous limits and repeated limits.

Theorem 7: Let f(x, y) be a real-valued function such that $\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$. If both the repeated limits $\lim_{x \to a} \left(\lim_{y \to b} f(x, y) \right)$ and $\lim_{y \to b} \left(\lim_{x \to a} f(x, y) \right)$ exist, then each one of these limits is equal to L.

We are not giving the proof of this theorem here, as it is beyond the scope of this course.

See if you can solve these exercises now.

E9) For the function $f(x, y) = \frac{xy}{x^2 + y^2}$, prove that the simultaneous limit does not exist at (0, 0), while the two repeated limits exist and are equal.

E10) For the function $f(x, y) = \frac{y - x}{y+x}\,\frac{1+x^2}{1+y^2}$,

show that $\lim_{x \to 0} \left(\lim_{y \to 0} f(x, y)\right) = -1$ and $\lim_{y \to 0} \left(\lim_{x \to 0} f(x, y)\right) = 1$.

Apply Theorem 7 to decide the existence of the simultaneous limit as $(x, y) \to (0, 0)$.

E11) Let $f(x, y) = \begin{cases} 1+xy, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$

Then prove that

$$\lim_{x \to 0} \left[\lim_{y \to 0} f(x, y)\right] = 1 = \lim_{y \to 0} \left[\lim_{x \to 0} f(x, y)\right]$$

but $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist.

This brings us to the end of the unit. Let us briefly recall what we have covered in it.

4.6 SUMMARY

In this unit we have

1) Defined the limit of a function $f: \mathbf{R}^n \to \mathbf{R}^m$.

   $\lim_{x \to 0} f(x) = L$ if $\forall\, \varepsilon > 0, \exists\, \delta > 0$, such that

   $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

2) Discussed repeated limits and their connection with the simultaneous limit.

3) Defined the concept of continuity for functions of several variables:

   $f : \mathbf{R}^n \to \mathbf{R}^m$ is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$.

   Alternatively, f is continuous at a, if $\forall\, \varepsilon > 0, \exists\, \delta > 0$, s.t.
   $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

4) Seen that the consideration of the continuity of vector-valued functions reduces to the consideration of the continuity of real-valued functions of several variables.

5) Stated many results about the continuity of functions from $\mathbf{R}^n \to \mathbf{R}$:
   about the algebra of continuous functions,
   about the continuity of the composite of two functions,
   about the sign of the values of a continuous function f in a neighbourhood of a, s.t. $f(a) \neq 0$.

4.7 SOLUTIONS AND ANSWERS

E1) Let f(x) be defined in a neighbourhood S(a, h) of the point a except possibly at a. Suppose $L \neq M$, then $|L - M| > 0$.

Since, $\lim_{x \to a} f(x) = L$, if we take $\varepsilon = \frac{|L - M|}{2}$, then $\exists\, \delta_1 > 0$,

$\delta_1 < h$ such that

$|x - a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$.

Similarly, since $\lim_{x \to a} f(x) = M$, $\exists\, \delta_2 > 0$, $\delta_2 < h$ such that

$|x-a| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$.

Choose $\delta = \min\{\delta_1, \delta_2\}$. If $|x-a| < \delta$, then $|x-a| < \delta_1$ and

$|x-a| < \delta_2$. Also $\delta < h$ since $\delta_1 < h$ and $\delta_2 < h$. So
$|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$ and $|f(x) - M| < \varepsilon$.

Then
$$|L - M| = |L - f(x) + f(x) - M|$$
$$= |L - f(x)| + |f(x) - M|$$
$$< 2\varepsilon = |L - M|.$$

Thus $|L - M| < |L - M|$. This is a contradiction. Hence our assumption that $|L - M| > 0$ is wrong.

E2) a) Put $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \left| \frac{r\cos\theta .\ r\sin\theta}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}} \right|$$

$$= \left| \frac{r^2 \cos\theta \sin\theta}{r} \right|$$

$$\le r = \sqrt{x^2+y^2}$$

Now if $|x| < \frac{\varepsilon}{\sqrt{2}}$, $|y| < \frac{\varepsilon}{\sqrt{2}}$, then $\sqrt{x^2+y^2} < \varepsilon$ and therefore

$$\left| \frac{x}{\sqrt{x^2+y^2}} \right| < \varepsilon.$$

Thus $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \left| \frac{xy}{\sqrt{x^2+y^2}} \right| = 0.$

b) Put $x = r\cos\theta$, $y = r\sin\theta$. Then

$$\left| \frac{x^2y^2}{\sqrt{x^2+y^2}} \right| = \left| \frac{r^2\cos^2\theta .\ r^2\sin^2\theta}{\sqrt{r^2(\cos^2\theta + \sin^2\theta)}} \right|$$

$$= \left| \frac{r^4 \cos^2\theta \ \sin^2\theta}{r} \right|$$

$$\le r^3 = (x^2+y^2)^{3/2}$$

Now if $|x| < \frac{\sqrt[3]{\varepsilon}}{\sqrt{2}}$, $|y| < \frac{\sqrt[3]{\varepsilon}}{\sqrt{2}}$, then

$$(x^2+y^2)^{3/2} < \left(\frac{\varepsilon^{2/3}}{2} + \frac{\varepsilon^{2/3}}{2} \right)^{3/2} = (\varepsilon^{2/3})^{3/2} = \varepsilon.$$

Therefore, $\left| \frac{x^2y^2}{\sqrt{x^2+y^2}} \right| < \varepsilon.$

That is, $\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{x^2y^2}{\sqrt{x^2+y^2}} = 0$

c) Since

$$\lim_{(x,y,z) \to (0,1,2)} (x^2 + 3xyz - 5z^2) = -20$$

and $\lim\limits_{(x,y,z) \to (0,1,2)} (xy^3 + 5z^2 - 3xy + x^3) = 20,$

using algebra of limits, we get

$$\lim_{(x,y,z) \to (0,1,2)} \left(\frac{x^2 + 3xyz - 5z^2}{xy^3 + 5z^2 - 3xy + x^3} \right)$$

$$= \frac{\lim\limits_{(x,y,z) \to (0,1,2)} (x^2 + 3xyz - 5z^2)}{\lim\limits_{(x,y,z) \to (0,1,2)} (xy^3 + 5z^2 - 3xy + x^3)} = \frac{-20}{20} = -1$$

d) Since

$$\lim_{(x,y)\to(0,0)} x\sin y = 0 \text{ and } \lim_{(x,y)\to(0,0)} (2x^2+1) = 1 \neq 0,$$

using algebra of limits, we get

$$\lim_{(x,y)\to(0,0)} \left(\frac{x\sin y}{2x^2+1}\right) = \frac{\lim_{(x,y)\to(0,0)} x\sin y}{\lim_{(x,y)\to(0,0)} (2x^2+1)} = \frac{0}{1} = 0.$$

E3) a) $f(x,y) = \dfrac{xy}{x^2+y^2}$

If we put $y = \phi_1(x) = mx$, then

$$f(x, \phi_1(x)) = \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}$$

This value is different for different m which shows that the function has different limits in different directions. Therefore the limit does not exist.

b) In view of Corollary 1, it is enough to prove that there exist real-valued functions $\phi_1(x)$ and $\phi_2(x)$ such that

$\lim_{x\to 0} \phi_1(x) = 0 = \lim_{x\to 0} \phi_2(x)$

and $\lim_{x\to 0} f(x, \phi_1(x)) \neq \lim_{x\to 0} f(x, \phi_2(x))$

Let $\phi_1(x) = x^2$ and $\phi_2(x) = x - x^2$. Then

$\lim_{x\to 0} \phi_1(x) = \lim_{x\to 0} \phi_2(x) = 0$

Also, $\lim_{x\to 0} f(x, \phi_1(x)) = \lim_{x\to 0} \dfrac{x^2}{x^2+x^2} = \lim_{x\to 0} \dfrac{1}{2} = \dfrac{1}{2}$

and $\lim_{x\to 0} f(x, \phi_2(x)) = \lim_{x\to 0} \dfrac{x^2}{x^2+x-x^2}$

$= \lim_{x\to 0} x = 0.$

Thus, $\lim_{x\to 0} f(x, \phi_1(x)) \neq \lim_{x\to 0} f(x, \phi_2(x))$

Therefore, by Corollary 1, the limit does not exist.

c) Let $\phi_1(x) = m_1 x$ and $\phi_2(x) = m_2 x$.

$\lim_{x\to 0} \phi_1(x) = \lim_{x\to 0} \phi_2(x) = 0.$

Now, $\lim_{x\to 0} f(x, \phi_1(x)) = \dfrac{x^2 - m_1^2 x^2}{x^2+m_1^2 x^2} + \dfrac{2.x.m_1 x}{x^2+m_1^2 x^2}$

$$= \frac{1-m_1^2}{1+m_1^2} + \frac{2m_1}{1+m_1^2}$$

$$= \frac{1+2m_1-m_1^2}{(1+m_1^2)}$$

Similarly $\lim_{x\to 0} f(x, \phi_2(x)) = \dfrac{1+2m_2-m_2^2}{(1+m_2^2)}$

Then, $\lim_{x\to 0} f(x, \phi_1(x)) \neq \lim_{x\to 0} f(x, \phi_2(x))$. Check with $M_1 = 1$, $M_2 = -1$

Hence, the limit does not exist.

E4) a) We are asked to find δ such that, if $|x| < \delta$, $|y| < \delta$ and $|z| < \delta$, then $|x^2+y^2+z^2| < 0.01$.
Now, $|x| < \delta$, $|y| < \delta$ and $|z| < \delta \Rightarrow x^2+y^2+z^2 < 3\delta^2$.
Then if we choose δ such that $3\delta^2 < 0.01$, then we are through. For example, we can take $\delta = 0.05$.

b) If $|x| < \delta$, $|y| < \delta$, then $|xy| < \delta^2$. Then we can take any δ such that $\delta^2 < 0.0004$. For example, $\delta = 0.01$.

E5) a) We first note that to prove that a function is discontinuous at a point it is enough to show that the limit in any particular direction exists and is not equal to f(0, 0).

Let $\phi_1(x) = x$. Then

$$\lim_{x\to 0} f(x, \phi_1(x)) = \lim_{x\to 0} \frac{x^3}{x^3+x^2} = \lim_{x\to 0} \frac{x}{x+1} = 0 \neq 2 = f(0,0).$$

Therefore f is not continuous at (0,0).

b) You have already seen in Example 1 that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \left(y \sin \frac{1}{x} + x \sin \frac{1}{y} \right) = 0.$$

But $f(0,0) = 1$. Therefore, the function is not continuous at (0,0).

E6) Let $a = (a_1, a_2, \ldots, a_n) \in \mathbf{R}^n$. We have to show that $\lim_{x\to a} \pi_j(x) = \pi_j(a)$.
Let $\varepsilon > 0$ be given.

Consider $|\pi_j(x) - \pi_j(a)| = |\pi_j(x_1, \ldots, x_n) - \pi_j(a_1, a_2, \ldots, a_n)|$
$= |x_j - a_j|$.

Take $\delta_i = \frac{\varepsilon}{\sqrt{n}}$, for $i = 1, 2, \ldots, n$. Then,

$0 < |x_i - a_i| < \delta_i, 1 \le i \le n \Rightarrow |\pi_j(x) - \pi_j(a)| = |x_j - a_j| < \delta_j < \varepsilon$.
Therefore, by Theorem 2,

$$\lim_{x\to a} \pi_j(x) = \pi_j(a).$$

This shows that π_j is continuous.

E7) a) $|x \sin y + y \sin z + z \sin x| \le |x| + |y| + |z|$

Then $|x| < \frac{\varepsilon}{3}$, $|y| < \frac{\varepsilon}{3}$ and $|z| < \frac{\varepsilon}{3}$

$\Rightarrow |x\sin y + y\sin z + z\sin x| < \varepsilon$.

Thus, applying Theorem 2,

$$\lim_{\substack{x\to 0\\ y\to 0\\ z\to 0}} (x\sin y + y\sin z + z\sin x) = 0.$$

This shows that the function is continuous at (0,0,0).

b) Using algebra of limits,

$$\lim_{(x,y,z)\to(0,0,0)} (e^x \cos y + e^y \cos z + e^z \cos x) =$$

$$\lim_{(x,y,z)\to(0,0,0)} e^x . \lim_{(x,y,z)\to(0,0,0)} \cos y + \lim_{(x,y,z)\to(0,0,0)} e^y .$$
$$\lim_{(x,y,z)\to(0,0,0)} \cos z + \lim_{(x,y,z)\to(0,0,0)} e^z . \lim_{(x,y,z)\to(0,0,0)} \cos x$$

$= 1+1+1 = 3$, since $\lim_{t\to 0} e^t = e^0 = 1$ and $\lim_{t\to 0} \cos t = \cos 0 = 1$.

c) Let $f(x,y,z) = \ln(1+x^2+y^2+z^2)$.
Then $f = g \circ h$ where $h(x,y,z) = 1 + x^2+y^2+z^2$ and $g(t) = \ln t$.
Clearly, the function h is continuous at (0,0,0) and g is continuous at $h(0,0,0) = 1$. Therefore, by applying Theorem 6, f is continuous at (0,0,0).

d) $f(x_1, x_2, \ldots, x_n) = |x_1| + |x_2| + \ldots + |x_n|$.
Let $f_i(x_1, x_2, \ldots, x_n) = |x_i|$, $i = 1, 2, \ldots, n$.
Then f_i is a real-valued function of several variables. Also, $f_i = g \circ \pi_i$ for each $i = 1, 2, \ldots, n$ where $g(t) = |t|$. For each i, the function π_i is continuous at $(0,0,\ldots,0)$ and g is continuous at $\pi_i(0,0,\ldots,0) = 0$. Therefore by applying Theorem 6, f_i is continuous for $i = 1, 2, 3, \ldots, n$. Then, using algebra of limits, we get that f is continuous at $(0,0,\ldots,0)$.

E8) Suppose that each f_j is continuous at a. Then given $\varepsilon > 0$, there exists real numbers $\delta_i > 0$, $1 \le i \le n$, such that

$|x-a| < \delta_i \Rightarrow |f_i(x) - f_i(a)| < \frac{\varepsilon}{\sqrt{n}}$.

Let $\delta = \min\{\delta_1, \delta_2, \ldots\ldots, \delta_n\}$. Then

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| = \sqrt{\sum_{i=1}^{n} (f_i(x) - f_i(a))^2} < \varepsilon.$$

This shows that f is continuous at a.

E9) We have already seen in E3) that the simultaneous limit of the function

$f(x,y) = \frac{xy}{x^2+y^2}$ does not exist. The repeated limits

$\lim_{x\to 0}\left[\lim_{y\to 0} \frac{xy}{x^2+y^2}\right]$ and $\lim_{y\to 0}\left[\lim_{x\to 0} \frac{xy}{x^2+y^2}\right]$ exist and are equal to 0,

since $\lim_{x\to 0} \frac{xy}{x^2+y^2} = 0$ and $\lim_{y\to 0} \frac{xy}{x^2+y^2} = 0$.

E10) $\lim_{x\to 0} \left(\frac{y-x}{y+x} \frac{1+x^2}{1+y^2}\right) = \frac{y}{y(1+y^2)} = \frac{1}{1+y^2}$

Therefore, $\lim_{y\to 0}\left[\lim_{x\to 0} \frac{y-x}{y+x} \frac{1+x^2}{1+y^2}\right] = \lim_{y\to 0} \frac{1}{1+y^2} = 1.$

Now,

$\lim_{y\to 0}\left(\frac{y-x}{y+x} \frac{1+x^2}{1+y^2}\right) = \frac{-x(1+x^2)}{x} = -(1+x^2)$, and therefore,

$\lim_{x\to 0}\left[\lim_{y\to 0}\left(\frac{y-x}{y+x} \cdot \frac{1+x^2}{1+y^2}\right)\right] = -1.$

This shows that the two repeated limits exist. Now, by applying Theorem 7, if the simultaneous limit

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{(y-x)(1+x^2)}{(y+x)(1+y^2)}$ exists, then the two repeated limits have to be equal.

But we know that the repeated limits are unequal. Hence simultaneous limit does not exist in this case.

E11) $\lim_{y\to 0} f(x,y) = 1$ and then

$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = 1.$

Similarly, $\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = 1.$

Consider $y = \phi_1(x) = x$. Then we get

$\lim_{x\to 0} f(x, \phi_1(x)) = \lim_{x\to 0} (1+x^2) = 1$

and when we take $y = \phi_2(x) = 0$, then we get

$\lim_{x\to 0} f(x, \phi_2(x)) = 0$, since $y=0 \Rightarrow xy = 0 \Rightarrow f(x,y) = 0$ for every x.

This shows that the simultaneous limit does not exist.

UNIT 5 FIRST ORDER PARTIAL DERIVATIVES AND DIFFERENTIABILITY

Structure

5.1 INTRODUCTION

You are already familiar with the concept of a derivative of a real-valued function of a real variable (Calculus, Unit 3). In this unit, we shall study this concept for functions of several variables. You have seen that the notions of limit and continuity can be easily extended to these functions. But the definition of the derivative of a real-valued function of a real variable cannot be applied, as it is, to a real-valued function f(x) of n variables (n > 1). This is because for any $h \in \mathbf{R}^n$, $h \neq 0$, the quotient $\frac{f(x+h) - f(x)}{h}$ does not make sense as the division by a vector in $\mathbf{R}^n$ is not possible.

However, if we examine the definition of derivative more closely, we realise that a function of a single variable is differentiable at a point if and only if the two directional derivatives, i.e., the right hand derivative and the left hand derivative, exist and are equal at that point. You will see later, in Unit 7, that this concept of directional derivative can be generalised to functions of several variables. The only hitch is that in $\mathbf{R}^n$, we have to deal with infinitely many directions. However, in the beginning we shall confine ourselves to the special directions which are parallel to the coordinate axes. This leads us to the notion of partial derivatives.

In this unit we'll discuss the concept of partial derivatives of a function of several variables in detail. Note that this notion of partial derivatives does not fully generalise the concept of derivative of a real valued function of a real-variable. Later in this unit we introduce the concept of differentiability for functions of several variables, and discuss the relationship between differentiability, continuity and the existence of partial derivatives.

Throughout this unit by the word 'function' we shall mean a real-valued function of several variables, i.e., a function from $D \to \mathbf{R}$, where D is a subset of $\mathbf{R}^n$, $n > 1$. We shall give the definitions for general n, but most of the time during our discussion we shall confine ourselves to the case $n = 2$, i.e., to real-valued functions of two variables. We'll briefly discuss the case $n = 3$ also.

Objectives

After reading this unit, you should be able to

- define partial derivatives of the first order for a function of several variables,
- partially differentiate a given real-valued function of several real variables with respect to any particular variable,
- give the geometrical interpretation of first order partial derivatives of functions of two variables,

- decide whether a given function of two or more variables is differentiable or not,
- give examples to establish relationships between the continuity, differentiability and existence of partial derivatives at a point for a function of several variables.

5.2 FIRST ORDER PARTIAL DERIVATIVES

In this section we shall see what the partial derivative of a function at a point means. We already know how to define the derivative of a function of a single variable. We'll use this knowledge in defining the partial derivatives of functions of several variables.

5.2.1 Definition and Examples

Consider a function $f : D \to \mathbf{R}$, where $D \subseteq \mathbf{R}^n$. Let $(x_1, x_2, \ldots, x_n)$ be an **interior point** of D, i.e., there exists an open sphere with centre $(x_1, x_2, \ldots, x_n)$ contained in D. Then for each i, $1 \le i \le n$, we can construct a real valued function of a real variable from this function f in the following manner.

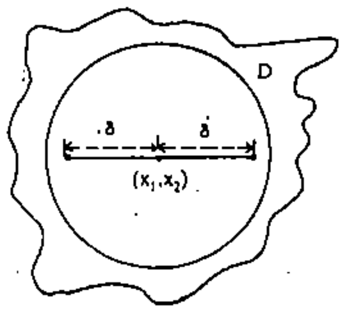


Fig. 1

Choose a small number $\delta > 0$ such that the point $(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) \in D$ for all $h \in]-\delta, \delta[$. Such a δ exists since $(x_1, x_2, \ldots, x_n)$ is an interior point of D. We have shown this for n = 2 in Fig. 1. Now we can define $f_i :]-\delta, \delta[\to \mathbf{R}$ such that

$$f_i(h) = \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

for all $h \in]-\delta, \delta[$, $h \neq 0$.

Now if $\lim_{h\to 0} f_i(h)$ exists, then we say that f has a first order partial derivative with respect to the ith variable x_i at the point $(x_1, x_2, \ldots, x_n)$ and the value of $\lim_{h\to 0} f_i(h)$ is called ith first order partial derivative of f at the point $(x_1, x_2, \ldots, x_n)$.

You should note here that for $\lim_{h\to 0} f_i(h)$ to exist, it is enough that f be defined in a neighbourhood of the point under consideration.

More formally, we have the following definition.

Definition 1: Let $f : D \to \mathbf{R}$, where $D \subseteq \mathbf{R}^n$, and let $(x_1, x_2, \ldots, x_n)$ be an interior point of D. We say that the function f has ith partial derivative at the point $(x_1, x_2, \ldots, x_n)$ if

$$\lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h} \text{ exists.}$$

Whenever we say that a limit exists, we mean that a finite limit exists.

The value of this limit is called the **ith partial derivative** of f at the point $(x_1, x_2, \ldots, x_n)$.

There are different symbols available in literature to denote the partial derivatives of a given function. However, we shall use only the following symbols

$\frac{\partial f}{\partial x_i}$ is read as "del f by del x_i"

$$\frac{\partial f}{\partial x_i},\ f_{x_i},\ D_i f.$$

according to our convenience to denote the ith partial derivative of first order of f. In case we want to emphasise the point $(x_1, x_2, \ldots, x_n)$ at which the partial derivative has been calculated, we write

$$\frac{\partial f}{\partial x_i}(x_1, x_2, \ldots, x_n)$$

or $f_{x_i}(x_1, x_2, \ldots, x_n)$

or $D_i f(x_1, x_2, \ldots, x_n)$

You have seen in the Calculus course that we write $y = f(x)$ to express a real-valued function of a real variable. For functions of two variables it is customary to write $z = f(x,y)$, and the two partial derivatives of f at the point (x,y) are then denoted by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Remark 1 : i) Note that the notion of partial derivative of a function (like continuity) is local in character. That is, we check the existence of the partial derivative of a function at a point. Thus, when we say that a function has partial derivatives on a set A, we mean that the function has partial derivatives at each point of A.

ii) It is obvious from the definition of a partial derivative at a point that the function must be defined in a neighbourhood of the point. Thus, we can talk about the partial derivatives only at the interior points of the domain D. For example, if D is a disc in $\mathbf{R}^2$, then we cannot talk about the partial derivative of a point on the circumference of this disc.

iii) If f has a partial derivative at $(x_1, x_2, \ldots, x_n)$, then its value depends only on the values of f in an open sphere around the point. If the function is changed outside this sphere, it would not affect the value of the partial derivative.

We now give some examples to show how to obtain partial derivatives of given functions at given points. If we do not mention a specific point at which partial derivatives are to be calculated, we mean that they are to be calculated at a generic point (x,y) or (x,y,z) according as $n = 2$ or $n = 3$.

Example 1: Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function given by $f(x,y) = x^2+xy+y^3$. Let us find $f_x(x,y)$ and $f_y(x,y)$.

By definition,

$$
\begin{aligned}
f_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 + (x+h)\,y + y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h\to 0} \frac{x^2+2xh+h^2+xy+hy+y^3 - x^2 - xy - y^3}{h} \\
&= \lim_{h\to 0} (2x + h + y) \\
&= 2x+y
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
f_y(x,y) &= \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} \\
&= \lim_{k\to 0} \frac{x^2+x(y+k)+(y+k)^3 - x^2 - xy - y^3}{k} \\
&= \lim_{k\to 0} \frac{xk+3y^2k+3yk^2+k^3}{k} \\
&= \lim_{k\to 0} (x+3y^2+3yk+k^2) \\
&= x+3y^2
\end{aligned}
$$

When we are considering functions of two variables x and y, then for the increment in x we normally use the letter h and for the increment in y, the letter k. Similarly, when we are dealing with functions of three variables x, y and z, we use the letters p, q and r for increments in x, y and z, respectively. This is only a matter of convention, not a rule.

In the next example we consider a function of three variables.

Example 2 : Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a function defined by $f(x, y, z) = xy + yz + zx$. Let us find the partial derivatives at the point (a, b, c). By definition

$$f_x(a, b, c) = \lim_{p \to 0} \frac{f(a+p, b, c) - f(a, b, c)}{p}$$

$$= \lim_{p \to 0} \frac{(a+p)\,b + bc + c\,(a+p) - ab - bc - ca}{p}$$

$$= b + c$$

$$f_y(a, b, c) = \lim_{q \to 0} \frac{f(a, b+q, c) - f(a, b, c)}{q}$$

$$= \lim_{q \to 0} \frac{a(b+q) + (b+q)\,c + ca - ab - bc - ca}{q}$$

$$= a + c$$

$$f_z(a, b, c) = \lim_{r \to 0} \frac{f(a, b, c+r) - f(a, b, c)}{r}$$

$$= \lim_{r \to 0} \frac{ab + b\,(c+r) + (c+r)\,a - ab - bc - ca}{r}$$

$$= b + a$$

In the next example we shall calculate the partial derivatives of a function of n variables.

Example 3 : Let $f : \mathbf{R}^n \to \mathbf{R}$ be a function defined by
$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2$.

To find the partial derivative of f with respect to x_i at the point $(a_1, a_2, \ldots, a_n)$, we write

$$f_{x_i}(a_1, a_2, \ldots, a_n) = \lim_{h \to 0} \frac{f(a_1, a_2, \ldots, a_{i-1}, a_i + h, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{h}$$

$$= \lim_{h \to 0} \frac{a_1^2 + a_2^2 + \ldots + a_{i-1}^2 + (a_i + h)^2 + a_{i+1}^2 + \ldots + a_n^2 - a_1^2 - a_2^2 - \ldots - a_n^2}{h}$$

$$= \lim_{h \to 0} \frac{2a_i h + h^2}{h} = 2a_i.$$

Try to solve these exercises now.

E1) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be the constant function defined by

$f(x, y) = c$ for all (x, y). Show that $f_x(a, b) = 0 = f_y(a, b)$ for all points (a, b).

E2) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $f_x(0, 0)$ as well as $f_y(0, 0)$ does not exist.

E3) Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} x & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$.

From these examples and exercises you must have observed that $f_x(x, y)$ is nothing but the derivative of f(x, y) considered as a function of a single variable x, treating y as a constant. Similarly, $f_y(x, y)$ is nothing but the derivative of f(x, y) considering it as a function of the single variable y, and treating x as a constant. In general, we can say that $f_{x_i}(x_1, x_2, \ldots, x_n)$ is nothing but the derivative of f $(x_1, x_2, \ldots, x_n)$ with respect to x_i treating all the other variables as constants. Thus, for calculating partial derivatives, we can use our knowledge of calculating derivatives of functions of a single real variable and avoid the lengthy limiting process.

Here is an example to illustrate this.

Example 4 : Let us find the partial derivatives of the following functions.

i) $z = x^3 - 4x^2y^2 + 8y^2$

ii) $z = x \sin y + y \cos x$

iii) $z = x e^y + y e^x$.

In the Calculus course (Block 1) you have seen that polynomial, trigonometric and exponential functions of a single variable are differentiable.

We first note that in all the three cases, the functions involved are either polynomials or trigonometric or exponential functions. This ensures that the partial derivatives exist. By direct differentiation, we get

i) $\frac{\partial z}{\partial x} = 3x^2 - 8xy^2 \; ; \; \frac{\partial z}{\partial y} = -8x^2y + 16y$

ii) $\frac{\partial z}{\partial x} = \sin y - y \sin x \; ; \; \frac{\partial z}{\partial y} = x \cos y + \cos x$

iiii) $\frac{\partial z}{\partial x} = e^y + ye^x \; ; \; \frac{\partial z}{\partial y} = xe^y + e^x$

The calculation of partial derivatives is not always as simple as in these examples. In some exceptional cases, we have to use the limiting process as in the case of one variable. You will be able to recognise such cases with practice. Let us consider one such situation.

Example 5 : Suppose $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^4+y^4}, & (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

Let us find the two partial derivatives at the points (0, 0), (a, 0), (0, b) and (a, b), where $a \neq 0$, $b \neq 0$.

Now, by definition,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0.$$

$$f_x(a, 0) = \lim_{h \to 0} \frac{f(a+h, 0) - f(a, 0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$$f_y(a, 0) = \lim_{k \to 0} \frac{f(a, 0+k) - f(a, 0)}{k} = \lim_{k \to 0} \frac{\dfrac{ak}{a^4+k^4}}{k} = \frac{1}{a^3}$$

$$f_x(0, b) = \lim_{h \to 0} \frac{f(0+h, b) - f(0, b)}{h} = \lim_{h \to 0} \frac{\dfrac{bh}{b^4+h^4}}{h} = \frac{1}{b^3}$$

$$f_y(0, b) = \lim_{k \to 0} \frac{f(0, b+k) - f(0, b)}{k} = \lim_{k \to 0} \frac{0-0}{k} = 0.$$

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{(a+h)\,b}{(a+h)^4 + b^4} - \dfrac{ab}{a^4+b^4}}{h}$$

$$= \lim_{h \to 0} \frac{(ab+hb)(a^4+b^4) - (ab)(a^4+4a^3h+6a^2h^2+4ah^3+h^4+b^4)}{h(a^4+b^4)[(a+h)^4+b^4]}$$

$$= \lim_{h \to 0} \frac{b(a^4+b^4) - (ab)(4a^3+6a^2h+4h^2a+h^3)}{(a^4+b^4)[(a+h)^4+b^4]}$$

$$= \frac{b^5 - 3a^4b}{(a^4+b^4)^2}$$

$$f_y(a, b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a, b)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{a(b+k)}{a^4+(b+k)^4} - \frac{ab}{a^4 + b^4}}{k}$$

$$= \lim_{k \to 0} \frac{(ab+ak)(a^4+b^4) - (ab)(a^4+b^4+4b^3k+6b^2k^2+4bk^3+k^4)}{k(a^4+b^4)[a^4 + (b+k)^4]}$$

$$= \lim_{k \to 0} \frac{a(a^4+b^4) - (ab)(4b^3+6b^2k+4bk^2+k^3)}{(a^4+b^4)[a^4+(b+k)^4]}$$

$$= \frac{a^5 - 3ab^4}{(a^4+b^4)^2}.$$

Here, by direct differentiation, we could have obtained $f_x(a, b)$ and $f_y(a, b)$, $(a, b) \neq (0, 0)$, correctly, but not $f_x(0, 0)$ or $f_y(0, 0)$.

Can you see why? Since f is defined as a quotient of two polynomial functions for all $(x, y) \neq (0, 0)$, we can use direct differentiation to calculate partial derivatives at these points. But to calculate $f_x(0, 0)$ or $f_y(0, 0)$ we need to use $f(0, 0)$, which is not defined by the same quotient. Also note that after obtaining $f_x(a, b)$ and $f_y(a, b)$, we could have substituted $a = 0$ or $b = 0$ to get $f_x(0, b)$, $f_y(0, b)$, $f_x(a, 0)$ and $f_y(a, 0)$.

You must have come across functions from **R**→**R** which do not possess derivatives at some points. For example, $f(x) = |x|$ can not be differentiated at $x = 0$. Here is an example of a function of two variables whose partial derivatives fail to exist at some points.

Example 6 : If $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} \frac{x}{y} + \frac{y}{x}, & y \neq 0, x \neq 0 \\ 0 & , \text{otherwise,} \end{cases}$$

then $f_x(0, 1)$ and $f_y(1, 0)$ do not exist.
Let us prove this.

Here $\frac{f(0+h, 1) - f(0, 1)}{h} = \frac{h + \frac{1}{h} - 0}{h} = 1 + \frac{1}{h^2}$

and $\frac{f(1, 0+k) - f(1, 0)}{k} = \frac{\frac{1}{k} + k - 0}{k} = \frac{1}{k^2} + 1.$

Since $\lim_{h \to 0} \frac{1}{h^2} = \infty$, neither

$$\lim_{h \to 0} \frac{f(0+h, 1) - f(0, 1)}{h} \text{ nor } \lim_{k \to 0} \frac{f(1, 0+k) - f(1, 0)}{k}$$

exist. So f_x and f_y do not exist, respectively, at the points (0, 1) and (1, 0).

Why don't you try some exercises now?

E4) If $f(x, y) = 2x^2 - xy + 2y^2$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point (1, 2).

E5) Find all the first order partial derivatives of the following functions.

a) $\sin(x^2 - y)$

b) $\frac{1}{\sqrt{x+y^2+z^2+1}}$

c) $y \sin xz$

d) x^y

e) $x^3y + e^{xy^2}$

E6) Show that the functions $u = e^x \cos y$, $v = e^x \sin y$ satisfy the conditions
$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

E7) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two differentiable functions.
Let $F(x, y) = f(x) + g(y)$ for all x and y. Show that
$F_x(x, y) = f'(x)$ and $F_y(x, y) = g'(y)$.

E8) Let f and g be two real-valued functions for which $f_x(a, b)$ and $g_x(a, b)$ exist.
Show that $\frac{\partial (f+g)}{\partial x}$ exists at (a, b) and is equal to $f_x(a, b) + g_x(a, b)$.
Is the converse true?

Through these exercises you must have gained enough practice of calculating partial derivatives. In the next sub-section we shall try to interpret these geometrically.

5.2.2 Geometric Interpretation

In the case of a real-valued function f of one variable x, you know that the derivative $f'(x)$ gives the slope of the tangent to the curve $y = f(x)$ at a generic point (x, y). Now we shall try to visualise the partial derivatives of a real-valued function of two variables. Such a function, as you know, represents a surface in $\mathbb{R}^3$.

So, let f(x, y) be a real-valued function of two variables and let $S = \{(x, y, z) \mid z = f(x, y)\}$ be the surface represented by the function f(x, y) in $\mathbb{R}^3$. Suppose that f(x, y) has both the partial derivatives at a point (a, b) and let $c = f(a, b)$. Let P_1 be the point (a, b, c) on the surface S. Now the plane $y = b$, which is parallel to the XOZ plane and passes through P_1 will intersect the surface in a curve (see Fig. 2(a)). We are giving an enlarged version of the curve C in Fig. 2(b).

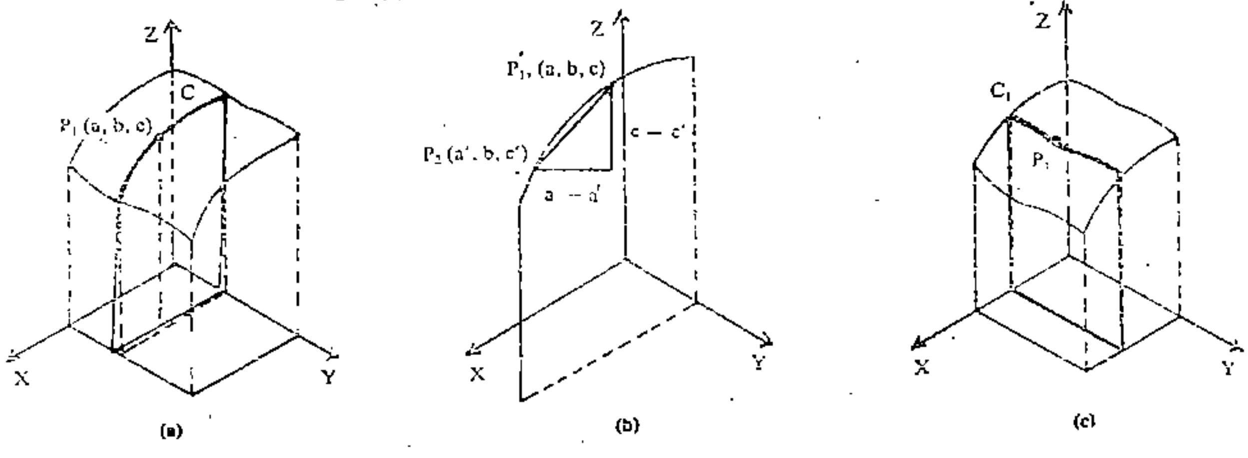


Fig. 2

Let P_2 be a point with coordinates (a', b', c') on the curve C close to P_1. Since P_2 lies on the surface S and the plane $y = b$, we have $b' = b$ and $c' = f(a', b)$. Clearly P_1P_2 is a line joining the points (a, c) and (a', c') in the plane $y = b$ and its slope is given by

$$= \frac{c' - c}{a' - a} = \frac{f(a', b) - f(a, b)}{a' - a}$$

$$= \frac{f(a+h, b) - f(a, b)}{h}, \text{ where we have put } a' = a+h.$$

As h approaches zero, the secant P_1P_2 approaches the tangent at the point P_1 to the

curve C. Consequently, $\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h}$, i.e., $\frac{\partial f}{\partial x}(a, b)$ or $\left(\frac{\partial z}{\partial x}\right)_{(a, b)}$ gives the slope of the tangent at the point (a, b, f(a, b)) to the curve C which is the intersection of the surface $z = f(x, y)$ and the plane $y = b$. Similarly, $\left(\frac{\partial z}{\partial y}\right)_{(a, b)}$ is the slope of the tangent at the point (a, b, c) to the curve C_1, which is the intersection of the surface $z = f(x, y)$ and the plane $x = a$ (see Fig. 2(c)). We use this fact in the following example.

Example 7 : Suppose we want to find the slopes of the tangents to the curves of intersection of the planes $x = 2$ and $y = 3$ and the surface $z = xy + 3x^2$ at the point (2, 3, 18).

We know that the slope of the tangent to the curve of intersection of the plane $x = 2$ and the surface $z = xy + 3x^2$ at the point (2, 3, 18) will be given by $\left(\frac{\partial z}{\partial y}\right)_{(2, 3)}$

Now, $\left(\frac{\partial z}{\partial y}\right)_{(2, 3)} = (x)_{(2, 3)} = 2.$

Therefore, the slope of the tangent at the point (2, 3, 18) to the curve of intersection of the plane $x = 2$ and the surface $z = xy + 3x^2$ is 2.

Similarly, since $\left(\frac{\partial z}{\partial x}\right)_{(2, 3)} = (y + 6x)_{(2, 3)} = 15$, therefore, the slope of the tangent at the point (2, 3, 18) to the curve of intersection of the plane $y = 3$ and the surface $z = xy + 3x^2$ is 15.

You should be able to do this exercise now.

E9) Find the slope of the tangent at the point (1, 2, 14) to the curve of intersection of the plane $y = 2$ and the surface $z = 2x^2 + 3y^2$.

You know that if a function f from $\mathbf{R} \to \mathbf{R}$ is differentiable at a point, it is also continuous at that point. In the next sub-section we shall see whether any such link exists between the continuity and the existence of partial derivatives of functions from $\mathbf{R}^2 \to \mathbf{R}$ or not.

5.2.3 Connection between Continuity and Partial Derivatives

If the two partial derivatives of a function f(x, y) exist, then what can we infer from this? Let's see. Suppose f(x, y) is a real-valued function having partial derivatives at a point (a, b). Then for $h \neq 0$,

$$f(a+h, b) - f(a, b) = \frac{f(a+h, b) - f(a, b)}{h} \times h$$

and therefore,

$$\lim_{h \to 0} [f(a+h, b) - f(a, b)] = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \cdot \lim_{h \to 0} h$$
$$= f_x(a, b) . 0$$
$$= 0.$$

Therefore, we can say that $f(a+h, b) \to f(a, b)$ as $h \to 0$, i.e., $f(x, y) \to f(a, b)$ as (x, y) approaches (a, b) along a line parallel to the x-axis. Similarly, the existence of the other partial derivative shows that $f(x, y) \to f(a, b)$ as (x, y) approaches (a, b) along a line parallel to the y-axis. The existence of $f_x(a, b)$ and $f_y(a, b)$ does not give us any further information. So we do not know whether the limit of $f(x, y)$ exists or not if $(x, y) \to (a, b)$ along any other path. But you have learnt in Unit 4 that $f(x, y)$ would be continuous at (a, b) if $f(x, y) \to f(a, b)$ as $(x, y) \to (a, b)$ along **any** path (which need not even be a straight line). Thus, it is clear from the above discussion that the mere existence of partial derivatives need not ensure the continuity of the function at that point. This, in fact, is the case as shown by the following example. Later on we shall see that if the partial derivatives satisfy some additional requirements, then their existence does imply continuity.

Example 8 : Let the function $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2+y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

You will see that both the partial derivatives of f at (0, 0), that is, $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not continuous at (0, 0).

Now, $$f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0, \text{ and}$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, 0+k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0.$$

So, f possesses both the first order partial derivatives at (0, 0). However, this function is not continuous at the point (0, 0) since $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist. (see E 3) a), Unit 4).

$f(x) = |x|$ is continuous on **R** but is not differentiable at $x = 0$.

We know that a real-valued continuous function of a real variable need not be differentiable. The same is true for functions of several variables. This means that a function of several variables which is continuous at a point need not have any of the partial derivatives at the point. The following example illustrates this fact.

Example 9 : Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by

$$f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|.$$

We'll show that f is continuous at (0, 0, 0) but does not possess any of the three first order partial derivatives at (0, 0, 0).

Now at the point (0, 0, 0), we have

$$f_1(h) = \frac{f(0+h, 0, 0) - f(0, 0, 0)}{h} = \frac{|h|}{h}.$$

Therefore, $$\lim_{h \to 0^+} f_1(h) = \lim_{\substack{h \to 0 \\ h>0}} \frac{|h|}{h} = \lim_{\substack{h \to 0 \\ h>0}} \frac{h}{h} = 1,$$

but $$\lim_{h \to 0^-} f_1(h) = \lim_{\substack{h \to 0 \\ h<0}} \frac{|h|}{h} = \lim_{\substack{h \to 0 \\ h<0}} \frac{-h}{h} = -1.$$

Hence, $\lim_{h \to 0} f_1(h)$ does not exist.

You can similarly check that

$\lim_{h \to 0} f_2(h)$ and $\lim_{h \to 0} f_3(h)$ also do not exist.

Thus, f does not possess any of the first order partial derivatives at the point (0, 0, 0). But this function is continuous at (0, 0, 0) as you have seen in E7) d), Unit 4.
If you have understood these examples, you should be able to solve the following exercises.

E10) Let $f(x, y) = \sqrt{x^2+y^2}$ for all $(x, y) \in \mathbf{R}^2$. Show that f is continuous at (0, 0), but $f_x(0, 0)$ as well as $f_y(0, 0)$ do not exist.

E11) Let $f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ 0, & xy = 0. \end{cases}$

Show that $f_x(0, 0)$ as well as $f_y(0, 0)$ exist. Also show that f is continuous at (0, 0).

E12) Show that the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x, y) = \begin{cases} \frac{x^2y}{x^4+y^2}, & \text{if } x^4+y^2 \neq 0 \\ 0, & \text{if } x = 0 = y \end{cases}$$

possesses first order partial derivatives everywhere including the origin but the function is discontinuous at the origin.

E13) Let $f(x, y) = \begin{cases} \frac{xy}{|y|}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$

a) Prove that f is continuous at (0, 0), and both $f_x(0, 0)$ and $f_y(0, 0)$ exist.

b) Show that $f_x(1, 0)$ exists, but $f_y(1, 0)$ does not.

In the above examples and exercises we have seen that the existence of partial derivatives does not imply continuity. However, if the partial derivatives satisfy some more conditions, then we can ensure continuity. You will study this in Theorem 3. In order to prove this theorem we need a simple result which follows easily from Lagrange's mean value theorem (Block 2, Calculus, also see margin remark). We first state this result.

Lagrange's m.v. theorem: Let f be a real-valued function continuous on [a, a+h] and differentiable on]a, a+h[. Then $\exists\ \theta \in\ ]0, 1[$, s.t. $f(a+h) - f(a) = hf'(a+\theta h)$.

Theorem 1 (Mean value theorem) : Let f be a real-valued function defined on a neighbourhood N of (a, b). If f_x exists at all points of N and f_y exists at the point (a, b), then

$$f(a+h, b+k) = f(a, b) + hf_x(a+\theta h, b+k) + k(f_y(a, b) + \eta)$$

for all real h, k such that (a+h, b+k) belongs to N where θ depends on h and k and $0 < \theta < 1$. Moreover, η is a function of k, which tends to 0 as $k \to 0$.

You can see that this is an extension of Lagrange's mean value theorem to functions from $\mathbf{R}^2$ to $\mathbf{R}$. Interchanging x and y in the hypothesis of this theorem we get another theorem, which we now state.

Theorem 2 : Let f be a real-valued function defined on a neighbourhood N of (a, b). If f_y exists at all points of N and f_x exists at (a, b), then for real h and k such that $(a+h, b+k) \in N$, we have

$$f(a+h, b+k) = f(a, b) + kf_y(a, b+\theta' k) + h(f_x(a, b) + \eta')$$

where θ' depends upon h and k, $0 < \theta' < 1$, η' is a function of h and tends to zero as $h \to 0$.
Both Theorem 1 and Theorem 2 are easy consequences of Lagrange's mean value theorem, but we shall not discuss their proofs here. We will now use these theorems to prove Theorem 3.

Theorem 3 : Let f be a real-valued function of two variables defined in a neighbourhood N of a point (a, b) such that one of the first order partial derivatives exists at all points $(x, y) \in N$ and is bounded on N, whereas the other partial derivative exists at the point (a, b). Then the function f is continuous at the point (a, b).

Proof : Without loss of generality we can assume that f_x exists for all $(x, y) \in N$ and is bounded on N, whereas f_y exists at the point (a, b). By Theorem 1, for all real h, k such that $(a+h, b+k) \in N$, we get

$$f(a+h, b+k) = f(a, b) + hf_x(a+\theta h, b+k) + k(f_y(a, b) + \eta), \qquad \ldots(1)$$

where $0 < \theta < 1$ and $\eta \to 0$ as $k \to 0$.
Since f_x is bounded on N, it follows that

$$\lim_{(h, k) \to (0, 0)} hf_x(a+\theta h, b+k) = 0.$$

Consequently, from (1) we get,

$$\lim_{(h, k) \to (0, 0)} f(a+h, b+k) = f(a, b)$$

Hence the function f is continuous at (a, b) and the proof is complete.

While proving Theorem 3 we had assumed that f_x exists at all points of N and is bounded on N, and f_y exists at (a, b). If instead we had assumed that f_y exists at all points of N and is bounded on N and f_x exists at (a, b), then we could have proved the result using Theorem 2.
Now here is a result which follows easily from Theorem 3.

Corollary 1 : Let f be a real-valued function of two variables defined in a neighbourhood N of a point (a, b), such that both the partial derivatives of f exist at all points of N and one of them is bounded on N. Then the function f is continuous everywhere on N.
Note that the conditions given in Theorem 3 are only sufficient and are not necessary. We have already seen in Example 9 that a function may be continuous even when none of the partial derivatives exist.
In the next example we use Corollary 1 to prove the continuity of the given function.

Example 10 : Is the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by $f(x, y) = ye^x$ continuous everywhere? Let us find out.
We first note that $f_x(x, y) = ye^x$ and $f_y(x, y) = e^x$.
Let (a, b) be any point in $\mathbf{R}^2$, and consider a neighbourhood

$$N = \{(x, y) \mid \sqrt{(x-a)^2 + (y-b)^2} < 1\}$$

of (a, b). Now, $f_x(x, y)$ and $f_y(x, y)$ exist at all points of N. Further, since

$$|x - a| \le \sqrt{(x-a)^2 + (y-b)^2},$$

therefore, for all $(x, y) \in N$, we have $|x - a| < 1$,

i.e., $a - 1 < x < a + 1$,

i.e., $e^{a-1} < e^x < e^{a+1}$.

So, f_y is bounded on N. Therefore, in view of Corollary 1, f is continuous at the point (a, b). Since (a, b) was any arbitrary point of $\mathbf{R}^2$, we can say that f is continuous everywhere on $\mathbf{R}^2$.

Try this exercise now.

E14) Use Corollary 1 to show that the following functions are continuous everywhere on $\mathbf{R}^2$.

a) $f(x, y) = xe^y$

b) $f(x, y) = 3xy$

In this section we have seen that the mere existence of partial derivatives does not imply continuity. This shows that the concept of partial derivatives does not generalise the concept of differentiation of functions from $\mathbf{R} \to \mathbf{R}$. In the next section we'll introduce a concept which is a generalisation of the concept of differentiation of real-valued functions of a single variable.

5.3 DIFFERENTIABILITY OF FUNCTIONS FROM $\mathbf{R}^2$ TO R

What do we mean when we say that a function f from **R** to **R** is differentiable at a point c?

Do you agree that f is differentiable at a point c if and only if there exists a constant A (depending on the function f and the point c) such that

$$f(c+h) - f(c) = Ah + h\eta(h)$$

where $\eta(h) \to 0$ as $h \to 0$?

While checking this you will find that A is nothing but $f'(c)$.

This definition of differentiability of a function of a single variable can be generalised in a natural way for functions of several variables. In this section we shall study the differentiability of real-valued functions of two variables. Here is its definition.

Definition 2 : Let f be a real-valued function defined in a neighbourhood N of a point (a, b). We say that the function f is **differentiable at (a, b)**, if

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k), \text{ where}$$

- h and k are real numbers such that $(a+h, b+k) \in N$,
- A and B are constants independent of h and k but dependent on the function f and the point (a, b),
- ϕ and ψ are two functions tending to zero as $(h, k) \to (0, 0)$.

We would like to make an important remark here.

Remark 2 : (i) In the literature you may find another definition of differentiability which we give below.

Let f be a real-valued function defined in a neighbourhood N of a point (a, b). Then the function f is said to be differentiable at the point (a, b), if there exist two constants A and B (depending on f and the point (a, b) only) such that

$$f(a+h, b+k) - f(a, b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h, k) \text{ where } \phi(h, k)$$

is a real-valued function such that $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

The equivalence of these two definitions can be established easily by using the identity

$$\sqrt{h^2+k^2} = h\left(\frac{h}{\sqrt{h^2+k^2}}\right) + k\left(\frac{k}{\sqrt{h^2+k^2}}\right)$$

ii) For a function f from **R** to **R**, if $f'(x_0)$ exists, then we can approximate $f(x) - f(x_0)$ by the linear function $(x - x_0) f'(x_0)$ near x_0. Similarly, if g(x, y) is differentiable at (a, b), then g(x, y) – g(a, b) can be approximated by the linear function (x – a) A + (y – b) B in a neighbourhood of the point (a, b).

We now illustrate the definition of differentiability with the help of a few examples.

Example 11 : Let $f(x, y) = x^2+y^2$. Then we can show that f is differentiable at any point (a, b).

For any two real numbers h and k, we have

$$\begin{aligned} f(a+h, b+k) - f(a, b) &= (a+h)^2 + (b+k)^2 - (a^2+b^2) \\ &= 2ah + 2bk + hh + kk \end{aligned}$$

If we set $A = 2a$, $B = 2b$, $\phi(h, k) = h$, $\psi(h, k) = k$, then $f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k)$, where A and B are constants independent of h and k, $\phi(h, k) \to 0$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$. Thus f is differentiable at the point (a, b).

Example 12 : Let $f(x, y) = \frac{x}{y}$. Then let us show that f is differentiable at all points (a, b) in the domain of definition of the function.
Since f is not defined for $y = 0$, we take $b \neq 0$. Let h and k be two real numbers such that $(a+h, b+k)$ is a point in a neighbourhood N of (a, b), which is contained in the domain of f. Then $b+k \neq 0$, and

$$
\begin{aligned}
f(a+h, b+k) - f(a, b) &= \frac{a+h}{b+k} - \frac{a}{b} \\
&= \frac{a}{b+k} - \frac{a}{b} + \frac{h}{b+k} \\
&= -\frac{ak}{b(b+k)} + \frac{h}{b+k} \\
&= -\frac{a}{b^2}\left[1 - \frac{k}{b+k}\right]k + \frac{h}{b}\left[1 - \frac{k}{b+k}\right] \\
&= \frac{1}{b}h - \frac{a}{b^2}k + h\left(\frac{-k}{b(b+k)}\right) + k\left(\frac{ak}{b^2(b+k)}\right)
\end{aligned}
$$

Set $A = \frac{1}{b}$, $B = -\frac{a}{b^2}$, $\phi(h, k) = \frac{-k}{b(b+k)}$, $\psi(h, k) = \frac{ak}{b^2(b+k)}$.

Then $f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k)$, where A and B are constants independent of h and k, $\phi(h, k) \to 0$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$. Hence, f is differentiable at the point (a, b).
Now here is an example of a function which is not differentiable.

Example 13 : We will prove that the function given by $f(x, y) = |x| + |y|$ is not differentiable at (0, 0). Suppose, if possible, that f is differentiable at (0, 0). Then

$$f(0+h, 0+k) - f(0, 0) = Ah + Bk + h\phi(h, k) + k\psi(h, k)$$

where A and B are constants, $\phi(h, k) \to 0$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.
Therefore, $|h| + |k| = Ah + Bk + h\phi(h, k) + k\psi(h, k)$.

Let $h = 0$ and $k > 0$. Then

$$k = Bk + k\psi(0, k),$$

$$\text{i.e., } 1 = B + \psi(0, k).$$

Taking limits on both sides as $(h, k) \to (0, 0)$, we get $B = 1$, because $\psi(0, k) \to 0$.

Now let $h = 0$ and $k < 0$. Then

$$-k = Bk + k\psi(0, k)$$

$$-1 = B + \psi(0, k)$$

Taking limits on both sides as $(h, k) \to (0, 0)$, we get $B = -1$, because $\psi(0, k) \to 0$.

Thus the assumption that the given function is differentiable at (0, 0) leads us to the contradiction $B = 1 = -1$. Hence, $|x| + |y|$ is not differentiable at (0, 0).
Now see if you can do these exercises.

E15) We have listed some results about the differentiability of real-valued functions of a real variable in the first column of the following table. Write analogous statements for real-valued functions of two variables in the second column, and check whether each of them is true.

One variable	Two variables
a) A constant function is differentiable everywhere	
b) If f is differentiable at $a \in \mathbf{R}$, then cf ($c \in \mathbf{R}$) is also differentiable at a.	
c) If f and g are differentiable at $a \in \mathbf{R}$, then $f \pm g$ is also differentiable at a.	
d) If f, g are differentiable at $a \in \mathbf{R}$ then fg is also differentiable at a.	

E16) Show that the function x^2+y+xy is differentiable at (0, 0).

E17) Show that cos (x+y) is differentiable at the point $\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$.

E18) Show that the following function f is not differentiable at (0,0)

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

In the case of a real-valued function of a real variable, continuity does not imply differentiability. The same is true for real-valued functions of two variables. Consider the function $f(x, y) = |x| + |y|$ of Example 13. According to E7) d) of Unit 4, it is continuous at (0, 0). In Example 13 we have seen that it is not differentiable at (0, 0). So continuity at a point does not imply differentiability at that point. However, every function which is differentiable at a point, is also continuous at that point. This is proved in the theorem that follows.

Theorem 4 : Let f be a real-valued function defined in a neighbourhood N of a point (a, b). If f is differentiable at (a, b), then f is continuous at (a, b).

Proof : Let h and k be two real numbers such that $(a+h, b+k) \in N$. Then differentiability of f at (a, b) implies that there exist two constants A, B and two functions $\phi(h, k)$, $\psi(h, k)$ such that

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k) \quad \ldots(2)$$

where $\phi(h, k) \to 0$, $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.
Now taking the limit on both sides of (2) as $(h, k) \to (0, 0)$, we get

$$\lim_{(h, k) \to (0, 0)} (f(a+h, b+k) - f(a, b)) = 0, \text{ or}$$

$$\lim_{(h, k) \to (0, 0)} f(a+h, b+k) = f(a, b)$$

This shows that the function f is continuous at (a, b). We can use this result to establish the non-differentiability of a function at a given point. For instance, we have seen in Sec. 4.4.2. that the function

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2+y^2}, & \text{if } x^2+y^2 \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is not continuous at (0, 0). Thus, in view of Theorem 4, we can conclude that this function is not differentiable at (0, 0).

You can now use Theorem 4 to solve this exercise.

E19) Show that the following functions are not differentiable at (0, 0) by showing that they are discontinuous at (0, 0).

a) $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 1 & (x, y) = (0, 0) \end{cases}$

b) $f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 2 & (x, y) = (0, 0) \end{cases}$

c) $f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{x - y} & x \neq y \\ 1 & x = y \end{cases}$

d) $f(x, y) = \begin{cases} \dfrac{x^5}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 3 & (x, y) = (0, 0) \end{cases}$

You can check that $A = f_x(a, b)$ and $B = f_y(a, b)$ in Examples 11, 12 and 13.

So far we have not said anything about the values of the constants occurring in the definition of differentiability of a function at a point. We shall now show (Theorem 5) that the constants A, B mentioned in Definition 2 are nothing but the two partial derivatives of the function under consideration at a point. This, in particular, would show that if a function is differentiable at a point, then it has both the partial derivatives at that point. But is the converse true? That is, if a function has both the partial derivatives at a point, can we conclude that it is differentiable there? No. The existence of partial derivatives, as we have seen in Example 8, does not guarantee even continuity. So, obviously, it cannot guarantee differentiability.
Look at the following theorem now.

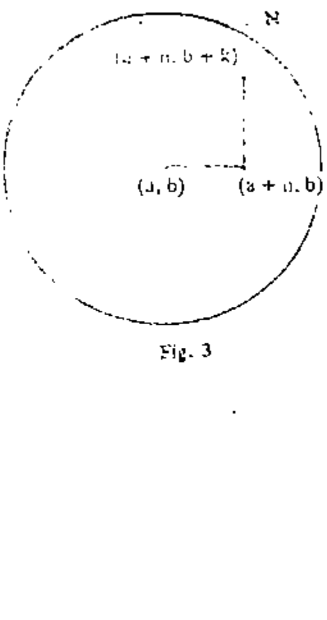


Fig. 3

Theorem 5 : Let f be a real-valued function defined in a neighbourhood N of the point (a, b). If f is differentiable at (a, b), then f possesses both the partial derivatives at (a, b).

Proof : Let h, k be real numbers such that $(a+h, b+k) \in N$. Since f is differentiable at the point (a, b), it follows that

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k).$$

where A and B are constants, $\phi(h, k) \to 0$, $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.
You can see from Fig. 3, that if $(a+h, b+k) \in N$, then $(a+h, b)$ and $(a, b+k)$ also belong to N. So if we let $k = 0$ in the above equation, then we get

$$f(a+h, b) - f(a, b) = Ah + h\phi(h, 0)$$

i.e., $$\frac{f(a+h, b) - f(a, b)}{h} = A + \phi(h, 0) \text{ for } h \neq 0.$$

Therefore, $$\lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = A,$$

i.e., $f_x(a, b) = A$.

Similarly, by setting $h = 0$ and proceeding as above, we can prove that $f_y(a, b) = B$.
This completes the proof of the theorem.
In view of Theorem 5, if f is differentiable at the point (a, b), then

$$f(a+h, b+k) - f(a, b) = hf_x(a, b) + kf_y(a, b) + h\phi(h, k) + k\psi(h, k)$$

where $\phi(h, k) \to 0$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

From this we see that for small values of h and k we can approximate $f(a+h, b+k) - f(a, b)$ by the expression $hf_x(a, b) + kf_y(a, b)$. This expression is given a special name, as you will now see.

Definition 3 : Let f (x, y) be a real-valued function defined in a neighbourhood of the point (a, b). If f (x, y) is differentiable at (a, b), then the linear function $T : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$T(h, k) = hf_x(a, b) + kf_y(a, b)$$

is called the **differential of f at (a, b).** It will be denoted by **df (a, b).**
We will now give an example to show that a function may possess partial derivatives and still not be differentiable.

Example 14 : If $f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

then f possesses both the first order partial derivatives at the point (0, 0), but is not differentiable at (0, 0).

Now, $\lim\limits_{h \to 0} \dfrac{f(0+h, 0) - f(0, 0)}{h} = \lim\limits_{h \to 0} \dfrac{h - 0}{h} = 1$

and $\lim\limits_{k \to 0} \dfrac{f(0, 0+k) - f(0, 0)}{k} = \lim\limits_{k \to 0} \dfrac{-k - 0}{k} = -1$

Therefore, both the first order partial derivatives exist at the point (0, 0) and $f_x(0, 0) = 1$, $f_y(0, 0) = -1$.
Suppose, if possible, that f is differentiable at (0, 0). Then, by Remark 2 (i),

$$f(0+h, 0+k) - f(0, 0) = hf_x(0, 0) + kf_y(0, 0) + \sqrt{h^2+k^2}\,\phi(h, k)$$

where $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Now, $\phi(h, k) = \dfrac{f(h, k) - h + k}{\sqrt{h^2+k^2}}$

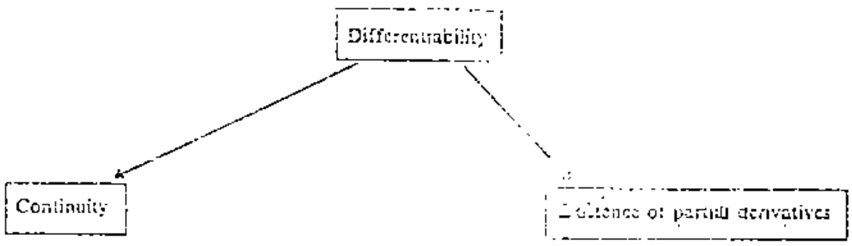
This means that $\lim\limits_{(h, k) \to (0, 0)} \dfrac{f(h, k) - h + k}{\sqrt{h^2+k^2}} = 0$

Now, if $h = r\cos\theta$, $k = r\sin\theta$, then

$$\frac{f(h, k) - h + k}{\sqrt{h^2+k^2}} = \cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$$

Therefore, $0 = \lim\limits_{(h, k) \to (0, 0)} \dfrac{f(h, k) - h + k}{\sqrt{h^2+k^2}} = \lim\limits_{r \to 0} (\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta)$ ...(3)

Now since the expression $\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta$ is independent of r, (3) implies that
$\cos^3\theta - \sin^3\theta - \cos\theta + \sin\theta = 0$ for all θ. But this is not true. So we have arrived at a contradiction, proving that the given function is not differentiable at (0, 0).
We now present the results of this section in the following chart.



But the arrows in this chart cannot be reversed. We shall now give (without proof) a sufficient set of conditions, which would ensure the differentiability of the function under consideration.

Theorem 6 : If f is a real-valued function defined in a neighbourhood of (a, b) such that

i) f_x is continuous at (a, b),

and ii) f_y exists at (a, b),

then f is differentiable at the point (a, b).

Similarly, the statement that f is differentiable at (a, b) if f_x exists at (a, b) and f_y is continuous at (a, b) is true. Thus, the continuity of one of the partial derivative and the existence of the other guarantees the differentiability of the function under consideration.

Now a function, both of whose partial derivatives are continuous, is given a special name. Here is the precise definition.

Definition 4 : A real-valued function f of two variables is said to be **continuously differentiable** at a point (a, b) if both the first order partial derivatives exist in a neighbourhood of (a, b) and are continuous at the point (a, b).

Note that the above definition requires that a neighbourhood of (a, b) should be contained in the domain D of the given function.

An immediate consequence of Definition 4 and Theorem 6 is the following.

Theorem 7 : A function, which is continuously differentiable at a point is differentiable at that point.

Now we shall illustrate the above discussion with some examples.

Example 15 : Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a function defined by

$$f(x, y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x = 0 = y \end{cases}$$

We'll show that f is differentiable at (0, 0).

We shall prove the result using Theorem 6.

$$\text{Now, } f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= 0$$

Similarly, $f_y(0, 0) = 0$ and for $(x, y) \neq (0, 0)$,

$$f_x(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

Using polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, we get

$$|f_x(x, y)| = r\,|(\cos^4\theta \sin\theta + 4\cos^2\theta \sin^3\theta - \sin^5\theta)|$$

$$\leq 6\sqrt{x^2+y^2}, \text{ since } \sin\theta \leq 1 \text{ and } \cos\theta \leq 1.$$

This can be made less than a given ε if we take $|x| < \dfrac{\varepsilon}{\sqrt{72}}$ and $|y| < \dfrac{\varepsilon}{\sqrt{72}}$.

This means that $\lim\limits_{(x, y) \to (0, 0)} f_x(x, y) = f_x(0, 0)$.

Thus, we can say that f_x is continuous at (0, 0). Consequently, f is differentiable at (0, 0) in view of Theorem 6.

Example 16 : We will now show that $f(x, y) = e^{x+y}\sin x + x^2 + 2xy$ is differentiable everywhere.

Since $f_x(x, y) = e^{x+y}\sin x + e^{x+y}\cos x + 2x + 2y$ and

$$f_y(x, y) = e^{x+y}\sin x + 2x$$

are continuous everywhere, it follows that f is differentiable everywhere in view of Theorem 6.

The following example shows that the conditions stated in Theorem 6 are sufficient but not necessary. That is, a function can be differentiable at a point even when none of the partial derivatives is continuous at that point.

Example 17 : Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ given by

$$f(x, y) = \begin{cases} x^2 \sin \frac{1}{x} + y^2 \sin \frac{1}{y}, & \text{if } xy \neq 0 \\ x^2 \sin \frac{1}{x}, & \text{if } x \neq 0 \text{ and } y = 0 \\ y^2 \sin \frac{1}{y}, & \text{if } x = 0 \text{ and } y \neq 0 \\ 0 & \text{, if } x = 0 = y \end{cases}$$

We'll prove that f is differentiable at (0, 0), but neither f_x nor f_y is continuous at (0, 0).

$$\text{Here } f_x(x, y) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\text{and } \ f_y(x, y) = \begin{cases} 2y \sin \frac{1}{y} - \cos \frac{1}{y} & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

Since $\lim\limits_{t \to 0} \cos \frac{1}{t}$ does not exist, and $\lim\limits_{t \to 0} t \sin \frac{1}{t} = 0$, it follows that

$\lim\limits_{(x, y) \to (0, 0)} f_x(x, y)$ and $\lim\limits_{(x, y) \to (0, 0)} f_y(x, y)$ do not exist.

This means that f_x and f_y are discontinuous at (0, 0).

Even though the limits of two given functions do not exist, the limit of their difference may exist. Therefore it is necessary to check that $\lim\limits_{t \to 0} t \sin \frac{1}{t}$ exists in addition to noting that $\lim\limits_{t \to 0} \cos \frac{1}{t}$ does not exist.

However, $f(h, k) - f(0, 0) = 0.h + 0.k + h\left(h \sin \frac{1}{h}\right) + k\left(k \sin \frac{1}{k}\right)$,

where $\lim\limits_{(h, k) \to (0, 0)} h \sin \frac{1}{h} = 0 = \lim\limits_{(h, k) \to (0, 0)} k \sin \frac{1}{k}$.

Therefore, f is differentiable at (0, 0).

See if you can solve these exercises now.

E20) Show that the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x, y) = \begin{cases} x \sin \frac{1}{x} + y \sin \frac{1}{y}, & xy \neq 0 \\ x \sin \frac{1}{x}, & y = 0, \ x \neq 0 \\ y \sin \frac{1}{y}, & x = 0, \ y \neq 0 \\ 0 & , \ x = 0 = y \end{cases}$$

is continuous but not differentiable at the origin.

E21) Check the following functions for continuity and differentiability at the origin.

a) $f(x, y) = \begin{cases} \frac{xy^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

b) $f(x, y) = \begin{cases} y \sin \frac{1}{x} & \text{if } x \neq 0 \\ & \text{if } x = 0 \end{cases}$

E22) Show that the following functions are differentiable everywhere.

a) $f(x, y) = e^{x+y}$

b) $f(x, y) = 2 \sinh x + 3 \cosh y$

In the next section we shall discuss the differentiability of functions of three or more variables.

5.4 DIFFERENTIABILITY OF FUNCTIONS FROM $R^n \to R$, $n > 2$

In the last section we defined and studied differentiability of real-valued functions of two variables. Most of the results established in the last section can be extended to real-valued functions of three or more variables. Let us start by defining differentiability of real-valued functions of three or more variables.

Definition 5 : Let f be a real-valued function defined in a neighbourhood of the point $a = (a_1, a_2, \ldots, a_n)$. The function f is said to be **differentiable** at the point a, if there exist constants $A_1, A_2, \ldots, A_n$ (depending on the function and the point a) such that

$$f(a_1 + h_1, \ldots, a_n + h_n) - f(a_1, \ldots, a_n)$$
$$= \sum_{i=1}^{n} h_i A_i + \sum_{i=1}^{n} h_i \phi_i (h_1, \ldots, h_n)$$

where each $\phi_i \to 0$ as $(h_1, h_2, \ldots, h_n) \to (0, 0, \ldots 0)$.
As in the case of two variables we have the following results.

i) $A_i = \frac{\partial f}{\partial x_i}$ at $(a_1, \ldots, a_n)$

ii) If f is differentiable at a, then f is continuous at a.

iii) f is differentiable at a if and only if

$$f(a+h) - f(a) = \sum_{i=1}^{n} h_i A_i + |h| \phi(h_1, h_2, \ldots, h_n)$$

where $\phi(h_1, h_2, \ldots, h_n) \to 0$ as $h \to 0$,

$h = (h_1, h_2, \ldots, h_n)$ and $|h| = \sqrt{\sum h_i^2}$

iv) If f is differentiable at a, then f has all the partial derivatives at a.

v) If the partial derivatives of f are continuous at a, then f is differentiable at a.

Don't worry, we don't expect you to prove them.
As for functions of two variables, we say that the linear function T defined by

$$T(h_1, h_2, \ldots, h_n) = \sum_{i=1}^{n} f_i(a) h_i$$

is the **differential** of f at a and denote it by **df (a)**.
Here are a few examples which will help you in understanding Definition 5.

Example 18 : Let $f(x, y, z) = x^2 + y^2 + z^2$. We'll prove that f is differentiable everywhere.

$f(x+h, y+k, z+l) - f(x, y, z) - hf_x(x, y, z) - kf_y(x, y, z) - lf_z(x, y, z)$

$= (x+h)^2 + (y+k)^2 + (z+l)^2 - x^2 - y^2 - z^2 - 2hx - 2ky - 2lz$

$= h^2 + k^2 + l^2$

$= h\,\phi_1(h, k, l) + k\phi_2(h, k, l) + l\phi_3(h, k, l)$

where $\phi_1(h, k, l) = h$, $\phi_2(h, k, l) = k$, $\phi_3(h, k, l) = l$,

and each one of them tends to 0 as $(h, k, l) \to (0, 0, 0)$.
Thus the given function is differentiable at every point.

Example 19 : Let $f(x, y, z) = |x+y+z|$. Then we'll show that f is differentiable at all those points (a, b, c) for which $a+b+c \neq 0$. We'll also show that f (x, y, z) is not differentiable at those points (a, b, c) for which $a+b+c = 0$.
Suppose that the point (a, b, c) is such that $a+b+c = 0$. Then, by definition,

$$f_x(a, b, c) = \lim_{p \to 0} \frac{f(a+p, b, c) - f(a, b, c)}{p},$$ provided the limit exists. But

$f(a+p, b, c) - f(a, b, c) = |a+p+b+c| - |a+b+c| = |p|$, since $a+b+c = 0$.
Therefore,

$$\lim_{p \to 0} \frac{f(a+p, b, c) - f(a, b, c)}{p} = \lim_{p \to 0} \frac{|p|}{p}.$$

Now we know that $\lim_{p \to 0} \frac{|p|}{p}$ does not exist. This means that $f_x(a, b, c)$ does not exist if $a+b+c = 0$. Hence, f is not differentiable at such a point.
Now, let's take the case when $a+b+c < 0$. Then there exists a neighbourhood N of (a, b, c) such that $(x, y, z) \in N$ implies that $x+y+z < 0$ because the function $(x, y, z) \to x+y+z$ is continuous everywhere (see Theorem 6 of Unit 4). Thus for those (p, q, r) for which $(a+p, b+q, c+r) \in N$, we have

$$f(a+p, b+q, c+r) - f(a, b, c) = -p - q - r$$
$$= p\, f_x(a, b, c) + q\, f_y(a, b, c) + r f_z(a, b, c),$$

$a+b+c < 0$ and $a+p+b+c < 0$. Therefore, $|a+b+c| = -a-b-c$ and $|a+p+b+c| = -a-p-b-c$.

because $$f_x(a, b, c) = \lim_{p \to 0} \frac{|a+p+b+c| - |a+b+c|}{p}$$
$$= \lim_{p \to 0} \frac{-p}{p} = -1.$$

Similarly,

$f_y(a, b, c) = -1 = f_z(a, b, c)$.

Thus, f is clearly differentiable at (a, b, c) if $a+b+c < 0$.
The remaining case i.e., when $a+b+c > 0$ is similar.

Now here are some exercises which you can try to solve.

E23) a) Show that every constant function is differentiable.

b) Let $f : \mathbf{R}^3 \to \mathbf{R}$ and $g : \mathbf{R}^3 \to \mathbf{R}$ be two functions such that each is differentiable at a point (a, b, c). Show that $f + g$ is also differentiable.

c) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a differentiable function. Prove that the function λf is differentiable, where λ is a constant.

E24) Show that the following functions from $\mathbf{R}^3 \to \mathbf{R}$ are differentiable everywhere.

a) $f(x, y, z) = x + 2y + 4z$

b) $f(x, y, z) = xy + yz + zx$

E25) Prove that the function $f : \mathbf{R}^3 \to \mathbf{R}$ defined by

$$f(x, y, z) = \frac{e^x + e^y}{z}$$ is not differentiable at (0, 0, 0).

Let us briefly recall what we have covered in this unit.

5.5 SUMMARY

In this unit we have extended the concept of derivatives to functions of several variables. In the process, we have

1) Defined first order partial derivatives of a function at a point. If $f : \mathbf{R}^n \to \mathbf{R}$ then

$$\lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}$$

if it exists, is called the ith partial derivative of f at $(x_1, x_2, \ldots, x_n)$.

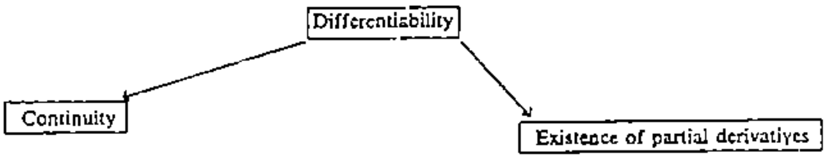
2) Discussed the methods to prove the existence of first order partial derivatives at a point and to evaluate them.

3) Given examples to establish that the existence of first order partial derivatives at a point need not imply the continuity of the function at that point.

4) Defined a differentiable function of several variables: f is differentiable at $a = (a_1, a_2, \ldots, a_n)$ if there exist constants $A_1, A_2, \ldots, A_n$ s.t.

$$f(a_1 + h_1, \ldots, a_n + h_n) - f(a_1, \ldots, a_n) = \sum_{i=1}^{n} h_i A_i + \sum_{i=1}^{n} h_i \phi_i (h_1, \ldots, h_n)$$

5) Brought out the connection between the existence of partial derivatives, differentiability and continuity of a function.



5.6 SOLUTIONS AND ANSWERS

E1) $f_x(a, b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$

Similarly $f_y(a, b) = 0$.

E2) By definition $f_x(0, 0) = \lim_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{1}{h}$

Since $\lim_{h \to 0} \frac{1}{h}$ does not exist, $f_x(0, 0)$ does not exist.

Similarly, $f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{1}{k}$

does not exist.

E3) $f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{h - 0}{h} = 1$

and $f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0.$

E4) The function $f(x, y) = 2x^2 - xy + 2y^2$ is a polynomial in x and y. Therefore the partial derivatives exist. By direct differentiation we get,

$\frac{\partial f}{\partial x} = 4x - y$; $\frac{\partial f}{\partial y} = -x + 4y.$

So, $\left(\frac{\partial f}{\partial x}\right)_{(1, 2)} = 4 - 2 = 2$; $\left(\frac{\partial f}{\partial y}\right)_{(1, 2)} = -1 + 8 = 7.$

E5) a) $\frac{\partial f}{\partial x} = 2x \cos (x^2 - y)$; $\frac{\partial f}{\partial y} = - \cos (x^2 - y)$

b) $\frac{\partial f}{\partial x} = - \frac{1}{2} \frac{1}{(x+y^2+z^2+1)^{3/2}}$

$\frac{\partial f}{\partial y} = - \frac{1}{2} \frac{2y}{(x+y^2+z^2+1)^{3/2}} = - \frac{y}{(x+y^2+z^2+1)^{3/2}}$

$\frac{\partial f}{\partial z} = - \frac{1}{2} \frac{2z}{(x+y^2+z^2+1)^{3/2}} = - \frac{z}{(x+y^2+z^2+1)^{3/2}}$

c) $\frac{\partial f}{\partial x} = yz \cos xz$; $\frac{\partial f}{\partial y} = \sin xz$; $\frac{\partial f}{\partial z} = xy \cos xz$

d) $\frac{\partial f}{\partial x} = yx^{y-1}$; $\frac{\partial f}{\partial y} = x^y \ln x$.

e) $\frac{\partial f}{\partial x} = 3x^2y + y^2e^{xy^2}$; $\frac{\partial f}{\partial y} = x^3 + 2xy\, e^{xy^2}$

E6) $u = e^x \cos y$, $v = e^x \sin y$. Since the functions involved in u and v are exponential and trigonometric functions, the partial derivatives exist. Then

$\frac{\partial u}{\partial x} = e^x \cos y$ and $\frac{\partial u}{\partial y} = -e^x \sin y$ and

$\frac{\partial v}{\partial x} = e^x \sin y$ and $\frac{\partial v}{\partial y} = e^x \cos y$. Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and

$\frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x}$.

E7) $$F_x (x, y) = \lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(y) - (f(x) + g(y))}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

$$F_y (x, y) = \lim_{k \to 0} \frac{F(x, y+k) - F(x, y)}{k}$$

$$= \lim_{k \to 0} \frac{f(x) + g(y+k) - (f(x) + g(y))}{k}$$

$$= \lim_{k \to 0} \frac{g(y+k) - g(y)}{k} = g'(y).$$

E8) By definition,

$$\left(\frac{\partial (f+g)}{\partial x}\right)_{(a,\,b)} = \lim_{h \to 0} \frac{(f+g)(a+h, b) - (f+g)(a, b)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(a+h, b) - f(a, b)}{h} + \frac{g(a+h, b) - g(a, b)}{h} \right]$$

$$= \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} + \lim_{h \to 0} \frac{g(a+h, b) - g(a, b)}{h}$$

$$= \left(\frac{\partial f}{\partial x}\right)_{(a,\,b)} + \left(\frac{\partial g}{\partial x}\right)_{(a,\,b)}$$

The converse is not necessarily true.
Consider the following functions f and g.

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

$$g(x, y) = \begin{cases} 0 & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$$

Then, $(f+g)(x, y) = 1$ for all (x, y).

In E2) it has been shown that $f_x(0, 0)$ does not exist.
Similarly, we can show that $g_x(0, 0)$ also does not exist.

However, $\frac{\partial}{\partial x}(f+g)$ exists at (0, 0) since $f+g$ is a constant function.

E9) The slope of the tangent at the point (1, 2, 14) to the curve of intersection of the plane $y = 2$ and the surface $z = 2x^2+3y^2$ is given by $\left(\frac{\partial z}{\partial x}\right)_{(1,2,14)}$

Now, $z = 2x^2+3y^2$. So $\frac{\partial z}{\partial x} = 4x$ and hence $\left(\frac{\partial z}{\partial x}\right)_{(1,2,14)} = 4$.
Thus, the slope is 4.

E10) To show the continuity at (0, 0) of f, let $\varepsilon > 0$ be given a real number.

Choose $\delta = \frac{\varepsilon}{\sqrt{8}}$. Then,

$$|x| < \frac{\varepsilon}{\sqrt{8}}, \ |y| < \frac{\varepsilon}{\sqrt{8}} = \sqrt{x^2+y^2} < \frac{\varepsilon}{2} < \varepsilon.$$

Therefore, $\lim\limits_{\substack{x\to 0\\ y\to 0}} \sqrt{x^2+y^2} = 0 = f(0, 0)$.

Hence, f is continuous at (0, 0). Now, by définition

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h\to 0} \frac{\sqrt{h^2}}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

Since, $\lim\limits_{h\to 0} \frac{|h|}{h}$ does not exist, $f_x(0, 0)$ does not exist.

Similarly, $f_y(0, 0)$ does not exist.

E11) $f_x(0, 0) = \lim\limits_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$

$$= \lim_{h\to 0} \frac{0 - 0}{h} = 0.$$

Similarly, $f_y(0, 0) = 0$. Now we have to show that f is continuous at (0, 0).
We have

$$\left|x \sin \frac{1}{x} + y \sin \frac{1}{y}\right| \leq |x| + |y|.$$

This implies that $\lim\limits_{x\to 0} \left(x \sin \frac{1}{x} + y \sin \frac{1}{y}\right) = (0, 0) - f(0, 0)$.

Therefore, f is continuous at (0, 0).

E12) $f_x(0, 0) = \lim\limits_{h\to 0} \frac{f(0+h, 0) - f(0, 0)}{h}$

$$= \lim_{h\to 0} \frac{f(h, 0) - 0}{h} = 0$$

Similarly, $f_y(0, 0) = 0$

Similarly, for points (x, y) such that xy = 0 we get that

$f_x(x, y) = f_y(x, y) = 0.$

Now, suppose $x \neq 0 \neq y$. Then both the partial derivatives at (x, y) exist since f (x, y) is a quotient of two differentiable functions in x and y. By direct differentiation

$$f_x(x, y) = \frac{(x^4+y^2) \cdot (2xy) - x^2y \cdot (4x^3)}{(x^4+y^2)^2}$$

$$= \frac{2x^5y + 2xy^3 - 4x^5y}{(x^4+y^2)^2}$$

and

$$f_y(x, y) = \frac{(x^4+y^2)(x^2) - x^2y(2y)}{(x^4+y^2)^2}$$

$$= \frac{x^6 - x^2y^2}{(x^4+y^2)^2}$$

But $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist, because when we put y = x,

$\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim\limits_{x \to 0} \frac{x^3}{x^4+x^2} = 0$, and when we put

$y = x^2$, $\lim\limits_{\substack{x \to 0 \\ y \to 0}} f(x, y) = \lim\limits_{x \to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$.

E13) a) Since $\left|\frac{xy}{y}\right| = \frac{|x|\ |y|}{|y|} = |x|$ for all $y \neq 0$,

$\lim\limits_{\substack{x \to 0 \\ y \to 0}} \frac{xy}{|y|} = 0 = f(0, 0)$. Hence f is continuous at (0, 0).

Also, $f_x(0, 0) = \lim\limits_{h \to 0} \frac{f(0+h, 0) - f(0, 0)}{h}$

$= \lim\limits_{h \to 0} \frac{0 - 0}{h} = 0.$

Similarly, $f_y(0, 0) = 0$.

b) $f_x(1, 0) = 0$. However, $f_y(1, 0)$ does not exist since

$\frac{f(1, k) - f(1, 0)}{k} = \frac{\frac{k}{|k|} - 0}{k} = \frac{1}{|k|}$ and $\lim\limits_{k \to 0} \frac{1}{|k|}$ does not exist.

E14) a) $f(x, y) = xe^y \Rightarrow f_x(x, y) = e^y$ and $f_y(x, y) = xe^y$. Proceeding as in Example 10, we can show that f_x is bounded in any neighbourhood of any point of $\mathbf{R}^2$. Therefore, by Corollary 1, f is continuous everywhere.

b) Now, $f(x, y) = 3xy \Rightarrow f_x(x, y) = 3y$ and $f_y(x, y) = 3x$. Both the partial derivatives f_x and f_y satisfy the conditions of Corollary 1. Therefore f is continuous everywhere.

E15) a) A constant function of two variables is differentiable everywhere.

b) If f is differentiable at $(a, b) \in \mathbf{R}^2$, then cf $(c \in \mathbf{R})$ is also differentiable at (a, b).

c) If f and g are differentiable at $(a, b) \in \mathbf{R}^2$, then $f \pm g$ is also differentiable at (a, b).

d) If f and g are differentiable at (a, b) then fg is also differentiable at (a, b).

Now we shall check the validity of these statements.

a) Let f (x, y) = c be a given constant function. Let (a, b) be any point of $\mathbf{R}^2$. Then

$$f(a+h, b+k) - f(a, b) = c - c = 0$$
$$= 0.h + 0.k + h\phi(h, k) + k\psi(h, k).$$

where $\phi(h, k) = 0 = \psi(h, k)$ for all h and k.
Since (a, b) was any arbitrary point of $\mathbf{R}^2$, f is differentiable everywhere.

b) Let f be differentiable at (a, b). Then there exist constants A and B and functions ϕ and ψ which tend to zero as $(h, k) \to (0, 0)$ such that

$$f(a+h, b+k) - f(a, b) = Ah + Bk + h\phi(h, k) + k\psi(h, k).$$

If $c \neq 0$, is a constant, then multiplying the above equation by c, we get

$$(cf)(a+h, b+k) - (cf)(a, b) = A'h + B'k + h\phi'(h, k) + k\psi'(h, k)$$

where $A' = cA$, $B' = cB$, $\phi' = c\phi$, $\psi' = c\psi$.

Now we have seen in Unit 4 that if ϕ and ψ tend to zero as $(h, k) \to (0, 0)$ then ϕ' and ψ' also tend to zero as $(h, k) \to (0, 0)$. Therefore cf is differentiable whenever c is a non-zero constant. If c is zero, then cf is a constant function taking every point to zero and hence is differentiable.

c) Since f is differentiable at (a, b) there exist constants A′ and B′ and functions $\phi'(h, k)$ and $\psi'(h, k)$ which tend to zero as $(h, k) \to (0, 0)$ such that

$$f(a+h, b+k) - f(a, b) = A'h + B'k + h\phi'(h, k) + k\psi'(h, k).$$

Similarly, since g is differentiable at (a, b) there exist constants A″ and B″ and functions $\phi''(h, k)$ and $\psi''(h, k)$ which tend to zero as $(h, k) \to (0, 0)$ such that

$$g(a+h, b+k) - g(a, b) = A''h + B''k + h\phi''(h, k) + k\psi''(h, k)$$

Then, $(f \pm g)(a+h, b+k) - (f \pm g)(a, b)$

$$= (A' \pm A'')h + (B' \pm B'')k + h[\phi'(h, k) \pm \phi''(h, k)]$$
$$+ k[\psi'(h, k) \pm \psi''(h, k)]$$
$$= Ah + Bk + h\phi(h, k) + k\psi(h, k)$$

where $A = A' \pm A''$, $B = B' \pm B''$, $\phi(h, k) = \phi'(h, k) \pm \phi''(h, k)$ and $\psi(h, k) = \psi'(h, k) \pm \psi''(h, k)$.

Now since A and B are constants and functions ϕ and ψ tend to zero as $(h, k) \to (0, 0)$, f+g as well as f − g is differentiable at (a, b).

d) Proceed as in c). Then

$$fg(a+h, b+k) - fg(a, b) = f(a+h, b+k)\, g(a+h, b+k)$$
$$- f(a+h, b+k)\, g(a, b) + f(a+h, b+k)\, g(a, b)$$
$$- f(a, b)\, g(a, b).$$

$$= f(a+h, b+k)[g(a+h, b+k) - g(a, b)] + g(a, b)[f(a+h, b+k) - f(a, b)]$$
$$= [f(a, b) + A'h + B'k + h\phi'(h, k) + k\psi'(h, k)][A''h + B''k$$
$$+ h\phi''(h, k) + k\psi''(h, k)] + g(a, b)[A'h + B'k + h\phi'(h, k) + k\psi'(h, k)]$$

$$= Ah + Bk + h\phi(h, k) + k\psi(h, k),$$

where $A = A''f(a, b) + A'g(a, b)$, $B = B''f(a, b) + B'g(a, b)$,

$$\phi(h, k) = [A' + \phi'(h, k)][A''h + B''k + h\phi''(h, k) + k\psi''(h, k)]$$
$$+ f(a, b)\phi''(h, k) + g(a, b)\phi'(h, k)$$

and
$$\psi(h, k) = [B' + \psi'(h, k)][A''h + B''k + h\phi''(h, k) + k\psi''(h, k)]$$
$$+ f(a, b)\psi''(h, k) + g(a, b)\psi'(h, k).$$

E16) Here $f(h, k) - f(0, 0) = h^2 + k + hk = 0.h + 1.k + h\phi(h, k) + k\psi(h, k)$ where $\phi(h, k) = h = \psi(h, k)$ for all h, k, and therefore, $\phi(h, k) \to 0$ and $\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$. Thus, f is differentiable at (0, 0).

E17) Let $f(x, y) = \cos(x+y)$. Consider

$$f\left(\frac{\pi}{4} + h, \frac{\pi}{4} + k\right) - f\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{4} + h + \frac{\pi}{4} + k\right) - \cos\left(\frac{\pi}{4} + \frac{\pi}{4}\right)$$

$$= \cos\left(\frac{\pi}{2} + h + k\right) - \cos\left(\frac{\pi}{2}\right)$$

$$= -\sin(h+k)$$

$$= -h - k + h\left[1 - \frac{\sin(h+k)}{h+k}\right] + k\left[1 - \frac{\sin(h+k)}{h+k}\right]$$

$$= Ah + Bk + h\phi(h, k) + k\psi(h, k),$$

where $A = -1$, $B = -1$, $\phi(h, k) = \psi(h, k) = 1 - \dfrac{\sin(h+k)}{(h+k)}$.

Now, $\lim\limits_{(h, k) \to (0, 0)} \phi(h, k)$

$$= \lim_{(h, k) \to (0, 0)} \psi(h, k) = 1 - \lim_{(h, k) \to (0, 0)} \frac{\sin(h+k)}{h+k}$$

$$= 1 - 1 = 0,$$

since $\lim\limits_{(h, k) \to (0, 0)} \dfrac{\sin(h+k)}{h+k} = \lim\limits_{t \to 0} \dfrac{\sin t}{t} = 1$ where $t = h+k$.

Therefore, f is differentiable at $\left(\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$

E18) Suppose, if possible, that f is differentiable at (0, 0). Then there exist constants A and B and functions ϕ and ψ which tend to zero as $(h, k) \to (0, 0)$, such that

$$f(0+h, 0+k) - f(0, 0) = Ah + Bk + h\phi(h, k) + k\psi(h, k),$$

i. e., $\dfrac{hk}{\sqrt{h^2+k^2}} = Ah + Bk + h\phi(h, k) + k\psi(h, k)$.

Put $h = 0$, $k \neq 0$. Then $0 = Bk + k\psi(0, k)$
i.e., $B + \psi(0, k) = 0$.
Taking the limit as $k \to 0$, we get $B = 0$.
Similarly, by putting $h \neq 0$, $k = 0$, we get $A = 0$.
Now if we let $h = k \neq 0$, then we get

$$\frac{h}{\sqrt{2}} = Ah + Bh + h\phi(h, h) + h\psi(h, h)$$

i.e., $\dfrac{1}{\sqrt{2}} = A + B + \phi(h, h) + \psi(h, h)$

and so taking limit as $h \to 0$ we get $A + B = \dfrac{1}{\sqrt{2}}$

which is impossible as $A = 0 = B$.
Thus we arrive at a contradiction and so f can not be differentiable at (0, 0).

E19) a) You have already seen in Example 3, Unit 4 that

$\lim\limits_{x \to 0} \dfrac{x^2 - y^2}{x^2 + y^2}$ does not exist at (0, 0). This shows that the function is discontinuous at (0, 0), and therefore not differentiable at (0, 0).

b) If we put $y = mx$, $f(x, y) = \frac{2}{1+m^2}$. This means that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along $y = mx$ is $\frac{2}{1+m^2}$, and this is different for different values of m.

Therefore, $\lim_{(x, y)\to(0, 0)} \frac{2xy}{x^2+y^2}$ does not exist and hence f is not continuous at (0, 0).

c) When we put $y = mx$, $m \neq 1$, then $\lim_{(x, y)\to(0, 0)} \frac{x^2+y^2}{x - y}$

$= \lim_{x\to 0} \frac{x(1+m^2)}{1 - m} = 0$. But $f(0, 0) = 1$.

Therefore, f is discontinuous at (0, 0).

d) $x^4+y^4 \leq x^4$

$\Rightarrow |f(x, y)| \leq \left|\frac{x^5}{x^4}\right| = |x|$

This shows that $\lim_{(x, y)\to(0, 0)} f(x, y) = 0$, which is different from $f(0, 0)$ $= 3$. Thus f is not continuous at (0, 0).

E20) Now $|f(x, y)| \leq |x| + |y|$ in all cases.

$\lim_{(x, y)\to(0, 0)} f(x, y) = f(0, 0)$ and hence f is continuous at (0, 0). Now, we have to show that f is not differentiable at (0, 0). In view of Theorem 5, it is enough to show that either f_x or f_y does not exist.

Now $f_x(0, 0)$ does not exist, since

$$\frac{f(h, 0) - f(0, 0)}{h} = \frac{h \sin \frac{1}{h} - 0}{h} = \sin \frac{1}{h}$$

and $\lim_{h\to 0} \sin \frac{1}{h}$ does not exist. Therefore f is not differentiable at (0, 0).

E21) a) Now $f_x(0,0) = 0 = f_y(0,0)$.

At $(x,y) \neq (0,0)$, $f_x(x, y) = \frac{y^5-x^2y^3}{(x^2+y^2)^2}$ and $f_y(x,y) = \frac{3x^3y^2+xy^4}{(x^2+y^2)^2}$

Putting $x = r\cos\theta$, $y = r\sin\theta$, we find that

$|f_x(x,y)| = r\,|\sin^5\theta - \cos^2\theta \sin^3\theta| \leq 2r = 2\sqrt{x^2+y^2}$

This implies that $\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$. Hence f_x is continuous at (0,0) and $f_y(0,0)$ exists. Thus f_x and f_y satisfy all conditions of Theorem 6. Therefore, f is differentiable at (0,0), and hence f is also continuous at (0,0).

b) Since $|f(x,y)| \leq |y|$, it can be easily shown that f is continuous at (0,0). Now, $f_x(0,0) = 0$ and $f_y(0,0) = 1$.

Suppose, if possible, that f is differentiable at (0,0). Then there exist functions ϕ and ψ, which tend to zero as $(h,k) \to (0,0)$, such that $f(h,k) - f(0,0) = 0.h + 1.k + h\phi(h,k) + k\psi(h,k)$.

Let $h = k \neq 0$. Then

$h \sin \frac{1}{h} = h + h\phi(h,h) + h\psi(h,h)$

i.e., $\sin \frac{1}{h} = 1 + \phi(h,h) + \psi(h,h)$

Therefore, $\lim_{h\to 0} \sin \frac{1}{h} = 1$, which contradicts the fact that $\lim_{h\to 0} \sin \frac{1}{h}$ does not exist.

E22) a) Since $f_x(x,y) = e^{x+y} = f_y(x,y)$ it can be easily seen that f_x and f_y are continuous everywhere. Therefore, f is differentiable everywhere.

b) $f_x(x,y) = 2\cosh x$ and $f_y(x,y) = 3\sinh y$.

So f_x and f_y are continuous everywhere and hence f is differentiable everywhere.

E23). a) Let $f(x,y,z) = k$ be a constant function. Then
$f(x+h_1, y+h_2, z+h_3) - f(x,y,z)$

$$= k - k = \sum_{i=1}^{3} h_i A_i + \sum_{i=1}^{3} h_i \phi_i$$

where $A_i = 0$, $i = 1, 2, 3$.

and $\phi_i(x,y,z) = 0$, $i = 1, 2, 3$.

So f is differentiable everywhere.

b) Since f is differentiable at (a, b, c) there exists a neighbourhood N_1 of (a, b, c) such that for all h_i, $i = 1, 2, 3$, such that $(a+h_1, b+h_2, c+h_3) \in N_1$, there exist constants A_i', $i = 1, 2, 3$, and functions $\phi_i' : \mathbf{R}^3 \to \mathbf{R}$, $i = 1, 2, 3$, which tend to zero as $(h_1, h_2, h_3) \to (0, 0, 0)$, such that

$$f(a+h_1, b+h_2, c+h_3) - f(a,b,c) = \sum_{i=1}^{3} h_i A_i' + \sum_{i=1}^{3} h_i \phi_i$$

Similarly, since g is differentiable at (a, b, c) there is a neighbourhood N_2 of (a, b, c), constants A_i'', $i = 1, 2, 3$, functions ϕ_i'', $i = 1, 2, 3$, such that

$$g(a+h_1, b+h_2, c+h_3) - g(a,b,c) = \sum_{i=1}^{3} h_i A_i'' + \sum_{i=1}^{3} h_i \phi_i''$$

Let $N = N_1 \cap N_2$. Then N is a neighbourhood of the point (a, b, c) such that for all h_i, $i = 1, 2, 3$, such that

$(a+h_1, b+h_2, c+h_3) \in N$, we have

$(f+g)(a+h_1, b+h_2, c+h_3) - (f+g)(a, b, c)$

$= \Sigma h_i (A_i' + A_i'') + \Sigma h_i (\phi_i' + \phi_i'')$

where $A_i' + A_i''$, $i = 1, 2, 3$, are constants and $\phi_i = \phi_i' + \phi_i''$, $i = 1, 2, 3$, are functions from $\mathbf{R}^3 \to \mathbf{R}$ which tend to zero as $(x,y,z) \to (a,b,c)$.

Hence f + g is differentiable at (a,b,c).

c) Can be proved similarly.

E24) Let (a,b,c) be any point and let the function $f(x,y,z) = x+2y+4z$ is defined in a neighbourhood N of (a,b,c). Let h_1, h_2, h_3 be numbers such that the point $(a+h_1, b+h_2, c+h_3)$ belongs to the neighbourhood N of the point (a,b,c). Then

$f(a+h_1, b+h_2, c+h_3) - f(a,b,c)$

$= (a+h_1) + 2(b+h_2) + 4(c+h_3) - a - 2b - 4c$

$= h_1 + 2h_2 + 4h_3$

$= A.h_1 + B.h_2 + C.h_3 + h_1\phi + h_2\psi + h_3\xi$

where $A = 1$, $B = 2$, $C = 4$ are constants and $\phi(x,y,z) = \psi(x,y,z) = \xi(x,y,z) = 0$. Hence f is differentiable at the point (a,b,c) and so everywhere.

b) $f(x,y,z) = xy + yz + zx$

Let (a,b,c) be any point.

$$f_x(a,b,c) = \lim_{h \to 0} \frac{f(a+h, b, c) - f(a,b,c)}{h}$$

$$= \lim_{h \to 0} \frac{(a+h)b + bc + c(a+h) - (ab+bc+ca)}{h}$$

$$= \lim_{h \to 0} \frac{bh + ch}{h}$$

$= b+c$

Similarly, $f_y(a,b,c) = c+a$,

$f_z(a,b,c) = a+b$.

That is, $f_x(a,b,c)$, $f_y(a,b,c)$, $f_z(a,b,c)$ exist and are continuous. Therefore f is differentiable at (a,b,c). This is true for all points of $\mathbf{R}^3$. Hence f is differentiable everywhere.

E25) To show that f is not differentiable at (0,0,0), it is enough to show that either f_x, f_y or f_z does not exist at (0,0,0).

$$f_z(0,0,0) = \lim_{r \to 0} \frac{f(0,0,0+r) - f(0,0,0)}{r}$$

$$= \lim_{r \to 0} \frac{2}{r^2}.$$

Since $\lim_{r \to 0} \frac{2}{r^2}$ does not exist, f_z does not exist at (0,0,0).

Hence f cannot be differentiable at (0,0,0).

UNIT 6 HIGHER ORDER PARTIAL DERIVATIVES

Structure

6.1 INTRODUCTION

In the last unit you studied partial derivatives of first order and differentiability of functions of several variables. You must have seen often, that partial derivatives of first order again define functions. For instance, if $f(x,y) = 3x^3+2xy^2+5y^2+6$, then $f_x(x,y) = 9x^2+2y^2$ and $f_y(x,y) = 4xy+10y$ are again real-valued functions of two variables with the domain $\mathbf{R}^2$. Thus we can talk of first order partial derivatives of these new functions. If we consider a function of two variables, there are two first order partial derivatives, which may give rise to four more partial derivatives, which might again turn out to be functions. If this chain continues, then we obtain higher order partial derivatives which constitute the subject matter of this unit. We shall be using these partial derivatives in the next block. In this unit you will study Euler's, Schwarz's and Young's theorems, which give some sets of conditions under which the mixed partial derivatives become equal.

Objectives

After studying this unit, you should be able to

- define and evaluate higher order partial derivatives,
- state and prove Euler's theorem,
- state Schwarz's and Young's theorems,
- decide about the commutativity of the operations of taking partial derivatives with respect to different variables.

6.2 HIGHER ORDER PARTIAL DERIVATIVES

In the introduction you have seen that the partial derivative f_x of the function $f(x, y) = 3x^3 + 2xy^2 + 5y^2 + 6$ is again a function of x and y. In general, let $D \subset \mathbf{R}^2$ and let $f : D \to \mathbf{R}$ have a first order partial derivative f_x at every point of D. Then we get a new function, say $g = f_x$, which is defined on D. This new function g may or may not possess first order partial derivatives. In case it does, then g_x and g_y are called the second order partial derivatives of f and are denoted by f_{xx} and f_{xy}, respectively. Similarly, if the function f has a first order partial derivative f_y at every point of D, then f_y defines a new function. And if this new function has first order partial derivatives, then we get two more second order partial derivatives, namely, f_{yx} and f_{yy}. Thus, if f(x, y) is a real-valued function defined in a neighbourhood of (a, b) having both the partial derivatives at all the points of the neighbourhood, then

$$f_{xx}(a, b) = \lim_{h \to 0} \frac{f_x(a+h, b) - f_x(a, b)}{h}$$

$$f_{xy}(a, b) = \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$f_{yx}(a, b) = \lim_{h \to 0} \frac{f_y(a+h, b) - f_y(a, b)}{h}$$

$$f_{yy}(a, b) = \lim_{k \to 0} \frac{f_y(a, b+k) - f_y(a, b)}{k}$$

provided each one of these limits exists.

We also denote the second order partial derivatives of f by

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} ; \; f_{xy} = \frac{\partial^2 f}{\partial y \, \partial x} ;$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \, \partial y} ; \; f_{yy} = \frac{\partial^2 f}{\partial y^2} .$$

If we want to indicate the particular point at which the second order partial derivatives are taken, then we write

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(a, b)}, \; \frac{\partial^2 f(a, b)}{\partial x^2}, \; f_{xx}(a, b), \; \left(\frac{\partial^2 f}{\partial x \, \partial y}\right)_{(a, b)}$$

$\frac{\partial^2 f(a, b)}{\partial x \, \partial y}$, $f_{xy}(a, b)$, and so on.

In a similar manner partial derivatives of order higher than two are defined. For example,

$$\frac{\partial^3 f}{\partial x \, \partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \, \partial y}\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] ;$$

i.e., $\frac{\partial^3 f}{\partial x \, \partial x \, \partial y}$ stands for the partial derivative of $\frac{\partial^2 f}{\partial x \, \partial y}$ with respect to x and is written as $\frac{\partial^3 f}{\partial^2 x \, \partial y}$.

Similarly, we can extend the idea of partial derivatives of higher orders to functions of more than two variables. In general, if f is a function of n variables $x_1, x_2, \ldots, x_n$ defined on $D \subset R^n$, then $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ denotes the second order partial derivative of f with respect to x_i and x_j, obtained by differentiating partially the partial derivative $\frac{\partial f}{\partial x_j}$ with respect to x_i. Further, $\frac{\partial^3 f}{\partial x_i \, \partial x_j \, \partial x_k}$ will denote the third order partial derivative of f with respect to the variables x_i, x_j and x_k, obtained by partial differentiation of $\frac{\partial^2 f}{\partial x_j \, \partial x_k}$ with respect to the variable x_i and so on.

In the following examples, we show how to calculate these higher order partial derivatives.

Example 1 : Let us find all the second order partial derivatives of the following functions:

i) $U(x, y) = x^3 + y^3 + 3axy$, a is constant,

ii) $U(x, y, z) = x^2 + yz + xz^3$.

Let's take these one by one.

i) Clearly, for $U(x, y) = x^3 + y^3 + 3axy$,

$\frac{\partial U}{\partial x} = 3x^2 + 3ay$ and $\frac{\partial U}{\partial y} = 3y^2 + 3ax$. Therefore,

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 + 3ay) = 6x,$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial}{\partial y}(3x^2 + 3ay) = 3a = \frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial}{\partial x}(3y^2 + 3ax) \text{ and}$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}(3y^2 + 3ax) = 6y.$$

ii) For $U(x, y, z) = x^2 + yz + xz^3$

$$\frac{\partial U}{\partial x} = 2x + z^3,\ \frac{\partial U}{\partial y} = z \text{ and } \frac{\partial U}{\partial z} = y + 3xz^2 \text{ Therefore,}$$

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial x}(2x + z^3) = 2,$$

$$\frac{\partial^2 U}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial y}(2x + z^3) = 0$$

$$\frac{\partial^2 U}{\partial z\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial}{\partial z}(2x + z^3) = 3z^2$$

$$\frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial}{\partial x}(z) = 0$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{\partial}{\partial y}(z) = 0$$

$$\frac{\partial^2 U}{\partial z\,\partial y} = \frac{\partial}{\partial z}(z) = 1$$

$$\frac{\partial^2 U}{\partial x\,\partial z} = \frac{\partial}{\partial x}(y + 3xz^2) = 3z^2$$

$$\frac{\partial^2 U}{\partial y\,\partial z} = \frac{\partial}{\partial y}(y + 3xz^2) = 1$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial}{\partial z}(y + 3xz^2) = 6xz.$$

Example 2 : If $f(x, y) = x^2 \tan^{-1}\frac{y}{x} - y^2 \tan^{-1}\frac{x}{y},\ x \neq 0,\ y \neq 0,$

we will prove that $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Here, $\frac{\partial f}{\partial y} = x^2.\ \frac{1}{1 + y^2/x^2}\cdot\frac{1}{x} - 2y\tan^{-1}\frac{x}{y} - y^2.\ \frac{1}{1 + x^2/y^2}\left(-\frac{x}{y^2}\right)$

$$= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\frac{x}{y}$$

And therefore,

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(x - 2y\tan^{-1}\frac{x}{y}\right)$$

$$= 1 - 2y.\ \frac{1}{1 + x^2/y^2}\cdot\frac{1}{y}$$

$$= 1 - \frac{2y^2}{x^2 + y^2}$$

$$= \frac{x^2 - y^2}{x^2 + y^2}$$

In the next example we go a step further and calculate a third order partial derivative.

Example 3 : If $u(x, y, z) = e^{xyz}$, then we can show that

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

Now $u(x, y, z) = e^{xyz}$. Therefore,

$$\frac{\partial u}{\partial z} = xye^{xyz},$$

$$\frac{\partial^2 u}{\partial y\, \partial z} = x\, e^{xyz} + x^2yz\, e^{xyz}, \text{ and}$$

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = e^{xyz} + xyze^{xyz} + 2xyz\, e^{xyz} + x^2y^2z^2\, e^{xyz}$$

$$= (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

We are sure you will be able to solve these exercises now.

E1) Find all the second order partial derivatives of the following functions.

a) $f(x, y) = \cos \frac{y}{x}$;

b) $f(x, y) = x^5 + y^4 \sin x^6$

c) $f(x, y, z) = \sin xy + \sin yz + \cos xz$

d) $f(x, y, z) = xyz^2 + xyz + x^3y$

E2) If $V(x, y, z) = (x^2+y^2+z^2)^{-1/2}$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

E3) Verify that $\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$ for each of the following functions.

a) $f(x, y) = x^3y + e^{xy^2}$

b) $f(x, y) = \tan(xy^3)$

E4) If $x^xy^yz^z = c$, show that at $x = y = z$, $\frac{\partial^2 z}{\partial x\, \partial y} = -(x \ln ex)^{-1}$.

(Hint : Take logarithms on both sides and differentiate.)

In Unit 5 you have seen that it is not always possible to find first order partial derivatives by direct differentiation (See Examples 5 and 6 of Unit 5). The same is true for higher order partial derivatives of some functions. This is illustrated by the following examples.

Example 4 : Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

We'll now evaluate the second order partial derivatives of f at (0, 0).

Since $f_{xx}(0, 0) = \lim_{h \to 0} \frac{f_x(h, 0) - f_x(0, 0)}{h}$, we have to first evaluate $f_x(h, 0)$ and $f_x(0, 0)$.

$$f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t}$$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

$$f_x(h, 0) = \lim_{t \to 0} \frac{f(h+t, 0) - f(h, 0)}{t}$$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Therefore,

$$f_{xx}(0, 0) = \lim_{h \to 0} \frac{0 - 0}{h} = 0.$$

Since $f_{xy}(0, 0) = \lim_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k}$, we must first evaluate $f_x(0, k)$.

Now, $f_x(0, k) = \lim_{t \to 0} \dfrac{f(t, k) - f(0, k)}{t}$

$$= \lim_{t \to 0} \frac{\dfrac{tk(t^2 - k^2)}{t^2 + k^2} - 0}{t}$$

$$= \lim_{t \to 0} \frac{k(t^2 - k^2)}{t^2 + k^2}$$

$$= -\frac{k^3}{k^2}$$

$$= -k.$$

Therefore, $f_{xy}(0, 0) = \lim_{k \to 0} \dfrac{-k - 0}{k} = -1.$

Since $f_{yx}(0, 0) = \lim_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h}$, we first evaluate $f_y(h, 0)$ and $f_y(0, 0)$.

Now, $f_y(0, 0) = \lim_{s \to 0} \dfrac{f(0, s) - f(0, 0)}{s} = \lim_{s \to 0} \dfrac{0 - 0}{s} = 0.$

$$f_y(h, 0) = \lim_{s \to 0} \frac{f(h, s) - f(h, 0)}{s}$$

$$= \lim_{s \to 0} \frac{\dfrac{hs(h^2 - s^2)}{h^2 + s^2} - 0}{s}$$

$$= \lim_{s \to 0} \frac{h(h^2 - s^2)}{h^2 + s^2}$$

$$= \frac{h^3}{h^2}$$

$$= h.$$

Therefore, $f_{yx}(0, 0) = \lim_{h \to 0} \dfrac{h - 0}{h} = 1.$

Since, $f_{yy}(0, 0) = \lim_{k \to 0} \dfrac{f_y(0, k) - f_y(0, 0)}{k}$, we first evaluate $f_y(0, k)$.

Now, $f_y(0, k) = \lim_{s \to 0} \dfrac{f(0, k+s) - f(0, k)}{s} = \lim_{s \to 0} \dfrac{0 - 0}{s} = 0.$

Therefore, $f_{yy}(0, 0) = \lim_{k\to 0} \frac{0-0}{k} = 0.$

Thus, you can see that to evaluate the partial derivatives of this function, we had to resort to the definition of partial derivatives, and direct differentiation was not possible.

In the next example we take up a function which is slightly more complicated.

Example 5 : Let us evaluate $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$, for the function f given by

$$f(x, y) = \begin{cases} (x^4 + y^4)\tan^{-1}(y^2/x^2), & x \neq 0 \\ \dfrac{\pi y^4}{2}, & x = 0 \end{cases}$$

We first note that

$$f_x(0, 0) = \lim_{t\to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0, \text{ and}$$

$$f_x(0, k) = \lim_{t\to 0} \frac{f(t, k) - f(0, k)}{t}$$

$$= \lim_{t\to 0} \frac{(t^4 + k^4)\tan^{-1}(k^2/t^2) - \pi k^4/2}{t}$$

We have applied L'Hopital's rule here, since $\frac{(t^4+k^4)\tan^{-1}(k^2/t^2) - \pi k^4/2}{t}$ is in the $\frac{0}{0}$ form as $t \to 0$.

By L'Hopital's rule, we have

$$f_x(0, k) = \lim_{t\to 0} \frac{4t^3 \tan^{-1}\dfrac{k^2}{t^2} + (k^4+t^4)\cdot\dfrac{1}{1+(k^4/t^4)}\left(-\dfrac{2k^2}{t^3}\right)}{1}$$

$$= \lim_{t\to 0} [4t^3 \tan^{-1}(k^2/t^2) - 2k^2 t]$$

$$= 0$$

Therefore, $f_{xy}(0, 0) = \lim_{k\to 0} \frac{0-0}{k} = 0$

$$f_y(0, 0) = \lim_{s\to 0} \frac{f(0, s) - f(0, 0)}{s}$$

$$= \lim_{s\to 0} \frac{(\pi s^4/2) - 0}{s}$$

$$= 0$$

Further, $f_y(h, 0) = \lim_{s\to 0} \frac{f(h, s) - f(h, 0)}{s}$

$$= \lim_{s\to 0} \frac{(h^4+s^4)\tan^{-1}(s^2/h^2) - 0}{s}$$

$$= \lim_{s\to 0} \frac{4s^3 \tan^{-1}(s^2/h^2) + (h^4+s^4)\cdot\left(\dfrac{1}{1+s^4/h^4}\right)(2s/h^2)}{1}$$

$$= \lim_{s\to 0} [4s^3 \tan^{-1}(s^2/h^2) + 2sh^2]$$

$$= 0.$$

Consequently, $f_{yx}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$

In Unit 5 you have seen some examples of functions whose partial derivatives f_{xy} do not exist (see Example 6 of Unit 5).

Here we will give you an example of a function whose first order partial derivatives exist, but higher order ones do not exist. From this example you will also see that the existence of a partial derivative of a particular order does not imply the existence of other partial derivatives of the same order.

Example 6 : Let us examine whether the second order partial derivatives of f at (0, 0) exist or not, if $f : \mathbf{R}^2 \to \mathbf{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{\sqrt{x^2 + y^2}}, & xy \neq 0 \\ 0, & xy = 0 \end{cases}$$

Now, $f_x(0, 0) = \lim_{t \to 0} \dfrac{f(t, 0) - f(0, 0)}{t}$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Similarly, for $h \neq 0$, $f_x(h, 0) = \lim_{t \to 0} \dfrac{f(h+t, 0) - f(h, 0)}{t}$

$$= \lim_{t \to 0} \frac{0 - 0}{t} = 0.$$

Therefore, $f_{xx}(0, 0) = \lim_{h \to 0} \dfrac{f_x(h, 0) - f_x(0, 0)}{h} = 0$

Now to check the existence of f_{xy}, we will have to see whether

$$\lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} \text{ exists or not.}$$

Therefore, let us find $f_x(0, k)$, for $k \neq 0$.

For $k \neq 0$, $f_x(0, k) = \lim_{t \to 0} \dfrac{f(t, k) - f(0, k)}{t}$

$$= \lim_{t \to 0} \frac{1}{t} \left[\frac{tk^2}{\sqrt{t^2 + k^2}} \right]$$

$$= \lim_{t \to 0} \frac{k^2}{\sqrt{t^2 + k^2}}$$

$$= \frac{k^2}{\sqrt{k^2}}$$

$$= |k|.$$

Now, $\lim_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k}$

$$= \lim_{k \to 0} \frac{|k|}{k},$$

which does not exist, showing that f_{xy} does not exist at (0, 0).

Now

$f_y(0, 0) = \lim_{s \to 0} \dfrac{f(0, s) - f(0, 0)}{s} = \lim_{s \to 0} \dfrac{0 - 0}{s} = 0,$

and for $h \neq 0$, $f_y(h, 0) = \lim_{s \to 0} \dfrac{f(h, s) - f(h, 0)}{s}$

$$= \lim_{s \to 0} \frac{\dfrac{hs^2}{\sqrt{h^2 + s^2}} - 0}{s}$$

$$= \lim_{s \to 0} \frac{hs}{\sqrt{h^2 + s^2}} = 0$$

Therefore, $f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$

Again, for $k \neq 0$, $f_y(0, k) = \lim_{s \to 0} \frac{f(0, k+s) - f(0, k)}{s} = \lim_{s \to 0} \frac{0 - 0}{s} = 0$

Therefore, $f_{yy}(0, 0) = \lim_{k \to 0} \frac{f_y(0, k) - f_y(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0.$

Thus, f_{xx}, f_{yy} and f_{yx} exist at (0, 0) and are equal to 0, while $f_{xy}(0, 0)$ does not exist.

See if you can solve these exercises now.

E5) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x, y) = \begin{cases} \frac{xy^5}{x^2 + y^4}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

E6) Examine the following functions for equality of f_{xy} and f_{yx} at (0, 0).

a) $$f(x, y) = \begin{cases} \frac{x^2 y^2}{\sqrt{x^4 + y^4}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

b) $$f(x, y) = \begin{cases} \frac{xy^3}{\sqrt{x^2 + y^4}}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$$

E7) Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$ for the function f defined by

$$f(x, y) = \begin{cases} xy, & \text{if } |y| \leq |x| \\ -xy, & \text{if } |y| > |x|. \end{cases}$$

The study of the above example and exercises must have convinced you that we have to be careful about the order of variables with respect to which higher order derivatives are taken. For instance, from Example 4 it is clear that f_{xy} need not be equal to f_{yx}. Example 6 goes a step further, where f_{yx} exists at (0, 0), while f_{xy} does not, showing that the question of their equality does not arise at all. If you look at the definitions of f_{xy} and f_{yx} at a point (a, b) more carefully, you would see why the expectation of the equality $f_{xy}(a, b) = f_{yx}(a, b)$ is farfetched. By definition

$$f_{xy}(a, b) = \lim_{k \to 0} \frac{f_x(a, b+k) - f_x(a, b)}{k}$$

$$= \lim_{k \to 0} \left[\frac{1}{k} \left\{ \lim_{h \to 0} \frac{f(a+h, b+k) - f(a, b+k)}{h} - \lim_{h \to 0} \frac{f(a+h, b) - f(a, b)}{h} \right\} \right]$$

$$= \lim_{k \to 0} \left[\lim_{h \to 0} \left\{ \frac{f(a+h, b+k) - f(a, b+k) - f(a+h, b) - f(a, b)}{hk} \right\} \right],$$

Similarly,

$$f_{yx}(a, b) = \lim_{h \to 0} \left[\lim_{k \to 0} \left\{ \frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) - f(a, b)}{hk} \right\} \right]$$

and we have already seen in Unit 4 that repeated limits are not equal, in general.

In the next section we will study the conditions under which these mixed partial derivatives become equal.

6.3 EQUALITY OF MIXED PARTIAL DERIVATIVES

We shall now give a set of sufficient conditions which would ensure that the order of the variables with respect to which higher order partial derivatives are taken is immaterial. In other words, if a function f satisfies these conditions, then its mixed partial derivatives will be equal.

Theorem 1 : Let f(x, y) be a real-valued function such that the two second order partial derivatives f_{xy} and f_{yx} are continuous at a point (a, b). Then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Proof : The continuity of f_{xy} and f_{yx} at (a, b) implies that f_x, f_y, f_{xy} and f_{yx} exist in a neighbourhood, say D of (a, b).
Consider the expression

$$\psi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b),$$

which is defined for all those real numbers h, k for which $(a+h, b+k) \in D$.

Let I_h denote the closed interval [a, a+h] or [a+h, a] according as $h > 0$ or $h < 0$. Let G(x) be a real-valued function defined on the closed interval I_h by

$$G(x) = f(x, b+k) - f(x, b)$$

so that $G(a+h) - G(a) = \psi(h, k)$. Since, for all x in I_h, the points (x, b+k) and (x, b) belong to D, it follows that $f_x(x, b+k)$ and $f_x(x, b)$ exist for all $x \in I_h$. Now we can write

$$G'(x) = f_x(x, b+k) - f_x(x, b).$$

Therefore, the function G(x) is differentiable on the closed interval I_h. Thus G(x) satisfies the requirements of Lagrange's mean value theorem, and we get

$$\psi(h, k) = G(a+h) - G(a) = h\,G'(a+\theta h)$$
$$= h\,[f_x(a+\theta h, b+k) - f_x(a+\theta h, b)], \quad \ldots\ldots\ldots\ldots(1)$$

where $0 < \theta < 1$.

Now we define a function $F : I_k \to \mathbf{R}$ by

$$F(t) = f_x(a+\theta h, t),$$

where I_k is the closed interval [b, b+k] or [b+k, b] according as $k > 0$ or $k < 0$. Since f_{xy} exists on D, it follows that the function F is differentiable on I_k. Therefore, by Lagrange's mean value theorem, we get

$F(b+k) - F(b) = kF'(b+\theta' k)$ for some θ', $0 < \theta' < 1$. This means that

$$f_x(a+\theta h, b+k) - f_x(a+\theta h, b) = k\,f_{xy}(a+\theta h, b+\theta' k)$$

Using Equation (1), we get

$$\psi(h, k) = hk\,f_{xy}(a+\theta h, b+\theta' k)$$

and consequently

$$\lim_{\substack{h\to 0\\ k\to 0}} \frac{\psi(h, k)}{hk} = \lim_{\substack{h\to 0\\ k\to 0}} f_{xy}(a+\theta h, b+\theta' k)$$
$$= f_{xy}(a, b)$$

as f_{xy} is given to be continuous at (a, b).

Starting with the function

$$H(y) = f(a+h, y) - f(a, y)$$

for $y \in I_k$ and proceeding exactly as above we can prove that

$$\lim_{\substack{h\to 0\\ k\to 0}} \psi(h, k) = f_{yx}(a, b)$$

and conclude $f_{xy}(a, b) = f_{yx}(a, b)$.

(L. Euler (1707-1783))

This result was proved by L. Euler around 1734, when he was working on some problems in hydrodynamics. Later the German mathematician Hermann Amandus Schwarz (1843-1921) proved another theorem about the equality of mixed partial derivatives. ThePEL- conditions in Schwarz's theorem are less restrictive than those in Euler's theorem (Theorem 1). We give only the statement of Schwarz's theorem here.

Theorem 2 : (Schwarz's Theorem) : Let $f(x, y)$ be a real-valued function defined in a neighbourhood of (a, b) such that

i) f_y exists on a certain neighbourhood of (a, b).

ii) f_{xy} is continuous at (a, b).

Then f_{yx} exists at (a, b) and $f_{yx}(a, b) = f_{xy}(a, b)$.

We now give an example to illustrate this.

Example 7 : Let us evaluate f_{xy} at a point (x, y) for the function f defined by $f(x, y) = x^4 + x^2y^2 + y^6$. Then we'll use Schwarz's theorem to evaluate f_{yx} at the point (x, y).

By direct differentiation you can see that

$f_x(x, y) = 4x^3 + 2xy^2$. Therefore, $f_{xy}(x, y) = 4xy$.

Since $4xy$ is a polynomial, f_{xy} is a continuous function.

Further, $f_y(x, y) = 2x^2y + 6y^5$ exists. Hence f satisfies the conditions of Schwarz's theorem and so $f_{yx}(x, y) = f_{xy}(x, y) = 4xy$.

Are you ready for an exercise now?

E8) Evaluate f_{xy} at a point (x, y) for each of the following functions.

a) $f(x, y) = x^2 + xy + y^2$

b) $f(x, y) = e^x \cos y - e^y \sin x$

c) $f(x, y) = \dfrac{x^2-y^2}{x^2+y^2}$, $x \neq 0, y \neq 0$

Verify that each of these functions satisfies the requirements of Schwarz's theorem and hence evaluate $f_{yx}(x, y)$.

In Euler's Theorem we assume that both the mixed partial derivatives are continuous, whereas in Schwarz's theorem we assume that only one of them, say f_{xy} is continuous, and that f_y exists. But even though the conditions of Schwarz's theorem are less strict, these are still not necessary for the equality of mixed partial derivatives. In other words, we can have functions whose mixed partial derivatives at some point are equal, but which do not satisfy the requirements of Schwarz's theorem. We give one such function in the following example.

Example 8 : Consider the function f defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & x = 0 = y \end{cases}$$

We will show that $f_{xy}(0, 0) = f_{yx}(0, 0)$, even though f does not fulfil the requirements of Schwarz's theorem.

$$\text{Now, } f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{0 - 0}{h}$$

$$= 0.$$

Also, for $y \neq 0$,

$$f_x(0, y) = \lim_{h \to 0} \frac{f(h, y) - f(0, y)}{h}$$

$$= \lim_{h \to 0} \frac{h^2y^2}{h^2+y^2} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{hy^2}{h^2+y^2}$$

$$= 0.$$

Therefore, $f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$

$$= 0.$$

Similarly, you can check that

$f_y(0, 0) = 0$ and for $x \neq 0$, we have

$$f_y(x, 0) = \lim_{k \to 0} \frac{f(x, k) - f(x, 0)}{k}$$

$$= \lim_{k \to 0} \frac{x^2k^2}{x^2+k^2} \cdot \frac{1}{k}$$

$$= 0.$$

From this we get

$$f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(x, 0) - f_y(0, 0)}{h}$$

$$= 0.$$

Hence, we have shown that

$f_{xy}(0, 0) = f_{yx}(0, 0)$.

We'll now show that the conditions of Schwarz's theorem are **not** satisfied. Now, for $x \neq 0$, $y \neq 0$, we can find the partial derivatives of f at (x, y) by differentiating directly. Thus,

$$f_x(x, y) = \frac{\partial}{\partial x}\left[\frac{x^2y^2}{x^2+y^2}\right]$$

$$= \frac{2(x^2+y^2)\,xy^2 - 2x^3y^2}{(x^2+y^2)^2}$$

$$= \frac{2xy^4}{(x^2+y^2)^2}$$

Further, $f_{xy}(x, y) = \frac{\partial}{\partial y}\left[\frac{2xy^4}{(x^2+y^2)^2}\right]$

$$= \frac{8x(x^2+y^2)^2y^3 - 8xy^5(x^2+y^2)}{(x^2+y^2)^4}$$

$$= \frac{8xy^3\,(x^2+y^2)\,[x^2+y^2-y^2]}{(x^2+y^2)^4}$$

$$= \frac{8x^3y^3}{(x^2+y^2)^3}$$

Now, $\lim_{(x, y) \to (0, 0)} \frac{8x^3y^3}{(x^2+y^2)^3}$ does not exist. Put $y = mx$ in $\frac{8x^3y^3}{(x^2+y^2)^3}$ and take the limit as $x \to 0$. You will find that the limit is different for different values of m. This means that $\lim_{(x, y) \to (0, 0)} f_{xy}(x, y)$ does not exist, which implies that f_{xy} is not continuous at (0, 0).

There is another criterion which tells us when f_{xy} equals f_{yx} at a particular point. We state this also without proof.

Theorem 3 (Young's Theorem) : Let f(x, y) be a real-valued function defined in a neighbourhood of a point (a, b) such that both the first order partial derivatives f_x and f_y are differentiable at (a, b). Then $f_{xy}(a, b) = f_{yx}(a, b)$.

As in the case of Schwarz's theorem the conditions stated in Young's theorem are less strict than in Theorem 1. However, these are not necessary for the equality of mixed partial derivatives. Our next example illustrates this fact.

Example 9 : Consider the function f in Example 8.

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0 & , x = y = 0. \end{cases}$$

We have seen that $f_x(0, 0) = 0$ and $f_{xy}(0, 0) = 0$.

You can easily check that $f_x(h, 0) = 0$. Now we'll prove that f_x is not differentiable at (0, 0). For this, let us start with the assumption that f_x is differentiable at (0, 0). Then there exist functions $\phi(h, k)$ and $\psi(h, k)$, such that

$$f_x(h, k) - f_x(0, 0) = h\, f_{xx}(0, 0) + k\, f_{xy}(0, 0) + h\phi(h, k) + k\psi(h, k) \qquad \ldots(2)$$

and $\phi(h, k) \to 0$ as $(h, k) \to (0, 0)$,

$\psi(h, k) \to 0$ as $(h, k) \to (0, 0)$.

Now let us calculate $f_{xx}(0, 0)$.

$$f_{xx}(0, 0) = \lim_{h \to 0} \frac{f_x(h, 0) - f_x(0, 0)}{h}$$

$$= 0.$$

Therefore, (2) becomes

$$f_x(h, k) = h\phi(h, k) + k\psi(h, k).$$

or, $$\frac{2hk^4}{(h^2+k^2)^2} = h\,\phi(h, k) + k\,\psi(h, k).$$

Now if we put $h = r\cos\theta$ and $k = r\sin\theta$ we get

$$2\cos\theta \sin^4\theta = \cos\theta\, \phi(r\cos\theta, r\sin\theta) + \sin\theta\, \psi(r\cos\theta, r\sin\theta) \qquad \ldots(3)$$

Now, if $r \to 0$, $r\cos\theta \to 0$ and $r\sin\theta \to 0$.
This means as $r \to 0$, $h \to 0$ and $k \to 0$, and therefore,

$\phi(r\cos\theta, r\sin\theta) \to 0$ and $\psi(r\cos\theta, r\sin\theta) \to 0$.

Thus, taking the limit of (3) as $r \to 0$, we get

$2\cos\theta \sin^4\theta = 0$, for all θ.

But this is impossible. Hence f_x is not differentiable. Thus this function f does not satisfy the requirements of Young's theorem, even though we have $f_{xy}(0, 0) = f_{yx}(0, 0)$.

For most of the functions that we come across, all the partial derivatives are continuous, and therefore the value of the mixed partial derivatives does not change when there is a change in the order of variables with respect to which the partial derivatives are taken. Let us look at a few more examples.

Example 10 : Consider the function $f : R^3 \to R$ defined by

$$f(x, y, z) = \begin{cases} \dfrac{x^3y - xy^3 + yz^2}{x^2+y^2+z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

We'll show that $f_{xy}(0, 0, 0) \neq f_{yx}(0, 0, 0)$, whereas $f_{xz}(0, 0, 0) = f_{zx}(0, 0, 0)$.

Let us first calculate $f_{xy}(0, 0, 0)$. For this we need to evaluate $f_x(0, 0, 0)$ and $f_x(0, k, 0)$. Now

$$f_x(0, 0, 0) = \lim_{p \to 0} \frac{f(p, 0, 0) - f(0, 0, 0)}{p} = \lim_{p \to 0} \frac{0-0}{p} = 0 \text{ and}$$

$$f_x(0, k, 0) = \lim_{p \to 0} \frac{f(p, k, 0) - f(0, k, 0)}{p}$$

$$= \lim_{p \to 0} \frac{\frac{p^3k - pk^3}{p^2+k^2} - 0}{p}$$

$$= \lim_{p \to 0} \frac{p^2k - k^3}{p^2+k^2} = -k.$$

Therefore, $f_{xy}(0, 0, 0) = \lim_{k \to 0} \frac{f_x(0, k, 0) - f_x(0, 0, 0)}{k}$

$$= \lim_{k \to 0} \frac{-k-0}{k}$$

$$= -1.$$

Next, we'll evaluate $f_{yx}(0, 0, 0)$. For this we need $f_y(h, 0, 0)$ and $f_y(0, 0, 0)$.

Now, $f_y(0, 0, 0) = \lim_{q \to 0} \frac{f(0, q, 0) - f(0, 0, 0)}{q} = 0$, and

$$f_y(h, 0, 0) = \lim_{q \to 0} \frac{f(h, q, 0) - f(h, 0, 0)}{q}.$$

$$= \lim_{q \to 0} \frac{\frac{h^3q - hq^3}{h^2+q^2} - 0}{q}$$

$$= \lim_{q \to 0} \frac{h^3 - hq^2}{h^2+q^2}$$

$$= h.$$

Therefore,

$$f_{yx}(0, 0, 0) = \lim_{h \to 0} \frac{f_y(h, 0, 0) - f_y(0, 0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h-0}{h} = 1$$

So $f_{xy}(0, 0, 0) \neq f_{yx}(0, 0, 0)$

Now, $f_z(0, 0, 0) = \lim_{r \to 0} \frac{f(0, 0, r) - f(0, 0, 0)}{r} = 0$, and

$$f_z(h, 0, 0) = \lim_{r \to 0} \frac{f(h, 0, r) - f(h, 0, 0)}{r} = 0$$

Therefore, $f_{zx}(0, 0, 0) = \lim_{h \to 0} \frac{f_z(h, 0, 0) - f_z(0, 0, 0)}{h} = 0$

Also $f_x(0, 0, r) = \lim_{p \to 0} \frac{f(p, 0, r) - f(0, 0, r)}{p} = 0$

This means, $f_{xz}(0, 0, 0) = \lim_{r \to 0} \frac{f_x(0, 0, r) - f_x(0, 0, 0)}{r} = 0$

Hence $f_{xz}(0, 0, 0) = f_{zx}(0, 0, 0)$.

Here is another example to show that the conditions in Theorem 1 are not necessary for the equality of mixed partial derivatives.

Example 11 : Let us show that $f_{xy}(0, 0, 0) = f_{yx}(0, 0, 0)$, but neither f_{xy} nor f_{yx} is continuous at (0, 0, 0) for the function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f(x, y, z) = \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, & x \neq 0, y \neq 0, z \neq 0, \\ 0 & , \text{otherwise} \end{cases}$$

Now, $f_x(0, 0, 0) = \lim_{h \to 0} \frac{f(h, 0, 0) - f(0, 0, 0)}{h} = 0$, and

$$f_x(0, k, 0) = \lim_{h \to 0} \frac{f(h, k, 0) - f(0, k, 0)}{h} = 0$$

$\lim_{h \to 0} \frac{-1}{y^2 h}$ does not exist because

$\lim_{h \to 0^+} \frac{-1}{y^2 h} = -\infty$ and

$\lim_{h \to 0^-} \frac{-1}{y^2 h} = \infty$

Therefore, $f_{xy}(0, 0, 0) = 0$.

Similarly we can show that $f_{yx}(0, 0, 0) = 0$. However, for $y \neq 0$, $z \neq 0$,

$\lim_{h \to 0} \frac{f_y(h, y, z) - f_y(0, y, z)}{h} = \lim_{h \to 0} \frac{-1/y^2}{h}$ does not exist, and we conclude that $f_{yx}(0, y, z)$ does not exist. Since in any neighbourhood of (0, 0, 0) there exist points (0, y, z) with $y \neq 0$, $z \neq 0$, it follows that f_{yx} is not defined in any neighbourhood of (0, 0, 0), and hence f_{yx} cannot be continuous at (0, 0, 0). In a similar way, we can show that f_{xy} is not continuous at (0, 0, 0).

You can try this exercise now.

E9) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be defined by

$$f(x, y, z) = \begin{cases} \frac{x}{y} + \frac{y}{z}, & y \neq 0, z \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

Show that at the origin f_{xy}, f_{yx}, f_{xz} and f_{zx} all exist, but neither f_{zy} nor f_{yz} exists.

Now let us briefly recall what we have covered in this unit.

6.4 SUMMARY

In this unit, we have

1) Introduced partial derivatives of order more than one.
2) Evaluated these higher order partial derivatives for various functions.
3) Studied examples of functions which show that, in general, the two partial derivatives of order more than one obtained by changing the order of variables are not equal in value, even if both exist.
4) Applied the following three sets of sufficient conditions which ensure the equality of $f_{xy}(a, b)$ and $f_{yx}(a, b)$.
   - Euler's theorem says that if f_{xy} and f_{yx} are both continuous at a point (a, b), then

     $f_{xy}(a, b) = f_{yx}(a, b)$.
   - Schwarz's theorem tells us that if f_{xy} is continuous at (a, b), and if f_y exists at (a, b), then $f_{yx}(a, b) = f_{xy}(a, b)$.
   - Young's theorem says that if f_x and f_y are differentiable at (a, b) then $f_{xy} = f_{yx}(a, b)$.
5) Seen, through some examples, that the conditions stated in the above three theorems are only sufficient and not necessary.

6.5 SOLUTIONS AND ANSWERS

E1) a) $f(x, y) = \cos \frac{y}{x}$. Then

$$f_x = -\sin\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) = \frac{y}{x^2} \sin \frac{y}{x}$$

$$f_y = -\sin\left(\frac{y}{x}\right) \cdot \left(\frac{1}{x}\right) = -\frac{1}{x} \sin \frac{y}{x}$$

$$f_{xx} = \frac{y}{x^2} \cos\left(\frac{y}{x}\right) \cdot \left(-\frac{y}{x^2}\right) + \left(-\frac{2y}{x^3}\right) \sin \frac{y}{x}$$

$$= -\frac{y^2}{x^4} \cos \frac{y}{x} - \frac{2y}{x^3} \sin \frac{y}{x}$$

$$f_{yx} = \frac{1}{x^2} \sin \frac{y}{x} + \frac{y}{x^2} \left(\cos \frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$= \frac{1}{x^2} \sin \frac{y}{x} + \frac{y}{x^3} \cos \frac{y}{x}$$

$$f_{xy} = \frac{1}{x^2} \sin \frac{y}{x} - \frac{1}{x} \left(\cos \frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{x^2} \sin \frac{y}{x} + \frac{y}{x^3} \cos \frac{y}{x}$$

$$f_{yy} = -\frac{1}{x^2} \cos \frac{y}{x}$$

b) $f(x, y) = x^5 + y^4 \sin(x^6)$

$\therefore f_x = 5x^4 + 6x^5y^4 \cos(x^6)$

$f_y = 4y^3 \sin x^6$

$f_{xx} = 20x^3 + 30x^4y^4 \cos(x^6) - 36x^{10}y^4 \sin(x^6)$

$f_{yx} = 24x^5y^3 \cos(x^6) = f_{xy}$

$f_{yy} = 12y^2 \sin(x^6)$

c) $f(x, y, z) = \sin xy + \sin yz + \cos xz$

$\therefore f_x = y \cos xy - z \sin xz$

$f_y = x \cos xy + z \cos yz$

$f_z = y \cos yz - x \sin xz$

$f_{xx} = -y^2 \sin xy - z^2 \cos xz$

$f_{yx} = \cos xy - xy \sin xy = f_{xy}$

$f_{zx} = -\sin xz - xz \cos xz = f_{xz}$

$f_{yy} = -x^2 \sin xy - z^2 \sin yz$

$f_{zy} = \cos yz - yz \sin yz = f_{yz}$

$f_{zz} = -y^2 \sin yz - x^2 \cos xz$

d) $f(x, y, z) = xyz^2 + xyz + x^3y$

$f_x = yz^2 + yz + 3x^2y$

$f_y = xz^2 + xz + x^3$

$f_z = 2xyz + xy$

$f_{xx} = 6xy$

$f_{yx} = z^2 + z + 3x^2 = f_{xy}$

$f_{zx} = 2yz + y = f_{xz}$

$f_{zz} = 2xy$

$f_{yy} = 0$

$f_{zy} = 2xz + x = f_{yz}$

E2) $v(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$

$$\therefore \frac{\partial v}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3}{2}\frac{2x}{(x^2+y^2+z^2)^{5/2}} \cdot x$$

$$= -\frac{1}{(x^2+y^2+z^2)^{3/2}} + \frac{3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{-(x^2+y^2+z^2)+3x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$$

Similarly,

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{3/2}}\ ;\ \frac{\partial^2 v}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\text{So, } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

E3) a) $f(x, y) = x^3y + e^{xy^2}$

$$\frac{\partial f}{\partial x} = 3x^2y + y^2 e^{xy^2}$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 3x^2 + 2y\, e^{xy^2} + 2xy^3 e^{xy^2}$$

$$\frac{\partial f}{\partial y} = x^3 + 2xy\, e^{xy^2}$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 3x^2 + 2y\, e^{xy^2} + 2xy^3 e^{xy^2}$$

$$\text{So, } \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}$$

b) $f(x, y) = \tan(xy^3)$

$$\therefore \frac{\partial f}{\partial x} = y^3 \sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial y\, \partial x} = 3y^2 \sec^2(xy^3) + 6xy^5 \sec^2(xy^3)\tan(xy^3)$$

$$\frac{\partial f}{\partial y} = 3xy^2 \sec^2(xy^3)$$

$$\frac{\partial^2 f}{\partial x\, \partial y} = 3y^2 \sec^2(xy^3) + 6xy^5 \sec^2(xy^3)\tan(xy^3)$$

$$\text{So, } \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 f}{\partial x\, \partial y}$$

E4) $x^x y^y z^z = c$

Taking logarithms on both sides, we get

$x \ln x + y \ln y + z \ln z = \ln c.$

Differentiating with respect to y, treating z as a function of x and y, we get

$$\ln y + y \cdot \frac{1}{y} + \left[\ln z + z\frac{1}{z}\right]\frac{\partial z}{\partial y} = 0$$

$$\therefore \frac{\partial z}{\partial y} = -\frac{\ln y + 1}{\ln z + 1}$$

$$\therefore \frac{\partial^2 z}{\partial x\, \partial y} = \frac{\ln y + 1}{(\ln z + 1)^2} \cdot \frac{1}{z} \frac{\partial z}{\partial x} = -\frac{\ln ey \ln ex}{z (\ln ez)^3}$$

Putting $x = y = z$, we get

$$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{1}{x \ln ex} = -(x \ln ex)^{-1}$$

E5) $f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = 0$. Similarly, $f_y(0, 0) = 0$,

and for $k \neq 0$, $f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{k} = 1$

So, $f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 1.$

Also, for $h \neq 0$, $f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k}$

$$= \lim_{k \to 0} \frac{hk^4}{h^2 + k^4} = 0$$

So, $f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 0.$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E6) a) $f_x(0, 0) = 0$.

For $k \neq 0$, $f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} \frac{hk^2}{\sqrt{h^4 + k^4}} = 0.$

So, $f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 0$

Similarly, $f_{xy}(0, 0) = 0$.

Hence, $f_{xy}(0, 0) = f_{yx}(0, 0)$

b) $f_x(0, 0) = 0 = f_y(0, 0)$

For $k \neq 0$, $f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} \frac{k^3}{\sqrt{h^2 + k^4}} = k.$

So, $f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = 1$

For, $h \neq 0$, $f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \to 0} \frac{hk^2}{\sqrt{h^2 + k^4}} = 0$

So, $f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = 0$

Consequently, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E7) $f_x(0, 0) = 0 = f_y(0, 0)$

For, $k \neq 0$, $f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h}$

$$= \lim_{h \to 0} \frac{-hk - 0}{h}$$ (since k is fixed we can suppose that $|h| < |k|$)

$$= -k.$$

For, $h \neq 0$, $f_y(h, 0) = \lim_{k \to 0} \dfrac{f(h, k) - f(h, 0)}{k}$

$= \lim_{k \to 0} \dfrac{hk - 0}{k}$ (since h is fixed we can suppose that $|k| < |h|$)

$= h.$

Therefore, $f_{yx}(0, 0) = \lim_{k \to 0} \dfrac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k \to 0} \dfrac{-k - 0}{k} = -1$

and $f_{xy}(0, 0) = \lim_{h \to 0} \dfrac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \dfrac{h - 0}{h} = 1$

Hence, $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

E8) a) $f(x, y) = x^2 + xy + y^2$. Then

$f_y(x, y) = x + 2y$

$f_{xy}(x, y) = 1$

Clearly f_y exists everywhere and f_{xy} is continuous being a constant function. This shows that f satisfies the requirements of Schwarz's theorem. Hence, by Schwarz's theorem, f_{yx} exists and $f_{yx} = f_{xy} = 1$.

b) $f(x, y) = e^x \cos y - e^y \sin x$

$\therefore f_y(x, y) = -e^x \sin y - e^y \sin x$

and $f_{xy}(x, y) = -e^x \sin y - e^y \cos x$

It is easy to see that f_y exists and f_{xy} is continuous. So, in view of Schwarz's theorem, f_{yx} exists and

$f_{yx}(x, y) = f_{xy}(x, y) = -e^x \sin y - e^y \cos x.$

c) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$, $x \neq 0$, $y \neq 0$.

$$f_y(x, y) = \frac{-4x^2 y}{(x^2 + y^2)^2}$$

$$f_{xy}(x, y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

Since f_y exists and f_{xy} is continuous at all points (x, y), where $x \neq 0$, $y \neq 0$, by Schwarz's theorem, we have

$$f_{yx}(x, y) = f_{xy}(x, y) = \frac{8xy(x^2 - y^2)}{(x^2 + y^2)^3}$$

E9) Now $f_x(0, 0, 0) = f_y(0, 0, 0) = f_z(0, 0, 0) = 0$.

For $k \neq 0$, $f_x(0, k, 0) = \lim_{h \to 0} \dfrac{f(h, k, 0) - f(0, k, 0)}{h} = 0$

Therefore, $f_{yx}(0, 0, 0) = \lim_{k \to 0} \dfrac{f_x(0, k, 0) - f_x(0, 0, 0)}{k} = 0.$

Similarly, $f_{xy}(0, 0, 0) = f_{yz}(0, 0, 0) = f_{zx}(0, 0, 0) = 0.$

For $r \neq 0$, $f_y(0, 0, r) = \lim_{k \to 0} \dfrac{f(0, k, r) - f(0, 0, r)}{k}$

$= \lim_{k \to 0} \dfrac{k/r}{k}$

$= \lim_{k \to 0} \dfrac{1}{r} = \dfrac{1}{r}$

So, $f_{zy}(0, 0, 0) = \lim_{r \to 0} \frac{f_y(0, 0, r) - f_y(0, 0, 0)}{r} = \lim_{r \to 0} \frac{1}{r^2}$

But $\lim_{r \to 0} \frac{1}{r^2}$ does not exist.

Hence, $f_{zy}(0, 0, 0)$ does not exist.

Since $f_z(0, k, 0) = \lim_{r \to 0} \frac{f(0, k, r) - f(0, k, 0)}{r} = k \lim_{r \to 0} \frac{1}{r^2}$,

therefore, $f_z(0, k, 0)$ does not exist as $\lim_{r \to 0} \frac{1}{r^2}$ does not exist.

Hence $f_{yz}(0, 0, 0) = \lim_{k \to 0} \frac{f_z(0, k, 0) - f_z(0, 0, 0)}{k}$ does not exist.

UNIT 7 CHAIN RULE AND DIRECTIONAL DERIVATIVES

Structure

7.1 INTRODUCTION

You are already familiar with the chain rule which is used for evaluating the derivative of a function of a function (Calculus, Unit 3). In this unit we shall study the chain rule to evaluate the partial derivatives of functions of several variables where each variable itself is a function of several independent variables. Let u, v, w,.... be functions of a single variable t, then f (u, v, w,....) is a function F(t) of the single variable t. We shall describe how to find the 'total derivative' F′(t). Using these results we shall prove Euler's theorem about homogeneous functions. Finally, we introduce the concept of directional derivatives and discuss its connection with partial derivatives which you have studied in Unit 5.

Objectives

After reading this unit, you should be able to

- define and evaluate the total derivative of a function using the chain rule,
- use the various forms of the chain rule for functions of several variables to solve problems,
- define and identify homogeneous functions,
- state, prove and use Euler's theorem for homogeneous functions,
- calculate directional derivatives of given functions,
- establish a relationship between directional and partial derivatives.

7.2 CHAIN RULE

In this section we describe the chain rule which enables us to calculate the derivatives of functions of several variables where each variable itself is a function of an independent variable. Recall that you have learnt the chain rule for functions of one variable in your calculus course. The rule says that, suppose we have a function

$$y = f(x),$$

where x is a function of t, say

$$x = g(t),$$

then y also may be regarded as a function of t, say y = F(t), and we have

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Or, in other words, $F'(t) = f'(x)\, g'(t) = f'(g(t))\, g'(t)$.

Now we extend the chain rule for functions of one variable to functions of several variables. We have defined the composite of functions of several variables in Sec. 3.3. There you have seen that there are several ways of forming a composite function. For instance,

Case 1 : Let $f(x, y) = x^2+xy+y^2$ be a function from $\mathbf{R}^2 \to \mathbf{R}$, $g(t) = \sin t$ be a function from $\mathbf{R} \to \mathbf{R}$. Then the composite function $g \circ f$ defined by

$$g \circ f(x, y) = g(f(x, y)) = \sin(x^2 + xy + y^2)$$

is a function from $\mathbf{R}^2 \to \mathbf{R}$.

Case : Consider the function $\phi(x, y) = x^y + y^x$ from $\mathbf{R}^2 \to \mathbf{R}$ and the function $g(t) = (\sin t, \tan t)$ from $\mathbf{R} \to \mathbf{R}^2$. Then the composite function $\phi \circ g$ defined by

$$\phi \circ g(t) = \phi(g(t)) = \phi(\sin t, \tan t) = (\sin t)^{\tan t} + (\tan t)^{\sin t}$$

is a function from $\mathbf{R} \to \mathbf{R}$.

Since there are many ways of forming composite functions, we will have to derive chain rule separately for each type. In Theorem 1 below, we shall derive the chain rule for finding the derivatives of composite functions in Case 1. Later in Theorem 2, we derive the chain rule for Case 2. Let us now state and prove Theorem 1.

Theorem 1 : Let $f(x, y)$ be a real-valued function having continuous first order partial derivatives at a point (a, b), and let g be a real-valued function of a real variable which is differentiable at the point $f(a, b)$. Then the composite function $\phi = g \circ f$ has first order partial derivatives at (a, b) and

$$\phi_x(a, b) = g'(f(a, b))\, f_x(a, b)$$

$$\phi_y(a, b) = g'(f(a, b))\, f_y(a, b).$$

Proof : First of all note that the function $f(x, y)$ is defined in a neighbourhood of (a, b) and the function g is defined in an open interval
$I = \,]f(a, b) - \delta, f(a, b) + \delta[$ for some $\delta > 0$. Since $f(x, y)$ possesses continuous partial derivatives at (a, b), it follows that it is continuous at (a, b).
Consequently, there exists a neighbourhood N of (a, b) such that for $(x, y) \in N$, the real number $f(x, y) \in I$. This means that the composite function $\phi = g \circ f$ is defined in the neighbourhood N of (a, b) and we can talk about its partial derivatives at (a, b).
Here $\phi = g \circ f$ is a real-valued function of two variables (also see Fig. 1).

A function $f : \mathbf{R}^n \to \mathbf{R}$ is continuous at a if $\forall\, \varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$. Equivalently, f is continuous at a if for every neighbourhood I of $f(a)$, there exists a neighbourhood N of a s.t. $x \in N \Rightarrow f(x) \in I$.

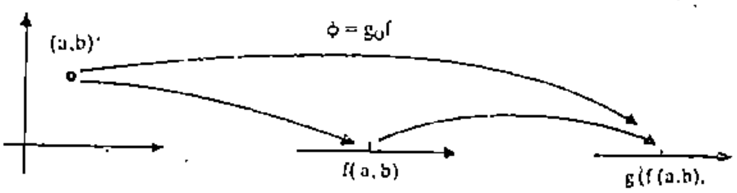


Fig. 1

Let us first find the partial derivative of ϕ with respect to x.

Since the function g is differentiable at $f(a, b)$ there exists a function $\psi(k)$ such that

$$g(f(a, b) + k) - g(f(a, b)) = kg'(f(a, b)) + k\psi(k) \quad \ldots(1)$$

where $\psi(k) \to 0$ as $k \to 0$ (See Sec. 5.3 of Unit 5) $\quad \ldots(2)$

Choose $k = k(h) = f(a+h, b) - f(a, b),\ h \neq 0,$

Now to find the partial derivative of ϕ w.r.t. x we have to find

$\lim\limits_{h \to 0} \dfrac{\phi(a+h, b) - \phi(a, b)}{h}$. So, let us consider the quotient

$\dfrac{\phi(a+h, b) - \phi(a, b)}{h}$. Now

$$\frac{\phi(a+h, b) - \phi(a, b)}{h} = \frac{g(f(a+h, b)) - g(f(a, b))}{h}$$

$$= \frac{g(f(a, b)+k) - g(f(a, b))}{h}$$

$$= \frac{g'(f(a, b)\,k + k\psi(k)}{h}, \text{ by (1).}$$

Substituting for k given in (2), we get

$$\frac{\phi(a+h, b) - \phi(a, b)}{h} = g'(f(a, b)). \frac{f(a+h, b) - f(a, b)}{h} + \frac{f(a+h, b) - f(a, b)}{h}\psi(k) \quad ...(3)$$

From (2) we can also see that, $k(h) \to 0$ as $h \to 0$, because $f(x, y)$ is continuous at (a, b). This in turn implies $\psi(k) \to 0$ as $h \to 0$. Consequently, the last term in (3) tends to zero as $h \to 0$, and we get

$$\phi_x(a, b) = g'(f(a, b))\, f_x(a, b).$$

The proof of the remaining part is similar,

We illustrate this theorem with an example.

Example 1 : Let us consider the composite function $\phi(x, y) = \sin(x^2+xy+y^2)$, given in Case 1 and calculate $\frac{\partial \phi}{\partial x}(a, b)$ and $\frac{\partial \phi}{\partial y}(a, b)$. Here $\phi = g \circ f$ where $f(x, y) = x^2+xy+y^2$ and $g(t) = \sin t$. Both functions f and g satisfy the requirements of Theorem 1.

So by Theorem 1,

$$\frac{\partial \phi}{\partial x}(a, b) = g'(f(a, b)) . \frac{\partial f}{\partial x}(a, b)$$
$$= \cos(a^2 + ab + b^2) . (2a + b)$$
$$= (2a + b)\cos(a^2 + ab + b^2)$$
$$\frac{\partial \phi}{\partial y}(a, b) = g'(f(a, b))\ \frac{\partial f}{\partial y}(a, b)$$
$$= \cos(a^2 + ab + b^2) . (a + 2b)$$
$$= (a + 2b)\cos(a^2 + ab + b^2)$$

You can try this exercise now.

E1) Let $f(x, y) = x^2+3xy+y^2$ and $g(t) = \cos t$. Find the partial derivatives of $\phi = g \circ f$ at $\left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{\pi}{2}}\right)$

In the following theorem we state the chain rule for Case 2. The proof of the theorem is beyond the scope of this course.

Theorem 2 : If f (t) and g (t) are two real-valued functions which are differentiable at a point t_0 and if $\phi(x, y)$ is a real-valued function of two variables, which is differentiable at the point $f(t_0), g(t_0)$, then the function

$F(t) = \phi(f(t), g(t))$ is differentiable at t_0 and

$F'(t_0) = f'(t_0)\,\phi_x(f(t_0), g(t_0)) + g'(t_0)\,\phi_y(f(t_0), g(t_0))$

Fig. 2 gives an illustration of the functions considered in Theorem 2.

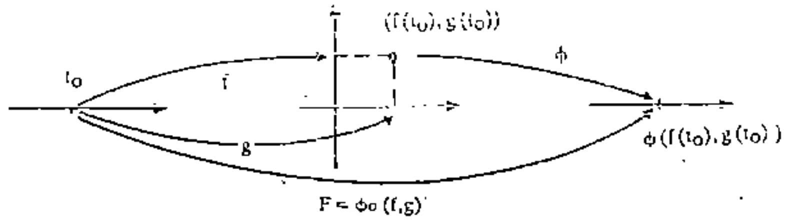


Fig. 2

If we write

$x = f(t)$, $y = g(t)$, $z = F(t) = \varphi(f(t), g(t)) = \phi(x, y)$, then the result of the above theorem can be written as

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

This is known as the **chain rule for partial derivatives.** The derivative $\frac{dz}{dt}$ is also known as the **total derivative of z.**

A similar result holds for functions of n variables where $n > 2$. We have the result: Suppose z is a function of n variables $x_1, x_2, \ldots\ldots, x_n$ and each x_i is a function of t, then

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x}\frac{dx_i}{dt},$$

provided z is differentiable at the point $(x_1(t), x_2(t), \ldots\ldots, x_n(t))$, and each x_i is differentiable at t.

Let us look at some examples.

Example 2 : Let us find the total derivative of the function $f(x, y) = x^2y - 2x + 3y - 4$, where $x = t-2$ and $y = t^2$.

You can easily verify that all the requirements of Theorem 2 are satisfied. Therefore, we have

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= (2xy-2)(1) + (x^2+3)(2t)\\ &= [2(t-2)t^2-2] + [(t-2)^2+3](2t)\\ &= 2t^3-4t^2-2+2t^3-8t^2+14t\\ &= 4t^3-12t^2+14t-2\end{aligned}$$

In the next example we consider a function of three variables.

Example 3 : Consider the function $f : \mathbf{R}^3 - \mathbf{R}$ defined by $f(x, y, z) = xy + yz + zx$, where $x = t$, $y = e^t$, $z = e^{-t}$. To find the total derivative of this function, we apply the chain rule for n variables and get

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}\\ &= (y+z)(1) + (x+z)e^t - (x+y)e^{-t}\\ &= e^t+e^{-t} + (t+e^{-t})e^t - (t+e^t)e^{-t}\\ &= (1+t)e^t + (1-t)e^{-t}\end{aligned}$$

Example 4 : Let us find the total derivative of the function $z = xy$ where $x = \cos$ $x = \cos t$, $y = t^2$.

By the chain rule we get

$$\frac{dz}{dt} = y(-\sin t) + x \cdot 2t$$

Note that we can also write

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\\ &= x\frac{dy}{dt} + y\frac{dx}{dt}\end{aligned}$$

This formula is familiar to you. This is nothing but the product rule of differentiation for functions of one-variable.

Note that in Examples 2, 3 and 4, instead of using the chain rule, we could have first substituted for x, y and/or z in terms of t, and then differentiated the resulting function w.r.t. t. Thus, for the function in Example 3, namely

$f(x, y, z) = xy+yz+zx$, where $x = t$, $y = e^t$, $z = e^{-t}$, we can write

$f(t) = te^t+e^t e^{-t} + e^{-t} t$

$= te^t + 1 + te^{-t}$

And therefore, $f'(t) = (1+t) e^t + (1-t) e^{-t}$.

You can see that this is the same as the total derivative which we have calculated in Example 3 by the chain rule.

You might be wondering why we have done so much work to discover the additional (complicated !) method for finding $\frac{dz}{dt}$. There are several reasons.

- First of all, it may not be always possible to express x or y explicitly in terms of t.
- Secondly, substituting for x and y could make the expression of z very complicated.

Thus, the evaluation of $\frac{dz}{dt}$ may become very lengthy and tedious. In our formula, we are carrying out the calculations in bits which are usually simpler than the calculations involved in the evaluation of $\frac{dz}{dt}$ after z has been expressed as a function of t.

We illustrate this by means of an example.

Example 5 : Suppose we want to find the derivative of $(\sin t)^{\tan t} + (\tan t)^{\sin t}$.
Let $F(t) = (\sin t)^{\tan t} + (\tan t)^{\sin t}$
We rewrite F(t) as $F(t) = \phi(f(t), g(t))$ where

$\phi(x, y) = x^y + y^x$, $x = f(t) = \sin t$ and $z = g(t) = \tan t$.

Then, ϕ, f and g satisfy the conditions of Theorem 2. Therefore, by chain rule

$$\frac{dF}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt}$$

$$= \left[yx^{y-1} + (\ln y)\, y^x\right]\cos t + \left[(\ln x)\, x^y + xy^{x-1}\right]\sec^2 t$$

$$= (\tan t)\frac{(\sin t)^{\tan t}}{\sin t}\cos t + (\ln \tan t)(\tan t)^{\sin t}(\cos t)$$

$$+ (\ln \sin t)(\sin t)^{\tan t}(\sec^2 t) + \sin t\,\frac{(\tan t)^{\sin t}}{\tan t}(\sec^2 t)$$

$$= \left[1 + \sec^2 t \ln \sin t\right](\sin t)^{\tan t} + \left[\cos t \ln \tan t + \sec t\right](\tan t)^{\sin t}$$

Can you imagine how long it would have taken had we done this without writing F as a composite function?

Now if you have gone through the examples carefully, you should be able to do these exercises.

E2) Find the total derivative with respect to t in each of the following cases:

a) $z = x^2 + 3xy+y^2$ if $x = 2\cos\frac{\pi t}{8}$, $y = 3 + \sin\frac{\pi t}{8}$.

b) $z = \frac{2x+3}{3y-2}$ if $x = e^t+t$, $y=e^{-t} - t$.

c) $u = xyz$ if $x = e^t$, $y = e^{-t}$, $z = t$.

d) $u = x^2+y^2+z^2+w^2$ if $x = t^2+1$, $y=2t$, $z = e^t$, $w = t^5$.

E3) Find $\frac{d}{dt}$ in each of the following cases.

a) $z = \ln (x^2+3xy)$, $x = e^t$, $y = e^{-t}$

b) $z = \tan^{-1} \frac{y}{x}$, $x = \ln t$, $y = e^t$

c) $w = e^{xy^2+yz}$, $x = t \cos t$, $y = t \sin t$, $z = \cos t + \sin t$

E4) Find the derivatives of the following functions using the concept of total derivative.

a) $t^{3\sin t}+(\sin t)^{t^2}$

b) $t^{2t}+(t+1)^{t^2}$

c) $e^{t^4} + t^{4\cos t}$

The chain rule is very useful for finding the slope of a curve given by an implicit function. Even though you have already studied the differentiation of an implicit function in Calculus, we shall briefly explain what is meant by an implicit function.

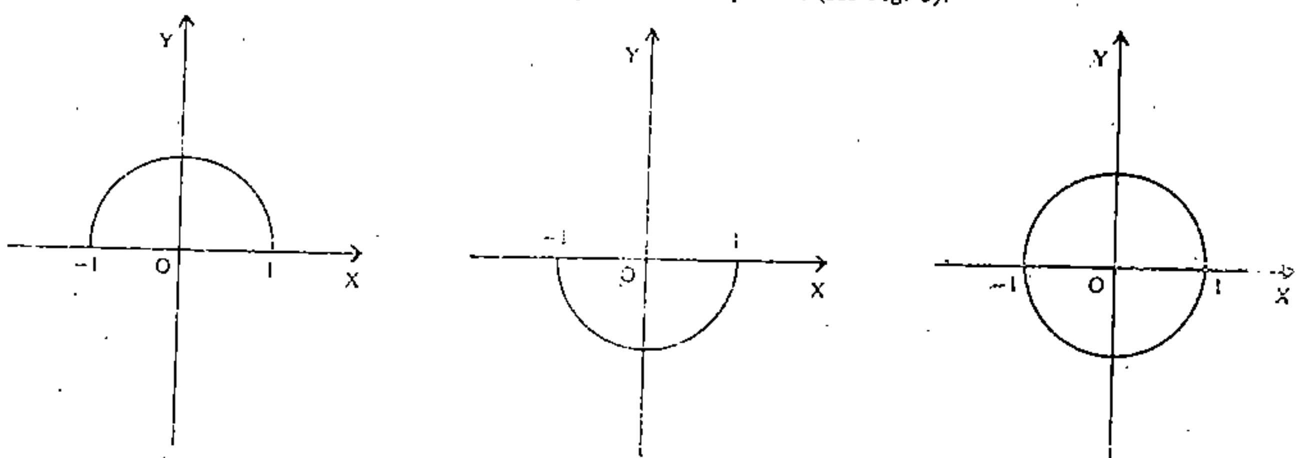
Many times we come across equations of the type $x + e^{xy} + 3xy = 0$. Given any value of x, there exists a unique value of y such that the above equation is satisfied. Thus, y is a function of x, but we cannot express it explicitly, i.e., we cannot express it in the form $y = f(x)$. In such a situation we say that y is an **implicit function** of x, defined implicitly by the given equation.

We can also apply the chain rule to get the total derivatives of some composite functions of two variables when these variables are implicitly connected.

Let us consider the equation $x^2 + y^2 - 1 = 0$. Then given any value of x, there exist two values of y which satisfy the above equation. Therefore, we cannot find a single f (x) such that $y = f(x)$, and $(x, f(x))$ gives all the points satisfying the above equation. In fact, there are two functions

$$y_1 = \sqrt{1 - x^2} \text{ and } y_2 = -\sqrt{1 - x^2}$$

which together give all the points (x, y) satisfying the above equation (see Fig. 3).



(a) (b) (c)

Fig. 3

Thus, given a real-valued function F (x, y) of two variables, we cannot hope to find a single function f (x) such that $F(x, y) = F(x, f(x)) = 0$ for all x. Later in Unit 10 you will see that under suitable conditions, given any point x_0, there does exist a function $\phi(x)$ defined in a neighbourhood of x_0 such that $F(x, \phi(x)) = 0$ for all x in the above mentioned neighbourhood. In such a situation, using the chain rule we get

$$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

In the Calculus course you have differentiated implicit functions without being aware of the exact rule involved.

and therefore,

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}, \text{ if } \frac{\partial F}{\partial y} = \neq 0$$

and $\frac{dy}{dx}$ at any point gives the slope of the plane curve given by the equation $F(x, y) = 0$.

Note that we have found $\frac{dy}{dx}$ without explicitly knowing y in terms of x.

Now let us work out some examples to illustrate this method.

Example 6 : Suppose y is an implicit function of x defined by the equation $ax^2+2hxy+by^2 = 1$ and $hx+by \neq 0$. Let us find $\frac{dy}{dx}$.

We let $f(x, y) = ax^2+2hxy+by^2 - 1$, so that $f(x, y) = 0$ represents the given implicit function.

Now, $\frac{\partial f}{\partial y} = 2hx + 2by \neq 0$, since we know that $hx + by \neq 0$.

Therefore, by Formula (*), we get

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{2ax+2hy}{2hx+2by} = -\frac{ax+hy}{hx+by}.$$

In the next example we prove a simple result for implicit functions of three variables.

Example 7 : Let $f(x, y, z) = 0$ be an equation in three variables x, y, z such that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are non-zero. We will show that

$$\left(\frac{dy}{dx}\right)_z \left(\frac{dx}{dz}\right)_y \left(\frac{dz}{dy}\right)_x = -1,$$

where $\left(\frac{dy}{dx}\right)_z$ denotes the derivative of y with respect to x when z is treated as a constant, and so on.

We first note that when z is treated as a constant, we can think of y as an implicit function of x determined by the equation $f(x, y, z) = 0$.

Therefore, by Formula (*) we get

$$\left(\frac{dy}{dx}\right)_z = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

Similarly, $$\left(\frac{dx}{dz}\right)_y = -\frac{\partial f/\partial z}{\partial f/\partial x}$$

and $$\left(\frac{dz}{dy}\right)_x = -\frac{\partial f/\partial y}{\partial f/\partial z}$$

Consequently,

$$\left(\frac{dy}{dx}\right)_z \left(\frac{dx}{dz}\right)_y \left(\frac{dz}{dy}\right)_x = -1.$$

In the next example we will consider a composite function of two variables, when these variables are implicitly connected, i.e., occur in an implicit equation.

Example 8 : Let us find $\frac{du}{dx}$ for $u = \sin(x^2 + y^2)$ where x and y satisfy the equation $a^2x^2 + b^2y^2 = c^2$.

Here u is a function of two variables x and y, where x = x and y is an implicit function of x given by $a^2x^2 + b^2y^2 = c^2$. Then u, regarded as a function of a single variable x satisfies all the conditions of Theorem 2. Hence we can write

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx} \qquad \ldots(4)$$

Now we use Theorem 1 to find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. Note that u is the composite of two functions $g(x, y) = x^2 + y^2$ and $f(t) = \sin t$. Clearly the function $u = \sin(x^2+y^2)$ satisfies the requirements of Theorem 1, since x^2+y^2 has continuous partial derivatives of first order and the function sin t is differentiable everywhere. Therefore, by Theorem 1,

$$\frac{\partial u}{\partial x} = 2x\cos(x^2+y^2), \quad \frac{\partial u}{\partial y} = 2y\cos(x^2+y^2)$$

Finally, to obtain $\frac{dy}{dx}$, we write $\phi(x, y) = a^2x^2+b^2y^2 - c^2$. Then by Formula (*),

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{a^2x}{b^2y}, \; y \neq 0.$$

Substituting the expressions for $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{dy}{dx}$ in (4),

we get,

$$\frac{du}{dx} = 2x\cos(x^2+y^2) + 2y\cos(x^2+y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2\left(1 - \frac{a^2}{b^2}\right)x\cos(x^2+y^2).$$

Why don't you try these exercises?

E5) If $y^x + x^y = a^b$, show that

$$\frac{dy}{dx} = -\frac{y^x \ln y + yx^{y-1}}{xy^{x-1} + x^y \ln x}$$

E6) If $f(x, y) = 0$, $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial \phi}{\partial y}$$

(Hint : Find $\frac{dy}{dx}$ and $\frac{dz}{dy}$ to calculate $\frac{dz}{dx}$)

E7) If A, B, C are the angles of a triangle such that $\sin^2 A + \sin^2 B + \sin^2 C = k$, prove that

$$\frac{dA}{dB} = \frac{\tan C - \tan B}{\tan A - \tan C}$$

E8) Find $\frac{du}{dx}$ in each of the following problems.

a) $u = x^2 - xy + y^2$, $y = 3x+2$
(Here u is a function of x and y, where x and y are functions of x ; x = x and y = 3x+2)

b) $u = x^2 - y^3$, $y = \ln x$

c) $u = x \ln xy$, where $x^3+y^3+3xy = 1$.

So far we have been discussing the derivatives of composite functions in some special cases. Now we shall introduce you to the most general form of the chain rule. But before stating the result, let us look at the definition of differentiability of a vector-valued function.

Let g be a vector-valued function defined in a neighbourhood N of a point $a \in \mathbf{R}^n$ with values in $\mathbf{R}^m$. We have seen that g can be expressed as $g(x) = (g_1(x), \ldots, g_m(x))$ for $x \in \mathbf{N}$, where $g_1, \ldots, g_m$ are real-valued functions determined uniquely by g. The **vector-valued function g** is said to be **differentiable at a** if each g_i is differentiable at a.

Now we state the theorem.

Theorem 3 (Chain Rule) : Let $g = (g_1, \ldots, g_m)$ be a vector-valued function of n variables with values in $\mathbf{R}^m$, which is differentiable at a point $a \in \mathbf{R}^n$. Let f be a real-valued function of m variables, having continuous first order partial derivatives at the point $g(a) = (g_1(a), \ldots, g_m(a))$ (also see Fig. 4). Then the composite function $\phi = f \circ g$ from $\mathbf{R}^n \to \mathbf{R}$ has first order partial derivatives at a point a in $\mathbf{R}^n$, given by

$$D_j \phi(a) = \sum_{k=1}^{m} D_k f(g_1(a), \ldots, g_m(a)) D_j(g_k(a)), \; j = 1, 2, \ldots, n,$$

where $\quad D_j = \frac{\partial}{\partial x_j}$ for $j = 1, 2, \ldots, n$.

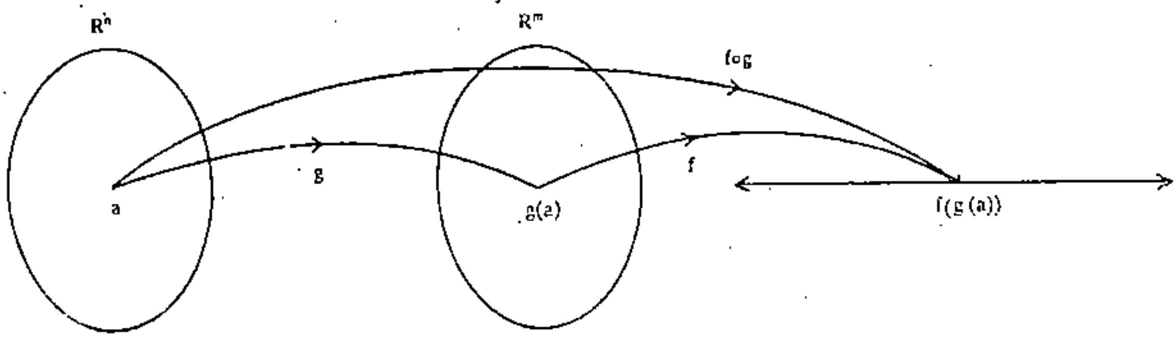


Fig. 4

This expression may appear difficult to you. But if you look at the expressions for the cases $n=1$, $m=2$ and $n=2$, $m=1$ given in Remark 1 (i) and (ii) below, you will get a clearer picture. Though we have stated the theorem for the general case, we shall consider examples of functions of two and three variables only. For this purpose, we restate Theorem 3 for the case $n=m=2$ in Remark 1 (iii).

Remark 1 : (i) If $n = 2$, $m = 1$, then we get Theorem 1. This is because, in this case, g is a function from $\mathbf{R}^2 \to \mathbf{R}$, f is a real-valued function, and $\phi = f \circ g$ is a function from $\mathbf{R}^2 \to \mathbf{R}$. Then by Theorem 3,

$$D_1\phi(a) = D_1 f(g(a)) D_1 g(a)$$

$$D_2\phi(a) = D_2 f(g(a)) D_2 g(a).$$

But, $D_1\phi = \frac{\partial \phi}{\partial x} = \phi_x$, $D_1 g = g_x$,

$$D_2\phi = \frac{\partial \phi}{\partial y} = \phi_y \text{ and } D_2 g = g_y.$$

Therefore, $\phi_x(a, b) = f'(g(a, b))\, g_x(a, b)$ and

$$\phi_y(a, b) = f'(g(a, b))\, g_y(a, b),$$

which is what Theorem 1 states.

(ii) If $n = 1$, $m = 2$, then we get Theorem 2. In this case $g = (g_1, g_2)$ is a vector-valued function from $\mathbf{R}$ to $\mathbf{R}^2$ and f is a real-valued function defined on $\mathbf{R}^2$. Then $\phi = f \circ g$ is a real-valued function defined on $\mathbf{R}$. From Theorem 3 we can write

$$D\phi(a) = D_1 f(g_1(a), g_2(a)) D(g_1(a)) + D_2 f(g_1(a), g_2(a)) D(g_2(a))$$

$$= f_x(g_1(a), g_2(a))\, g_1'(a) + f_y(g_1(a), g_2(a))\, g_2'(a),$$

which is what Theorem 2 states.

(iii) If $n = 2$, $m = 2$, then the statement of Theorem 3 becomes:
Let $g = (g_1, g_2)$ be a vector-valued function of two variables. Let f be a real-valued function of two variables, having continuous partial derivatives at the point $g(a) = (g_1(a), g_2(a))$. Then the composite function $\phi = f \circ g$ from $\mathbb{R}^2 \to \mathbb{R}$ has first order partial derivatives at the point a given by

$$D_1\phi(a) = D_1 f(g_1(a), g_2(a))\, D_1g_1(a) + D_2 f(g_1(a), g_2(a))\, D_1g_2(a) \quad \ldots(5)$$

$$D_2\phi(a) = D_1 f(g_1(a), g_2(a))\, D_2g_1(a) + D_2 f(g_1(a), g_2(a))\, D_2g_2(a) \quad \ldots(6)$$

If we put $x = f(u, v)$, $y = g(u, v)$ and $z = \phi(u, v)$ in the expressions (5) and (6), then z can be thought of as a function of the two variables x and y.
Then, using Theorem 3, we can write

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

This formula is very convenient in the two variables' case as you will see in the following examples.

Example 9 : Let $f(x, y) = x^3 - 3xy^2$, $x = s^2 - t^2$, $y = 2st$. Let us find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$.

By Remark 1 (iii),

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

$$= (3x^2 - 3y^2)(2s) + (-6xy)(2t)$$

$$= 6s\,[(s^2 - t^2)^2 - 4s^2t^2] - 12t\,(s^2 - t^2)(2st)$$

$$= 6s^5 + 6st^4 - 36s^3t^2 - 24s^3t^2 + 24st^4$$

$$= 6s\,(s^4 - 10s^2t^2 + 5t^4)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$= (3x^2 - 3y^2)(-2t) + (-6xy)(2s)$$

$$= -6t\,[(s^2 - t^2)^2 - 4s^2t^2] - 12s\,(s^2 - t^2)(2st)$$

$$= -30s^4t + 60s^2t^3 - 6t^5$$

$$= 6t\,(t^4 - 10s^2t^2 + 5s^4)$$

In the next example we consider a function of three variables, each of which is a function of two variables.

Example 10 : Suppose we want to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = u^2 + v^2 + w^2$, where $u = ye^x$, $v = xe^{-y}$, $w = \frac{y}{x}$.

As in the case of Remark 1 (iii), we can write

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial x}, \text{ and}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

Therefore,

$$\frac{\partial z}{\partial x} = 2u.\, ye^x + 2v.e^{-y} + 2w.\left(-\frac{y}{x^2}\right)$$

$$= 2y^2e^{2x} + 2xe^{-2y} - \frac{2y^2}{x^3}, \text{ and}$$

$$\frac{\partial z}{\partial y} = 2ue^x + 2v(-xe^{-y}) + 2s\left(\frac{1}{x}\right)$$

$$= 2ye^{2x} - 2x^2e^{-2y} + \frac{2y}{x^2}.$$

We now give an example which deals with higher order partial derivatives.

Example 11 : Let $x = r\cos\theta$, $y = r\sin\theta$ and V be a continuously differentiable function of x and y, whose partial derivatives are also continuously differentiable. We can show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}.$$

Here V is a function of x and y, where x and y are again functions of r and θ. Therefore, using the chain rule, we get

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial r}$$

$$= \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y} \qquad \ldots(7)$$

Now, $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are functions of x and y where both x and y are functions of r and θ. Therefore, using the chain rule once again, we get

$$\frac{\partial^2 V}{\partial r^2} = (\cos\theta)\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial x}\right) + \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial V}{\partial y}\right)$$

$$= (\cos\theta)\left(\frac{\partial^2 V}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 V}{\partial y\,\partial x}\frac{\partial y}{\partial r}\right) + (\sin\theta)\left(\frac{\partial^2 V}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 V}{\partial y^2}\frac{\partial y}{\partial r}\right)$$

$$= (\cos\theta)\left[(\cos\theta)\frac{\partial^2 V}{\partial x^2} + (\sin\theta)\frac{\partial^2 V}{\partial y\,\partial x}\right]$$

$$+ (\sin\theta)\left[(\cos\theta)\frac{\partial^2 V}{\partial x\,\partial y} + (\sin\theta)\frac{\partial^2 V}{\partial y^2}\right]$$

$$= \cos^2\theta\frac{\partial^2 V}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 V}{\partial x\,\partial y} + \sin^2\theta\frac{\partial^2 V}{\partial y^2} \qquad \ldots(8)$$

Note that $\frac{\partial^2 V}{\partial x\,\partial y} = \frac{\partial^2 V}{\partial y\,\partial x}$, since V satisfies the conditions of Theorem 1 of Unit 6.

Similarly,

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= (-r\sin\theta)\frac{\partial V}{\partial x} + (r\cos\theta)\frac{\partial V}{\partial y}, \text{ and therefore}$$

$$\frac{\partial^2 V}{\partial \theta^2} = \frac{\partial}{\partial \theta}\left(\frac{\partial V}{\partial \theta}\right)$$

$$= (-r\cos\theta)\frac{\partial V}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 V}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 V}{\partial y\,\partial x}\frac{\partial y}{\partial \theta}\right)$$

$$+ (-r\sin\theta)\frac{\partial V}{\partial y} + (r\cos\theta)\left(\frac{\partial^2 V}{\partial x\,\partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 V}{\partial y^2}\frac{\partial y}{\partial \theta}\right)$$

Or,

$$\frac{1}{r}\frac{\partial^2 V}{\partial \theta^2} = -\cos\theta\frac{\partial V}{\partial x} - \sin\theta\frac{\partial V}{\partial y} - (\sin\theta)\left(-r\sin\theta\frac{\partial^2 V}{\partial x^2} + r\cos\theta\frac{\partial^2 V}{\partial x\,\partial y}\right)$$

$$+ (\cos\theta)\left(-r\sin\theta\,\frac{\partial^2 V}{\partial x\,\partial y} + r\cos\theta\,\frac{\partial^2 V}{\partial y^2}\right)$$

$$= -\frac{\partial V}{\partial r} + r\sin^2\theta\,\frac{\partial^2 V}{\partial x^2} - 2r\sin\theta\cos\theta\,\frac{\partial^2 V}{\partial x\,\partial y} + r\cos^2\theta\,\frac{\partial^2 V}{\partial y^2}$$

after substituting from Equation (7). Consequently,

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r} = \sin^2\theta\,\frac{\partial^2 V}{\partial x^2} - 2\sin\theta\cos\theta\,\frac{\partial^2 V}{\partial x\,\partial y} + \cos^2\theta\,\frac{\partial^2 V}{\partial y^2} \qquad ...(9)$$

Adding (8) and (9), we get,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2}$$

Here are some exercises for you.

E9) Find $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$ for each of the following functions.

a) $u = x^2+xy+y^2$, $x = r+s$, $y = r-s$

b) $u = \tan^{-1}\frac{y}{x}$, $x = r+s$, $y = rs$

c) $u = \cos xy$, $x = r^2 s$, $y = e^{rs}$

E10) If $u = f(y-z, z-x, x-y)$, then prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

E11) Find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ for the following functions.

a) $w = \frac{x+y}{z}$, $x = r-2s+t$, $y = 2r+s-2t$, $z = r^2+s^2+t^2$

b) $w = xy+yz+zx$, $x = r\cos s$, $y = r\sin t$, $z = st$

E12) If $z = f(u, v)$, where $u = e^x\cos y$ and $v = e^x \sin y$, obtain $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ and show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (u^2+v^2)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2}\right)$$

In the next section we shall discuss an application of the chain rule.

7.3 HOMOGENEOUS FUNCTIONS

In this section we will consider a special class of functions of several variables, called homogeneous functions. We shall mainly describe a theorem known as Euler's theorem, which characterises homogeneous functions using the techniques of the chain rule given in the last section. But what is a homogeneous function? Let us see.

You have come across polynomials of the type $ax+by$, $2x^2+3xy+5y^2$ in various contexts. Note that each term in $ax+by$ is of degree 1, while each term in $2x^2+3xy+5y^2$ is of degree 2. Suppose we replace x by tx and y by ty in the first polynomial. Then we get $atx+bty = t(ax+by)$. Similarly replacing x by tx and y by ty in the second polynomial, we get

$$2t^2x^2 + 3tx\,ty + 5t^2y^2 = t^2(2x^2+3xy+5y^2).$$

These are the simplest examples of homogeneous polynomials of degree 1 and 2, respectively. More generally, a polynomial with real coefficients in two variables x and y is called a **homogeneous polynomial of degree h,** if each term in the polynomial is of degree h. The most general polynomial of degree h in x, y is

$$p(x, y) = \sum_{\substack{\lambda+\mu=h \\ \lambda\geq 0,\ \mu\geq 0}} a_{\lambda\mu}\, x^{\lambda} y^{\mu}$$

Here also we note that if we replace x by tx and y by ty, then we get $p(tx, ty) = t^h p(x, y)$. Thus, if p(x, y) is a homogeneous polynomial of degree h, then p(x, y) gets multiplied by t^h if x is replaced by tx and y is replaced by ty for any real number t. This phenomenon makes sense for functions other than polynomials too.
For instance, if $f(x, y) = \sqrt{x^2+y^2}$, then

$f(tx, ty) = tf(x, y)$ for all $t > 0$.

We call $\sqrt{x^2+y^2}$ a homogeneous function of degree 1. More formally, we have the following definition.

Definition 2 : Let D be a subset of $\mathbf{R}^n$ such that if $(x_1, x_2,\ldots, x_n) \in D$, then $(tx_1, tx_2,\ldots, tx_n) \in D$ for all $t > 0$. A function $f : D \to \mathbf{R}$ is said to be a **homogeneous function of degree h**, h being a real number, if

$f(tx_1, tx_2, \ldots, tx_n) = t^h f(x_1, x_2, \ldots, x_n)$ for all points $(x_1, x_2, \ldots, x_n) \in D$ and all $t > 0$.

Let us look at some examples.

Example 12: Let us show that the following functions are homogeneous functions.

i) $f(x, y) = \tan \dfrac{y}{x}$

ii) $f(x, y) = \sqrt[3]{x^4+3y^4}$

iii) $f(x, y) = \dfrac{\sin\left(\dfrac{x^2y}{x^3+y^3}\right)}{\ln\left(\dfrac{x+y}{x}\right)}$

iv) $f(x, y, z) = \dfrac{xy^2+yz^2+zx^2}{x+y+z}$

Let us take these one by one.

i) Replacing x by tx and y by ty when t is a positive real number, we get

$$f(tx, ty) = \tan\frac{ty}{tx} = \tan\frac{y}{x} = t^0 f(x, y).$$

Thus, f(x, y) is a homogeneous function of two variables of degree zero.

ii) $$f(tx, ty) = \sqrt[3]{t^4x^4+3t^4y^4} = (t^4)^{1/3}\sqrt[3]{x^4+3y^4} = t^{4/3} f(x, y)$$

Thus, f(x, y) is a homogeneous function of two variables of degree 4/3.

iii) $$f(tx, ty) = \frac{\sin\left(\dfrac{t^2x^2 \cdot txy}{t^3x^3+t^3y^3}\right)}{\ln\left(\dfrac{tx+ty}{tx}\right)}, \quad t > 0$$

$$= \frac{\sin\left(\dfrac{t^3x^2y}{t^3(x^3+y^3)}\right)}{\ln\left(\dfrac{x+y}{x}\right)} = \frac{\sin\left(\dfrac{x^2y}{x^3+y^3}\right)}{\ln\left(\dfrac{x+y}{x}\right)}$$

$$= t^0 f(x, y).$$

Thus, f(x, y) is a homogeneous function of degree zero.

iv) $f(tx, ty, tz) = \dfrac{tx.\ t^2y^2 + ty.\ t^2z^2 + tz.\ t^2x^2}{tx+ty+tz}$, $t > 0$

$$= \frac{t^3(xy^2+yz^2+zx^2)}{t(x+y+z)}$$

$$= t^2 f(x, y, z).$$

Thus, f(x, y, z) is a homogeneous function of three variables of degree two.

Did you notice that the degree of homogeneity for the functions in i), iii) and iv) is an integer, whereas the degree of the function in ii) is not an integer?

You should be able to do these exercises now.

E13) Which of the following functions are homogeneous? If a function is homogeneous, determine the degree of homogeneity.

a) $f(x, y, z, u, v, w) = \dfrac{xu + yv + zw}{\sqrt{x^2+y^2+z^2}\sqrt{u^2+v^2+w^2}}$

b) $f(x, y) = \max\left\{\dfrac{x}{y}, y\right\}$

c) $f(x, y) = \dfrac{\sin x}{\sin y}$

d) $f(x, y) = x^{1/3}\, y^{-4/3}$

e) $f(x, y) = 3x^2y + xy^2 - \pi y^3$

f) $f(x, y) = x^2y + 2xy^2 - xy - 4y^3$

We shall now state Euler's theorem, which gives a beautiful characterisation of homogeneous functions. For the sake of simplicity we shall study this theorem only for the case n = 2.

Theorem 5 (Euler's Theorem) : Let D be a subset of $\mathbf{R}^2$ such that

i) for any (x, y) $\in$ D, there exists an open disc of radius r, with centre (x, y) contained in D, and

ii) for any point (x, y) $\in$ D, the point (tx, ty) $\in$ D for all $t > 0$.

Let $f : D \to \mathbf{R}$ be a function having continuous partial derivatives of first order at all points of D. Then f(x, y) is a homogeneous function of degree h if and only if

$a f_x(a, b) + b f_y(a, b) = h f(a, b)$ for any point (a, b) in D.

Proof : Suppose f(x, y) is a homogeneous function of degree h. Define a function $F :]0, \infty[\to \mathbf{R}$ by

$F(t) = f(at, bt) = f(u(t), v(t))$,

where (a, b) is any point of D and $u(t) = at$ and $v(t) = bt$.

Regarding F as a function of two variables, x and y, where $x = u(t) = at$, $y = v(t) = bt$, you can check that F satisfies all the conditions of Theorem 2. Thus, we get

$$F'(t) = u'(t)\, f_x(u(t), v(t)) + v'(t)\, f_y(u(t), v(t))$$
$$= a\, f_x(u(t), v(t)) + b\, f_y(u(t), v(t)), \text{ since } u'(t) = a \text{ and } v'(t) = b.$$
$$= a\, f_x(at, bt) + b\, f_y(at, bt). \qquad \ldots(10)$$

But f(x, y) is a homogeneous function of degree h. Therefore

$F(t) = f(at, bt) = t^h f(a, b)$

and $F'(t) = h\, t^{h-1} f(a, b)$. ...(11)

Equating (10) and (11) we get

$a f_x(at, bt) + b\, f_y(at, bt) = h t^{h-1} f(a, b)$ for all $t > 0$. ...(12)

Putting $t=1$ in (12) we get

$a f_x(a, b) + b\, f_y(a, b) = hf(a, b)$.

Conversely, suppose that the function f(x, y) satisfies the relation

$a\, f_x(a, b) + b\, f_y(a, b) = h\, f(a, b)$ for all (a, b) in D. ...(13)

For the function F(t) defined above, we have

$F'(t) = a\, f_x(at, bt) + b\, f_y(at, bt)$,

where (a, b) is any point of D. Since $(a, b) \in D$ implies that $(at, bt) \in D$, it follows from (12) and (13) that

$at f_x(at, bt) + bt f_y(at, bt) = hf(at, bt)$.

Consequently, $tF'(t) = hf(at, bt) = h\, F(t)$

or $F'(t) = \dfrac{h}{t} F(t)$

Consider the function $\phi(t) = t^{-h} F(t)$ for $t > 0$.

Clearly $\phi'(t) = t^{-h} F'(t) - h\, t^{-h-1} F(t)$

$$= t^{-h}\left[F'(t) - \frac{h}{t} F(t)\right]$$
$$= 0 \quad \text{for all } t > 0.$$

Therefore, $\phi(t)$ is a constant function for all $t > 0$.

But $\phi(1) = F(1) = f(a, b)$. Therefore,

$\phi(t) = f(a, b)\ \forall\ t > 0$

i.e., $t^{-h} F(t) = f(a, b)$ for all $t > 0$

i.e., $F(t) = t^h f(a, b)$.

Thus, $f(at, bt) = t^h f(a, b)$ for any point $(a, b) \in D$. This means that f is a homogeneous function of degree h.

Remark 2 : If we write $z = f(x, y)$, then by Euler's theorem f(x, y) is a homogeneous function of degree n if and only if

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz.$$

This relation is known as **Euler's relation.**

Try this exercise before proceeding further.

E14) If D is a subset of $\mathbf{R}^2$ which satisfies (i) and (ii) of Theorem 5, and if $f : D \to \mathbf{R}$ is a homogeneous function of degree n, which has continuous second order partial derivatives, then $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are homogeneous functions of degree n−1.

This exercise leads us to the following result.

Corollary 1 : Let $f : D \to \mathbf{R}$ where D is a subset of $\mathbf{R}^2$ as mentioned in the statement of Theorem 5. If f is a homogeneous function of degree n and if f has continuous partial derivatives of second order at all points of D, then

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

for all points $(x, y) \in D$, where $z = f(x, y)$.

Proof : Since z is a homogeneous function of degree n and has continuous second order partial derivatives at all points of D, it follows that both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are homogeneous functions of degree n–1 (see E 14), and have continuous partial derivatives of first order at all points of D. Thus, applying Euler's theorem to the functions $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, we obtain

$$x \frac{\partial^2 z}{\partial x^2} + y \frac{\partial^2 z}{\partial y \, \partial x} = (n-1) \frac{\partial z}{\partial x} \qquad \ldots(14)$$

and

$$x \frac{\partial^2 z}{\partial x \, \partial y} + y \frac{\partial^2 z}{\partial y^2} = (n-1) \frac{\partial z}{\partial y} \qquad \ldots(15)$$

But $\frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial^2 z}{\partial y \, \partial x}$, in view of Schwarz's theorem (Theorem 2, Unit 6).

Therefore, multiplying (14) by x and (15) by y and adding we get

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = (n-1) \left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right)$$

But $\left(x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \right) = nz$. Therefore,

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \, \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

We shall now illustrate Euler's theorem with some examples.

Example 13 : We'll first show that the function $\frac{xy}{x+y}$, $x > 0$, $y > 0$ satisfies the requirements of Euler's theorem, and then verify Euler's relation by direct computation.

Let $D = \{(x, y) \mid x > 0, y > 0\}$, and $f : D \to R$ be defined by $f(x, y) = \frac{xy}{x+y}$.

Then

i) $(x, y) \in D \Rightarrow (tx, ty) \in D$ for all $t > 0$

ii) if $(a, b) \in D$, then the disc of radius $r = \frac{1}{2} \min\{a, b\}$ with centre (a, b) is contained in D. See Fig. 5

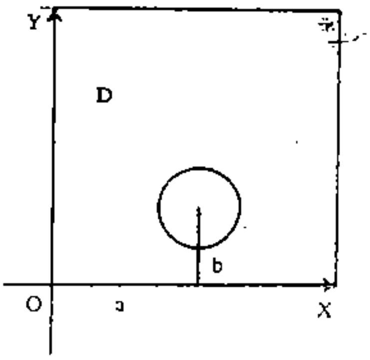


Fig. 5

Now the given function is a homogeneous function of degree 1, because

$$f(tx, ty) = \frac{t^2 xy}{t(x+y)} = tf(x, y).$$

Further, for any point $(x, y) \in D$, a simple calculation shows that

$$f_x(x, y) = \frac{y^2}{(x+y)^2} \text{ and } f_y(x, y) = \frac{x^2}{(x+y)^2}, \text{ and}$$

these are clearly continuous on D. Thus, all the requirements of Euler's theorem are satisfied.

To verify Euler's relation we have to prove that

$$x f_x(x, y) + y f_y(x, y) = 1.f(x, y)$$

Now,

$$x f_x(x, y) + y f_y(x, y) = x \frac{y^2}{(x+y)^2} + y \frac{x^2}{(x+y)^2}$$

$$= xy \left[\frac{x+y}{(x+y)^2} \right]$$

$$= \frac{xy}{(x+y)}$$

$$= 1. \quad f(x, y)$$

This proves Euler's relation.

In the next two examples we consider inverse trigonometric functions.

Example 14 : For the function $z = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ defined on

$D = \{ (x, y) \mid 0 < x < y \}$, we'll prove that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

We first note that D satisfies conditions (i) and (ii) of Euler's theorem. Let

$$f(x, y) = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}.$$

Then since

(i) $(tx, ty) \in D$ for all $(x, y) \in D$, $t > 0$, and

(ii) $f(tx, ty) = t^0 f(x, y)$ for all $(x, y) \in D$, we can say that $z = f(x, y)$ is a homogeneous function of degree 0.

Further, $$f_x(x, y) = \frac{\partial z}{\partial x} = \frac{1}{y} \left(\frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \right) - \frac{y}{x^2} \left(\frac{1}{1 + \frac{y^2}{x^2}} \right)$$

Since $(x, y) \in D$ is such that $0 < x < y$, f_x is defined and continuous for all points of D. Similarly

$$f_y(x, y) = \frac{\partial z}{\partial x} = \left(\frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \right) \left(-\frac{x}{y^2} \right) + \left(\frac{1}{1 + \frac{y^2}{x^2}} \right) \left(\frac{1}{x} \right)$$

$$= -\frac{x}{y^2} \left(\frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \right) + \frac{1}{x} \left(\frac{1}{1 + \frac{y^2}{x^2}} \right)$$

is defined and continuous for all points of D. Thus by Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

Example 15 : If $u = \sin^{-1} \frac{x^2 + y^2}{x + y}$, $0 < x < 1$, $0 < y < 1$, let us prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$$

Let $D = \{(x, y) \mid x > 0, y > 0\}$ and $f : D \to \mathbf{R}$ be defined by

$$f(x, y) = \frac{x^2 + y^2}{x + y}.$$

Then the function $z = f(x, y)$ is a homogeneous function of degree 1 and satisfies the requirements of Euler's theorem. Therefore,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z \text{ for all } (x, y) \in D. \quad \ldots(16)$$

Now take $D' = \{(x, y) \mid 0 < x < 1, 0 < y < 1\}$. Then $D' \subset D$ and hence Equation (16) is true in particular for all $(x, y) \in D'$. Also, for all $(x, y) \in D'$ we have

$\sin u = \frac{x^2+y^2}{x+y} = z.$

Therefore, $\frac{\partial z}{\partial x} = \cos u \frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y} = \cos u \frac{\partial u}{\partial y}$. Consequently, substituting these values in (16) we get

$$\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \cos u = \sin u$$

or $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u.$

Why don't you try some exercises now?

E15) If $z = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$, show that $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{1}{20} z.$

E16) Verify Euler's relation for the functions

a) $u = \frac{x^3+y^3}{x+y}$;

b) $u = \tan^{-1} \frac{y}{x}$

by direct calculation.

E17) If $z = \sin^{-1} \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, then show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 0.$$

E18) If $z = \tan^{-1} \frac{x^3+y^3}{x+y}$, then show that

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \sin 2x.$$

In the next section we shall study yet another way of defining the derivative of a function of several variables.

7.4 DIRECTIONAL DERIVATIVES

We will now introduce you to the concept of directional derivatives. First we shall explain the concept for $\mathbb{R}^2$. You will see that the partial derivatives of a function f (x, y) which you have been studying so far can be considered as directional derivatives in the directions of x and y axes.

You are already familiar with the term 'unit vector'. From Block 1, Sec. 3.2 you know that a vector (a, b) is a unit vector if $|(a, b)| = 1$. Recall that the points $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are known as unit vectors in the direction of the x and y axes, respectively.

Note that there is a one-to-one correspondence between vectors in the Cartesian plane and points in $\mathbb{R}^2$ as mentioned in Sec. 3.2.

In general, any unit vector in $\mathbb{R}^2$ is given by the vector $(\cos \theta, \sin \theta)$ where θ is the angle which the unit vector makes with the positive direction of the x-axis. If we put $\theta = 0$, then we get the unit vector $e_1 = (\cos 0, \sin 0) = (1, 0)$ in the direction of the x-axis. When we put $\theta = \pi/2$, we get the unit vector $e_2 = (0, 1)$ in the direction of the y-axis. Here we will always denote a unit vector by 'v'.

Now let $a = (a_1, a_2)$ be any point of $\mathbb{R}^2$ and $v = (\cos \theta, \sin \theta)$ be any unit vector in $\mathbb{R}^2$. Then the set $\{a + tv : t \in \mathbb{R}\} = \{(a_1 + t \cos \theta, a_2 + t \sin \theta), t \in \mathbb{R}\}$ gives all points on the line joining a and a + v. By varying 'v' we get all lines through a, i.e., lines in the direction of all the unit vectors (see Fig. 6).

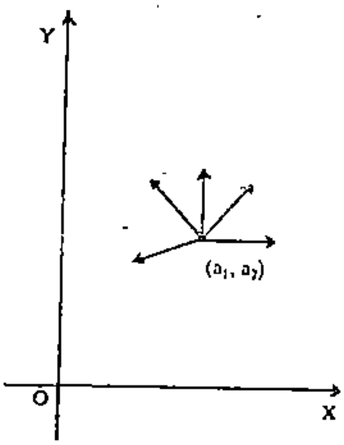


Fig. 6

Now we are in a position to define directional derivatives.

Definition 3 : Let f (x, y) be a real-valued function defined on an open disc S (a, r) with centre a = (a_1, a_2) in $\mathbf{R}^2$ and let v = (cos θ, sin θ) be a unit vector. If

$$\lim_{t\to 0} \frac{f(a_1 + t\cos\theta, a_2 + t\sin\theta) - f(a_1, a_2)}{t}$$

exists, then we say that f has a directional derivative at a in the direction of v and the value of the limit is called the **directional derivative** of f at a in the direction of v. We denote the directional derivative of f at the point a in the direction v = (cos θ, sin θ) by $f_v(a)$ or $D_\theta f(a)$.

Remark 3 : (i) Note that for |t| < r, the point

$(a_1 + t\cos\theta, a_2 + t\sin\theta) \in S(a, r)$,

which is the domain of f. Hence the function

$$\phi(t) = \frac{f(a_1 + t\cos\theta, a_2 + t\sin\theta) - f(a_1, a_2)}{t}$$

is defined for all t such that |t| < r. Therefore we can talk about its limit as $t \to 0$.

All points a + tv, where |t| < r, belong to S(a, r), since |a + tv − a| = |tv| = |t| |v| = |t| < r. In particular, if v = (cos θ, sin θ), then $(a_1 + t\cos\theta, a_2 + t\sin\theta) \in S(a, r)$.

ii) The directional derivative in the direction of v = (cos θ, sin θ) when θ = 0 is

$$\lim_{t\to 0} \frac{f(a_1 + t, a_2) - f(a_1, a_2)}{t}.$$

This is nothing but f_x, the partial derivative with respect to x. Thus, the directional derivative in the direction of the x axis is the same as the partial derivative w.r.t. x.

iii) Similarly when $\theta = \frac{\pi}{2}$, we get the directional derivative in the direction of the y-axis, given by

$$\lim_{t\to 0} \frac{f(a_1, a_2 + t) - f(a_1, a_2)}{t},$$

which is nothing but the partial derivative f_y.

iv) We can give a geometric interpretation of the directional derivatives at a point a. In Sec. 5.2.2 you have already seen the geometric interpretation of the partial derivatives f_x and f_y, which are, respectively, the directional derivatives in the directions θ = 0 and $\theta = \frac{\pi}{2}$. f_x (a, b) gives the slope of the tangent at the point (a, b, f (a, b)) to the curve which is the intersection of the surface z = f(x, y) and the plane x = a. Directional derivatives have a similar interpretation which we now give.

Consider a point a = (a_1, a_2) and the vector v = (cos θ, sin θ). The directional derivative in the direction of v represents the slope of the curve C which is the

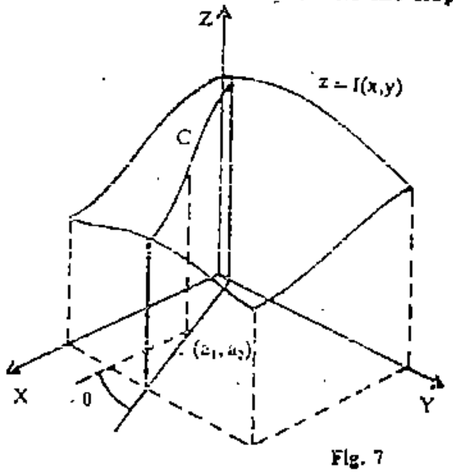


Fig. 7

intersection of the surface $z = f(x, y)$ with the plane parallel to the z axis containing the line through (a_1, a_2) in the direction of the unit vector v. See Fig. 7.

v) In the vector notation the directional derivative in the direction v is written as

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}.$$

Using Remark 3(v), we now extend the notion of directional derivatives to functions of several variables.

Definition 4 : Let f(x) be a real-valued function of n-variables defined in an open sphere S(a, r) with centre $a = (a_1, a_2, \ldots, a_n)$ and radius r. Let $v = (v_1, \ldots, v_n)$ be a unit vector in $\mathbf{R}^n$, i.e., $\Sigma v_i^2 = 1$. If

$$\lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

exists, then we say that f has a directional derivative at a in the direction of the unit vector v. The value of the limit is called the **directional derivative of f at a in the direction of v.** We shall denote the directional derivative by $f_v(a)$.

Note that as in the case of two variables, all the points $a+tv$ with $|t| < r$ belong to S(a, r), since

$$|a+tv - a| = |tv| = |t| < r.$$

Hence the function $\phi(t) = \frac{f(a+tv) - f(a)}{t}$ is defined for all t such that $|t| < r$, and therefore we can talk about its limit as $t \to 0$.

We shall now give some examples.

Example 16 : Let $f(x, y) = \begin{cases} \frac{x^3 - y^3}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Then let us show that f(x, y) has directional derivatives in all directions at (0, 0) and each one of them is a linear combination of $f_x(0, 0)$ and $f_y(0, 0)$.

Let $v = (\cos\theta, \sin\theta)$ be any unit vector. Then

$$\phi(t) = \frac{f(t\cos\theta, t\sin\theta) - f(0, 0)}{t}$$

$$= \frac{\dfrac{t^3\cos^3\theta - t^3\sin^3\theta}{t^2\cos^2\theta + t^2\sin^2\theta} - 0}{t}$$

$$= \cos^3\theta - \sin^3\theta,$$

and therefore $\lim_{t \to 0} \phi(t) = \cos^3\theta - \sin^3\theta$.

Consequently, $f_v(0, 0) = \cos^3\theta - \sin^3\theta$.
Now in Example 14 of Unit 5 we have already seen that for this function,

$$f_x(0, 0) = 1 \text{ and } f_y(0, 0) = -1.$$

Therefore, we can write

$$f_v(0, 0) = \cos^3\theta - \sin^3\theta$$

$$= \cos^3\theta \cdot f_x(0, 0) + \sin^3\theta \cdot f_y(0, 0).$$

This shows that at (0, 0) the directional derivative f_v is a linear combination of f_x and f_y.

Now you can do these exercises easily.

E19) Find the directional derivative of each of the following functions at the specified point and for the specified direction.

a) $u = x^2 - xy + y^2$, $(3, 1)$, $\theta = \frac{\pi}{3}$

b) $u = e^y \cos x$, $\left(\frac{\pi}{2}, 0\right)$, $\theta = -\frac{\pi}{6}$

E20) Prove that the function,

$$f(x, y) = \begin{cases} \frac{2xy^2}{x^2+y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

has all the directional derivatives at (0, 0).

E21) Find the directions in which the function f defined by

$$f(x, y) = \begin{cases} \frac{2xy^2}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & \text{otherwise}. \end{cases}$$

has directional derivatives at (0, 0).

We did not discuss the concept of a directional derivative earlier and instead, lai all the stress on the study of partial derivatives, which are, in fact, a particular case of directional derivatives. This was because in most of the cases, the existen of partial derivatives implies the existence of all the directional derivatives. Moreover, if we know the partial derivatives at a point, then all the directional derivatives can be calculated easily. In fact, in Theorem 6 we will prove, for $n =$ that the directional derivatives at a point are linear combinations of the partial derivatives at that point.

This result is also true for $n > 2$.

Theorem 6 : If a real-valued function $f(x, y)$ is differentiable at a point (a, b), th all the directional derivatives of f exist at (a, b). Further, they are linear combinations of $f_x(a, b)$ and $f_y(a, b)$.

Proof : The hypothesis that $f(x, y)$ is differentiable at (a, b) implies that $f(x, y)$ defined in a neighbourhood N of (a, b). Let $(\cos \alpha, \sin \alpha)$ be any unit vector in $\mathbf{R}^2$. If N is an open disc of radius r, then the points $(a+\delta \cos \alpha, b+\delta \sin \alpha)$ for all δ with $|\delta| < r$ belong to N as we have observed earlier. (See margin remark c p. 88). Since $f(x, y)$ is differentiable at (a, b),

$$f(a+\delta \cos \alpha, b+\delta \sin \alpha) - f(a, b)$$

$$= \delta \cos \alpha\, f_x(a, b) + \delta \sin \alpha\, f_y(a, b) + \delta \cos \alpha\, \phi_1(\delta, \alpha) + \delta \sin \alpha\, \phi_2(\delta, \alpha), \quad \ldots(.$$

where ϕ_1 and ϕ_2 are functions of δ and α which tend to 0 as $(\delta \cos \alpha, \delta \sin \alpha) \to$ Now as $\delta \to 0$, $(\delta \cos \alpha, \delta \sin \alpha) \to 0$ which, in turn, implies that the last two terms in (17) tend to zero. This is because

$$|\cos \alpha\, \phi_1 + \sin \alpha\, \phi_2| \leq |\phi_1| + |\phi_2|,$$

and both $\phi_1, \phi_2 \to 0$ as $(\delta \cos \alpha, \delta \sin \alpha) \to 0$. Thus, we have

$$\lim_{\delta \to 0} \frac{f(a+\delta \cos \alpha, b+\delta \sin \alpha) - f(a, b)}{\delta}$$

$$= \cos \alpha\, f_x(a, b) + \sin \alpha\, f_y(a, b).$$

Hence $f(x, y)$ has all the directional derivatives at (a, b) and

$$f_\alpha(a, b) = D_\alpha f(a, b) = \cos \alpha\, f_x(a, b) + \sin \alpha\, f_y(a, b)$$

for any unit vector $(\cos \alpha, \sin \alpha)$.

Corollary 2 : If $f(x, y)$ has continuous partial derivatives of first order at (a, b), then $f(x, y)$ has all the directional derivatives at (a, b) and

$D_\alpha f(a, b) = \cos \ldots$

for any unit vector $(\cos \alpha, \sin \alpha)$

Proof : The hypothesis implies ... Theorem 7, Unit 5). Therefore the result ...

We give an example now which shows ... compute the directional derivatives easily by applying Theorem 7.

Example 17 : Let $f(x, y) = 4x^2 - xy + 3y^2$. ... find the directional derivative of f at $(1, -2)$ in the direction $\alpha = \frac{\pi}{3}$.

We first note that the given function $f(x, y) = 4x^2 - xy + 3y^2$ is differentiable at $(1, -2)$, since it is a polynomial in x and y. Also

$f_x(x, y) = 8x - y$ and $f_y(x, y) = -x + 6y$

Therefore, $f_x(1, -2) = 8 + 2 = 10$ and $f_y(1, -2) = -1 - 12 = -13$.

Then by applying Theorem 7 we get the directional derivative

$$D_{\pi/3} f(1, -2) = \cos \frac{\pi}{3} f_x(1, -2) + \sin \frac{\pi}{3} f_y(1, -2)$$

$$= \frac{1}{2} \cdot 10 + \frac{\sqrt{3}}{2} (-13)$$

$$= 5 - \frac{13\sqrt{3}}{2}$$

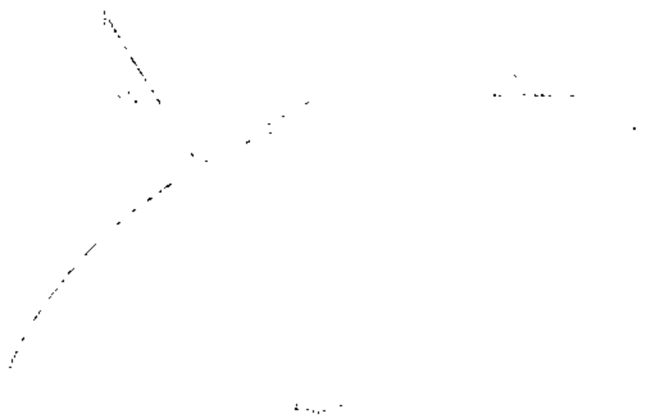
$$= \frac{10 - 13\sqrt{3}}{2}$$

Among all the directional derivatives of a ... function, those in the directions of the x and y axes, i.e. the partial derivatives ... play an important role. Because of this ... the gradient of the function ... It is

$\nabla f = (f_x, f_y)$

... that the function ... derivatives ... Geometrically, we ... following way ...

... the function ... level curve. Then the ... (see Fig. ...)



The notion of the gradient ... in the study of electro magnetic phenomena ... go into the details of the study of the ... we are not ... vector calculus, which you may not be ... techniques of ...

... don't you try some exercises now?

E22) Find the directional derivatives of each of the following functions at the specified point for the specified direction, using Theorem 7.

a) $u = x^2+y^2-4, \quad (2, -1), \theta = \frac{\pi}{4}$

b) $u = e^{x+y} - e^{-x-y}, \quad (\ln 3, \ln 2), \theta = \frac{\pi}{4}$

c) $u = \tan x + \sec y, \quad \left(\frac{\pi}{4}, \frac{\pi}{3}\right), \theta = \frac{\pi}{2}$

E23) Let $f(x, y)$ be a real-valued function which is defined in a neighbourhood of (a, b). Let $F(t) = f(a+t\cos\theta, b+t\sin\theta)$ for any θ. Prove that $f(x, y)$ has a directional derivative at (a, b) in the direction $v = (\cos\theta, \sin\theta)$ if and only if $F(t)$ is differentiable at 0 and the directional derivative $D_\theta f(x, y)$ at (a, b) is given by $F'(0)$.

E24) If $f(x, y)$ has all the directional derivatives at a point (a, b), then does it imply that f is continuous? Give reasons.

Now let us briefly recall the main points discussed in this unit.

7.5 SUMMARY

In this unit, we have:

1) Used the following forms of chain rule to differentiate composite functions:

Rule 1: If $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R}^2 \to \mathbf{R}$ and $\phi = f \circ g$, then the partial derivatives of the composite function $\phi = f \circ g : \mathbf{R}^2 \to \mathbf{R}$ are given by

$\phi_x (a, b) = g'(f(a, b)) f_x (a, b)$

$\phi_y (a, b) = g'(f(a, b)) f_y (a, b)$

Rule 2: If $f: \mathbf{R}^2 \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}^2$ and $\phi = f \circ g$, then

$\phi'(t_0) = f'(t_0) \phi_x (f(t_0), g(t_0)) + g'(t_0) \phi_y (f(t_0), g(t_0))$

Rule 3: If $f: \mathbf{R}^2 \to \mathbf{R}^2$, $g: \mathbf{R}^2 \to \mathbf{R}^2$ and $\phi = f \circ g$, then

$\phi_x (a) = D_1 \phi(a) = D_1 f(g_1 (a), g_2 (a)) D_1 g_1(a) + D_2 f(g_1 (a), g_2 (a)) D_1 g_2 (a)$

$\phi_y (a) = D_2 \phi(a) = D_1 f(g_1 (a), g_2 (a)) D_2 g_1 (a) + D_2 f(g_1 (a), g_2 (a)) D_2 g_2 (a)$

2) Studied the notion of the total derivative of functions of several variables. Suppose z is a real-valued function of n variables $x_1, x_2, \ldots, x_n$ where each x_i is a function of t, then the total derivative of z is given by

$$\frac{dz}{dt} = \sum_{i=1}^{n} \frac{\partial z}{\partial x_i} \frac{dx_i}{dt}$$

3) Defined homogeneous functions of several variables : f is a homogeneous function of degree n if

$f(tx_1, tx_2, \ldots, tx_n) = t^n f(x_1, x_2, \ldots, x_n) \quad \forall\, t > 0$, (n is a real number).

4) Proved and applied Euler's theorem for homogeneous functions.

5) Defined and evaluated the directional derivatives of a two-variable function

$$f_v(a_1, a_2) = D_\theta f(a_1, a_2) = \lim_{t \to 0} \frac{f(a_1+t\cos\theta, a_2+t\sin\theta) - f(a_1, a_2)}{t}$$

where $a = (a_1, a_2)$ and $v = (\cos\theta, \sin\theta)$.

6) Established a relationship between the directional derivatives and the partial derivatives of a function of two variables:

$D_\theta f(a_1, a_2) = \cos\theta\, f_x (a_1, a_2) + \sin\theta\, f_y (a_1, a_2)$.

7.6 SOLUTIONS AND ANSWERS

E1) a) By applying Theorem 1, we have

$$\frac{\partial\phi}{\partial x}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right) = g'\left[f\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)\right]\cdot\frac{\partial f}{\partial x}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)$$

$$= -\sin\left(\frac{\pi}{2}+3\,\frac{\pi}{2}+\frac{\pi}{2}\right)\times\left(2\sqrt{\frac{\pi}{2}}+3\sqrt{\frac{\pi}{2}}\right)$$

$$= -5\sqrt{\frac{\pi}{2}}$$

$$\frac{\partial\phi}{\partial y}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right) = g'\left[f\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)\right]\times\frac{\partial f}{\partial y}\left(\sqrt{\frac{\pi}{2}},\sqrt{\frac{\pi}{2}}\right)$$

$$= -\left(3\sqrt{\frac{\pi}{2}}+2\sqrt{\frac{\pi}{2}}\right)$$

$$= -5\sqrt{\frac{\pi}{2}}$$

E2) a) $$\frac{dz}{dt} = \frac{-\pi}{8}(2x+3y)\sin\frac{\pi t}{8} + \frac{\pi}{8}(3x+2y)\cos\frac{\pi t}{8}$$

$$= \frac{\pi}{8}\left[12\cos\frac{\pi t}{8} - 13\sin\frac{\pi t}{8} + 3\left(\cos^2\frac{\pi t}{8} - \sin^2\frac{\pi t}{8}\right)\right]$$

b) $$\frac{dz}{dt} = \frac{2}{3y-2}(e^t+1) + (2x+3)\left(-\frac{3}{(3y-2)^2}\right)(-e^{-t}-1)$$

$$= \frac{(2e^t+2)(3e^{-t}-3t-2) + (2e^t+2t+3)(3e^{-t}+3)}{(3e^{-t}-3t-2)^2}$$

$$= \frac{17-6te^t+6te^{-t}+2e^t+15e^{-t}}{(3e^{-t}-3t-2)^2}$$

c) $$\frac{du}{dt} = yz.\,e^t - xz\,e^{-t} + xy.\,1$$

$$= t.e^{-t}.\,e^t - te^t\,e^{-t} + e^t\,e^{-t}$$

$$= 1.$$

d) $$\frac{du}{dt} = 2x.\,2t + 2y.2 + 2z.\,e^t + 2w.\,5t^4$$

$$= 2(t^2+1)\,2t + 2(2t).\,2 + 2e^t.\,e^t + 2t^5.\,5t^4$$

$$= 10t^9+4t^3+12t+2e^{2t}$$

E3) a) $$\frac{dz}{dt} = \frac{1}{x^2+3yx}.\,(2x+3y)\,e^t + \frac{1}{x^2+3xy}.\,3x(-e^{-t})$$

$$= \frac{(2e^t+3e^{-t})\,e^t - 3e^t\,e^{-t}}{e^{2t}+3e^t.\,e^{-t}}$$

$$= \frac{2e^{2t}}{e^{2t}+3}$$

b) $$\frac{dz}{dt} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right).\frac{1}{t} + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right).e^t$$

$$= \frac{-e^t\frac{1}{t} + e^t.\ln t}{(\ln t)^2+e^{2t}} = \frac{e^t(t\ln t - 1)}{t\,[(\ln t)^2+e^{2t}]}$$

c) $$\frac{dw}{dt} = e^{xy^2+yz}.\,y^2(\cos t - t\sin t) + e^{xy^2+yz}(2xy+z)(\sin t + t\cos t)$$

$$+ e^{xy^2+yz}.y(-\sin t + \cos t)$$

$$= e^{[t^3\cos t\sin^2 t + t\sin t(\cos t+\sin t)]}$$

$$[t^2\sin^2 t\cos t - t^3\sin^3 t + (2t^2\sin t\cos t + \sin t + \cos t)$$

$$(\sin t + t\cos t) - t\sin^2 t + t\sin t\cos t]$$

E4) a) Let $f(x, y) = x^y + y^x$, where $x = t^3$, $y = \sin t$ so that the function f v considered as a function of t gives us the function whose derivative we wish to find out. Thus,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= [yx^{y-1} + \ln x.\,y^x]\,3t^2 + [\ln y.\,x^y + xy^{x-1}]\cos t$$

$$= [(\sin t)\,t^{3(\sin t-1)} + \ln t^3.\,(\sin t)^{t^3}]\,3t^2$$

$$+ [\ln\sin t.\,t^{3\sin t} + t^3(\sin t)^{t^3-1}]$$

b) Let $f(x, y) = x^{y-1} + y^x$, where $x = t^2$, $y = t+1$

Therefore, $$\frac{df}{dt} = [(y-1)x^{y-2} + y^x\ln y]\,2t + [\ln x.\,x^{y-1} + xy^{x-1}]$$

$$= [t.t^{2(t-1)} + (t-1)^{t^2}\ln(t+1)]\,2t$$

$$+ [\ln t^2.\,t^{2t} + t^2(t+1)^{t^2-1}]$$

$$= 2t^{2t} + 2t(t+1)^{t^2}\ln(t+1) + t^{2t}\ln t^2 + t^2(t+1)^{t^2-1}$$

c) Let $f(x, y) = e^x + x^y$, where $x = t^4$, $y = \cos t$.

Then, $$\frac{df}{dt} = [e^x + yx^{y-1}](4t^3) + (x^y\ln x)(-\sin t)$$

$$= 4t^3 e^{t^4} + 4t^3\cos t\;t^{4(\cos t-1)} - \sin t.\,t^{4\cos t}\ln t^4$$

E5) Let $f(x, y) = y^x + x^y - a^b = 0$.

$$\therefore \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{y^x\ln y + yx^{y-1}}{xy^{x-1} + x^y\ln x}$$

E6) $$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y},\quad \frac{dz}{dy} = -\frac{\partial\phi/\partial y}{\partial\phi/\partial z}$$

$$\therefore \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{\partial\phi/\partial y}{\partial\phi/\partial z}.\frac{\partial f/\partial x}{\partial f/\partial y}$$

$$\therefore \frac{\partial f}{\partial y}\frac{\partial\phi}{\partial z}\frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{\partial\phi}{\partial y}$$

E7) Since A, B, C are the angles of a triangle,

$A+B+C = \pi$. If we let

$$f(A, B) = \sin^2 A + \sin^2 B + \sin^2 C - k$$
$$= \sin^2 A + \sin^2 B + \sin^2(\pi - A - B) - k$$
$$= \sin^2 A + \sin^2 B + \sin^2(A+B) - k,$$

then $f(A, B) = 0$. So we can think of A as an implicit function of B.

Consequently,

$$\frac{dA}{dB} = -\frac{\partial f/\partial B}{\partial f/\partial A}$$

$$= -\frac{2\sin B\cos B + 2\sin(A+B)\cos(A+B)}{2\sin A\cos A + 2\sin(A+B)\cos(A+B)}$$

$$= -\frac{\sin 2B + \sin 2(A+B)}{\sin 2A + \sin 2(A+B)}$$

$$= -\frac{\sin 2B - \sin 2C}{\sin 2A - \sin 2C}, \text{ since } A + B + C = \pi$$

$$= -\frac{2\sin(B-C)\cos(B+C)}{2\sin(A-C)\cos(A+C)}$$

$$= -\frac{\cos A\,(\sin B\cos C - \sin C\cos B)}{\cos B\,(\sin A\cos C - \sin C\cos A)}$$

$$= -\frac{\tan B - \tan C}{\tan A - \tan C}$$

$$= -\frac{\tan C - \tan B}{\tan A - \tan C}$$

E8) In all these problems (a), (b) and (c), u can be treated as a function of x and y where x = x and y is an implicit function of x and therefore

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\frac{dx}{dx} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

a) $\frac{du}{dx} = 2x - y + (-x+2y)(3) = 5y - x$

b) $\frac{du}{dx} = 2x - 3y^2\,\frac{1}{x} = \frac{2x^2 - 3y^2}{x}$

c) $\frac{du}{dx} = \ln xy + 1 + \frac{x}{y}\,\frac{dy}{dx}$

If $\phi(x, y) = x^3 + y^3 + 3x^2y - 1$, then

$$\frac{dy}{dx} = -\frac{\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{3x^2 + 6xy}{3y^2 + 3x^2} = -\frac{x^2 + 2xy}{y^2 + x^2}$$

Therefore, $\frac{du}{dx} = 1 + \ln xy - \frac{x}{y}\,\frac{x^2 + 2xy}{y^2 + x^2}$

$$= 1 + \ln xy - \frac{x^2(x+2y)}{y(y^2+x^2)}$$

E9) a) $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = (2x+y)(1) + (x+2y)(1)$

$= 3x + 3y$

$= 3(r+s) + 3(r-s) = 6r$

$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (2x+y)(1) + (x+2y)(-1)$

$= x - y$

$= r + s - (r - s) = 2s.$

b) $\frac{\partial u}{\partial r} = \frac{1}{1 + y^2/x^2}\left(-\frac{y}{x^2}\right).1 + \frac{1}{1 + y^2/x^2}\left(\frac{1}{x}\right)s$

$$= \frac{-rs + (r+s)s}{(r+s)^2 + r^2s^2} = \frac{s^2}{(r+s)^2 + r^2s^2}$$

$$\frac{\partial u}{\partial s} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right).1 + \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right).r$$

$$= \frac{-y+xr}{x^2+y^2} = \frac{r^2}{(r+s)^2+r^2s^2}$$

c) $$\frac{\partial u}{\partial r} = -y \sin xy.\ 2rs - x \sin xy.\ se^{rs}$$

$$= [-2rs\, e^{rs} - r^2s^2e^{rs}] \sin (r^2se^{rs})$$

$$= -(2+rs)\, rs\, e^{rs} \sin (r^2 s\, e^{rs})$$

$$\frac{\partial u}{\partial s} = -y \sin xy.\ r^2 - x \sin xy.\ re^{rs}$$

$$= -[r^2+r^3 s]\, e^{rs} \sin (r^2 se^{rs})$$

$$= -(r+s)\, r^2e^{rs} \sin (r^2se^{rs})$$

E10) Let $u = f(t, r, s)$ where $t = y-z$, $r = z-x$, $s = x-y$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial f}{\partial t}\cdot 0 + \frac{\partial f}{\partial r}\,.\,(-1) + \frac{\partial f}{\partial s}(1) = \frac{\partial f}{\partial s} - \frac{\partial f}{\partial r}$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial t}\cdot 1 + \frac{\partial f}{\partial r}\cdot 0 + \frac{\partial f}{\partial s}(-1) = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}$$

$$\frac{\partial u}{\partial z} = \frac{\partial f}{\partial t}\cdot(-1) + \frac{\partial f}{\partial r}\cdot 1 + \frac{\partial f}{\partial s}\cdot 0 = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

E11) a) $$\frac{\partial w}{\partial r} = \frac{1}{z}\cdot 1 + \frac{1}{z}\cdot 2 - \frac{x+y}{z^2}\, 2r$$

$$= \frac{3(r^2+s^2+t^2) - 2r\,(r-2s+t+2r+s-2t)}{(r^2+s^2+t^2)^2}$$

$$= \frac{3s^2+3t^2-3r^2+2rs+2rt}{(r^2+s^2+t^2)^2}$$

$$\frac{\partial w}{\partial s} = \frac{1}{z}\cdot(-2) + \frac{1}{z}\cdot 1 - \frac{x+y}{z^2}\cdot 2s$$

$$= \frac{-(r^2+s^2+t^2) - 2s\,(r-2s+t+2r+s-2t)}{(r^2+s^2+t^2)^2}$$

$$= \frac{s^2-r^2-t^2+2st-6rs}{(r^2+s^2+t^2)^2}$$

$$\frac{\partial w}{\partial t} = \frac{1}{z}\cdot 1 + \frac{1}{z}(-2) - \frac{x+y}{z^2}\cdot 2t$$

$$= \frac{-(r^2+s^2+t^2) - 2t(r-2s+t+2r+s-2t)}{(r^2+s^2+t^2)^2}$$

$$= \frac{t^2-r^2-s^2+2ts-6rt}{(r^2+s^2+t^2)^2}$$

b) $$\frac{\partial w}{\partial r} = (y+z)\cos s + (x+z)\sin t + (x+y)\,.\,0$$

$$= (r\sin t+st)\cos s + (r\cos s+st)\sin t$$

$$= 2r\sin t\cos s + st\,(\cos s + \sin t)$$

$$\frac{\partial w}{\partial x} = (y+z)(-r\sin s) + (x+z)\cdot 0 + (x+y)\,t$$

$$= (r\sin t + st)(-r\sin s) + (r\cos s + r\sin t)\,t$$

$= -r^2 \sin t \sin s - rst \sin s + rt \cos s + rt \sin t$

$\frac{\partial w}{\partial t} = (y+z).\ 0 + (x+z)\ (tr\cos t) + (x+y) \cdot s$

$= (r\cos s + st)\ (tr\cos t) + (r\cos s + r\sin t)s$

$= r^2 \cos s \cos t + rst \cos t + rs \cos s + rs \sin t.$

E12) $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}(e^x \cos y) + \frac{\partial f}{\partial v}(e^x \sin y)$

$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u}(-e^x \sin y) + \frac{\partial f}{\partial v}(e^x \cos y)$

$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}(e^x \cos y)^2 + \frac{\partial f}{\partial u}(e^x \cos y) + \frac{\partial^2 f}{\partial v^2}(e^x \sin y)^2 + \frac{\partial f}{\partial v}(e^x \sin y)$

$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}(-e^x \sin y)^2 + \frac{\partial f}{\partial u}(-e^x \cos y) + \frac{\partial^2 f}{\partial v^2}(e^x \cos y)^2 + \frac{\partial f}{\partial v}(-e^x \sin y)$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial u^2} e^{2x} + \frac{\partial^2 f}{\partial v^2} e^{2x}$

$= (u^2+v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right)$, since $u^2 + v^2 = e^{2x}$

E13) a) $f(tx, ty, tz, tu, tv, tw) = \frac{t^2 (xu+yv+zw)}{t^2 \sqrt{x^2+y^2+z^2}\ \sqrt{u^2+v^2+w^2}}$

$= \frac{(xu+yv+zw)}{\sqrt{x^2+y^2+z^2}\ \sqrt{u^2+v^2+w^2}}$

$= t^0 f(x, y, z, u, v, w).$

$\therefore$ f is a homogeneous of degree 0.

b) For $t > 0$, $f(tx, ty) = \max\left\{\frac{tx}{ty}, ty\right\}$

$= \max\left\{\frac{x}{y}, ty\right\}$

which need not be equal to $t \max\left\{\frac{x}{y}, y\right\} = t f(x, y)$.

check with $x = 2$, $y = 1$ and $t = 2$,

$\therefore$ f is not homogeneous.

c) Since $f(tx, ty) = \frac{\sin tx}{\sin ty} \neq t^n \frac{\sin x}{\sin y} = t^n f(x, y)$, for any n,

f is not homogeneous.

Check with $x = \pi$, $y = \frac{\pi}{2}$, $t = \frac{1}{2}$.

d) $f(tx, ty) = (tx)^{1/3}(ty)^{-5/3}$

$= t^{-4/3} x^{1/3} y^{-5/3}$

$= t^{-4/3} f(x, y)$

Thus, f is a homogeneous function of degree -4/3.

e) $f(tx, ty) = t^3(3x^2y + xy^2 - \pi y^3)$

$= t^3 f(x, y)$

Thus, f is a homogeneous function of degree 3.

f) f is not homogeneous.

E14) Since f is a homogeneous function of degree n, and has continuous first order partial derivatives, we can apply Euler's theorem to f. Therefore, w get,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Differentiating the above equation with respect to x, we get

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial f}{\partial x} + y\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = n\frac{\partial f}{\partial x}$$

Since f admits continuous partial derivatives of second order, in view of Schwarz's theorem (see Unit 6), we get

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

Consequently, we have

$$x\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (n-1)\frac{\partial f}{\partial x}$$

Therefore, by using Euler's theorem we get that $\frac{\partial f}{\partial x}$ is a homogeneous function of degree n−1. Similarly, we can show that $\frac{\partial f}{\partial y}$ is also a homogeneous function of degree n−1.

E15) Note that z is a homogeneous function of degree $\frac{1}{20}$ and so Euler's theorem gives $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{20}z$.

E16) a) Note that u is a homogeneous function of degree 2. We have to verify Euler's relation $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$. Now,

i) $$\frac{\partial u}{\partial x} = \frac{3x^2(x+y) - (x^3+y^3)}{(x+y)^2} = \frac{2x^3+3x^2y-y^3}{(x+y)^2}$$

ii) $$\frac{\partial u}{\partial y} = \frac{3y^2(x+y) - (x^3+y^3)}{(x+y)^2} = \frac{-x^3+3xy^2+2y^3}{(x+y)^2}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^4+2x^3y+2xy^3+2y^4}{(x+y)^2}$$

$$= 2\cdot\frac{x^3+y^3}{(x+y)} = 2u.$$

b) u is a homogeneous function of degree 0. Now,

$$\frac{\partial u}{\partial x} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+y^2/x^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}.$$ Thus

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.u.$$ This proves Euler's relation.

E17) Let $D = \{(x, y) \mid x > 0, y > 0\}$

Then D satisfies the requirements (i) and (ii) of Euler's theorem. Moreov the function

$$z(x, y) = \sin^{-1}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)$$

is a homogeneous function of degree 0 and has continuous partial deriva on D. Therefore by applying Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

E18) Let $D = \{(x, y) \mid x \neq y\}$ and $f : D \to \mathbb{R}$ such that

$$f(x, y) = \frac{x^3+y^3}{x-y}$$

Then f is a homogeneous function of degree 2 and has continuous partial derivatives of first order.

Therefore, in view of Euler's Theorem, $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 2f$ ...(*)

Now we have $\tan z = f$. Then $\dfrac{\partial f}{\partial x} = \sec^2 z \cdot \dfrac{\partial z}{\partial x}$, $\dfrac{\partial f}{\partial y} = \sec^2 z \dfrac{\partial z}{\partial y}$

Substituting for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in (*), we get

$$x \sec^2 z \frac{\partial z}{\partial x} + y \sec^2 z \frac{\partial z}{\partial y} = 2 \tan z$$

i.e., $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \sin 2z.$

E19) a) $$\phi(t) = \frac{f\left(3+t\cos\frac{\pi}{3},\ 1+t\sin\frac{\pi}{3}\right) - f(3, 1)}{t}$$

$$= \frac{1}{t}\left[\left(3 + t\cos\frac{\pi}{3}\right)^2 - \left(3 + t\cos\frac{\pi}{3}\right)\left(1 + t\sin\frac{\pi}{3}\right) + \left(1 + t\sin\frac{\pi}{3}\right)^2 - 3^2 + 3 - 1\right]$$

$$= t\left(1 - \sin\frac{\pi}{3}\cos\frac{\pi}{3}\right) + 5\cos\frac{\pi}{3} - \sin\frac{\pi}{3}$$

$$\therefore D_{\pi/3} f(3, 1) = \lim_{t\to 0}\phi(t) = 5\cos\frac{\pi}{3} - \sin\frac{\pi}{3} = \frac{5-\sqrt{3}}{2}$$

b) $$\phi(t) = \frac{f\left(\frac{\pi}{2} + t\cos\left(-\frac{\pi}{6}\right),\ 0 + t\sin\left(-\frac{\pi}{6}\right)\right) - f\left(\frac{\pi}{2}, 0\right)}{t}$$

$$= -\frac{1}{t}e^{-t\sin\frac{\pi}{6}}\sin\left(t\cos\frac{\pi}{6}\right)$$

$$\therefore D_{-\pi/6} f\left(\frac{\pi}{2}, 0\right) = \lim_{t\to 0}\phi(t) = -\lim_{t\to 0}\frac{e^{-t\sin\pi/6} \times \sin\left(t\cos\frac{\pi}{6}\right)}{t}$$

$$= -\cos\frac{\pi}{6}\lim_{t\to 0}e^{-t\sin\frac{\pi}{6}} \times \lim_{t\to 0}\frac{\sin\left(t\cos\frac{\pi}{6}\right)}{t\cos\frac{\pi}{6}}$$

$$= \lim_{t\to 0}\frac{\sin\left(t\cos\frac{\pi}{6}\right)}{t\cos\frac{\pi}{6}}$$

$$= -\frac{\sqrt{3}}{2}$$

E20) Now, $\phi(t) = \dfrac{f(t\cos\theta, t\sin\theta) - f(0, 0)}{t}$

$$= \frac{1}{t}\left[\frac{2t^3\cos\theta\sin^2\theta}{t^2\cos^2\theta + t^4\sin^4\theta}\right]$$

$$= \frac{2\cos\theta\sin^2\theta}{\cos^2\theta + t^2\sin^4\theta}$$

$$\therefore \lim_{t\to 0}\phi(t) = \begin{cases} 2\sin\theta\tan\theta, & \text{if } \cos\theta \neq 0 \\ 0, & \text{if } \cos\theta = 0 \end{cases}$$

Hence f has directional derivatives in all directions.

E21) $\phi(t) = \dfrac{f(t\cos\theta, t\sin\theta) - f(0, 0)}{t}$

$$= \frac{1}{t}\,\frac{2t^2\sin\theta\cos\theta}{t^2(\sin^2\theta + \cos^2\theta)} = \frac{\sin 2\theta}{t}$$

Then $\lim_{t\to 0}\phi(t)$ exists if and only if $\sin 2\theta = 0$, i.e.,

$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

Consequently, f has directional derivatives at (0, 0) in the directions given by $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

E22) Note that the functions given in all the three problems are differentiable at the points specified and hence we can apply Theorem 6 in each of these cases.

a) $D_{\pi/4}\,u(2, -1) = (u_x)_{(2,-1)}\cos\frac{\pi}{4} + (u_y)_{(2,-1)}\sin\frac{\pi}{4}$

$$= 4\cdot\frac{1}{\sqrt{2}} - 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}.$$

b) $D_{\pi/4}\,u(\ln 3, \ln 2) = (u_x)_{(\ln 3, \ln 2)}\cos\frac{\pi}{4} + (u_y)_{(\ln 3, \ln 2)}\sin\frac{\pi}{4}$

$$= \left[e^{(\ln 3 + \ln 2)} + e^{(-\ln 3 - \ln 2)}\right]\frac{1}{\sqrt{2}}$$

$$+ \left[e^{(\ln 3 + \ln 2)} + e^{(-\ln 3 - \ln 2)}\right]\frac{1}{\sqrt{2}}$$

$$= \frac{37\sqrt{2}}{6}$$

c) $D_{\pi/2}\,u\left(\frac{\pi}{4}, \frac{\pi}{3}\right) = (u_x)_{(\pi/4, \pi/3)}\cos\frac{\pi}{2} + (u_y)_{(\pi/4, \pi/3)}\sin\frac{\pi}{2}$

$$= (\cos y - \tan y)_{(\pi/4, \pi/3)}$$

$$= 2\sqrt{3}$$

E23) Suppose f is defined in an ϵ-neighbourhood N of (a, b). Let N′ be an ϵ-neighbourhood of 0. Then

$|t - 0| < \epsilon \Rightarrow |t| < \epsilon \Rightarrow |(a + t\cos\theta, b + t\sin\theta) - (a, b)| < \epsilon$

Thus, $(a + t\cos\theta, b + t\sin\theta)$ belongs to N.

Since f is defined on N, we get that the function

$F(t) = f(a + t\cos\theta, b + t\sin\theta)$

is defined in the ϵ-neighbourhood N′ of 0 for any fixed θ. Now,

$$\phi(t) = \frac{f(a+t\cos\theta, b+t\sin\theta) - f(a, b)}{t}$$

$$= \frac{F(t) - F(0)}{t}$$

Therefore, $\lim_{t\to 0} \phi(t)$ exists if and only if F(t) is differentiable at 0. Now the function f has directional derivative at (a, b) in the direction of θ if and only if $\lim_{t\to 0} \phi(t)$ exists. Hence, we get that the function f(x, y) has directional derivative at (a, b) in the direction of θ if and only if the corresponding F(t) is differentiable at 0 and the directional derivative is given by F′(0).

E24) If f(x, y) has all the directional derivatives at a point (a, b), then it is not necessary that f is also continuous at (a, b). (See E 20). We have shown over there that f has all the directional derivatives at (0, 0). By putting $y^2 = mx$, it can be checked easily that f is not continuous at (0, 0).

NOTES

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UTTAR PRADESH
RAJARSHI TANDON OPEN UNIVERSITY

UGMM-07

Advanced Calculus

Block

3

APPLICATIONS OF PARTIAL DERIVATIVES

BLOCK 3 APPLICATIONS OF PARTIAL DERIVATIVES

In the last block of this course, we defined partial derivatives of all orders for functions of several variables. We also showed the connection between partial derivatives, differentiability and continuity. We stated sufficient conditions which ensured the equality of mixed partial derivatives. We also defined homogeneous functions and used our knowledge of partial derivatives to obtain some simple results about homogeneous functions.

In this block we apply the results obtained so far to prove some more important results about functions of several variables.

In Unit 8, we define relative maxima and minima for real-valued functions of two variables and obtain a necessary condition for the existence of relative extrema. This condition is similar to the condition obtained for functions of a single variable. We also obtain a set of sufficient conditions for determining the nature of stationary points (critical points in the one-variable case). This condition is parallel to the second derivative test.

In Unit 9, we introduce Jacobians, a notion which really has no analogue in the calculus of functions of one variable. Jacobians play an important role in the whole theory of functions of several variables. Since we have confined our study to functions of two variables, the significance of the role of Jacobians is not that apparent here. But Jacobians will be of great use in the next block, where we shall introduce the theory of integration of functions of two and three variables.

In the last unit, we state two very important theorems, namely, the implicit function theorem and the inverse function theorem. We prove implicit function theorem only in the simplest case. We also prove a necessary and sufficient condition for the functional dependence of functions of two variables.

In this block we have proved most of the results for functions of two variables only. The aim was to avoid significant steps getting obscured in technical details of the proofs. In fact, we have included only those results about functions of higher variables which would be needed in the next block.

Notations and Symbols

$P_n(x)$	n th Taylor Polynomial
$R_{n+1}(x)$	Remainder after (n+1) terms in Taylor's expansion
$\|X\|$	Determinant of the matrix X
$JF = \frac{\partial(f,g)}{\partial(x,y)}$	Jacobian of F = (f, g) w.r.t. x and y

UNIT 8 TAYLOR'S THEOREM

Structure

8.1 INTRODUCTION

In this unit we state, without proof, Taylor's Theorem (about approximating a function by polynomials) for real-valued functions of several variables. This theorem is the principal tool for finding out the points of relative maxima and minima for these functions. We also discuss briefly Lagrange's method of multipliers, which enables us to locate the stationary points when the variables are not free but are subject to some additional conditions.

In this unit we will be dealing with functions of two variables. Even though the results are true for any number of variables, their proof involves techniques which are not easy to understand at this level. So, for the sake of simplicity, we confine our attention to the two-variable case.

We start our discussion with the one variable case.

Objectives

After studying this unit, you should be able to

- find the Taylor polynomials for functions of one or two variables,
- state and apply Taylor's theorem for functions of one and two variables,
- locate the stationary points of functions,
- use the second derivative test to find the nature of stationary points,
- use the technique of Lagrange's multipliers in locating the stationary points of functions of two variables.

8.2 TAYLOR'S THEOREM

Taylor (1685-1731)

In the calculus course you have seen (Unit 6) that if we know the values of a function of one variable and its derivatives at 0, then we can find an expression for the value of the function at a nearby point. We can derive a similar expression for functions of two variables using partial derivatives. This expression was first derived by Brook Taylor, an English mathematician of the eighteenth century. We shall first discuss Taylor's theorem for functions of a single variable.

8.2.1 Taylor's Theorem for Functions of One Variable

You will agree when we say that polynomials are by far the simplest functions in calculus. We can evaluate the value of a polynomial at a point by using the four basic operations of addition, multiplication, subtraction and division. However, the situation in the case of functions like e^x, $\ln x$, $\sin x$, etc., is not so simple. These functions occur so frequently in all branches of mathematics, that approximate values of these functions have been tabulated extensively. The main tool for this purpose has been to find polynomials which approximate these functions in a neighbourhood of the point under consideration.

you are already familiar with Lagrange's mean value theorem. This theorem states that if

f(x) is differentiable in some neighbourhood N of the point x_0, then we have

$$f(x) = f(x_0) + (x - x_0) f'(\xi)$$

for all x such that $[x_0, x]$ or $[x, x_0]$ is contained in N. Here ξ is a point lying between x_0 and x.

If f is twice differentiabele in N, then, again applying mean value theorem to the function f' we can go a step further and write

$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2} f''(\xi)(x - x_0)^2$ is some point in N lying between x_0 and x.

Thus, the constant polynomial $f(x_0)$ approximates f(x) in N in the first case, while the polynomial $f(x_0) + (x - x_0) f'(x_0)$ approximates f(x) in N in the second case. The difference between the actual value and the approximated value is called the error tem.

The error term in the first case is $f'(\xi)(x - x_0)$, and in the second case it is $\frac{1}{2} f''(\xi)(x - x_0)^2$. We can estimate these error terms is f' and f'' are bounded.

Taylor's theorem tells us that if a function f(x) has derivatives of all orders upto n + 1) in a neighbourhood of x_0, then we can find plynomials $P_0(x), \ldots\ldots\ldots, P_n(x)$ of degree $0, \ldots\ldots, n$, respectively, such that the error term

$$f(x) - P_r(x)$$

is a polynomial of degree less than or equal to r + 1. Note that here we consider the polynomial 0 also as a polynomial of degree zero, which is not the usual practice. We have done this for the sake of uniformity of expression. In order to state the precise result, we start with the following definition.

Definition 1 : Let f(x) be a real-valued function having derivatives upto order $n \geq 1$ at the point x_0. A polynomial P (x) is said to be the **r^{th} Taylor polynomial** of f(x) at x_0, if

(i) the degree of $P(x) \leq r$, $r \leq n$

(ii) $P^{(j)}(x_0) = f^{(j)}(x_0)$ for $0 \leq j \leq r$.

where $P^{(0)}(x_0) = P(x_0)$ and $f^{(0)}(x_0) = f(x_0)$.

Recall that a polynomial P (x) is an expression that can be written as

$$P(x) = c_0 + c_1 x + c_2 x^2 + \ldots\ldots\ldots + c_n x^n, \quad \ldots(1)$$

where $c_0, c_1, \ldots\ldots\ldots\ldots, c_n$ are real numbers. Apart from these there are expressions like

$$P(x) = c_0 + c_1(x - x_0) + \ldots\ldots\ldots + c_n(x - x_0)^n, \quad \ldots(2)$$

where $x_0, c_1, c_2, \ldots\ldots\ldots$ are real numbers and $x_0 \neq 0$, which are also called polynomials. You can easily see that (2) can be rewritten in the form (1) by expanding the powers $(x - x_0)^2, \ldots\ldots\ldots, (x - x_0)^n$. We also call the expression in (1) a polynomial at zero and that in (2), a polynomial at x_0.

Now we state and prove a theorem which tells us that Taylor polynomials of a given function are unique. It also tells us how to find out the Taylor polynomials of given function.

Theorem 1 : Let $a_0, \ldots\ldots\ldots\ldots, a_r$ be any r + 1 real numbers. Then there exists a unique polynomial P (x) such that

(i) The degree of $P(x) \leq r$

(ii) $P_{(j)}(x_0) = a_j$, $0 \leq j \leq r$.

where x_0 is any fixed real number.

Moreover, $P(x)=\sum_{m=0}^{r}\frac{a_m}{m!}(x=x_0)^m$

Proof : We can write a polynomial at x_0 as

$P(x)=b_0+b_1(x-x_0)+\ldots\ldots b_r(x-x_0)^r$. ...(3)

where $b_0\ldots\ldots\ldots b_r$ are real numbers. now we have detemine $b_0\ldots\ldots\ldots b_r$ such that $P^{(j)}(x_0)=a_j$ for $0\le r$. If we differentiate the expression in (3) j times, then we get

$$p^{(j)}(x)=\sum_{k=j}^{r}k(k-1)\ldots(k-j+1)b_k(x-x_0)^{k-j},\ 1\le j\le r.$$

and therefore

$P^{(j)}(x_0)=j!b_j,\ 1\le j\le r.$

Thus,

$$b_j=\frac{P^{(j)}(x_0)}{j\ !},\ 1\ \le\ j\ \ge n.$$

Also

$$b_0=P(x_0)=\frac{P^{(0)}(x_0)}{0\ !}.$$

Hence,

$$b_j=\frac{P^{(j)}(x_0)}{j\ !}.\ \text{ for } 0\le j\le r \qquad ...(4)$$

Substituting for b_j s in (3), we get.

$$P(x)=\sum_{j=0}^{r}\frac{P^{(j)}(x_0)\ (x-x_0)^j}{j\ !}=\sum_{j=0}^{r}\frac{a_i}{j!}(x-x_0)^j \qquad ...(5)$$

Note that the polynomial P(x) will be of degree r if and only if $a_r\neq 0$. Now by (4) we can concluded that the polynomial is unique.

The following corollary of Theorem tells us how to find the Taylor polynomials of a given fuction.

Corollary 1 : If f(x) is a real-valued function having derivatives of all orders upto n $(n\ge 1)$, then the m^{th} Taylor polynomial fo f(x) at x_0 is given by

$$P_m(x)=\sum_{j=0}^{m}\frac{f^{(j)}(x_0)\ (x-x_0)^j}{j\ !},\ 0\le m\le n.$$

Proof : Let us take $a_j=f^{(j)}(x_0)$ $0<k<m$; in Theorem 1 Then the m^{th} Taylor polynomial of f, if it exists, must be in the form of Equation (5).Thus.

$$P_m(x)=\sum_{j=0}^{m}\frac{f^{(j)}(x_0)\ (x-x_0)^j}{j\ !},\ 0\le m\le n$$

The above discussion shows that the Taylor polynomials of a given function can be found step by step using the relation

$$P_{m+1}(x)=P_m(x)+\frac{f^{(m+1)}(x_0)}{(m\ +\ 1)\ !}(x-x_0)^{m+1}$$

Moreover, if $P_m(x)$ is the m^{th} Taylor polynomial of f(x) at x_0, then you can check that the derivative of $P_m(x)$ at x_0 is the $(m-1)^{th}$ Taylor polynomial of f (x) at x_0.

Let us consider some examples now.

Example 1 : Let us find the Taylor polynomials of

$f(x)=x^3-2x^2+4x+1$ at $x=3$.

We apply Theorem 1 with $x_0=3$.

Since $f^{(0)}(3) = f(3) = 22$,

$f^{(1)}(3) = 19$,

$f^{(2)}(3) = 14$,

$f^{(3)}(3) = 6$ and

$f^{(r)}(3) = 0$ for all $r > 3$, we get

$P_0(x) = 22,\ P_1(x) = 22 + \frac{19}{1!}(x-3)$,

$P_2(x) = 22 + \frac{19}{1!}(x-3) + \frac{14}{2!}(x-3)^2$,

$P_3(x) = 22 + \frac{19}{1!}(x-3) + \frac{14}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3$ and

$P_r(x) = P_3(x)$ for all $r > 3$.

Example 2 : Let us find the fourth Taylor polynomial

$T_4(x)$ of $f(x) = \sqrt{1+x}$ at $x = 0$.

We have $f'(x) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$, $f^{(3)}(x) = \frac{3}{8}(1+x)^{-5/2}$ and

$f^{(4)}(x) = -\frac{15}{16}(1+x)^{-7/2}$

Therefore, $f(0) = 1,\ f'(0) = \frac{1}{2},\ f''(0) = -\frac{1}{4},\ f^{(3)}(0) = \frac{3}{8},\ f^{(4)}(0) = -\frac{15}{16}$.

The desired polynomial is

$$T_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$= 1 + \frac{1}{2}x - \frac{1}{2!}\cdot\frac{1}{4}x^2 + \frac{1}{3!}\cdot\frac{3}{8}x^3 - \frac{1}{4!}\cdot\frac{15}{16}x^4$$

Example 3 : Let us find $T_8(x)$ for $\cos x$ at $x_0 = \pi$.

Now $\cos \pi = -1$ and the first eight derivatives of $\cos x$ at π are

0, 1, 0, –1, 0, 1, 0, –1.

Dropping the terms with coefficients 0, we have the polynomial

$$T_8(x) = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!}$$

Example 4 : Let us find $T_5(x)$ at $x_0 = 0$ for f, where $f(x) = \frac{1}{1-x} = (1-x)^{-1}$

Computing the derivatives, we obtain

$f'(x) = (1-x)^{-2},\ f''(x) = 2(1-x)^{-3}$

$f^{(3)}(x) = 3.2\,(1-x)^{-4}$, $f^{(4)}(x) = 4!\,(1-x)^{-5}$

$f^{(5)}(x) = 5!\,(1-x)^{-6}$

Thus, the successive derivatives of f at 0, in order, are

1!, 2!, 3!, 4!, 5!,...............

Since f(0) = 1, we obtain

$$T_5(x) = 1 + x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 + \frac{5!}{5!}x^5$$

$$= 1 + x + x^2 + x^3 + x^4 + x^5$$

Now you can try these exercises.

E1) Find the n^{th} Taylor polynomial of the function e^x at $x = 2$.

E2) Find the 6^{th} Taylor polynomial of sin x at x = 0.

E3) Find the r^{th} Taylor polynomials of the following functions at the indicated point and for the indicated value of r.

a) $x^2 - 3x + 4$, $a = -2$, $r = 2$

b) $x^4 - 5x^3 + 3$, $a = 1$, $r = 4$

E4) Find a polynomial f(x) of degree 2 that satisfies $f(1) = 2$, $f'(1) = -1$ and $f''(1) = 2$.

We now state Taylor's theorem which gives us the connection between a function and its Taylor polynomials at a point.

Theorem 2 (Taylor's Theorem) : Let f be a real-valued function defined on the open interval]a, b[. Suppose f has derivatives of all orders upto and including n + 1 in the interval]a, b[. Let x_0 be any point of the interval]a, b[. Then for any x in]a, b[,

$$f(x) = f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \ldots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c), \qquad \ldots(6)$$

where c is a point between x_0 and x.

The expression on the right hand side of (6) is called **Taylor's expansion of f(x) at x_0**.

We won't give the details of the proof of this theorem here.

But we wish to indicate that it can be proved by applying Rolle's theorem to the function

$\phi(X) = f(X) + \frac{(x-X)}{1!} f'(X) + \ldots\ldots + \frac{(x-X)^n}{n!} f^{(n)}(X) + (x-X)^{n+1} A$, defined on the interval $[x_0, x]$ or $[x, x_0]$ according as $x_0 < x$ or $x < x_0$, where A is a constant so determined that $\phi(x_0) = \phi(x)$.

The point c in Theorem 2 comes from the application of Rolle's theorem to the function $\phi(X)$ and therefore we can only assert its existence and not the exact location.

Now we rewrite Equation (6) in the form

$$f(x) = P_n(x) + R_{n+1}(x),$$

where $P_n(x)$ is the n^{th} Taylor polynomial of f(x) at x_0 and

$$R_{n+1}(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(c).$$

Then $R_{n+1}(x)$ depends on x, x_0 and n. We call $R_{n+1}(x)$ the **Lagrange's form of remainder** after n+1 terms in the Taylor's expansion of f(x) at x_0.

If we write x as x_0+h, then Taylor's expansion becomes

$$f(x_0 + h) = \sum_{r=0}^{n} f^{(r)}(x_0) \frac{h^r}{r!} + \frac{h^{n+1}}{n!} f^{(n+1)}(x_0 + \theta h),$$

where $0 < \theta < 1$, and θ is a real number depending on x_0 and n.

Note that if f(x) is a polynomial of degree m, then $f^{(r)}(x) = 0$ for $r > m$. Therefore, $R_{n+1}(x) = 0$ for all x and x_0, provided $n \geq m$. Thus in this case, finding Taylor's expansion of f (x) upto m+1 terms at x_0 is equivalent to expressing f(x) as a polynomial in $x-x_0$ with coefficients from **R**.

By estimating the remainder $R_{n+1}(x)$, we can find how close is f(x) to its n^{th} Taylor polynomial.

In Calculus (Unit 6) you have learnt how to write Taylor's or Maclaurin's series of a given function. At that time you were cautioned that these series need not be valid for a given function. In fact, the Taylor series of a function has a very close connection with its finite Taylor's expansion. Let us see.

Suppose f(x) has derivatives of all orders at x_0. If

$$f(x) = P_n(x) + R_{n+1}(x)$$

is Taylor's expansion of f at a point x_0, and if we are able to prove that

$$\lim_{n\to\infty} R_{n+1}(x) = 0,$$

then we say that **the Taylor's series of f(x) at x_0 converges to the given function** for all those x for which $\lim_{n\to\infty} R_{n+1}(x) = 0$.

Moreover, in that case we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

The coefficient $\frac{f^{(n)}(x_0)}{n!}$ is called **n-th Taylor's coefficient** in the Taylor's expansion of f(x) at x_0.

We now illustrate Taylor's theorem with the help of a few examples.

Example 5 : Let us apply Taylor's Theorem to the function $f(x) = e^x$ about $x = 0$ in the interval $]-1, 1[$.

You know from Calculus that the function $f(x) = e^x$ is continuous everywhere on the real line, and

$$f(x) = f'(x) = \ldots\ldots\ldots = f^{(n)}(x) = \ldots\ldots\ldots = e^x.$$

Thus, derivatives of f of all orders exist and are continuous in the interval $]-1, 1[$. Then by Taylor's theorem, given any $x \in]-1, 1[$ and $n \in N$, there exists a point c between 0 and x such that

$$e^x = T_n(x) + R_{n+1}(x),$$

where

$$T_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}$$

Now can we say anything particular about the remainder $R_{n+1}(x)$ in the above example ? Let's see.

$$\left|R_{n+1}(x)\right| = \left|\frac{x^{n+1}}{(n+1)!}e^c\right|$$

$$\leq e^c \left|\frac{1}{(n+1)!}\right|, \text{ since } |x| < 1.$$

Now since c lies between 0 and x, we have $e^c < e^{|x|}$.

Then $\left|R_{n+1}(x)\right| < e^{|x|}\left|\frac{1}{(n+1)}\right|$

Now $\frac{1}{(n+1)!}$ can be made as small as we like by choosing n sufficiently large,

i.e., $\lim_{n\to\infty} \frac{1}{(n+1)!} = 0.$

This means that $R_{n+1}(x) \to 0$ as $n \to \infty$.

Thus, for the function $f(x) = e^x$ we can write

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} + \ldots\ldots$$

Let us consider another example.

Example 6 : Let us find Taylor's expansion of $f(x) = \ln(1 + x)$ for $x \in\ ]-1, 1[$ at $x = 0$.

You know from Calculus (Unit 6) that

$$f^{(n)}(x) = \frac{(-1)^{n-1}\,(n-1)!}{(1+x)^n},$$

so that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$

Hence,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots\ldots + \frac{(-1)^{n-1}x^n}{n} + R_{n+1}(x),$$

where $R_{n+1}(x) = \frac{(-1)^n}{n+1}\ \frac{1}{(1+\xi)^{n+1}}\ x^{n+1}$.

Clearly for $x \in\ ]-1, 1[$,

$$\left|R_{n+1}(x)\right| \le \frac{1}{n+1}.$$

This shows that $\lim\limits_{n \to \infty} R_{n+1}(x) = 0$.

Thus, $\ln(1+x) = \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{n!}$ for any $x \in\ ]-1, 1[$.

Try the following exercises now.

E5) Obtain the Taylor's expansion of $f(x) = \frac{1}{1+x}$ about $x = 0$ in the interval $]-\frac{1}{2}, 1[$.

E6) Obtain the Taylor's expansion of $f(x) = \sin x$ about $x = \frac{\pi}{6}$.

So far we have seen how to find Taylor's expansions for functions of a single variable. In the next sub-section, we shall discuss Taylor's theorem for functions of two variables.

8.2.2 Taylor's Theorem For Functions of Two Variables

In this sub-section we extend Taylor's theorem to functions of two variables. For this let us first extend the notion of Taylor polynomials to functions of two variables. You have already seen the definition of a polynomial in n variables in Sec. 3.3. Here we will discuss the polynomials in two variables in detail.

Definition 2 : Let x and y denote two variables. Then an expression of the form $a_{jk}x^jy^k$, where j and k are non-negative integers and $a_{jk} \in R$, is called a **monomial**. The integer $j + k$ is called the **degree** of the monomial.

For example, x^2y^3 is a monomial of degree 5,

x^4 is a monomial of degree 4,

y^7 is a monomial of degree 7,

A polynomial in x and y is nothing but a finite sum of monomials. We now give a formal definition.

Definition 3 : A polynomial in two variables in x and y with coefficients in R is an expression of the type

$$P(x,y) = a_{00} + (a_{10}x + a_{01}y) + (a_{20}x^2 + a_{11}xy + a_{02}y^2) + \ldots +$$
$$(a_{i0}x^i + a_{(i-1)1}x^{i-1}y + \ldots + a_{0i}y^i) + \ldots + (a_{n0}x^n + \ldots + a_{0n}y^n),$$

where a_{ij}'s are real numbers.

Here you can note that we have grouped together the monomials having the same degree. In the first bracket each term is a monomial of degree 1. In the second, each is a monomial of degree 2, and so on.

For example, $P(x,y) = 1 + 2xy + x^2y$ is a polynomial in two variables. This polynomial is a sum of three monomials, having degree 0,2 and 3, respectively. The number 3, which is the maximum of these numbers is called the degree of this polynomial.

In general, we have the following definition.

Definition 4 : The highest degree of the monomials present in a polynomial P(x,y) is called the **degree of P(x,y).**

You can now easily do this exercise.

E7) Find the degree of the following polynomials:

a) $1 + y + x^2y + xy^2 + y^5$

b) $2 + x^3 + y^3$

c) $7 + x + xy + x^3y + x^4$

Now we give the definition of the n^{th} Taylor polynomial of a function of two variables.

Definition 5: Let f(x,y) be a real-valued function of two variables. Assume that it has continuous partial derivatives of all types of orders less than or equal to n in some neighbourhood of a point (x_0, y_0). Then

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(x_0 - y_0)\right](x-x_0)^i (y-y_0)^j$$

is called the **n^{th} Taylor Polynomial** of f at (x_0, y_0).

In particular, if f(x,y) is a polynomial of degree n, then all partial derivatives of order m for m > n will be zero. Therefore

$T_m(x,y) = T_n(x,y)$ for all $m \ge n$.

Further, as in the case of one variable, you can see that $T_n(x,y)$ at (0,0) is equal to f(x,y).

Again, from the definition, you can see that

$$T_{n+1}(x,y) = T_n(x,y) + \sum_{\substack{i,j=0\\ i+j=n+1}} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right](x-x_0)^i (y-y_0)^j;$$

so that the Taylor polynomials of a given function f(x,y) can be computed step by step. We show this by an example.

Example 7 : Let us find the Taylor polynomials of the function $P(x, y) = 1 + 2xy + x^2y$ at (1,1).

We first note that

$P(1,1) = 4$ and therefore, $T_0(x,y) = P(1,1) = 4$.

$$\frac{\partial P}{\partial x} = 2y + 2xy,\ \left(\frac{\partial P}{\partial x}\right)_{(1,1)} = 4.$$

$$\frac{\partial P}{\partial y} = 2x + x^2,\ \left(\frac{\partial P}{\partial y}\right)_{(1,1)} = 3$$

Therefore, $T_1(x,y) = T_0(x,y) + \frac{(x-1)}{1!}\frac{\partial P}{\partial x}(1,1) + \frac{(y-1)}{1!}\frac{\partial P}{\partial y}(1,1)$

$$= 4 + 4(x-1) + 3(y-1).$$

$$\frac{\partial^2 P}{\partial x^2} = 2y, \quad \frac{\partial^2 P}{\partial x\, \partial y} = 2 + 2x, \quad \frac{\partial^2 P}{\partial y^2} = 0$$

Therefore, $T_2(x,y) = T_1(x,y) + \frac{(x-1)^2}{2!}\frac{\partial^2 P}{\partial x^2}(1,1) + \frac{(x-1)(y-1)}{1!\,1!}\frac{\partial^2 P}{\partial x\,\partial y}(1,1)$

$$+ \frac{(y-1)^2}{2!}\frac{\partial^2 P}{\partial y^2}(1,1)$$

$$= 4 + 4(x-1) + 3(y-1) + (x-1)^2 + 4(x-1)(y-1).$$

Since $\frac{\partial^3 P}{\partial x^3} = 0$, $\frac{\partial^3 P}{\partial x^2\,\partial y} = 2$, $\frac{\partial^3 P}{\partial x\,\partial y^2} = 0$ and $\frac{\partial^3 P}{\partial y^3} = 0$, we get that

$$T_3(x,y) = T_2(x,y) + (x-1)^2(y-1)$$

You will agree that $T_r(x,y) = T_3(x,y)$ for all $r \geq 3$.

Here is another example.

Example 8 : Let us find the Taylor polynomial $T_3(x,y)$ for the function $\sin(x+y)$ at (0, 0).

We write $f(x,y) = \sin(x+y)$. It is clear that f has continuous partial derivatives of all orders. Let us compute these derivatives at (0,0). We get

$$\frac{\partial f}{\partial x}(x,y) = \cos(x+y) = \frac{\partial f}{\partial y}(x,y)$$

Therefore $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 1.$

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \frac{\partial^2 f}{\partial x\,\partial y}(0,0) = \frac{\partial^2 f}{\partial y^2}(0,0) = -\sin(x+y)\Big|_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial x^3}(0,0) = \frac{\partial^3 f}{\partial x^2\,\partial y}(0,0) = \frac{\partial^3 f}{\partial x\,\partial y^2}(0,0) = \frac{\partial^3 f}{\partial y^3}(0,0) = -\cos(x+y)\Big|_{(0,0)} = -1$$

Thus, the third Taylor polynomial of $\sin(x+y)$ at (0,0) is

$$T_3(x,y) = \sum_{\substack{i,j \geq 0 \\ i+j \leq 3}} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j}(0,0)\right] x^i y^j$$

$$= \frac{1}{0!\,0!} f(0,0) + \frac{y}{1!\,0!}\frac{\partial f}{\partial y}(0,0) + \frac{x}{1!\,0!}\frac{\partial f}{\partial x}(0,0)$$

$$+ \frac{x^2}{2!\,0!}\frac{\partial^2 f}{\partial x^2}(0,0) + \frac{xy}{1!\,1!}\frac{\partial^2 f}{\partial x\,\partial y}(0,0) + \frac{y^2}{0!\,2!}\frac{\partial^2 f}{\partial y^2}(0,0) + \ldots\ldots$$

$$= \frac{x}{1!} + y - \frac{1}{3!}x^3 - \frac{1}{2!\,1!}x^2 y - \frac{1}{1!\,2!}xy^2 - \frac{1}{3!}y^3.$$

When simplified it takes the form

$$T_3(x,y) = (x+y) - \frac{1}{3!}(x^3 + 3x^2 y + 3xy^2 + y^3)$$

$$= (x+y) - \frac{(x+y)^3}{3!}.$$

Try these exercises now.

E8) Find the second Taylor polynomial of e^{x+y} at (0,0).

E9) Find the Taylor polynomials of $f(x,y) = 2 + x^2 + y^2$ at (1,1).

E10) Let $f(x,y)$ be a polynomial of degree 2. Prove that $T_2(x,y)$ at (0,0) is equal to $f(x,y)$.

Now let us consider a function $f(x,y)$ of two variables. Assume that f has continuous partial derivatives of all orders less than or equal to n, for some integer n, in a neighbourhood of a point (x_0, y_0). Then the n^{th} Taylor polynomial

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!} \cdot \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j$$

has the same value as $f(x,y)$ at (x_0,y_0), and the same partial derivatives of all orders $\le n$ as f at (x_0, y_0). As in the case of one variable, we would naturally like to know whether we can approximate f by the corresponding Taylor polynomials. Put differently, we would like to have some information about the function

$$R_{n+1}(x,y) = f(x,y) - T_n(x,y).$$

An analogue of Taylor's theorem which we state now, provides us some information about the function $R_{n+1}(x, y)$.

Theorem 4: (Taylor's Theorem For Function of Two Variables):

Let f be a real-valued function of two variables x and y with continuous partial derivatives of orders $\le n+1$ in some neighbourhood $S(\bar{x}, r)$ of $\bar{x} = (x_0, y_0)$. Then for a given $(x,y) \ne (x_0, y_0)$ in $S(\bar{x}, r)$, there exists a point (c_1,c_2) on the line segment joining (x_0, y_0) and (x, y), such that

$$f(x,y) = T_n(x,y) + R_{n+1}(x,y), \qquad \dots (7)$$

where

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(x_0,y_0)} (x-x_0)^i (y-y_0)^j,$$

and

$$R_{n+1}(x,y) = \sum_{i+j=n+1} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j}\right)_{(c_1,c_2)} (x-x_0)^i (y-y_0)^j$$

This means $R_{n+1}(x, y) = \frac{1}{(n+1)!} \left(\frac{\partial^{n+1} f}{\partial x^{n+1}}\right)_{(c_1,c_2)} (x-x_0)^{n+1}$

$$+ \frac{1}{n!\,1!} \left(\frac{\partial^{n+1} f}{\partial x^n \partial y}\right)_{(c_1,c_2)} (x-x_0)^n (y-y_0)$$

$$+ \frac{1}{(n-1)!\,2!} \left[\frac{\partial^{n+1} f}{\partial x^{n-1} \partial y^2}\right]_{(c_1,c_2)} (x-x_0)^{n-1} (y-y_0)^2 +$$

$$\dots\dots + \left[\frac{1}{(n+1)!} \frac{\partial^{n+1} f}{\partial y^{n+1}}\right]_{(c_1,c_2)} (y-y_0)^{n+1}.$$

So you can see that $R_{n+1}(x, y)$ involves all the $(n+1)^{th}$ order partial derivatives of f evaluated at the point (c_1, c_2).

The right hand side of (7) is called the n^{th} Taylor expansion of f at (x_0, y_0). This expansion may seem a little complicated to you. But don't let it scare you. You will soon see that in this course you need to consider only the second Taylor expansions of functions. If you look at the expression for $R_3(x,y)$, you will see that it contains powers of $(x-x_0)$ and $(y-y_0)$. Now if we take the point (x, y) close enough to (x_0, y_0), then $(x-x_0)$ and $(y-y_0)$ will be very small. Therefore, we can get a good enough approximation of $f(x,y)$ by a second degree polynomial. Of course, $f(x,y)$ can be approximated as closely as we like by a polynomial by choosing n sufficiently large.

For future use, we write the expression for $T_2(x,y)$ and the second Taylor expansion of $f(x,y)$ at (x_0,y_0) explicitly :

$$f(x,y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)\right] +$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x-x_0)^2 + \frac{2\partial^2 f}{\partial x \partial y}(x_0, y_0)(x-x_0)(y-y_0) +\right.$$

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0)\ (y - y_0)^2\Bigg] + R_2(x,y).$$

Now consider this example.

Example 9 : Suppose we want to find the second Taylor expansion of the function $f(x,y) = \ln(1 + x + 2y)$ for points close to $(2,1)$.

Let us compute the partial derivatives one by one. We have

$$f(2,1) = \ln 5$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+x+2y}, \quad \frac{\partial f}{\partial x}(2,1) = \frac{1}{5}$$

$$\frac{\partial f}{\partial y} = \frac{2}{1+x+2y}, \quad \frac{\partial f}{\partial y}(2,1) = \frac{2}{5}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-1}{(1+x+2y)^2}, \quad \frac{\partial^2 f}{\partial x^2}(2,1) = \frac{-1}{25}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-4}{(1+x+2y)^2}, \quad \frac{\partial^2 f}{\partial y^2}(2,1) = \frac{-4}{25}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{2}{(1+x+2y)^2}, \quad \frac{\partial^2 f}{\partial x \partial y}(2,1) = \frac{-2}{25}$$

Then the second Taylor expansion is given by

$$f(x,y) = \ln 5 + \left[\frac{1}{5}(x - x_0) + \frac{2}{5}(y - y_0)\right] + \frac{1}{2}\Bigg[\left(-\frac{1}{25}\right)(x - x_0)^2 +$$

$$2\left(-\frac{2}{25}\right)(x - x_0)(y - y_0) + \left(-\frac{4}{25}\right)(y - y_0)^2\Bigg]$$

Why don't you try some exercises now?

E11) Find the second Taylor expansion for the function [illegible]
about $(1, \frac{\pi}{2})$.

E12) Find an approximation to the function $f(x,y) = e^x$ [illegible] by a second-degree polynomial near $(0,0)$.

In the next section we will discuss maxima and [illegible] of functions of two variables. You have already studied maxima and minima for functions of one variable in Block [illegible] of your Calculus course. There you have used the first and second derivative tests to locate the local maxima and local minima. You will see that the definitions of the maximum and minimum of functions of two variables are similar to those in the case of one variable. We will obtain a set of necessary conditions for the existence of maxima and minima, which are similar to the one variable case. Taylor's expansion of functions which you studied in this section will enable us in deriving a set of sufficient conditions for determining the points of maxima and minima. This is an analogue of the second derivative test which you have studied for functions of one variable.

8.3 MAXIMA AND MINIMA

This section deals with the concept of maxima and minima for functions of two variables. You know from the one-variable case that the study of maxima and minima (or extrema) is useful in graphing a function.

As in the one-variable case, we shall be interested in studying the local extrema of a function, rather than its absolute extrema. So let us take up the study of local or relative extrema, i.e., points of relative maxima or relative minima for functions of two variables.

8.3.1 Local Extrema

We shall first try to understand the concepts of local maximum and minimum for functions of two variables.

Let us start with some simple functions.

Consider the function

$$f(x,y) = (x + 1)^2 + (y - 3)^2 - 1$$

Now $f(-1,3) = -1$.

Since $(x + 1)^2$ and $(y - 3)^2$ are always positive for $x \neq -1$, and $y \neq 3$,

we have $(x + 1)^2 + (y - 3)^2 - 1 > -1$ for $(x, y) \neq (-1, 3)$.

That is, $f(x, y) \geq f(-1, 3)$ for all (x, y).

In this case we say that f has a minimum at $(-1, 3)$. See Fig. 1.

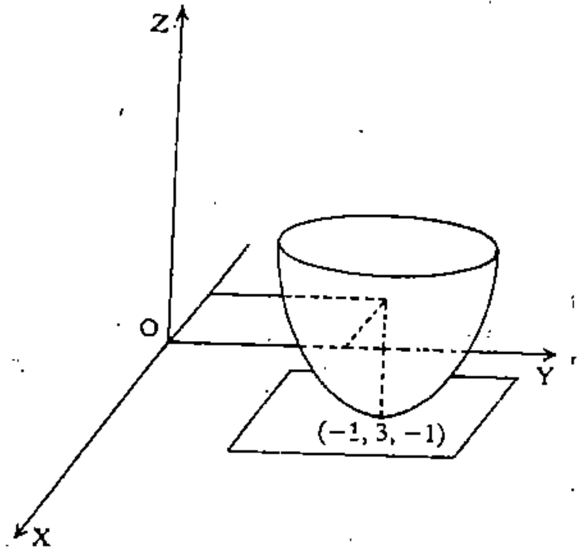


Fig. 1

You can see that the tangent plane to the surface at $(-1, 3, -1)$ is horizontal.

Now consider another function defined by

$$f(x, y) = \frac{1}{2} - \sin(x^2 + y^2)$$

Here $f(0,0) = \frac{1}{2}$. Take the circle $x^2 + y^2 = \frac{\pi}{6}$ with centre $(0,0)$. Then for any $(x, y) \neq (0, 0)$ inside the circle, we have

$$\sin(x^2 + y^2) > 0$$

and therefore

$$f(x, y) = \frac{1}{2} - \sin(x^2 + y^2) < \frac{1}{2} = f(0, 0).$$

Thus $f(x,y) \leq f(0,0)$ for all (x, y) in the circle. Note that $f(x,y)$ can be greater than $\frac{1}{2}$ for (x, y) outside the circle.

In this case we say that f has a local maximum at $(0,0)$.

This leads us to the following definition:

Definition 6: Suppose f is a real-valued function of two variables. We say that the function f has a **local maximum** at a point $P(x_0, y_0)$ if there exists some open disc

$S(\bar{x}, r)$, where $\bar{x} = (x_0, y_0)$ and $r > 0$, contained in the domain of definition of f such that for all $(x, y) \in S(\bar{x}, r)$ we have

$f(x, y) \leq f(x_0, y_0)$.

Now we are sure you will be able to define a local minimum on your own. See E 13). Don't forget to tally your definition with the one given in Sec. 8.6.

E13) Define a local minimum of a function of two variables.

Recall that in the case of functions of one variable, we took an open interval instead of an open disc. Thus, our notion of local maximum and local minimum for functions of two variables is a natural generalisation of the notion in the one-variable case.

Remark 1: i) If $f(x,y) \leq f(x_0, y_0)$ for all $(x, y) \in D$, the domain of f, then we say that the point (x_0,y_0) is the **global or absolute maximum** of f. We can similarly define a global minimum.

ii) The maximum and minimum values of a function are called extrema of the function; we say that a function has an extremum at a given point if the function has a maximum or a minimum at that point.

If a function has an absolute maximum or minimum at a point (x_0,y_0) and this point is such that there is a neighbourhood of (x_0,y_0) contained in the domain of f, then (x_0,y_0) is a relative maximum or relative minimum of the function f. But the converse may not be true.

In the case of functions of a single variable you know that the derivative of a function vanishes (if it exists) at each local maximum and minimum. There is a similar result for functions of two variables also. We present this result in the next theorem.

Theorem 5: Let f be a function of two variables. Suppose f has an extremum at some point (x_0,y_0) and the partial derivatives of f exist at that point. Then

$$\frac{\partial f}{\partial x}(x_0, y_0) = 0 = \frac{\partial f}{\partial y}(x_0, y_0).$$

Proof : Let us assume that f(x,y) has a relative maximum at $\bar{x} = (x_0, y_0)$. Then f(x,y) is defined in an open disc $S = S(\bar{x}, r)$, $r > 0$ and $f(x,y) \leq f(x_0, y_0)$ for all $(x,y) \in S$. Thus, there exist open intervals I_1 and I_2,

$I_1 =]x_0 - r, x_0 + r[\;;\; I_2 =]y_0 - r, y_0 + r[$

such that $x \in I_1 \Rightarrow (x,y_0) \in S$ and $y \in I_2 \Rightarrow (x_0,y) \in S$

Now consider the function g_1 defined on I_1 by

$g_1(x) = f(x,y_0)$.

Then g_1 is a function of one variable.

Similarly, the function g_2 defined on I_2 by $g_2(y) = f(x_0, y)$ is also a function of one variable. We can see from the definitions of g_1 and g_2 that

$g_1(x) = f(x,y_0) \leq f(x_0,y_0) = g_1(x_0)$ for all $x \in I_1$, and

$g_2(y) = f(x_0,y) \leq f(x_0,y_0) = g_2(y_0)$ for all $y \in I_2$.

This means that the functions g_1 and g_2 have relative maxima at the points x_0 and y_0, respectively.

Now we are given that the partial derivatives of f exist at (x_0, y_0). This means that g_1 and g_2 are differentiable at x_0 and y_0, respectively. Thus,

$g_1'(x_0) = f_x(x_0, y_0) = 0$ and

$g_2'(y_0) = f_y(x_0, y_0) = 0$,

because you already know that if a one-variable function has relative extremum at a point and is differentiable at that point, then its derivative vanishes at that point.

If f(x,y) has a relative minimum, then g_1 and g_2 also have relative minima at x_0 and y_0, respectively, and we have the same conclusion as above.

We can use this theorem to check whether a given function has an extremum at some point or not. All we have to do is to see whether its partial derivatives vanish at that point (if they exist). We shall illustrate this fact with some examples.

Example 10 : Let us check whether the function given by

$$f(x,y) = x^2 - 2x + \frac{y^2}{4}$$

has maximum or minimum values.

The given function $f(x, y) = x^2 - 2x + \frac{y^2}{4}$ is differentiable everywhere. According to Theorem 5, first we have to find out the points (x, y) such that $f_x(x, y) = 0 = f_y(x, y)$.

Now

$$f_x(x, y) = 2x - 2,\ f_y(x, y) = \frac{y}{2}.$$

Therefore, $f_x(x, y)$ and $f_y(x, y)$ will vanish only when x = 1 and y = 0. Therefore, the point (1, 0) is the only possible point where f can have a maximum or minimum value. Now let us see whether (1, 0) is a maximum or a minimum point for f.

We rewrite f(x, y) as

$$f(x, y) = x^2 - 2x + \frac{y^2}{4}$$

$$= x^2 - 2x + 1 + \frac{y^2}{4} - 1$$

$$= (x - 1)^2 + \frac{y^2}{4} - 1$$

This shows that, $f(x, y) \geq -1 = f(1,0)$ for all (x, y).

Thus, the function has a global minimum at (1, 0). The minimum value is $f(1, 0) = -1$. The function has no maximum values.

Thus, if $\frac{\partial f}{\partial x} \neq 0$ or $\frac{\partial f}{\partial y} \neq 0$ at some point, then we can straightaway say that the function does not have an extremum at that point. But if $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ at some point, then this does not imply that the function has an extremum at that point. It is possible that all the first order partial derivatives of a function are zero at some point (x_0, y_0), but still, that point is not an extremum point for that function. That is, the converse of Theorem 5 is not true. We illustrate this with an example.

Example 11 : Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f(x,y) = 1 - x^2 + y^2.$$

For this function we have

$$\frac{\partial f}{\partial x} = -2x \text{ and } \frac{\partial f}{\partial y} = 2y.$$

Therefore, $\frac{\partial f}{\partial x}(0, 0) = 0 = \frac{\partial f}{\partial y}(0,0)$.

Now, let us check whether f has an extremum at (0, 0). We have $f(0, 0) = 1$, $f(x_1, 0) < 1$ and $f(0, y_1) > 1$ for all non-zero x_1 and y_1. But in any neighbourhood of (0,0), we can find points of the type $(x_1, 0)$ and $(0, y_1)$. Thus, there exists no neighbourhood N of (0,0) for which $f(x, y) \leq f(0, 0)$ or $f(x, y) \geq f(0, 0)$ for all $(x, y) \in N$. (also see Fig. 2).

Thus, (0, 0) is neither a maximum nor a minimum point for f, though both the partial derivatives of f vanish at (0, 0).

In Fig. 2 you can see the graph of f.

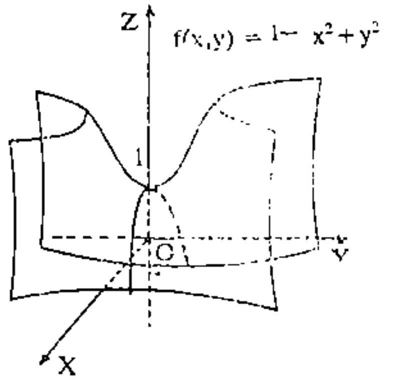
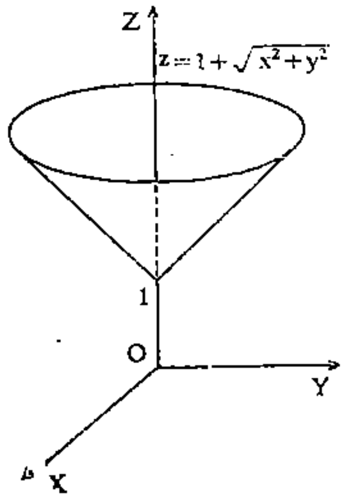


Fig. 2

Sometimes it may happen that the partial derivatives of a function do not exist at a point, but, still the function has an extremum at that point. This is the case with the function in our next example.

Example 12: Consider the function given by

$$f(x, y) = 1 + \sqrt{x^2 + y^2}$$



In Fig. 3 you can see the graph of this function. Obviously, f has a minimum at (0, 0). Now let us try to calculate $\frac{\partial f}{\partial x}$ (0, 0) and $\frac{\partial f}{\partial y}$ (0, 0).

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{1 + \sqrt{h^2} - 1}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

But we know that $\lim_{h \to 0} \frac{|h|}{h}$ does not exist. Hence $\frac{\partial f}{\partial x}$ (0, 0) does not exist. You can similarly check that $\frac{\partial f}{\partial y}$ (0, 0) also does not exist.

See if you can solve these exercises now.

E14) Find the points at which the partial derivatives of the function

$$f(x, y) = \frac{x}{x^2 + y^2 - 4}$$ vanish.

E15) Show that the function f(x, y) = [illegible] has a global minimum.

(Hint : Complete the squares involving x and y.)

Until now we have seen that if a function f has an extremum at (a, b), then either

i) the partial derivatives of f do not exist at (a, b) or

ii) $\frac{\partial f}{\partial x}(a, b) = 0 = \frac{\partial f}{\partial y}(a, b)$.

Henceforth in this unit we will be concerned only with functions whose partial derivatives exist. So the second condition becomes a necessary condition for the existence of extrema. Because of the importance of this condition, we give a special name to the points satisfying it.

Definition 7 : Let f be a function of two variables. A point (x, y) is said to be a **stationary point** of f if both the partial derivatives are zero at (x, y).

Now you will agree that if f has an extremum at (a, b), then (a, b) is a stationary point of f. But all stationary points of a function need not be its points of extrema. You have seen such a situation in Example 11.

You can now try this exercise.

E16) Find the stationary points of the following functions:

a) $f(x, y) = 1 + x^2 - y^3$

b) $f(x, y) = (x + y) e^{-xy}$

Now let us look at some stationary points which are not points of extrema. Suppose (a, b) is a stationary point of a function $f(x, y)$ but is not its point of extremum. Still it is possible that one of the functions f(x, b) or f(a, y), where a and b are fixed can have a maximum at (a, b), while the other has a minimum at this point. For example, let us look at the surface given in Fig. 4 by the equation

$$f(x, y) = y^2 - x^2.$$

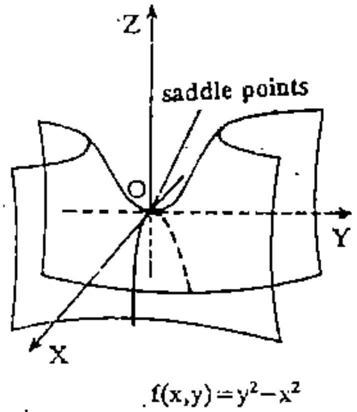


Fig. 4

(0,0) is a stationary point of f(x,y)

Look at the surface along x and y axes. If we fix y = 0, then $f(x, 0) = -x^2$ considered as a function of one variable, has a maximum at 0. But f does not have a maximum at (0,0). Thus if we fix x = 0 , then $f(0, y) = y^2$ has a minimum at 0, but f does not have a minimum at (0, 0). Such stationary points are termed as **saddle points.** From Fig. 4, you can see that the graph of f resembles a saddle around the stationary point (0, 0). That is why f does not have an extremum at (0, 0), even though $f_x(0, 0) = f_y(0, 0) = 0$

We say that a function f has a saddle point at (x_0, y_0), if there is a disc centred at (x_0, y_0) such that

i) f assumes its maximum value on one diameter of the disc only at (x_0,y_0).

ii) f assumes its minimum value at another diameter of the disc only at (x_0,y_0).

Thus, we have seen that a stationary point may not be a point of extremum. In the next sub-section we try to find conditions under which a stationary point is a point of either maximum or minimum.

8.3.2 Second Derivative Test For Local Extrema

In this sub-section, we derive a method using which we can test whether a point is a point of maximum or minimum. You will see that this test involves second derivatives.

You may recall that for the case of one variable also, we have a second derivative test for testing maxima and minima. According to this test, if we have a function f of one variable such that $f'(x_0) = 0$, then f has

a local minimum at x_0 if $f''(x_0) > 0$

a local maximum at x_0 if $f''(x_0) < 0$.

We have a similar test for two variables. But the test is not as easy as in the case of one variable.

You are already familiar with homogeneous functions (see ...). In order to find a set of sufficient conditions for determining the nature of stationary points, we shall need a simple result about the sign of values assumed by homogeneous polynomials in two variables of degree two. We shall call a homogeneous polynomial of degree two in n variables with real coefficients, a **real quadratic form** in n variables. A quadratic form in two variables is also called a **binary form or a binary quadratic form.** Thus, a binary form is an expression of the type

$ax^2 + bxy + cy^2$, where a, b, c are real numbers.

Now we state and prove a theorem which says that we can determine the sign of a quadratic form by looking at its coefficients.

Theorem 6 : Let q(x, y) be a binary quadratic form. Then

i) $b^2 - 4ac = 0 \Rightarrow q(\alpha, \beta) \geq 0 \ \forall \alpha, \beta \in R$ or $q(\alpha, \beta) \leq 0 \ \forall \alpha, \beta \in R$.

ii) $b^2 - 4ac > 0 \Rightarrow q(x, y)$ takes positive as well as negative values.

iii) $b^2 - 4ac < 0$ and $a > 0$ or $c > 0 \Rightarrow q(\alpha, \beta) > 0 \ \forall \alpha, \beta \in R, (\alpha, \beta) \neq (0, 0)$.

iv) $b^2 - 4ac < 0$ and $a < 0$ or $c < 0 \Rightarrow q(\alpha, \beta) < 0 \ \forall \alpha, \beta \in R, (\alpha, \beta) \neq (0, 0)$.

Proof : i) if $b^2 - 4ac = 0$, then both a and c cannot be zero. Let us assume without loss of generality that $a \neq 0$. Then

$$q(x, y) = a\left(x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right)$$

$$= a\left[\left(x + \frac{b}{2a}y\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)y^2\right]$$

$$= a\left(x + \frac{b}{2a}y\right)^2$$

This means that $q(\alpha, \beta)$ has the sign of a, if it is non-zero.

In fact, we obtain that $q(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in R$ if $a > 0$ and $q(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in R$ if $a < 0$. (Note that in this case there exist $\alpha, \beta \in R$ such that $(\alpha, \beta) \neq (0,0)$ and $q(\alpha, \beta) = 0$. For example, $\alpha = -\frac{b}{2a}$ and $\beta = 1$.)

By interchanging the roles of x and y, we can prove that

$q(\alpha, \beta) \geq 0$ for all $\alpha, \beta \in R$ if $c > 0$

and $q(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in R$ if $c < 0$.

ii) $b^2 - 4ac > 0$. If both a and c are zero, then $q(x, y) = bxy$, and therefore $q(1, -1) = -b$, $q(-1, -1) = b$. This shows that q(x, y) assumes both positive and negative values.

Now suppose $a \neq 0$. Then

$$q(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)y^2\right]$$

If $\alpha = -\frac{b}{2a}$ and $\beta = 1$, then $q(\alpha, \beta) = -\frac{b^2 - 4ac}{4a}$.

Also, if $\alpha_1 = 1$ and $\beta_1 = 0$, then $q(\alpha_1, \beta_1) = a$.

Thus, $q(\alpha, \beta)$ and $q(\alpha_1, \beta_1)$ have opposite signs. This proves Case ii) if $a \neq 0$.

Similarly we can prove Case ii) when $c \neq 0$.

(iii) and (iv) : If $b^2 - 4ac < 0$, then neither a nor c can be zero.

Since $a > 0 \iff c > 0$, it is enough to prove the result in case $a > 0$ or $a < 0$. As before, we write

$$q(x, y) = a\left[\left(x + \frac{b}{2a}y\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right)y^2\right]$$

Then for $\alpha, \beta \in \mathbf{R}$, the expression inside the bracket is positive for $(\alpha, \beta) \neq (0, 0)$ and is zero for $(\alpha, \beta) = (0, 0)$.

Thus, if $a > 0$, $q(\alpha, \beta) \geq 0$ for every $\alpha, \beta \in \mathbf{R}$ and

if $a < 0$, $q(\alpha, \beta) \leq 0$ for every $\alpha, \beta \in \mathbf{R}$.

This completes the proof of Theorem 6.

Now we use this theorem to obtain a sufficient condition to determine the nature of stationary points.

Theorem 7 : Suppose $f(x, y)$ is a function of two variables such that $f(x, y)$ has continuous partial derivatives upto order two in a disc N containing the point (x_0, y_0). Suppose that the point (x_0, y_0) is a stationary point of $f(x, y)$, that is,

$$\frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = 0.$$

Further, let $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = a$, $\frac{2\partial^2 f}{\partial x\, \partial y}(x_0, y_0) = b$ and $\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = c$.

Then,

i) $f(x,y)$ has a minimum (maximum) at (x_0, y_0) if $b^2 - 4ac < 0$ and $a > 0$

$(a < 0)$ or $c > 0$ $(c < 0)$.

ii) $f(x, y)$ has neither maximum nor minimum at (x_0, y_0) if $b^2 - 4ac > 0$, i.e., (x_0, y_0) is a saddle point.

Proof : Let us first consider the functions

$$g_1 = \frac{\partial^2 f}{\partial x^2}, \quad g_2 = \frac{\partial^2 f}{\partial x\, \partial y}, \quad g_3 = \frac{\partial^2 f}{\partial y^2}$$

These functions are given to be continuous at $(x_0, y_0) \in N$, and

$$g_1(x_0, y_0) = a,\ 2g_2(x_0, y_0) = b,\ g_3(x_0, y_0) = c.$$

This implies that the function $g_2^2 - g_1 g_3$ is also continuous on N, and

$$4(g_2^2 - g_1 g_3)(x_0, y_0) = b^2 - 4ac.$$

Therefore, by Theorem 6 of Unit 4, there exists a neighbourhood N_1 of (x_0, y_0) contained in N such that the function $g_2^2 - g_1 g_3$ will have the same sign as $(g_2^2 - g_1 g_3)(x_0, y_0)$ on N_1.

Similarly,

(i) Corresponding to g_1 we can choose a neighbourhood N_2 of (x_0, y_0) contained in N such that the function g_1 will have the same sign as $g_1(x_0, y_0)$ in N_2.

(ii) Corresponding to g_2 we can choose a neighbourhood N_3 of (x_0, y_0) contained in N such that the function g_2 will have the same sign as $g_2(x_0, y_0)$ in N_3.

Let $N_0 = N_1 \cap N_2 \cap N_3$. Then f satisfies all the hypotheses of Taylor's theorem in N_0. Therefore by second Taylor expansion we have

$$f(x_0 + h, y_0 + k) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}(x_0, y_0)(x_0 + h - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y_0 + k - y_0)\right)$$

$$+\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^2}(\xi,\eta)\,(x_0+h-x_0)^2+\frac{2\partial^2 f}{\partial x\,\partial y}(\xi,\eta)\,(x_0+h-x_0)\,(y_0+k-y_0)\right.$$

$$\left.+\frac{\partial^2 f}{\partial y^2}(\xi,\eta)\,(y_0+k-y_0)^2\right],$$

where h and k are numbers such that (x_0+h, y_0+k) belongs to N_0 and (ξ,η) is a point on the line segment joining (x_0, y_0) and (x_0+h, y_0+k).

Since (x_0, y_0) is a stationary point we have

$$\frac{\partial f}{\partial x}(x_0, y_0)=0=\frac{\partial f}{\partial y}(x_0, y_0).$$

Therefore,

$$f(x_0+h, y_0+k)-f(x_0, y_0)=\frac{1}{2}\,[\,g_1{}^2(\xi,\eta)\,h^2+2g_2(\xi,\eta)\,hk+g_2{}^2(\xi,\eta)\,k^2\,]$$

$$=q\,(h,k)\text{, say} \qquad \text{... (8)}$$

Then q (h, k) is a quadratic form in h and k.

Now let us take up the cases (i) and (ii), one by one.

Case (i) : Suppose $b^2-4ac<0$ and $a>0$. Then $(g_2^2-g_1\,g_3)\,(x_0, y_0)=\dfrac{b^2-4ac}{4}<0.$

Thus, $g_2^2-g_1g_3$ will be negative at all points in N_0, in particular, at $(\xi,\eta)\in N_0$. That is,

$$(g_2^2-g_1g_3)\,(\xi,\eta)<0.$$

Also, since $a=g_1\,(x_0, y_0)>0$, using the same argument we can conclude that

$$g_1\,(\xi,\eta)>0.$$

Therefore, by Theorem 6, the quadratic expression $q\,(h,k)\geq 0$.

This shows that

$$f(x_0+h, y_0+k)\geq f(x_0, y_0)$$

for all h, k such that $(x_0+h, y_0+k)\in N_0$. Hence f has a local minimum at (x_0, y_0). Similarly, we can show that if $c>0$, then f has a local minimum at (x_0, y_0). Using a similar argument we can prove that f has a local maximum if $a<0$ or $c<0$.

Case ii) Suppose $b^2-4ac>0$.

$b^2-4ac=4\,(g_2^2-g_1\,g_3)\,(x_0, y_0)>0$ implies that $g_2^2-g_1\,g_3$ is positive at all points in N_0.

Then $(g_2^2-g_1\,g_3)\,(\xi,\eta)>0.$

Therefore, by condition (iii) of Theorem 6, q (h, k) takes both positive and negative values. Hence from Equation (8), we conclude that f does not have an extremum in this case. This proves Case (ii) of the theorem.

Here, you may wonder why we don't consider the case of $b^2-4ac=0$. If $b^2-4ac=0$, then we cannot conclude that $g_2^2-g_1\,g_3=0$. In fact, the quadratic form could have different signs at different points of N_0. Thus, the case $b^2-4ac=0$ is a doubtful case.

Now we are giving a few examples to show how Theorem 7 can be applied to determine the points of extrema.

Example 13 : Let us find the local extrema of the function f(x, y), where

$$f(x,y)=x^2-2xy+2y^2-2x+2y+4$$

Here we have ,

$$\frac{\partial f}{\partial x}(x, y)=2x-2y-2,\text{ and}$$

$$\frac{\partial f}{\partial y}(x, y)=-2x+4y+2.$$

For both the partial derivatives to be zero, we must have

$2x - 2y - 2 = 0$ and

$-2x + 4y + 2 = 0.$

Adding these equations, we find that

$2y = 0$ or $y = 0$.

Then we must have $x = 1$. Thus, $(1, 0)$ is the only stationary point.

Computing the second derivative, we get

$$a = \frac{\partial^2 f}{\partial x^2}(1, 0) = 2, \quad b = 2\frac{\partial^2 f}{\partial x \partial y}(1, 0) = 4 \text{ and}$$

$$c = \frac{\partial^2 f}{\partial y^2}(1, 0) = 4.$$

Therefore, $b^2 - 4ac = -16 < 0$.

Since $a = 2 > 0$, by Theorem 7, f has a local minimum at $(1, 0)$.

Example 14 : We shall show that the function

$$f(x, y) = x^2 - 2xy + y^2 + x^3 - y^3 + 2x^7$$

has neither maximum nor minimum at $(0, 0)$.

Clearly $f_x(0, 0) = 0 = f_y(0, 0)$. i.e., $(0, 0)$ is a stationary point.

Moreover, $a = \frac{\partial^2 f}{\partial x^2}(0,0) = 2$. $c = \frac{\partial^2 f}{\partial y^2}(0, 0) = 2$, $b = \frac{2\partial^2 f}{\partial x \partial y}(0, 0) = -4$, so that $b^2 - 4ac = 0$, showing that Theorem 7 is not applicable. But

$$f(x, y) = (x - y)^2 + (x - y)(x^2 + xy + y^2) + 2x^7$$

Therefore,

$$f(x, x) = 2x^7.$$

Consequently in every neighbourhood of $(0, 0)$, there exist points (x, y) with $y = x$ such that $f(x, x)$ assumes both positive and negative values, showing that $f(x, y)$ has neither maximum nor minimum at $(0, 0)$.

Now we give an example of a function, where $b^2 - 4ac = 0$ at (x_o, y_o), but the function has an extremum at (x_o, y_o).

Example 15: Consider the function $f(x, y) = y^2 + x^2y + x^4$

Here $\frac{\partial f}{\partial x}(0, 0) = 0 = \frac{\partial f}{\partial y}(0, 0)$.

$$\frac{\partial^2 f}{\partial x^2}(0, 0) = 0, \quad \frac{\partial^2 f}{\partial y^2}(0,0) = 2, \quad \frac{\partial^2 f}{\partial x \partial y}(0, 0) = 0.$$

Therefore, $4\left(\frac{\partial^2 f}{\partial x \, \partial y}\right)^2 - 4\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) = 0$ at $(0, 0)$.

But $f(x, y) = (y + \frac{x^2}{2})^2 + \frac{3}{4}x^4$. Therefore,

$f(x, y) \geq 0 = f(0, 0)$ for all x, y, showing that $(0, 0)$ is the minimum of $f(x, y)$.

Now we shall see an application of the concept of maxima and minima.

Example 16: Of all the triangles of a fixed perimeter, let us find the one with maximum area.

If the sides are x, y, z, then the area A is given by the formula

$$A^2 = s(s - x)(s - y)(s - z),$$

where $s = \frac{1}{2}(x + y + z)$ is the semi-perimeter.

Thus,

$$2s = x + y + z, \text{ or}$$

$$s - z = x + y - s$$

$$\therefore \quad A^2 = s(s-x)\ (s-y)\ (x+y-s).$$

Here s is a constant and x and y are variables. Therefore, in order to maximize A, it is sufficient to maximize

$$f(x, y) = (s-x)\ (s-y)\ (x+y-s).$$

Now,

$$\frac{\partial f}{\partial x} = (s-y)\ (2s-2x-y)$$

$$\frac{\partial f}{\partial y} = (s-x)\ (2s-2y-x)$$

$$\frac{\partial^2 f}{\partial x^2} = -2(s-y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3s + 2x + 2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2\ (s-x).$$

Since in a triangle, $s \neq x$ and $s \neq y$,

the equations $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ imply that

$$2x + y = 2s, \quad x + 2y = 2s.$$

Consequently,

$$x = y = \frac{2}{3}\ s.$$

This gives the stationary points of f.

For these values of x, y, we get

$$a = c = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = -\frac{2s}{3} < 0, \text{ and}$$

$$b = 2\frac{\partial^2 f}{\partial x \partial y} = -\frac{2s}{3}.$$

$$b^2 - 4ac = \frac{4s^2}{9} - \frac{16s^2}{9} = -\frac{4s^2}{3} < 0,$$

These conditions ensure a maximum. Now when $x = y = \frac{2s}{3}$, we get

$$c = 2s - x - y = \frac{2s}{3}.$$

Hence x = y = z, i.e., the triangle is equilateral.

In the next section you will study a method for finding the maximum or minimum of functions of several variables subject to certain constraints. But first it is time to solve some exercises.

E17) Find the stationary points and the local extreme values for the following functions:

a) $f(x, y) = x^2 + 2y^2 - x$

b) $f(x, y) = x^2 + y^3 + 3xy^2 - 2x$

c) $f(x, y) = y + x \sin y$

E18) Discuss the behaviour of the function

$$f(x, y) = 2\cos(x + y) + e^{xy}$$

at the origin.

E19) Let n be an integer, $n \geq 2$, and let $f(x, y) = ax^n + cy^n$, where $ac \neq 0$.

a) Find the stationary points of f.

b) Find the local extreme values, given that

(i) $a > 0, c > 0$, (ii) $a < 0, c < 0$.

8.4 LAGRANGE'S MULTIPLIERS

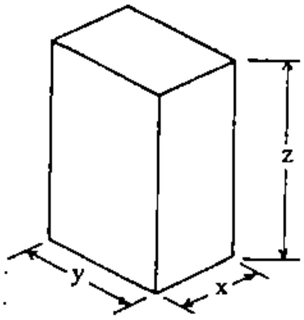


Fig. 5

Suppose we want to construct a closed box in the form of a parallelopiped of maximum volume using a piece of tin of area A. Let x, y, z denote the length, width and height of the box, respectively. Then the problem reduces to finding the maximum of the function

$$f(x, y, z) = x\,y\,z,$$

given that $2xy + 2xz + 2yz = A$ ... (9)

In Fig. 5, you can see a closed box. xyz is the volume of this box and $2xy + 2xz + 2yz$ is its surface area.

In this section we shall study such problems for functions of two variables, where the variables satisfy some side conditions as in (9). That is, we will discuss a method to find out the maximum and minimum values of a function

$$z = f(x, y),$$

given that x and y are connected by an equation

$$g(x, y) = 0.$$

If we could eliminate one variable from the equation $z = f(x, y)$ with the help of the relation $g(x, y) = 0$, then z would become a function of one variable. Then we can easily find its extreme values.

So, the problem reduces to finding the maximum and minimum values for a function of one variable.

Here is an example to illustrate this.

Example 17: Suppose we want to find the extreme values of the function $f(x, y) = x^2 + 2y^2 - x$ on the unit circle $x^2 + y^2 = 1$.

We first use the constraint $x^2 + y^2 = 1$ to reduce the function

$$f(x, y) = x^2 + 2y^2 - x \quad \text{... (10)}$$

to a one-variable function.

Thus, we get

$$f(x, y) = x^2 + 2(1 - x^2) - x$$
$$= 2 - x^2 - x.$$

Here we have a one-variable function, say $g(x) = 2 - x^2 - x$, defined on the interval $[-1, 1]$. Now we shall find out the points of extrema for g(x). By solving $g'(x) = -2x - 1 = 0$, we get that $x = -\frac{1}{2}$ is a stationary point of g(x). Then to check whether $x = -\frac{1}{2}$ is a maximum or a minimum, we calculate

$$g''(x) = -2$$

Thus, $g''(-\frac{1}{2}) < 0$. Therefore, by the second derivative test for one variable we get that $x = -\frac{1}{2}$ is a point of maximum for g. g has no minimum. Now we substitute the value $x = -\frac{1}{2}$ in (10). Then we have

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4} \text{ and } y = \pm\frac{\sqrt{3}}{2}.$$

Therefore we conclude that the function has a maximum at two points $(-\frac{1}{2}, +\frac{\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$. Also $f(-\frac{1}{2}, \frac{\sqrt{3}}{2}) = \frac{1}{4} + \frac{3}{2} + \frac{1}{2} = \frac{9}{4} = f(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$

Thus, the maximum value of the function on the unit circle is $\frac{9}{4}$.

You must have found this example quite easy to follow. But it is not always feasible to use this procedure. The reduction of the given function to a function of one variable using the given constraint might prove to be quite cumbersome or sometimes might not be possible at all.

We now present an alternative method which is often more convenient. This method is known as the method of Lagrange's multipliers.

Suppose we want to maximize or minimize a function $z = f(x, y)$ subject to the condition $g(x, y) = 0$. Theoretically, z is a function of a single variable (say x) and at the extreme values $\frac{dz}{dx} = 0$, i.e.,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \qquad \text{... (11)}$$

From the relation $g(x, y) = 0$, we find that at the extrema, we must have

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0 \qquad \text{... (12)}$$

Multiplying Equation (12) by an undetermined multiplier λ and adding this to Equation (11), we obtain

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial y}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)\frac{\partial y}{\partial x} = 0 \qquad \text{... (13)}$$

Choose λ so that the coefficient of $\frac{dy}{dx}$ vanishes in (13).

(This is possible and would become clear after you have studied the implicit function theorem in Unit 10.)

Hence at the points of extrema, we have

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0 \qquad \text{... (14)}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0 \qquad \text{... (15)}$$

$$g(x, y) = 0 \qquad \text{... (16)}$$

From these equations, we can determine the three unknowns x, y, λ. The values of x, y give us the coordinates of the stationary points. The role of λ is over and we don't need it any more. We may add here that each stationary point so determined need not be a maximum or minimum. Sometimes, we can determine their nature by simple observation of the equation $z = f(x,y)$. In some cases we can apply the second derivative test, by eliminating the dependent variable.

In fact, you can observe that equations (14), (15) and (16) are obtained by equating the partial derivatives of the function

$$F(x, y, \lambda) = f(x, y) + \lambda\, g(x, y).$$

Lagrange (1736-1813)

to 0, treating x, y and λ as independent variables. We can explain this in a simple way as follows :

Suppose we are given the function f(x, y), whose extrema are to be found subject to the constraint g(x, y) = 0. We form the auxiliary function

$$F(x, y, \lambda) = f(x, y) + \lambda\, g(x, y), \qquad \ldots (17)$$

where λ is to be determined. Then we find the three partial derivatives of F(x, y, λ) and equate these to 0. Then we solve the three equations. The values of (x, y) thus obtained are the stationary points of the given function under the given constraint.

The number λ is called **Lagrange's multiplier** after Joseph Louis Lagrange, a leading mathematician of the 18th century.

Here are some examples to illustrate the procedure.

Example 18 : Let us find the largest and smallest values of f(x, y) = x + 2y on the circle $x^2 + y^2 = 1$.

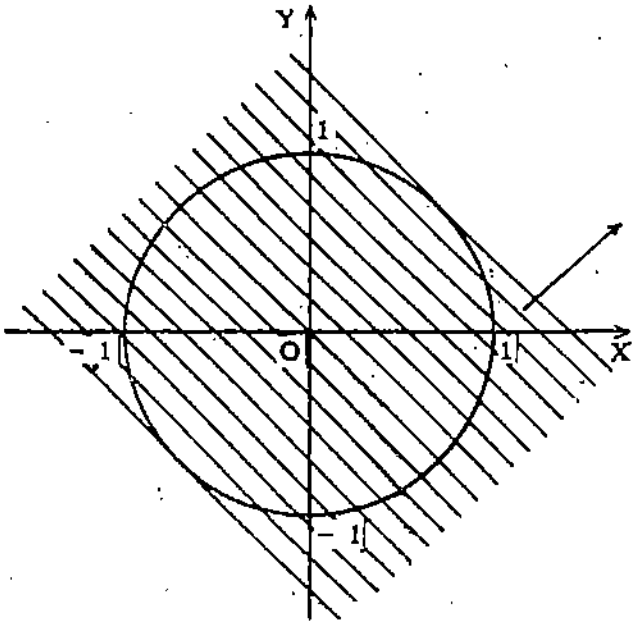


Fig. 6

In Fig. 6 you can see that f takes its maximum at a point in the first quadrant and its minimum at a point in the third quadrant.

Here

$$f(x, y) = x + 2y \text{ and } g(x, y) = x^2 + y^2 - 1.$$

Now $\frac{\partial f}{\partial x} = 1$, $\frac{\partial g}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2$ and $\frac{\partial g}{\partial y} = 2y$.

The auxiliary function is

$$F(x, y, \lambda) = (x + 2y) + \lambda\,(x^2 + y^2 - 1)$$

Therefore, to find stationary points we have to solve the system of equations,

$$1 + \lambda\, 2x = 0$$

$$2 + \lambda\, 2y = 0$$

$$x^2 + y^2 = 1$$

Solving the first two equations, we get

$$x = \frac{1}{2\lambda}, \quad y = \frac{1}{\lambda} \text{ and } \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{\lambda}\right)^2 = 1.$$

Using the third equation, we get

$$\lambda^2 = \frac{5}{4}, \lambda = \pm \frac{1}{2}\sqrt{5}.$$

The value $\lambda = \frac{1}{2}\sqrt{5}$ gives

$$x = \frac{1}{\sqrt{5}}, y = \frac{2}{\sqrt{5}}, f(x, y) = \frac{5}{\sqrt{5}} = \sqrt{5}.$$

The value $\lambda = -\frac{1}{2}\sqrt{5}$ yields.

$$x = -\frac{1}{\sqrt{5}}, \quad y = -\frac{2}{\sqrt{5}}, f(x, y) = -\frac{5}{\sqrt{5}} = -\sqrt{5}.$$

Thus, the stationary points are $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)$.

Since $f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \sqrt{5}$ and $f\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right) = -\sqrt{5}$, we get that the largest value is $\sqrt{5}$ and the smallest value is $-\sqrt{5}$.

Example 19 : Suppose we want to find the extreme values of the function f (x, y) = xy on the surface g (x, y), where

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

We first write the auxiliary function

$$F(x, y, \lambda) = xy + \lambda\left(\frac{x^2}{8} + \frac{y^2}{2} - 1\right)$$

Now we have to solve the system of equations

$$y + \frac{\lambda x}{4} = 0$$

$$x + \lambda y = 0$$

and

$$x^2 + 4y^2 = 8$$

$g(x, y) = 0 \iff x^2 + 4y^2 = 8.$

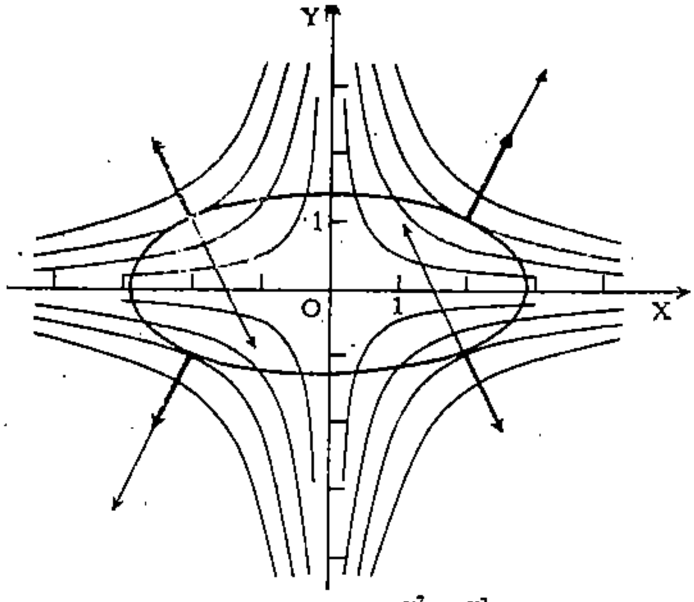
From the first two equations we get $x = \frac{\lambda^2}{4} x$, or $\lambda = \pm 2$.

Then $x = \pm 2y$. Substituting this in the third equation, we get

$$4y^2 + 4y^2 = 8 \Rightarrow y = \pm 1.$$

Correspondingly, we get $x = \pm 2$. Thus, the extreme values are obtained at the four points (2,1), (2, −1), (−2,1) and (−2, −1). The distinct values at these points are given by f(x,y) = 2 and f(x, y) = −2. Therefore, the maximum value is 2 and the minimum value is −2.

Note that f(x, y) = xy represents a hyperboloid and $g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0$ represents an ellipse. In Figure 7, you can see the points of extrema of f, subject to the condition that g (x, y) = 0.



$f(x,y) = xy, \; g(x,y) = \frac{x^2}{8} + \frac{y^2}{2} - 1$

Fig.7

Example 20 : Let us find the right angled triangle of perimeter 1 with the largest area. Suppose ABC is a right angled triangle with perimeter 1 (See Fig. 8)

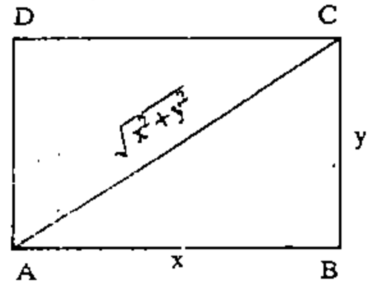


Fig. 8

Let the sides of the triangle be x, y, $\sqrt{x^2+y^2}$. Complete the rectangle ABCD with AC as diagonal. Then

$$f(x, y) = \text{area of } \Delta\, ABC = \frac{1}{2}\; x\; y.$$

The perimeter of the triangle $= x + y + \sqrt{x^2 + y^2}$.

We know that

$$x + y + \sqrt{x^2 + y^2} = 1.$$

We have to find the maximum of f (x, y) subject to the condition that
$g(x, y) = x + y + \sqrt{x^2 + y^2} = 1.$

Let us form the system of equations for this f and g. We get

$$\frac{1}{2}\; y + \lambda \left[1 + \frac{x}{\sqrt{x^2 + y^2}} \right] = 0$$

$$\frac{1}{2}\; x + \lambda \left[1 + \frac{y}{\sqrt{x^2 + y^2}} \right] = 0$$

$$x + y + \sqrt{x^2 + y^2} - 1 = 0.$$

From the first two equations we have

$$\frac{\frac{1}{2}y}{1 + \frac{x}{\sqrt{x^2+y^2}}} = \frac{\frac{1}{2}x}{1 + \frac{y}{r\sqrt{x^2+y^2}}}$$

or $\dfrac{y}{\sqrt{x^2+y^2}+x} = \dfrac{y}{\sqrt{x^2+y^2}+y}$

or $\dfrac{y}{1-y} = \dfrac{x}{1-x}$

or $y - xy = x - xy$

or $x = y$.

Therefore, the sides are x, x, and $\sqrt{2}\,x$,

such that $x + x + \sqrt{2}\,x = 1$, i.e., $(2 + \sqrt{2})x = 1$.

i.e. $x = \dfrac{1}{2+\sqrt{2}}$

Thus, the required sides are $\dfrac{1}{2+\sqrt{2}}$, $\dfrac{1}{2+\sqrt{2}}$ and $\dfrac{1}{1+\sqrt{2}}$

The following exercises will give you enough practice of solving problems by using the method of Lagrange's multipliers.

E20) Find the maximum value of $f(x, y) = 3xy$ on $2x + y = 8$.

E21) Of all the pairs of numbers whose sum is 70, find the one that has the maximum product.

E22) Find the minimum value of the function $f(x, y) = x + y^2$ on $2x^2 + y^2 = 1$.

E23) Find the point on the parabola $y - x^2 = 0$ that maximizes the function $f(x, y) = 2x - y$.

Now let us quickly recall the points covered in this unit.

8.5 SUMMARY

In this unit, we have

1) Defined Taylor polynomials of any order for functions of two variables.

2) Stated Taylor's theorem for functions of two variables.

   The second Taylor expansion:

$$f(x, y) = f(x_0, y_0) + \left[\frac{\partial f(x_0, y_0)}{\partial x}(x - x_0) + \frac{\partial f(x_0, y_0)}{\partial y}(y - y_0)\right]$$

$$+ \frac{1}{2}\left[\frac{\partial^2 f(x_0, y_0)}{\partial x^2}(x - x_0)^2 + 2\frac{\partial^2 f(x_0, y_0)}{\partial x\,\partial y}(x - x_0)(y - y_0) + \frac{\partial^2 f(x_0, y_0)}{\partial x^2}(y - y_0)^2\right]$$

$$+ R_3(x, y)$$

3) Defined the terms 'local maxima', 'local minima' and 'stationary point' for functions of two variables and discussed the relationship between them.

4) Derived the second derivative test for points of extrema:

   i) $f(x, y)$ has a minimum (maximum) if $b^2 - 4ac < 0$ and $a > 0$ $(a < 0)$.

   ii) $f(x, y)$ has neither maximum nor minimum if

   $b^2 - 4ac > 0$.

5) Used the method of Lagrange multipliers to find the maximum and minimum values of a function of two variables.

8.6 SOLUTIONS AND ANSWERS

E1) $f(x) = e^x$. The different derivatives of f are

$$f'(x) = e^x, f''(x) = e^x, \ldots\ldots, f^n(x) = e^x$$

Thus, $f'(2) = e^2$, $f''(2) = e^2$, ... $f^n(2) = e^2$

Then the n^{th} Taylor polynomial is given by

$$P_n(x) = e^2 + \frac{e^2(x-2)}{1!} + \frac{e^2(x-2)^2}{2!} + \ldots + \frac{e^2(x-2)^n}{n!}$$

$$= e^2\left[1 + \frac{(x-2)}{1!} + \frac{(x-2)^2}{2!} + \ldots + \frac{(x-2)^n}{n!}\right]$$

E2) $f(x) = \sin x$. The different derivatives of f are given by

$f(x) = \sin x$, $f'(x) = \cos x$

$f^{(2)}(x) = -\sin x$, $f^{(3)}(x) = -\cos x$

$f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$

and $f^{(6)}x = -\sin x$.

Thus, $f(0) = 0$, $f^1(0) = 1$, $f^2(0) = 0$, $f^{(3)}(0) = -1$, $f^{(4)}(0) = 0$, $f^{(5)}(0) = 1$

and $f^{(6)}(x) = 0$.

This shows that the even derivatives are zero whereas the odd derivatives are alternatively + 1 and − 1.

Thus, the 5th Taylor polynomial of f is given by

$P_6(x) = x - x^3 + x^5$.

E3) a) Let $f(x) = x^2 - 3x + 4$. Then

$f'(x) = 2x - 3$

and $f''(x) = 2$.

Therefore, $f(-2) = 14$, $\frac{f'(-2)}{1!} = -7$,

and $\frac{f''(-2)}{2!} = \frac{2}{2} = 1$.

Hence the polynomial is

$$P_2(x) = 14 - 7(x+2) + 1(x+2)^2$$

b) $P_4(x) = -1 - 11(x-1) + 9(x-1)^2 - (x-1)^3 + (x-1)^4$.

E4) By Theorem 1, there exists a unique polynomial P(x) given by

i) $$P(x) = \sum_{m=0}^{2} \frac{f^m(1)}{m!}(x - x_0)^m$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= 2 - 1(x-1) + \frac{2}{2}(x-1)^2$$

$= 2 - x + 1 + (x^2 - 2x + 1)$

$= x^2 - 3x + 4$

E5) Here $f(x) = \frac{1}{1+x}$, $\quad f(0) = 1$

$f'(x) = \frac{-1}{(1+x)^2}$, $\quad f'(0) = -1$

$f''(x) = \frac{1.2}{(1+x)^3}$, $\quad f''(0) = 1.2$

$f'''(x) = \frac{(-1)(-2)(-3)}{(1+x)^4}$, $\quad f'''(0) = (-1)^3\, 1 \times 2 \times 3$

$f^{n}(x) = \frac{(-1)(-2)\ldots(-n)}{(1+x)^{n+1}}$, $\quad f^{n}(0) = (-1)^n\, 1 \times 2 \times \ldots \times n$

The function f(x) and its derivatives of different orders are continuous in $]-\frac{1}{2}, 1[$. Therefore, by Taylor's theorem

$$f(x) = 1 - x + \frac{2}{2!}x^2 + \ldots + \frac{(-1)^n\, 1 \times 2 \times \ldots \times n}{n!}x^n$$

$$+ \frac{(-1)^{n+1}\, 1 \times 2 \times \ldots \times (n+1)}{(n+1)!\,(1+c)^{n+1}}$$

where c is a point between 0 and x.

E6) We have calculated derivatives upto order 6 of the function $f(x) = \sin x$ in E2). That pattern shows that the derivatives of even order are alternatively sin x and – sin x whereas the derivatives of odd orders are alternatively – cos x and + cos x.

Therefore,

$f^{(2k)}(x) = (-1)^k \sin x$, $f^{(2k+1)}(x) = (-1)^k \cos x$

Also the derivatives of any order are continuous and $|f^n(x)| \le 1$ for any n and any x. Therefore, by Taylor's theorem, we have

$$\sin x = \sin\frac{\pi}{6} + \cos\frac{\pi}{6}\left(x - \frac{\pi}{6}\right) - \sin\frac{\pi}{6}\left(x - \frac{\pi}{6}\right)^2 - \cos\frac{\pi}{6}\left(x - \frac{\pi}{6}\right)^3$$

$$+ \ldots + \frac{f^{n+1}(c)}{(n+1)!}\left(x - \frac{\pi}{6}\right)^{n+1}$$

where c is a point between $\frac{\pi}{6}$ and x.

E7) a) 5

b) 3

c) 4

E8) $f(x, y) = e^{x+y}$

$f(0, 0) = 1$

$f_x(x, y) = f_y(x, y) = f_{xy}(x, y) = f_{xx}(x, y) = f_{yy}(x, y) = e^{x+y}$.

Thus, $f_x(0, 0) = f_y(0, 0) = f_{xy}(0, 0) = f_{xx}(0, 0) = f_{yy}(0, 0) = 1$.

Hence the 2nd Taylor polynomial $P_2(x, y)$ of f is

$$P_2(x, y) = 1 + [x + y] + \frac{1}{2}[x^2 + 2xy + y^2]$$

$$= 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2$$

E9) $f(x, y) = 2 + x^3 + y^3$, $f(1, 1) = 4$

$f_x(x, y) = 3x^2$, $f_x(1, 1) = 3$

$f_y(x, y) = 3y^2$, $f_y(1,1) = 3$

$f_{xy}(x, y) = 0$, $f_{xy}(1, 1) = 0$

$f_{xx}(x, y) = 6x$, $f_{xx}(1, 1) = 6$

$f_{yy}(x, y) = 6y$, $f_{yy}(1, 1) = 6.$

The Taylor polynomials are given by

$P_0(x, y) = f(1, 1) = 4.$

$P_1(x, y) = 4 + 3(x-1) + 3(y-1) = 3x + 3y - 2.$

$$P_2(x, y) = P_1(x, y) + \frac{1}{2}[6(x-1)^2 + 6(y-1)^2]$$

$$= 3x + 3y - 2 + 3(x-1)^2 + 3(y-1)^2$$

$P_m(x, y) = P_2(x, y)$ for $m \geq 2.$

E10) Let $P(x, y) = a_{00} + a_{10}x + a_{01}y + [a_{20}x^2 + a_{11}xy + a_{02}y^2]$

$P(0, 0) = a_{00}.$

$$\frac{\partial P}{\partial x} = a_{10} + 2a_{20}x + a_{11}y, \left(\frac{\partial P}{\partial x}\right)_{(0,0)} = a_{10}$$

$$\frac{\partial P}{\partial y} = a_{01} + 2a_{02}y + a_{11}x, \left(\frac{\partial P}{\partial y}\right)_{(0,0)} = a_{01}$$

$$\frac{\partial^2 P}{\partial x^2} = 2a_{20}, \qquad \left(\frac{\partial^2 P}{\partial x^2}\right)_{(0,0)} = 2a_{20}$$

$$\frac{\partial^2 P}{\partial x \partial y} = a_{11}, \qquad \left(\frac{\partial^2 P}{\partial x \partial y}\right)_{(0,0)} = a_{11}$$

$$\frac{\partial^2 P}{\partial y^2} = 2a_{02}, \qquad \left(\frac{\partial^2 P}{\partial y^2}\right)_{(0,0)} = 2a_{02}$$

Thus the 2nd Taylor polynomial of P is

$$T_2(x, y) = a_{00} + a_{10}x + a_{01}y + \frac{1}{2}[2a_{20}x^2 + 2a_{11}xy + 2a_{02}y^2]$$

$$= a_{00} + a_{10}x + a_{01}y + [a_{20}x^2 + a_{11}xy + a_{02}y^2]$$

$$= P(x, y)$$

E11) $f(x, y) = xy^2 + \cos xy$

$$f(1, \frac{\pi}{2}) = \frac{\pi^2}{4}$$

$$\frac{\partial f}{\partial x} = y^2 - y \sin xy, \left(\frac{\partial f}{\partial x}\right)_{(1, \frac{\pi}{2})} = \frac{\pi^2}{4} - \frac{\pi}{2} = \frac{\pi^2 - 2\pi}{4}$$

$$\frac{\partial f}{\partial y} = 2xy - x \sin xy, \left(\frac{\partial f}{\partial y}\right)_{(1, \frac{\pi}{2})} = \pi - 1$$

$$\frac{\partial^2 f}{\partial x^2} = y^2 \cos xy, \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1, \frac{\pi}{2})} = 0.$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y - [\sin xy + yx \cos xy],$$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,\frac{\pi}{2})} = \pi - 1$

$\frac{\partial^2 f}{dy^2} = 2x - x^2 \cos xy, \frac{\partial^2 f}{\partial y^2}(1, \pi/2) = 2$

The second Taylor expansion is

$$f(x,y) = \frac{\pi^2}{4} + \left[\frac{(\pi^2 - 2\pi)}{4}(x-1) + (\pi - 1)(y - \frac{\pi}{2})\right.$$

$$\left. + \frac{1}{2}[0 + 2(\pi - 1)(x-1)(y - \frac{\pi}{2}) + 2(y - \frac{\pi}{2})^2\right].$$

E12) $f(x, y) = e^x \sin y$

$f(0, 0) = 0. (\frac{\partial f}{\partial x})_{(0,0)} = 0, (\frac{\partial f}{\partial y})_{(0,0)} = 1.$

$\left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = 0, \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = 1, \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = 0$

Then the required polynomial is $= y + xy$.

E13) Suppose f is a function of two variables. We say that the function f has a local minimum at $P(x_0, y_0)$ if there exists some open disc $S(P, r), r \in \mathbf{R}^+$, contained in the domain of f, such that for all $(x, y) \in S(P, r)$ we have

$f(x, y) \geq f(x_0, y_0)$.

E14) $f(x, y) = \frac{x}{x^2 + y^2 - 4}$

$\frac{\partial f}{\partial x} = \frac{[x^2 + y^2 - 4 - 2x^2]}{(x^2 + y^2 - 4)^2}$

$\frac{\partial f}{\partial y} = \frac{-2xy}{(x^2 + y^2 - 4)^2}$

$\frac{\partial f}{\partial x} = 0 \Rightarrow y^2 - x^2 - 4 = 0$

$\Rightarrow y^2 - (4 + x^2) = 0$

$\Rightarrow y^2 = 4 + x^2$ ....(*)

$\frac{\partial f}{\partial y} = 0 \Rightarrow 2xy = 0 \Rightarrow xy = 0$. This shows that either $x = 0$ or $y = 0$. But y cannot be zero because $y^2 > 0$ by Equation (*).

Therefore $x = 0$. Then $y^2 = 4$ and $y = \pm 2$

Hence the points are (0, 2) and (0, −2).

E15) $f(x, y) = x^2 + y^2 - 8x - 2y + 18$ is differentiable everywhere on the plane.

$\frac{\partial f}{\partial x} = 2x - 8, \frac{\partial f}{\partial y} = 2y - 2$

$\frac{\partial f}{\partial y} = 0 \Rightarrow x = 4, \frac{\partial f}{\partial y} = 0 \Rightarrow y = 1.$

Now check whether (4, 1) is a point of maxima or minima. Note that

$f(4, 1) = 16 + 1 - 32 - 2 + 18 = 1$

Now f(x, y) can be written as,

$f(x, y) = x^2 - 8x + y^2 - 2y + 18$

$= x^2 - 8x + 16 + y^2 - 2y + 1 + 1$

$= (x - 4)^2 + (y - 1)^2 + 1$

$\geq 1 = f(4, 1)$

Thus (4,1) is a global minimum for f.

E16) a) $f(x, y) = 1 + x^2 - y^3$. Then

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -3y^2$$

The stationary points are the solutions of

$$\frac{\partial f}{\partial x} = 2x = 0 \text{ and } \frac{\partial f}{\partial y} = -3y^2 = 0$$

This happens if and only if x = 0 and y = 0. Thus (0, 0) is the only stationary point of f.

b) $f(x, y) = (x + y) e^{-xy}$. Then

$$\frac{\partial f}{\partial x} = (x + y)(-y) e^{-xy} + e^{-xy}$$

$$= (-xy - y^2 + 1) e^{-xy}$$

$$\frac{\partial f}{\partial y} = (x + y)(-x) e^{-xy} + e^{-xy}$$

$$= (-xy - x^2 + 1) e^{-xy}$$

Then the stationary points are solutions of

$$-xy - y^2 + 1 = 0 \text{ and } -xy - x^2 + 1 = 0$$

Thus,

$$xy + y^2 = 1 \text{ and } xy + x^2 = 1.$$

This happens if and only if $x^2 = y^2$. Therefore $x = \pm y$. But $x = -y$ is not possible since otherwise $1 = xy + y^2 = -y^2 + y^2 = 0$. ∴ Substituting x= y in any of the earlier equations, we get that

$y = \pm \frac{1}{\sqrt{2}}$. Thus the points are

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

E17) a) The function $f(x, y) = x^2 + 2y^2 - x$ has continuous partial derivatives of any order, everywhere on the plane. $\frac{\partial f}{\partial x} = 2x - 1$, $\frac{\partial f}{\partial y} = 4y$.

The stationary points are given by

$2x - 1 = 0$ and $4y = 0$

i.e., $x = \frac{1}{2}$ and $y = 0$.

Thus $(\frac{1}{2}, 0)$ is the only stationary point of f. Now,

$$\frac{\partial^2 f}{\partial x^2} = 2, \therefore a = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(\frac{1}{2}, 0)} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0, \therefore b = 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(\frac{1}{2}, 0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2} = 4, \therefore c = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(\frac{1}{2}, 0)} = 4.$$

This shows that $b^2 - 4ac = -32 < 0$ and $a > 0$. Therefore, by Theorem 7 f has a minimum at $(\frac{1}{2}, 0)$.

The extreme value is $f(\frac{1}{2}, 0) = -\frac{1}{4}$.

b) $f(x, y) = x^2 + y^3 + 3xy^2 - 2x$.

$\frac{\partial f}{\partial x} = 2x + 3y^2 - 2$, $\frac{\partial f}{\partial y} = 3y^2 + 6xy$

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0 \Rightarrow 2x + 3y^2 - 2 = 0$ ......(*)

$3y^2 + 6xy = 0$ ......(**)

From (**), $3y(y + 2x) = 0 \Rightarrow y = 0$ or $y = -2x$ ......

Substituting $y = 0$ in (*), we get that $x = 1$.

Thus, $(1, 0)$ is a stationary point.

Now substituting $y = -2x$ in (*), we get

$2x + 12x^2 - 2 = 0 \Rightarrow x = \frac{1}{3}$ or $x = -\frac{1}{2}$.

When $x = \frac{1}{3}$, $y = \frac{-2}{3}$; when $x = -\frac{1}{2}$, $y = 1$.

Thus, the stationary points are $(1, 0)$, $(\frac{1}{3}, \frac{-2}{3})$ and $(\frac{1}{2}, 1)$.

We first check the point $(1, 0)$ for extreme values. We get

$\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial x \partial y} = 6y$, $\frac{\partial^2 f}{\partial y^2} = 6y + 6x$

Then, $a = \left(\frac{\partial^2 f}{\partial x^2}\right)_{(1,0)} = 2$, $b = \left(2\frac{\partial^2 f}{\partial x \partial y}\right)_{(1,0)} = 0$, $c = \left(\frac{\partial^2 f}{\partial y^2}\right)_{(1,0)} = 6$.

$b^2 - 4ac = -48 < 0$ and $a > 0$.

Therefore the function has a local minimum at $(1, 0)$.

The minimum value is $f(1, 0) = -1$

Now at $(\frac{1}{3}, -\frac{2}{3})$,

$a = 2$, $b = -8$, $c = -2$,

$b^2 - 4ac = 64 + 16 > 0$ and $a > 0$.

Therefore the function has no extremum at $(\frac{1}{3}, -\frac{2}{3})$.

Now at $(-\frac{1}{2}, 1)$,

$a = 2$, $b = 12$, $c = 3$.

$b^2 - 4ac = 12^2 - 24 > 0$.

$\therefore$ f has no extremum at $(-\frac{1}{2}, 1)$

c) $f(x, y) = y + x \sin y$

$\frac{\partial f}{\partial x} = \sin y = 0$, $\frac{\partial f}{\partial y} = 1 + x \cos y = 0$.

sin y = 0 implies that y = 2nπ or y = (2n + 1) π , n is an integer. Then cos ((2n +1) π) = −1 and cos 2nπ = 1. so x = 1, or x = −1. Thus the stationary points are (1, 2nπ) and (−1 , (2n + 1) π)

$$\frac{\partial^2 f}{\partial x^2} = 0, \frac{2\partial^2 f}{\partial x \partial y} = 2\cos y, \frac{\partial^2 f}{\partial y^2} = -x \sin y$$

Therefore, a = 0, b = ± 2 , c = 0.

and $b^2 - 4ac > 0$.

Thus, f has no extremum at any point.

E18) $f(x,y) = 2\cos(x + y) e^{xy}$

$$\frac{\partial f}{\partial x} = -2\sin(x + y) + ye^{xy}$$

$$\frac{\partial f}{\partial y} = -2\sin(x + y) + xe^{xy}$$

(0, 0) is a stationary point as both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ vanish at (0,0). Now

$$\frac{\partial^2 f}{\partial x^2} = -2\cos(x + y) + y^2 e^{xy}, \left(\frac{\partial^2 f}{\partial x^2}\right)_{(0,0)} = -2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2\cos(x + y) + (e^{xy} + yx e^{xy}), \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{(0,0)} = -1$$

$$\frac{\partial^2 f}{\partial y^2} = -2\cos(x + y) + x^2 e^{xy}, \left(\frac{\partial^2 f}{\partial y^2}\right)_{(0,0)} = -2$$

Then $b^2 - 4ac = -12 < 0$ and $a < 0$. Therefore, by Theorem 7, the function f has a local maximum at (0, 0).

E19) $f(x, y) = a x^n + c y^n, n \geq 2.$

$$\frac{\partial f}{\partial x} = na x^{n-1}, \frac{\partial f}{\partial y} = ncy^{n-1}$$

$\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ => $nax^{n-1} = 0$ and $ncy^{n-1} = 0$. This is possible only when x = 0 and y = 0, since $a \neq 0 \neq c$. Thus (0, 0) is the only stationary point of f.

Here we note that at (0, 0),

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y^2} = 0.$$

Therefore we cannot use Theorem 7. We have f (0, 0) = 0.

i) Now when a > 0 and c > 0, then

$$f(x, y) = ax^n + cy^n \geq 0 = f(0, 0), \text{ if n is even.}$$

This shows that if n is even, then the function has a minimum at (0, 0). The minimum value is 0.

If n is odd, in every neighbourhood of (0, 0) f takes on both positive and negative values

$$f(x, y) < 0 \text{ for } x < 0 \text{ and } y < 0, f(x, y) > 0 \text{ for } x > 0 \text{ and } y > 0.$$

Therefore, if n is odd, then (0, 0) is a saddle point of f.

ii) When a < 0, c < 0 , then $f(x, y) = ax^n + cy^n \leq 0 = f(0, 0)$, if n is even. Therefore, if n is even, f has a maximum at (0, 0). If n is odd, you can check that (0, 0) is a saddle point.

E20) $f(x, y) = 3xy$, $g(x, y) = 2x + y - 8$

$$\frac{\partial f}{\partial x} = 3y, \quad \frac{\partial g}{\partial x} = 2$$

$$\frac{\partial f}{\partial y} = 3x, \quad \frac{\partial g}{\partial y} = 1$$

Now we have to solve the system of equations

$$3y + 2\lambda = 0$$

$$3x + \lambda = 0$$

$$2x + y - 8 = 0$$

Then $y = -\frac{2}{3}\lambda$, $x = -\frac{1}{3}\lambda$. Substituting in the third equation we get

$$-\frac{2}{3}\lambda - \frac{2}{3}\lambda - 8 = 0$$

or $\frac{-4}{3}\lambda = 8$. Or, $\lambda = -6$.

Then $x = 2$, $y = 4$. $f(2, 4) = 24$

Thus, $(2, 4)$ is an extremum.

Now we shall check whether it is a maximum or minimum.

The point $(1, 6)$ also satisfies $g(x, y) = 0$, and

$$f(1, 6) = 18 \le f(2, 4)$$

Hence, $f(2, 4) = 24$ is the maximum value of f.

E21) Let x and y be the two numbers whose sum is 70 and whose product is maximum.

Let $f(x, y) = xy$ and $g(x, y) = x + y$. We have $x + y = 70$. To maximize $f(x, y)$ we use the method of Lagrange multipliers and solve the system of equations

$$y + \lambda = 0$$

$$x + \lambda = 0$$

$$x + y = 70.$$

Then $-2\lambda = 70$. Thus, $\lambda = -35$. Therefore $x = 35$ and $y = 35$ and the extremum value is $f(35, 35) = 1225$. Check that it is the maximum value.

E22) $f(x, y) = x + y^2$, $g(x, y) = 2x^2 + y^2 - 1$

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial g}{\partial x} = 4x$$

$$\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial g}{\partial x} = 2y$$

The system of equations is

$$1 + \lambda 4x = 0$$

$$2y + \lambda 2y = 0$$

$$2x^2 + y^2 - 1 = 0$$

$2y = -\lambda 2y \Rightarrow \lambda = -1$. Therefore $x = \frac{1}{4}$. Substituting this in the third equation, we get

$$y^2 = 1 - 2x^2 = 1 - \frac{1}{8} = \frac{7}{8} \quad \therefore \quad y = \pm\sqrt{\frac{7}{8}}.$$

The extreme value is given by

$$f\left(\frac{1}{4}, \sqrt{\frac{7}{8}}\right) = \frac{1}{4} + \frac{7}{8} = \frac{9}{8}$$. Check that this is the minimum value.

E23) The point is $(-1, 1)$.

UNIT 9 JACOBIANS

Structure

9.1 INTRODUCTION

The main purpose of this unit is to introduce you to the important notion of Jacobians. We show how Jacobians can be used to evaluate derivatives of functions defined implicitly. We also bring out the connection between functional dependence and Jacobians. For this we need an auxiliary but important result, that if all points of a domain are stationary points of a function, then the function must be constant. Naturally, we have to tell you what a domain is, and this is done in one of the sub-sections.

The real importance and usefulness of Jacobians will be clear to you in the next unit, where these will be used for the study of invertibility of functions and explicit determination of functions defined implicitly. You will also come across Jacobians in the next block, when you study change of variables in multiple integrals.

Objectives

After working through this unit you should be able to :

- calculate the Jacobians, whenever they exist, of functions of two or three variables,
- use the chain rule for Jacobians to calculate the Jacobians of many more functions and to find formulas for the derivatives of functions which are defined implicity,
- identify domains in R^2,
- determine whether two functions are dependent on each other or not.

9.2 JACOBIANS

Take a look at the following two situations for functions from $R \to R$.

1) Let us consider a real-valued function f defined on an open interval] a, b [. Now, if $f'(x_0) \neq 0$ for some $x_0 \in$] a, b [, then either $f'(x_0) > 0$ or $f'(x_0) < 0$. This means that f(x) is either strictly increasing or strictly decreasing in a neighbourhood of x_0. Therefore, we can say that there exists a real number $\delta > 0$ such that f is one-one on the open interval $]\, x_0 - \delta, x_0 + \delta\, [$. This means that the function f is invertible on this δ–neighbourhood of x_0.

2) Similarly, in the integral $\int_a^b f(x)\,dx$, if we make the substitution $x = \phi(t)$, then the integrand is replaced by $f(\phi(t))\,\phi'(t)$ and $\int_a^b f(x)\,dx = \int_\alpha^\beta f(\phi(t))\,\phi'(t)\,dt$, where $a = \phi(\alpha)$ and $b = \phi(\beta)$. (See Calculus, Sec. 11.3).

A natural question that arises is the following :

Suppose $u = f(x, y)$, $v = g(x, y)$, so that $(x, y) \to (u, v)$ is a mapping from $\mathbf{R}^2 \to \mathbf{R}^2$. What is the condition that the functions f, g should satisfy so that the transformation $(x, y) \to (u, v)$ is invertible in some open disc containing the point (x_0, y_0) ?

A similar question will arise when we have defined integrals of functions of two variables and would like to change the variables of integration.

In this section we introduce you to the notion of Jacobian, which has really no analogue in the theory of functions of one variable. However, Jacobians play the same role as the derivative in the two problems mentioned above. It has many other applications. We shall discuss some of these presently.

9.2.1 Definition and Examples

Before we give the precise definition, let us look at a specific example. Let

$$u = ax + by \quad \ldots (1)$$

$$v = cx + dy \quad \ldots (2)$$

Then the equations (1) and (2) define a linear transformation $\phi : (x, y) \to (u, v)$ from $\mathbf{R}^2$ to $\mathbf{R}^2$. Moreover, you know from your study of linear equations that the transformation ϕ is invertible if and only if the determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 .$$

But you can see that $a = \frac{\partial u}{\partial x}$, $b = \frac{\partial u}{\partial y}$, $c = \frac{\partial v}{\partial x}$, $d = \frac{\partial v}{\partial y}$.

Therefore, we can say that the transformation defined by (1) and (2) is invertible if and only if the determinant of the matrix

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

is non–zero.

Obviously, the above matrix and its determinant make sense even if u and v are not linear functions of x, y, but are any real-valued functions of x and y. We shall see later, in Unit 10, that if this matrix has non-zero determinant, then given any point $(x_0, y_0) \in \mathbf{R}^2$, there exists a neighbourhood N of (x_0, y_0) on which the transformation ϕ is invertible. Thus, this is the new notion we were looking for, which provides us some answer to the first question raised in the beginning of this section. The same notion would provide an answer to the second question also. But for this you have to wait till the next block. Now we give the precise definition.

Definition 1 : Let $f_1, \ldots, f_n$ be n real-valued functions of n variables $x_1, \ldots, x_n$, having first order partial derivatives at a point $a = (a_1, \ldots, a_n)$. Then the matrix

$$\begin{bmatrix} \frac{\partial f_1(a)}{\partial x_1} & \frac{\partial f_1(a)}{\partial x_2} & \cdots\cdots & \frac{\partial f_1(a)}{\partial x_n} \\ \frac{\partial f_2(a)}{\partial x_1} & \frac{\partial f_2(a)}{\partial x_2} & \cdots\cdots & \frac{\partial f_2(a)}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n(a)}{\partial x_1} & \frac{\partial f_n(a)}{\partial x_2} & \cdots\cdots & \frac{\partial f_n(a)}{\partial x_n} \end{bmatrix}$$

is called the **Jacobian matrix** of the functions $f_1, \ldots, f_n$ at $(a_1, \ldots, a_n)$. The determinant of this matrix is called the **Jacobian** of the functions at $(a_1, \ldots, a_n)$.

As in the case of partial derivatives, if we don't have any specific point $a = (a_1, \ldots, a_n)$ in mind, then the Jacobian matrix is written as

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ \vdots & \vdots & \vdots\ \vdots\ \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

and the Jacobian is denoted by $\dfrac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}$.

In Sec. 3.3 (Remark), we have seen that the functions $f_1, \ldots, f_n$ determine a unique function $f : \mathbf{R}^n \to \mathbf{R}^n$ so that $f(x) = f(x_1, \ldots, x_n) = (f_1(x), \ldots, f_n(x))$. The Jacobian matrix of $f_1, \ldots, f_n$ at a is also called the Jacobian matrix of f at a and is denoted by $\mathbf{J}_f(\mathbf{a})$.

Before we discuss some properties of Jacobians, we look at a few examples.

Jacobi (1804 – 1851)
Jacobians were introduced by the German mathematician Carl Gustav Jacob Jacobi.

Example 1 : Consider the transformation $(r, \theta) \to (x, y)$, given by

$$x = r\cos\theta$$
$$y = r\sin\theta \quad \ldots (3)$$

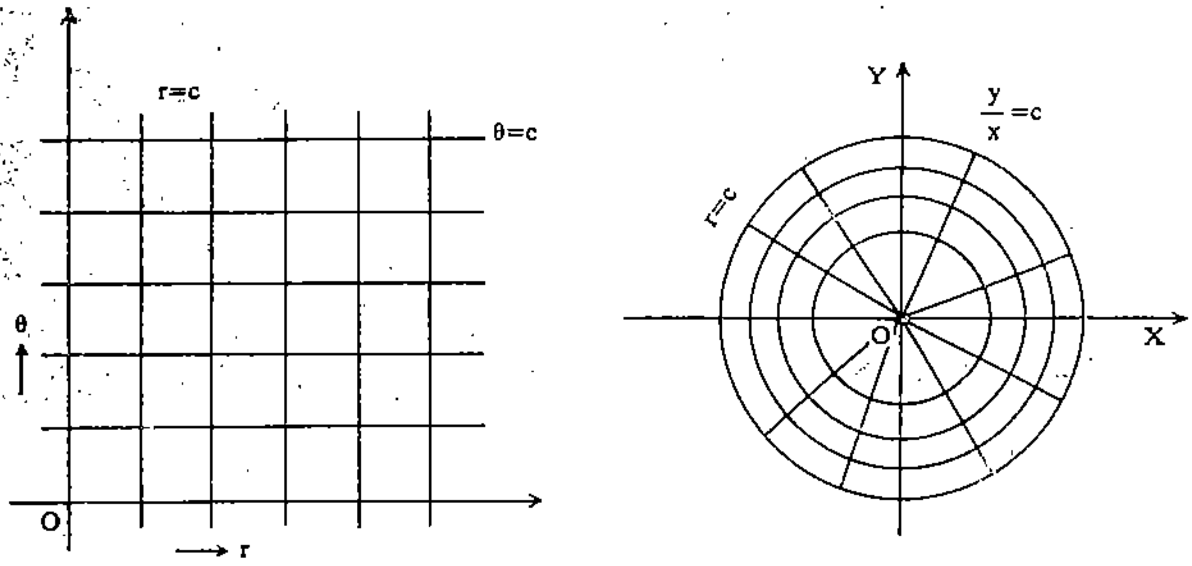
The Jacobian $\dfrac{\partial(x, y)}{\partial(r, \theta)}$ is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Let us try to geometrically interpret the transformation given by (3). We take r and θ as rectangular coordinates of the (r, θ) — plane as shown in Fig. 1 (a). The equation r = const. = c (say) represents a line parallel to the θ-axis. As c takes various values, we get a set of lines parallel to the θ-axis in the (r, θ) plane (see Fig. 1 (a)). But from Equation (3) we have that

$$x^2 + y^2 = r^2.$$

Now, $x^2 + y^2 = r^2$ represents a circle of radius r with centre at the origin O. Thus, as r varies, we get a set of concentric circles in the (x, y) – plane (see Fig. 1(b)). In other words, the transformation (3) transforms parallel lines to concentric circles.



(a) (b)

Fig. 1

What if θ is a constant ? If θ = c, a constant, then we get a straight line parallel to the r-axis. As c takes various values, we get a set of lines parallel to the r-axis in the (r, θ) – plane. However, from (3) again, we have that

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta$$

Thus, if θ = c, $0 \le c < 2\pi$, then $\frac{y}{x} = \tan \theta = \tan c$, and we get a line through the origin in the (x, y) – plane as shown in Fig. 1 (b). Further, if θ = c ($2\pi \le c < 4\pi$,) then c can be written as $c = 2\pi + c^*$, where $0 \le c^* < 2\pi$. Thus, $\frac{y}{x} = \tan \theta = \tan (2\pi + c^*) = \tan c^*$. Thus, for c lying between 2π and 4π, we can say that the strip in the (r, θ) – plane corresponding to θ = c ($2\pi \le c < 4\pi$) gets transformed into the same set of radial lines in the (x, y)–plane.

All this is true if $r \neq 0$. What happens if r = 0 ? It means that the Jacobian $\frac{\partial (x, y)}{\partial (r, \theta)}$ is zero. You can see that as r becomes smaller and smaller, the circles in the plane become smaller and smaller, because r represents the radii of these circles. Finally, in the limiting position, we will reach the origin which is a circle of radius 0. This happens when the line r = constant in the (r, θ) – plane coincides with the θ–axis.

In the next example, we obtain a Jacobian of three real-valued functions of three variables.

Example 2 : Consider the transformation

$$x = r \cos \theta \sin \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \phi$$

The Jacobian is equal to

$$\frac{\partial(x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \cos \theta \cos \phi & -r \sin \theta \sin \phi & r \cos \theta \cos \phi \\ \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos\phi \\ \cos \phi & 0 & -r \sin\phi \end{vmatrix}$$

$$= \cos\theta \sin\phi\,(-r^2 \cos\theta \sin^2\phi) + r \sin\theta \sin\phi\,(-r \sin\theta \sin^2\phi - r \sin\theta \cos^2\phi) - r^2 \cos^2\theta \cos^2\phi \sin\phi$$

$$= -r^2 \cos^2\theta \sin\phi\,(\sin^2\phi + \cos^2\phi) - r^2 \sin^2\theta \sin\phi\,(\sin^2\phi + \cos^2\phi)$$

$$= -r^2 \sin\phi$$

Geometrically, r is the distance of the point P(x, y, z) from the origin O,
θ is the geographic longitude, i.e., the angle between the xz– plane and the plane determined by P and the z-axis, and
ϕ is the polar distance, i.e., the angle between the radius vector OP and the positive z - axis. See Fig. 2.

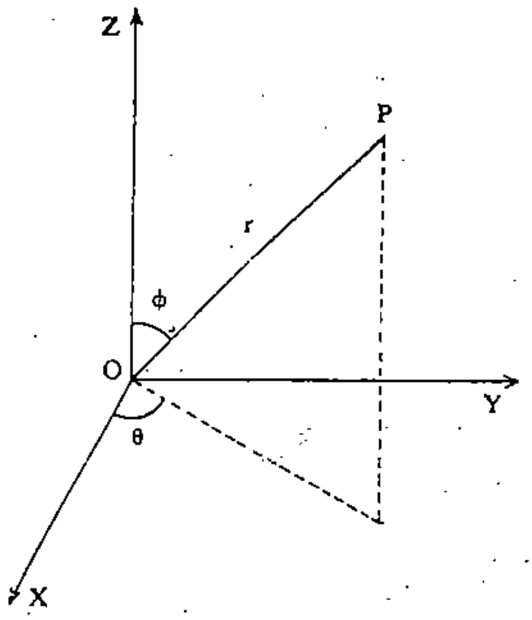


Fig. 2

Try these exercises now.

E1) Calculate the Jacobians for the following mappings. Also find out which of these are invertible.

a) $w = x - 2y,\ z = 2x + y$

b) $w = 2x - 3y,\ z = 5x + 7y.$

E2) Calculate the Jacobian for each of the following functions at the indicated point.

a) $F = (f, g)$, where $f(x, y) = \sin x$, $g(x, y) = \cos xy$ at $\left(\pi, \frac{\pi}{2}\right)$

b) $F = (f, g)$, where $f(x, y) = x - y^2 - 2x^2 y - x^4$,

$g(x, y) = y + x^2$

c) $F(x, y, z) = (\sin xyz, xz, c)$ at (x, y, z), where c is a constant.

d) $F(x, y, z) = (xz, xy, yz)$ at $(\pi, 2, 4)$

E3) Find the Jacobian matrix of the map

$w = x + y,\ z = x^2 y.$

Find all the points where its Jacobian is equal to zero.

If you have done these exercises, you must have become quite conversant with the computation of Jacobians. We'll now see how the Jacobian is useful in calculating the partial derivatives of implicit functions.

9.2.2 Partial Derivatives of Implicit Functions

You have seen in Unit 7 that it is possible to find the derivative $\frac{dy}{dx}$ for a function $f(x, y) = 0$, which defines y as a function of x implicitly. In many cases, it may not be possible to write y in terms of x explicitly. But yet, by using the total derivative, we can find the derivative $\frac{dy}{dx}$ without actually solving y as a function of x. In a similar way, it is possible to find the partial derivatives of implicit functions of two or more variables by the use of Jacobians. The following theorem gives us the rule for calculating these partial derivatives.

Theorem 2 : Suppose we are given two equations

$$\left.\begin{aligned} F(x,y,z,u,v) &= 0 \\ G(x,y,z,u,v) &= 0 \end{aligned}\right\} \qquad \ldots(4)$$

such that $\frac{\partial(F,G)}{\partial(u,v)} \neq 0$. Suppose, further, that the first order partial derivatives of F and G w.r.t. the five variables are continuous. Then

The condition $\frac{\partial(F,G)}{\partial(u,v)} \neq 0$ ensures that Equations (4) have a solution of the form $u = f(x, y)$, $v = g(x, y)$. We will talk about this, in detail, in Unit 10.

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \\ \frac{\partial v}{\partial x} &= -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} \end{aligned}\right\} \qquad \ldots(5)$$

A similar result holds for $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$.

Proof: Using the chain rule (Theorem 3 of Unit 7), we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial x} = 0$$

Solving these simultaneous equations, we get the result as stated in the theorem.

Proceeding exactly similarly we can get the expressions for $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$.

We are leaving it to you as an exercise. See E 5).

We will now use Theorem 2 to get the partial derivatives in our next example.

Example 3: Let us find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ for

$$F(x, y, u, v) = x^2 + ux + y^2 + v$$

$$G(x, y, u, v) = x + yu + v^2 + x^2v.$$

Note that here F and G are functions of four variables. Further, we have that

$$\frac{\partial F}{\partial x} = 2x + u,\ \frac{\partial F}{\partial u} = x \text{ and } \frac{\partial F}{\partial v} = 1$$

$$\frac{\partial G}{\partial x} = 1 + 2xv,\ \frac{\partial G}{\partial u} = y \text{ and } \frac{\partial G}{\partial v} = 2v + x^2$$

These are all continuous, being polynomials.

Thus, $$\frac{\partial(F,G)}{\partial(x,v)} = \begin{vmatrix} 2x+u & 1 \\ 1+2xv & 2v+x^2 \end{vmatrix} = 2xv + 2uv + 2x^3 + ux^2 - 1,$$

$$\frac{\partial (F, G)}{\partial (u, x)} = \begin{vmatrix} x & 2x + u \\ y & 1 + 2xv \end{vmatrix} = x + 2x^2 v - 2xy - uy, \text{ and}$$

$$\frac{\partial (F, G)}{\partial (u, v)} = \begin{vmatrix} x & 1 \\ y & 2v+x^2 \end{vmatrix} = 2xv + x^3 - y$$

Therefore, by (5) we get

$$\frac{\partial u}{\partial x} = -\frac{2xv + 2uv + 2x^3 + ux^2 - 1}{2xv + x^3 - y} \text{ and}$$

$$\frac{\partial v}{\partial x} = -\frac{x + 2x^2 v - 2xy - uy}{2xv + x^3 - y}$$

Now you should be able to solve these exercises easily.

E4) Let x, y, u, v be related by the equations

$xy + x^2u - vy^2 = 0$ and

$3x - 4uy - x^2v = 0$.

Find $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}$ without solving the equations explicitly.

E5) Suppose

$F(x, y, z, u, v) = 0$ and

$G(x, y, z, u, v) = 0$

are such that their first order partial derivatives are continuous and $\frac{\partial(F, G)}{\partial (u, v)} \neq 0$.

Compute $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$.

E6) x, y, z, u and v are related by the equations $F(x, y, z, u, v) = xy + yz + zu + uv = 0$ and $G(x, y, z, u, v) = x + y + z + u + v = 0$.

Compute $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial z}$.

In the next section we shall discuss the method of calculating the Jacobian of composite functions.

9.3 CHAIN RULE

Jacobians often behave like partial derivatives. In this section, we will state and prove a chain rule for Jacobians. The chain rule for Jacobians is similar to the chain rule for partial derivatives which you have studied in Unit 7. We will use the chain rule for partial derivatives to prove the rule for Jacobians.

Theorem 3 : (Chain Rule) : Let $f : \mathbf{R}^n \to \mathbf{R}^n$ and $g : \mathbf{R}^n \to \mathbf{R}^n$ be two differentiable functions and let $F = fog$. Then

$$J_F(x) = J_f(g(x))\, J_g(x) \qquad \ldots(6)$$

If $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n)$ and $F = (F_1, \ldots, F_n)$, then the chain rule stated above can also be formulated as follows :

$$\frac{\partial (F_1, \ldots, F_n)}{\partial (x_1, \ldots, x_n)} = \frac{\partial (f_1, \ldots, f_n)}{\partial (y_1, \ldots, y_n)} \cdot \frac{\partial (y_1, \ldots, y_n)}{\partial (x_1, \ldots, x_n)},$$

where $y_i = g_i(x)$. This formula resembles the chain rule for the derivative of a function.

As we shall be using the chain rule for $n = 2$ or 3, we give the proof for $n = 3$, the other case being similar.

Proof : The proof consists in writing down the product of the determinants on the right hand side of (6), and using the chain rule, given in Sec. 7.2 in Block 2, for partial derivatives. Indeed, starting from the right hand side of (6), we have

$$J_f(g(x))\,J_g(x) = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} \end{bmatrix} \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial f_1}{\partial y_3}\frac{\partial y_3}{\partial x_1} \ldots\ldots\ldots \\ \frac{\partial f_2}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_2}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial f_2}{\partial y_3}\frac{\partial y_3}{\partial x_1} \ldots\ldots\ldots \\ \frac{\partial f_3}{\partial y_1}\frac{\partial y_1}{\partial x_1} + \frac{\partial f_3}{\partial y_2}\frac{\partial y_2}{\partial x_1} + \frac{\partial f_3}{\partial y_3}\frac{\partial y_3}{\partial x_1} \ldots\ldots\ldots \end{bmatrix}$$

by the product rule for determinants. Now we will use the chain rule for partial derivatives as given in Theorem 3 of Unit 7 for the entries in the determinant on the right hand side. Since the functions involved are assumed to be differentiable, the conditions of Theorem 3, Unit 7 are satisfied. Thus,

$$J_f(g(x))\;J_g(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = J_F(x)$$

and this is what we wanted to prove.

Here are some examples to illustrate Theorem 3.

Example 4 : Using Theorem 3, let us find $\frac{\partial(x, y)}{\partial(z, w)}$ for the transformations $x = u^2 - v^2$, $y = 2uv$, where $u = z^3 - 3zw^2$ and $v = z - w$.

Here

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2 + v^2)$$

$$\frac{\partial(u, v)}{\partial(z, w)} = \begin{vmatrix} 3z^2 - 3w^2 & -6zw \\ 1 & -1 \end{vmatrix} = -3z^2 + 3w^2 + 6zw.$$

Therefore by Theorem 3,

$$\frac{\partial(x, y)}{\partial(z, w)} = 4(u^2 + v^2)(-3z^2 + 3w^2 + 6zw).$$

$$= 12(u^2 + v^2)(-z^2 + w^2 + 2zw).$$

Example 5 : Let $f(x, y) = (\sin x, \cos y)$ and $g(x, y) = (x^2, y^2)$. We can use the chain rule to calculate the Jacobian of $F = f \circ g$. Now suppose $g = (g_1, g_2)$, $f = (f_1, f_2)$ and $\xi = g_1(x, y)$, $\eta = g_2(x, y)$. Then, we get

$$\frac{\partial(g_1, g_2)}{\partial(x, y)} = 4xy \text{ and } \frac{\partial(f_1, f_2)}{\partial(\xi, \eta)} = -\cos\xi \sin\eta.$$

Thus, by chain rule

$$J_F(x, y) = -\cos x^2 \sin y^2 \cdot 4xy.$$

We could have calculated this Jacobian by direct computation also.

In fact, if $F = (F_1, F_2)$, then $F_1(x, y) = \sin x^2$ and $F_2(x, y) = \cos y^2$.

Therefore,

$$\frac{\partial(F_1, F_2)}{\partial(x, y)} = \begin{vmatrix} 2x\cos x^2 & 0 \\ 0 & -2y\sin y^2 \end{vmatrix} = -4xy\cos x^2 \sin y^2$$

Theorem 3 can also be used to calculate the Jacobian of the inverse of a transformation, if it exists. In fact, we have the following theorem.

Theorem 4 : Let $f = (f_1, \ldots\ldots, f_n) : D \to R^n$, where $D \subseteq R^n$, be differentiable. Suppose that f is invertible on D, and let f^{-1} be differentiable on the range of f. Then

$J_{f^{-1}}(y) = (J_f(x))^{-1}$, where $y = f(x)$.

Proof : If $F = f^{-1} \circ f$, then F is the identity mapping and therefore, $J_F(x) = 1$. Thus, by Theorem 3,

$$1 = J_{f^{-1}}(f(x))\, J_f(x), \text{ or}$$

$$J_{f^{-1}}(f(x)) = (J_f(x))^{-1}.$$

In the case of 3 variables, Theorem 4 becomes,

$$\frac{\partial(x_1, x_2, x_3)}{\partial(f_1, f_2, f_3)} = \left[\frac{\partial(f_1, f_2, f_3)}{\partial(x_1, x_2, x_3)}\right]^{-1} \text{ or } \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = \left[\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}\right]^{-1}$$

The following examples illustrate the utility of this theorem.

Example 6 : Let us find $\frac{\partial(r, \theta)}{\partial(x, y)}$, where $x = r\cos\theta$ and $y = r\sin\theta$.

We have seen in Example 1 that $\frac{\partial(x, y)}{\partial(r, \theta)} = r$. Therefore $\frac{\partial(r, \theta)}{\partial(x, y)} = \frac{1}{r}$ by Theorem 4 above.

This, however, is valid only if $r \neq 0$.

Example 7 : Let $f(x, y) = (x - y, x + y) = (\xi, \eta)$. Let us evaluate $\frac{\partial(x, y)}{\partial(\xi, \eta)}$.

Clearly, $\frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2.$

Therefore, by Theorem 4, we get $\frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{1}{2}$.

Example 8 : Suppose x and y are expressed in terms of t by $F(x, y, t) = 0$ and $G(x, y, t) = 0$. Assuming that these functions are differentiable, let us prove that

$$\frac{dx}{dt} = \frac{\partial(F, G)}{\partial(y, t)} \Big/ \frac{\partial(F, G)}{\partial(x, y)}, \text{ provided } \frac{\partial(F, G)}{\partial(x, y)} \neq 0.$$

Consider the equations

$F(x, y, t) = 0$, and

$G(x, y, t) = 0$.

We differentiate these with respect to t using the methods for finding the derivatives of functions defined implicity. Thus, we get

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0 \text{, and}$$

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} = 0.$$

Eliminating $\frac{dy}{dt}$ from these two equations we get

$$\left[\frac{\partial F}{\partial t}\frac{\partial G}{\partial y} - \frac{\partial G}{\partial t}\frac{\partial F}{\partial y}\right] + \frac{dx}{dt}\left[\frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y}\right] = 0$$

$$\text{or, } \left[-\frac{\partial(F, G)}{\partial(y, t)} + \frac{dx}{dt}\cdot\frac{\partial(F, G)}{\partial(x, y)}\right] = 0.$$

$$\text{Therefore, } \frac{dx}{dt} = \frac{\partial(F,G)}{\partial(y, t)} \Big/ \frac{\partial(F, G)}{\partial(x, y)}.$$

You can certainly do these exercises now.

E7) Find $\frac{dy}{dt}$ in Example 8.

E8) Calculate the Jacobian $\frac{\partial(x, y, z)}{\partial(r, \theta, z)}$ for $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

(r, θ, z are called the cylindrical coordinates).

E9) Verify the chain rule for the Jacobians for the following functions :

$u = e^{xy}$, $v = e^{yz}$, $w = e^{zx}$, $x = r$, $y = s^2$, $z = t^3$.

In this section, you have seen one use of Jacobians : to find the partial derivatives of implicit functions. You will see many more uses in the sequel, starting with the next section itself.

9.4 FUNCTIONAL DEPENDENCE

Suppose we are given n real-valued functions $f_1, \ldots, f_n$, $n \geq 2$ of several variables. We may want to know whether there exists some ' relationship' between these functions. For example, if $f_1(x, y, z) = \frac{x+y}{z}$, $f_2(x, y, z) = \frac{y+z}{x}$, $f_3(x, y, z) = \frac{y(x+y+z)}{xz}$, then you can check that $f_1 \cdot f_2 = f_3 + 1$. However, the situation may not be as simple as it is in this example. Therefore, we need to find some criterion which would ensure the existence of some relationship between the given functions. You will see that Jacobians provide us a necessary and sufficient condition for the existence of some relationship between the given functions. But before making precise as to what we mean by "relationship", we look at the following question and answer it satisfactorily. Apart from its own interest, it would be of use later on.

In Calculus (Unit 7) you have seen that if f is a real-valued function defined on an open interval, and if its derivative is zero at all points of the interval, then f is a constant function. Can we have an analogous result for functions of several variables ? In this section we shall see that those functions, all of whose partial derivatives vanish throughout some domain, are constant. But here the word "domain" has a different meaning. It is not just the set on which the function is defined. What, then, is this domain ? Let's see.

9.4.1 Domains in R^n

You would have observed that in Calculus, most of the time we assumed that function or functions under consideration were defined on open intervals. Similarly, for functions of two or three variables, we assumed that functions under consideration were defined on open discs or open spheres. All these subsets have two things in common.

1) All points of these sets are interior points [illegible] to the set then a whole neighbourhood of it is contained in the set.

2) It is possible to travel from one point of the set to another point of the set without leaving the set.

While the first statement is mathematically precise, the second one is not quite precise. However, intuitively we can say that given any two points in the set, there is a path completely lying in the set joining these points.

For example, the set $H = \{ (x, y) \in R^2 \mid x^2 - y^2 > 1 \}$ drawn in Fig. 3 below, obviously does not possess this property. You will agree that there is no way of "travelling" from the point P to the point Q without leaving the set H.

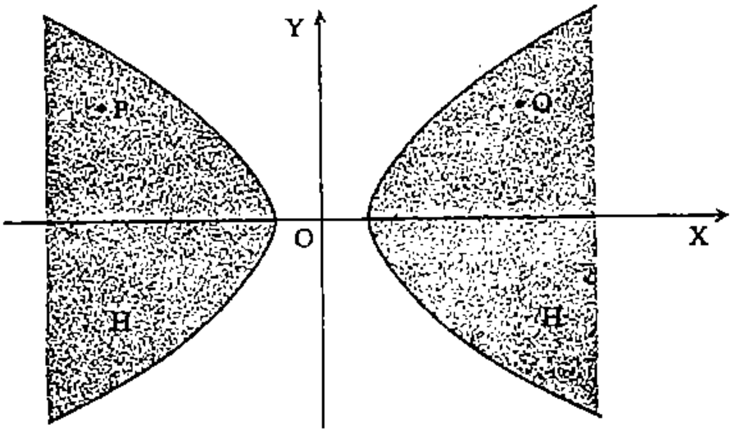


Fig. 3

On the other hand, interiors or exteriors of rectangles, circles, ellipses possess this property. We shall now define the concept of "travelling between" more precisely.

Definition 2 : Let $\alpha : [0, 1] \to R^n$ be the function defined by $\alpha(t) = (1 - t)\,x + t\,y$, where x, y are two given points of R^n. The set $\{ \alpha(t) \mid t \in [0, 1] \}$ is said to be the **line segment** joining x to y lying in the set R^n. x is called the **initial point** and y the **final point** of the line segment.

Observe that $\alpha(0) = x$ and $\alpha(1) = y$. Note also that this definition coincides with our intuitive idea of the line segment joining a pair of points in plane or space. See Fig. 4 (a). Sometimes the function α itself is called a line segment.

Definition 3: Let x, y, be two points in R^n. A polygonal arc joining x to y is nothing but a sequence $\alpha_0, \alpha_1, \alpha_2, \ldots\ldots, \alpha_r$ of line segments in R^n such that

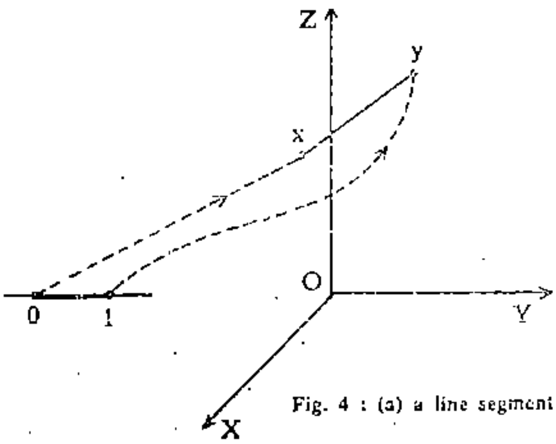
$\alpha_0(0) = x$, i.e., the initial point of α_0 is x.

$\alpha_r(1) = y$, i.e., the final point of α_r is y.

The final point of α_i is the initial point of α_{i+1} for each

$i = 0, 1, 2, \ldots.., r - 1.$

This definition agrees with our intuitive idea of a polygonal arc (see Fig. 4 (b)).



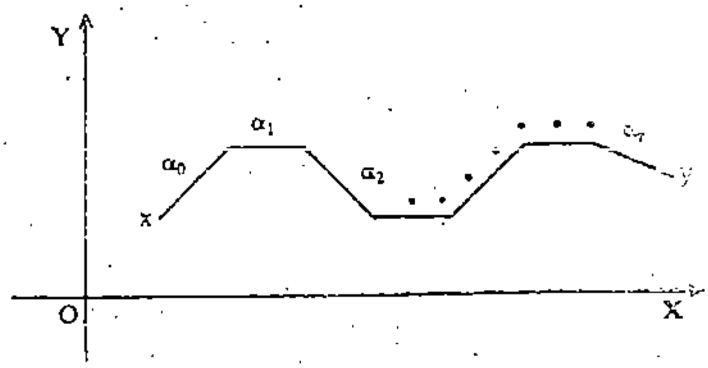


Fig. 4 : (a) a line segment in R^3 (b) a polygonal arc in R^2

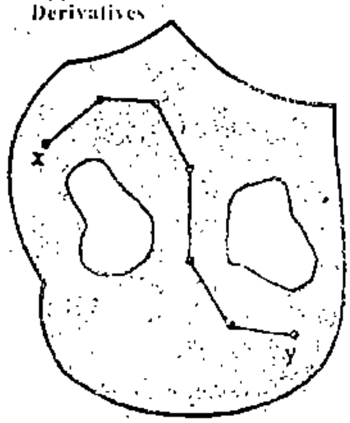


Fig. 5

Definition 4 : A subset S of R^n is said to be polygonally connected if, given any two points x, y in S, there is a polygonal arc lying in S joining x to y (See Fig. 5).

Interiors of rectangles and circles in the plane R^2 and interiors of spheres in R^3 are obviously polygonally connected. In fact, every pair of points in them can be joined by one line segment. Now we are in a position to define a domain.

Definition 5: A subset S of R^n will be said to be a **domain**, if S is polygonally connected and open, i.e., if for each point x in S there is a neighbourhood S (x, r) of x of radius r, which is contained in S.

Thus, we can say that S is a domain if

- S is polygonally connected, and
- every point of S is an interior point of S.

The set H in Fig. 3 is not a domain, since it is not polygonally connected. You can see that there can be no polygonal arc from P to Q lying within the set. However, every point of it is an interior point. The open disc $\{(x, y) \mid x^2 + y^2 < 1\}$ and the open rectangle $]a, b[\times]c, d[$ are domains in R^2.

S is open if each point of S is an interior point.

The closed disc $S = \{(x, y) \mid x^2 + y^2 \leq 1\}$ is not a domain because the points (x, y) for which $x^2 + y^2 = 1$, i.e., points on the boundary of the circle are not interior points of the set S. Note, however, that S is polygonally connected. In fact, every pair of points can be joined by a line segment in it (See Fig. 6). Similarly, the open sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 < 1\}$ and the open parallelopiped $]a, b[\times]c, d[\times]e, f[$ are domains in $\mathbf{R}^3$.

Can you do this exercise now ?

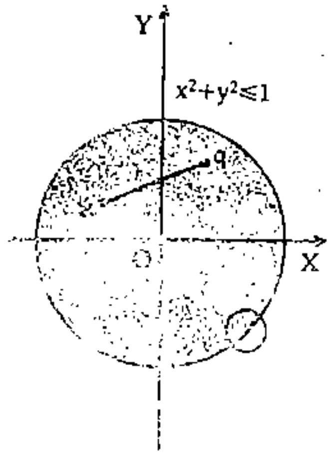


Fig. 6

E 10) Identify domains from the following sets :

a) $\{(x, y) \mid x = y\}$

b) $\{(x, y) \mid xy > 0\}$

c) $\{(x, y) \mid x > 1\}$

d) $\{(x, y) \mid x^2 + y \geq 0\}$

e) $\{(x, y, z) \mid x = y\}$

f) $\{(x, y, z) \mid x > 0, y > 0, z > 0\}$

Do not confuse the domain of function with a domain. The domain of a function need not be a domain in the sense of Definition 5. For example, the domain of the function $f(x, y) = \sin^{-1} x \sin^{-1} y$ is $[-1, 1] \times [-1, 1]$, which is not a domain in $\mathbf{R}^2$.

We are now ready to prove the result promised at the beginning of this section.

Theorem 5 : Let f (x, y) be a real-valued function defined on a domain D in $\mathbf{R}^2$. Let $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ on D. Then there is a constant c such that

$$f(x, y) = c \text{ for all } (x, y) \in D.$$

Proof : Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points of D such that the line segment P_1P_2 joining P_1 to P_2 lies in D. It is possible to choose such a pair of points since D is polygonally connected. For convenience of notation we shall write f (P) for f (x, y), if P is the point (x, y). By the mean value theorem for two variables (see margin remark), we have

$$f(P_2) - f(P_1) = (x_2 - x_1)\frac{\partial f}{\partial x}(P_1 + \theta(P_2 - P_1)) + (y_2 - y_1)\frac{\partial f}{\partial y}(P_1 + \theta(P_2 - P_1)), \quad \ldots(7)$$

where θ is a real number such that $0 < \theta < 1$.

The mean value theorem : $f(a + h, b + k) - f(a, b) = hf_x(a + \theta h, b + \theta k) + kf_y(a + \theta h, b + \theta k)$, where $0 < \theta < 1$.

Now $P_1 + \theta(P_2 - P_1) = \theta P_2 + (1-\theta) P_1$, lies on the segment $P_1 P_2$ and hence in D. Since, by assumption, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both zero on D, we have from (7) that

$$f(P_2) - f(P_1) = 0, \text{ or}$$

$$f(P_1) = f(P_2) = c, \text{ say.}$$

Every point P of D can be joined to P_1 by a polygonal arc in D.

Therefore by successive repetition of the above argument, we conclude that $f(P) = f(P_1) = c$ for all points P in D. The theorem is thus proved.

The assumption that D is polygonally connected is essential in Theorem 5. In fact, let us take D to be the union of two discs,

$S_1 = \{ (x, y) \mid x^2 + y^2 < 1 \}$ and $S_2 = \{ (x, y) \mid (x-3)^2 + y^2 < 1 \}$

Also see Fig. 7.

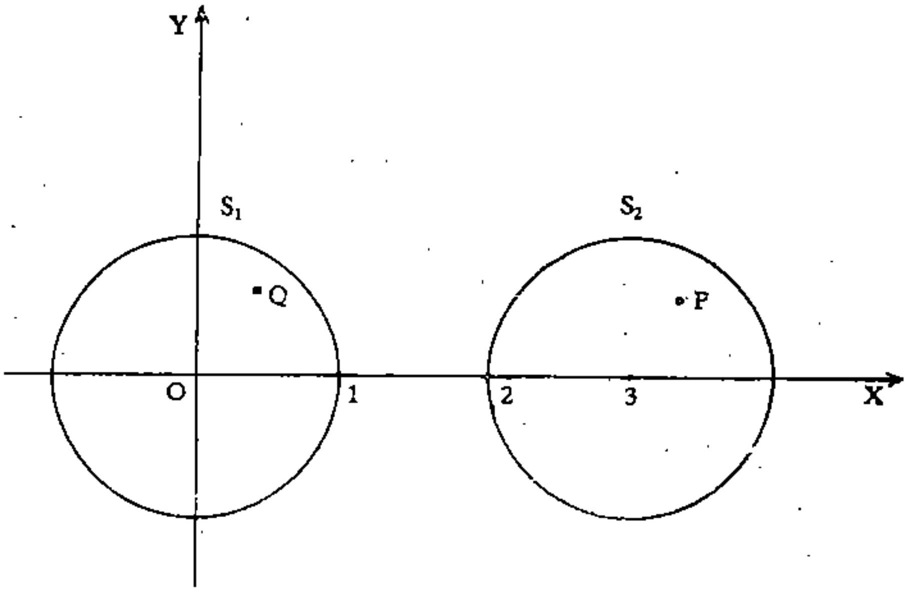


Fig. 7

Let us define a function $f : D \to \mathbf{R}$ such that

$$f(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S_1 \\ 2 & \text{if } (x, y) \in S_2. \end{cases}$$

Then both the partial derivatives of f vanish on D. But yet, it is not a constant function on D. Note that $D = S_1 \cup S_2$ is not polygonally connected, since it is not possible to travel from P to Q along a polygonal arc lying in D. A similar result is true for any number of variables, and can be proved by using the mean value theorem similar to the one used above. However, we don't give the proof here.

Recall that the statement of the analogous result for the one-variable case is :

If f is a function defined on an open interval I such that $f'(x) = 0$ for all $x \in I$, then f is a constant function on I.

So, you see, in this case also we insist on an open interval which is a domain. Can you think of an example to show that it is essential that the function be defined on an open interval ? Check with $f(x) = 0$ on $]0, 1[$ and $f(x) = 1$ on $]2, 3[$.

In the next sub-section we shall investigate how the question of dependence of various functions is connected with their Jacobian.

But try the following exercise first.

E11) Let f (x, y) be a real-valued function defined on a neighbourhood N of (a, b).

a) If $f_x (x, y) = 0$ for $(x, y) \in N$, then prove that f is a function of y alone.

b) If $f_y (x, y) = 0$ for all $(x, y) \in N$, then prove that f is a function of x alone.

9.4.2 Dependence

Before we give the precise definitions, we discuss a couple of examples.

Example 9 : Let

$$f(x, y) = e^x \sin y \text{ and}$$

$$g(x, y) = x + \ln \sin y$$

for $x \in \mathbf{R}$ and $0 < y < \pi$. Clearly, $D = \mathbf{R} \times]0, \pi[$, which is a strip bounded by the lines $y = 0$ and $y = \pi$, and is an open set. In fact, D is a domain. Moreover,

$$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ 1 & \cot y \end{vmatrix}$$

$$= e^x \cos y - e^x \cos y = 0$$

for all (x, y) in D.

Observe that in D, the functions f and g are related by the relation

$$\ln (f(x, y) - g(x, y)) = 0 \qquad \ldots(8)$$

In other words, g is simply the function ln f, a function of a function.

The relation (8) can also be put in the form

$$F(f(x, y), g(x, y)) = 0$$

for all (x, y) in D, where F is a real-valued function given by

$$F(u, v) = \ln u - v.$$

Note that $\frac{\partial F}{\partial u} \neq 0$ or $\frac{\partial F}{\partial v} \neq 0$ for any point (u, v) in the domain of F and, in particular, in the range of the mapping $(x, y) \to (f(x, y), g(x, y))$.

Example 10 : Let
$$u = 3x + 2y - z$$
$$v = x - 2y + z$$
$$w = x(x + 2y - z)$$

Then,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = x \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2 & 2 & -1 \end{vmatrix} = 0$$

for all $(x, y, z) \in \mathbf{R}^3$

You can check that $u^2 - v^2 = 8w$.

Thus, F (u, v, w) = 0 for every $(x, y, z) \in \mathbf{R}^3$, where $F(u, v, w) = u^2 - v^2 - 8w$. Note that one of $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v}$ and $\frac{\partial F}{\partial w}$ is different from zero at every point of $\mathbf{R}^3$.

This leads us to the definition below.

Definition 6 : The real-valued functions $f_1, f_2, \ldots, f_n$ of n variables $x_1, \ldots, x_n$, defined on an open subset D of $\mathbf{R}^n$ are said to be **functionally dependent** in D, if there exists a real-valued function F of n variables such that

i) at least one of $\frac{\partial F}{\partial u_i}$, $1 \le i \le n$, is different from zero at every point of the range of the mapping from $\mathbf{R}^n$ to $\mathbf{R}^n$ defined by $(x_1, x_2, \ldots, x_n) \to (u_1, u_2, \ldots, u_n)$, where $u_i = f_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$.

ii) $F(f_1(x), f_2(x), \ldots, f_n(x)) = 0$

for any $x = (x_1, x_2, \ldots, x_n) \in D$.

In the discussion that follows we shall confine our attention to the cases $n = 2$ and $n = 3$, as has been our practice so far.

In Example 9 we have seen that $e^x \sin y$ and $x + \ln \sin y$ are functionally dependent on $\mathbf{R} \times]0, \pi[$, whereas in Example 10 we have seen that the functions $3x + 2y - z$, $x - 2y + z$ and $x(x + 2y - z)$ are functionally dependent on $\mathbf{R}^3$.

In order to look at the geometrical significance of functional dependence, we take another example.

Example 11 : Consider the transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (u, v)$, where $u = \cos(x + y^2)$ and $v = \sin(x + y^2)$.

T is continuously differentiable since sine and cosine functions are so. We observe that T maps the entire plane onto the set of points on the circle $\{(u, v) \mid u^2 + v^2 = 1\}$ of radius 1. This circle has no interior points. Thus, T does not map neighbourhoods to neighbourhoods. Further, T is not 1 – 1 even on a neighbourhood. In fact, all the points on the parabola, $x + y^2 = c$ map into the same point $(\cos c, \sin c)$ (see Fig. 8). As c changes, these parabolas cover the entire plane. Thus, any neighbourhood contains points having the same image.

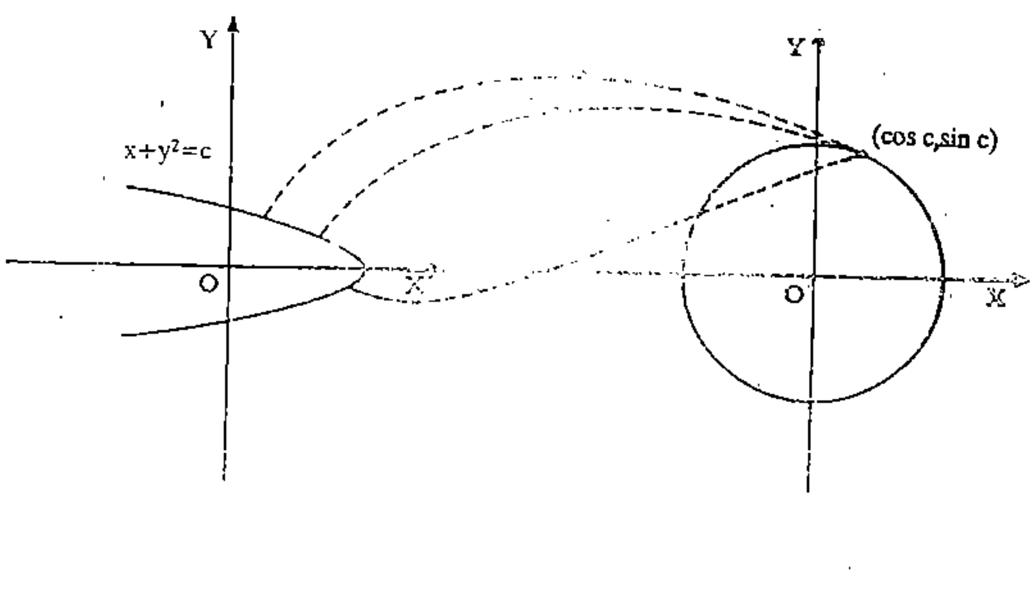


Fig. 8

All this behaviour is due to the fact that u and v are functionally dependent. In fact, $F(u, v) = u^2 + v^2 - 1 = 0$.

What about the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$?

We observe that

$$\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} -\sin(x + y^2) & -2y \sin (x + y^2) \\ \cos (x + y^2) & 2y \cos (x + y^2) \end{vmatrix}$$

$$= -2y \sin (x + y^2) \cos (x + y^2) + 2y \sin (x + y^2) \cos (x + y^2)$$

$$= 0$$

What we have seen here is not peculiar to this example. If u and v are two real-valued functions of two variables such that $\frac{\partial (u,v)}{\partial (x,y)} = 0$ in a neighbourhood of some point, then we shall see that u and v are functionally dependent and the transformation $(x, y) \rightarrow (u, v)$ maps a neighbourhood to a curve in $\mathbf{R}^2$. For example, the transformation $(x, y) \rightarrow (e^x \sin y, x + \ln \sin y)$ maps a neighbourhood to the curve $v - \ln u = 0$. The same is true for functions of any number of variables. For example, the transformation $(x, y, z) \rightarrow (u, v, w)$ in Example 10 maps a neighbourhood to the surface $u^2 - v^2 - 8w = 0$. For the complete proof of this result you will have to wait till the next unit. Here (Theorem 6) we shall prove that vanishing of the Jacobian in a neighbourhood is necessary for functional dependence.

Theorem 6 : If $u = f(x, y)$ and $v = g(x, y)$ are differentiable on an open subset D of $\mathbf{R}^2$, and are functionally dependent on D, then

$$\frac{\partial (u, v)}{\partial (x, y)} = 0 \text{ for all } (x, y) \text{ in D.}$$

Proof : By Definition 6 there exists a real-valued function F (u, v) such that

$$F(u, v) = F(f(x, y), g(x, y)) = 0 \qquad \ldots(9)$$

for all $(x, y) \in$ D and either $\frac{\partial F}{\partial u}$ or $\frac{\partial F}{\partial v} \neq 0$ at every point (u, v) of the range of the mapping $(x, y) \rightarrow (f(x,y), g(x,y))$.

We differentiate the relation (9) partially with respect to x and y by chain rule (see Unit 7) to obtain

$$\left.\begin{aligned} \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} &= 0 \\ \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} &= 0 \end{aligned}\right\} \qquad \ldots(10)$$

Since for any point $(x, y) \in$ D, either $\frac{\partial F}{\partial u}$ or $\frac{\partial F}{\partial v}$ is different from zero, the system of equations (10) implies that

Remember, $u = f(x, y)$ and $v = g(x, y)$.

$$\frac{\partial (f, g)}{\partial (x, y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = 0 \text{ at every point } (x, y) \in \text{ D.}$$

A similar result also holds for three functions of three variables. Its proof is exactly similar to the proof of Theorem 6.

Look at this example now.

Example 12 : Let us prove that the following functions are functionally dependent by finding a functional relation between them.

i) $f(x, y) = \ln x - \ln y \qquad \ldots(11)$

ii) $g(x, y) = \frac{x^2 + 3y^2}{2xy} \qquad \ldots(12)$

For convenience, we'll write f and g for f (x, y) and g (x, y).

The idea is to solve for x from (12) and substitute in (11), which will yield a relation between f and g.

Thus, from (12) we have that

$$2g.\,xy = x^2 + 3y^2$$

i.e., $x^2 - 2gxy + 3y^2 = 0$

or, $(x - gy)^2 - g^2y^2 + 3y^2 = 0$

i.e., $(x - gy)^2 = g^2y^2 - 3y^2 = (g^2 - 3)y^2$

Hence, $x - gy = \pm y\sqrt{g^2 - 3}$.

Let us consider

$$x - gy = +y\sqrt{g^2 - 3}.$$

Thus $x = y\left[g + \sqrt{g^2 - 3}\right]$.

Substituting for x in (11) we get

$$f = \ln\left[y\left(g + \sqrt{g^2 - 3}\right)\right] - \ln y$$

$$= \ln y + \ln\left(g + \sqrt{g^2 - 3}\right) - \ln y$$

i.e., $f = \ln\left(g + \sqrt{g^2 - 3}\right)$.

The functional relation F (u, v) = 0 that we are looking for, is

$$F(u, v) = u - \ln\left(v + \sqrt{v^2 - 3}\right) = 0.$$

You can easily verify that

$$F(f(x, y), g(x, y)) = 0.$$

You can also check that the Jacobian is zero.

We are leaving it to you as an exercise in E 12).

Observe that F is not identically zero.

You should be able to do these exercises now.

E12) a) Verify that the functions f and g of Example 12 actually satisfy the relation $F(u, v) = u - \ln\left(v + \sqrt{v^2 - 3}\right)$

b) Prove that $\frac{\partial(f, g)}{\partial(x, y)} = 0$ for all x, y for the functions f and g of Example 12.

E13) Prove that the following functions are functionally dependent.

$$f(x, y) = \frac{y}{x},\ g(x, y) = \frac{x - y}{x + y}$$

E14) Find the functional relation for the following pairs if such a relation exists.

a) $f(x, y) = \frac{x + y}{1 - xy}$, $g(x, y) = \frac{(x + y)(1 - xy)}{(1 + x^2)(1 + y^2)}$

b) $f(x, y) = \frac{x + y}{x}$, $g(x, y) = \frac{x + y}{y}$.

E15) Prove that the Jacobian of $u = x\cos y$, $v = x\sin y$ is non-zero throughout the domain $D = \{(x, y) \mid x > 0\}$.

Prove also that the map $T : D \to \mathbf{R}^2$ defined by $T(x, y) = (u, v)$ is not invertible.

E16) Show that the following functions satisfy the necessary condition for functional dependence.

a) $u = \frac{x}{y - z}$, $v = \frac{y}{z - x}$, $w = \frac{z}{x - y}$

b) $u = x^2y - xy^2 + xyz$

$v = xy + x - y + z$

$w = x^2 + y^2 + z^2 - 2yz + 2xz$

Let us now briefly recall what we have done in this unit.

9.5 SUMMARY

In this unit we have covered the following points.

1) We have discussed the invertibility of linear transformations from $\mathbf{R}^2$ to $\mathbf{R}^2$.

2) We have defined and computed Jacobians for functions of n variables. Thus, if $f_1, f_2, \ldots, f_n$ are n functions of n variables, which possess first order partial derivatives, then the Jacobian

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots\cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots\cdots & \frac{\partial f_2}{\partial x_n} \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots\cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

3) We have used the Jacobian to compute the partial derivatives of implicit functions.

4) We have studied the chain rule for Jacobians.

Thus,

$$\frac{\partial(u, v, w)}{\partial(r, s, t)} = \frac{\partial(u, v, w)}{\partial(x, y, z)} \frac{\partial(x, y, z)}{\partial(r, s, t)}$$

Where u, v, w are functions of x, y, z, and x, y, z are functions of r, s and t.

5) We have defined domains in $\mathbf{R}^n$.

A subset $D \subset \mathbf{R}^n$ is a domain if it is open and polygonally connected.

6) We have seen that if every point in a domain is a stationary point of a function, then the function must be a constant.

7) Finally, we have seen that the vanishing of the Jacobian $\frac{\partial(f, g)}{\partial(x, y)}$ on a domain is a necessary condition for the functional dependence of f and g in that domain.

9.6 SOLUTIONS AND ANSWERS

E1) a) $\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial y} = -2$, $\frac{\partial z}{\partial x} = 2$, $\frac{\partial z}{\partial y} = 1$.

$$\therefore \frac{\partial(w, z)}{\partial(x, y)} = \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} = 5 \neq 0.$$

b) $\frac{\partial(w, z)}{\partial(x, y)} = \begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix} = 29 \neq 0.$

Both are invertible.

E2) a) $\frac{\partial f}{\partial x} = \cos x \;\therefore\; \left.\frac{\partial f}{\partial x}\right|_{(\pi, \pi/2)} = \cos \pi = -1$

$\frac{\partial f}{\partial y} = 0$

$\frac{\partial g}{\partial x} = -y \sin xy \;\therefore\; \left.\frac{\partial g}{\partial x}\right|_{(\pi, \pi/2)} = -\frac{\pi}{2} \sin \frac{\pi^2}{2}$

$\frac{\partial g}{\partial y} = -x \sin xy \;\therefore\; \left.\frac{\partial g}{\partial y}\right|_{(\pi, \pi/2)} = -\pi \sin \frac{\pi^2}{2}$

$\therefore$ At $(\pi, \pi/2)$, $\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} -1 & 0 \\ -\frac{\pi}{2} \sin \frac{\pi^2}{2} & -\pi \sin \frac{\pi^2}{2} \end{vmatrix} = \pi \sin \frac{\pi^2}{2}.$

b) $\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

c) $J_F(x, y, z) = \begin{vmatrix} yz \cos xyz & xz \cos xyz & xy \cos xyz \\ z & 0 & x \\ 0 & 0 & 0 \end{vmatrix} = 0$

d) $J_F(x, y, z) = \begin{vmatrix} z & 0 & x \\ y & x & 0 \\ 0 & z & y \end{vmatrix} = 2xyz$

$\therefore J_F(\pi, 2, 4) = 16\pi.$

E3) Jacobian matrix $= \begin{bmatrix} 1 & 1 \\ 2xy & x^2 \end{bmatrix}$

Jacobian $= \begin{vmatrix} 1 & 1 \\ 2xy & x^2 \end{vmatrix} = x^2 - 2xy = 0 \Leftrightarrow x(x - 2y) = 0$

$\Leftrightarrow x = 0$ or $x = 2y$.

$\therefore$ The Jacobian is zero on the set $\{(x,y) \mid x = 0 \text{ or } x = 2y\}$.

E4) Let $F(x, y, u, v) = xy + x^2u - vy^2$ and $G(x, y, u, v) = 3x - 4uy - x^2v$.

Then $\frac{\partial F}{\partial x} = y + 2xu$, $\frac{\partial F}{\partial u} = x^2$, $\frac{\partial F}{\partial v} = -y^2$,

$\frac{\partial G}{\partial x} = 3 - 2xv$, $\frac{\partial G}{\partial u} = -4y$, $\frac{\partial G}{\partial v} = -x^2$

$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} x^2 & -y^2 \\ -4y & -x^2 \end{vmatrix} = -[x^4 + 4y^3].$

$\frac{\partial(F,G)}{\partial(x, v)} = \begin{vmatrix} y + 2xu & -y^2 \\ 3-2xv & -x^2 \end{vmatrix} = -x^2(y + 2xu) + y^2(3 - 2xv).$

$$\frac{\partial(F,G)}{\partial(u,x)} = \begin{vmatrix} x^2 & y+2xu \\ -4y & 3-2xv \end{vmatrix} = x^2(3-2xv) + 4y(y+2xu)$$

$$\therefore \frac{\partial v}{\partial x} = -\left[\frac{x^2(y+2xu) - y^2(3-2xv)}{x^4+4y^3}\right] \text{ and}$$

$$\frac{\partial u}{\partial x} = \left[\frac{x^2(3-2xv) + 4y(y+2xu)}{x^4+4y^3}\right]$$

E5). Differentiating F (x, y, z, u, v) = 0 and G (x, y, z, u, v) = 0 w.r.t. y, we get

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0$$

$$\frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} = 0$$

Solving these simultaneous equations we get

$$\frac{\partial u}{\partial y} = -\frac{\partial(F,G)}{\partial(y,v)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$$

and $\frac{\partial v}{\partial y} = -\frac{\partial(F,G)}{\partial(u,y)} \Big/ \frac{\partial(F,G)}{\partial(u,v)}$

Similarly obtain $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$.

E6) $$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} z+v & u \\ 1 & 1 \end{vmatrix} = z+v-u$$

$$\frac{\partial(F,G)}{\partial(y,v)} = \begin{vmatrix} x+z & u \\ 1 & 1 \end{vmatrix} = x+z-u$$

$$\frac{\partial(F,G)}{\partial(u,z)} = \begin{vmatrix} z+v & y+u \\ 1 & 1 \end{vmatrix} = z+v-y-u$$

$$\therefore \frac{\partial u}{\partial y} = \frac{u-x-z}{z+v-u}, \quad \frac{\partial v}{\partial z} = \frac{u+y-v-z}{z+v-u}$$

E7) From Example 8 we have

$$\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} = 0$$

$$\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} = 0.$$

Eliminating $\frac{dx}{dt}$ from these we get

$$\left[\frac{\partial F}{\partial y}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial y}\right]\frac{dy}{dt} + \left[\frac{\partial F}{\partial t}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial t}\right] = 0$$

$$\text{or } -\frac{\partial(F,G)}{\partial(x,y)}\frac{dy}{dt} - \frac{\partial(F,G)}{\partial(x,t)} = 0$$

$$\text{or } \frac{dy}{dt} = -\frac{\partial(F,G)}{\partial(x,t)} \Big/ \frac{\partial(F,G)}{\partial(x,y)}$$

E8) $\dfrac{\partial (x, y, z)}{\partial (r, \theta, z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$

E9) $\dfrac{\partial (u, v, w)}{\partial (x, y, z)} = \begin{vmatrix} ye^{xy} & xe^{xy} & 0 \\ 0 & ze^{yz} & ye^{yz} \\ ze^{zx} & 0 & xe^{zx} \end{vmatrix}$

$= 2xyz\, e^{(xy+yz+zx)}$

$\dfrac{\partial (x, y, z)}{\partial (r, s, t)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2s & 0 \\ 0 & 0 & 3t^2 \end{vmatrix} = 6st^2$

$u = e^{xy} = e^{rs^2}, v = e^{yz} = e^{s^2t^3}, w = e^{zx} = e^{rt^3}$

$\dfrac{\partial (u, v, w)}{\partial (r, s, t)} = \begin{vmatrix} s^2e^{rs^2} & 2rse^{rs^2} & 0 \\ 0 & 2st^3e^{s^2t^3} & 3s^2t^2e^{s^2t^3} \\ t^3e^{rt^3} & 0 & 3rt^2e^{rt^3} \end{vmatrix}$

$= 12rs^3t^5\, e^{(rs^2+s^2t^3+rt^3)}$

$\dfrac{\partial (u, v, w)}{\partial (x, y, z)} \cdot \dfrac{\partial (x, y, z)}{\partial (r, s, t)} = 2xyz\, e^{(xy+yz+zx)} \cdot 6st^2$

$= 12rs^3t^5 e^{(rs^2+s^2t^3+rt^3)}$

$= \dfrac{\partial (u, v, w)}{\partial (r, s, t)}$.

E10) c) and f) are domains.

a), b), d) and e) are not domains.

E11) a) For $y = y_0$, let $g_{y_0}(x) = f(x, y_0)$.

Then g_{y_0} is a function of a single variable.

Further, $g'_{y_0}(x) = f_x(x, y_0) = 0$ in an interval I which is the projection of N on the x–axis.

$\therefore$ $g_{y_0}(x)$ is a constant function on I.

$\therefore$ $g_{y_0}(x) = c_{y_0}$, say.

Thus, $f(x, y_0) = c_{y_0}$

In general, $f(x, y) = c_y$.

i. e., f (x, y) depends only on y, and hence is a function of y alone.

b) similar.

E12) a) $F(u, v) = u - \ln\left[v + \sqrt{v^2 - 3}\right]$

If $u = f(x, y)$, $v = g(x, y)$, then

$F(u, v) = f(x, y) - \ln\left[g(x, y) + \sqrt{g^2(x, y) - 3}\right]$

$$= \ln x - \ln y - \ln\left[\frac{x^2+3y^2}{2xy} + \sqrt{\frac{(x^2+3y^2)^2}{4x^2y^2} - 3}\right]$$

$$= \ln x - \ln y - \ln\left[\frac{x^2+3y^2}{2xy} + \frac{(x^2-3y^2)}{2xy}\right]$$

$$= \ln x - \ln y - \ln\left(\frac{x}{y}\right) = 0.$$

b) $$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{x} & -\frac{1}{y} \\ \frac{1}{2y} - \frac{3y}{2x^2} & -\frac{x}{2y^2} + \frac{3}{2x} \end{vmatrix}$$

$$= \frac{1}{x}\left[\frac{-x}{2y^2} + \frac{3}{2x}\right] + \frac{1}{y}\left[\frac{1}{2y} - \frac{3y}{2x^2}\right]$$

$$= \frac{-1}{2y^2} + \frac{3}{2x^2} + \frac{1}{2y^2} - \frac{3}{2x^2} = 0.$$

E13) $1 + g(x, y) = 1 + \frac{x-y}{x+y} = \frac{2x}{x+y}$

$1 - g(x, y) = 1 - \frac{x-y}{x+y} = \frac{2y}{x+y}$

$\therefore \frac{1 - g(x, y)}{1 + g(x, y)} = \frac{y}{x} = f(x, y).$

Hence f and g satisfy the functional relation

$F(f, g) = f - \frac{1-g}{1+g} = 0.$

$\therefore$ f and g are functionally dependent.

E14) a) $$\frac{\partial(f, g)}{\partial(x, y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{(1-xy)^2-(x+y)^2}{(1+x^2)^2(1+y^2)} & \frac{(1-xy)^2-(x+y)^2}{(1+x^2)(1+y^2)^2} \end{vmatrix}$$

$$= \frac{(1-xy)^2-(x+y)^2}{(1-xy)^2(1+x^2)(1+y^2)} \begin{vmatrix} 1+y^2 & 1+x^2 \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = 0$$

Hence f and g can be functionally dependent.

$$f = \frac{x+y}{1-xy}$$

Solving for x.

$(1 - xy)f = x + y$

$x(1 + fy) = f - y$

or $x = \frac{f-y}{1+fy}$.

Substitute in $g = \frac{(x+y)(1-xy)}{(1+x^2)(1+y^2)}$

$$g = \frac{\left[\frac{f-y}{1+fy} + y\right]\left[1 - \frac{(f-y)\,y}{(1+fy)}\right]}{\left[1 + \frac{(f-y)^2}{(1+fy)^2}\right](1+y^2)}$$

$$= \frac{[(f-y) + y\,(1+fy)]\ [(1+fy) - fy + y^2]}{[(1+fy)^2 + (f-y)^2]\ (1+y^2)}$$

$$= \frac{f\,(1+y^2)}{(f^2+1)\,(1+y^2)}.$$

$$g = \frac{f}{f^2+1} \quad \therefore \quad g - \frac{f}{f^2+1} = 0$$

$F(u, v) = u - \frac{v}{v^2+1} = 0$ is the required function.

$$\frac{\partial (f, g)}{\partial (x, y)} = \begin{vmatrix} -y/x^2 & 1/x \\ 1/y & -x/y \end{vmatrix}$$

$$= \frac{xy}{x^2y^2} - \frac{1}{xy} = 0 \ \forall\ x, y.$$

Hence f and g can be functionally dependent.

Now $f(x, y) = 1 + \frac{y}{x}$

$$\Rightarrow x = \frac{y}{f(x, y) - 1}$$

$$\Rightarrow g(x, y) = 1 + \frac{x}{y} = 1 + \frac{1}{f(x, y) - 1} = \frac{f(x, y)}{f(x, y) - 1}$$

Hence $g(x, y) - \frac{f(x, y)}{f(x, y) - 1} = 0.$

Hence g and f satisfy the relation

$$F(f, g) = g - \frac{f}{f-1} = 0$$

E15) $\frac{\partial (u, v)}{\partial (x, y)} = \begin{vmatrix} \cos y & -x \sin y \\ \sin y & x \cos y \end{vmatrix}$

$$= x \cos^2 y + x \sin^2 y.$$

$$= x$$

$\therefore \frac{\partial (u, v)}{\partial (x, y)}$ is non–zero on $D = \{(x, y) \mid x > 0\}$.

Now $T(x, y) = T(x, y + 2\pi)$, where $(x, y) \in D$.

$\therefore$ T is not (1–1) on D.

Hence it is not invertible.

UNIT 10 IMPLICIT AND INVERSE FUNCTION THEOREMS

Structure

10.1 INTRODUCTION

In this unit, we state two very important theorems, namely, the implicit function theorem and the inverse function theorem.

By now you are quite familiar with implicit functions. The implicit function theorem tells us that under certain conditions an implicit equation F(x,y) = 0 can be locally expressed explicitly as y = f(x).

In the last unit we saw that a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ is invertible if and only if its Jacobian is non-zero. Here we'll discuss the invertibility of non-linear transformations with the help of Jacobians.

In this unit you will find a lot of examples illustrating these two theorems. Please go through them carefully. You will get a better grasp of the theorems through them.

Objectives

After studying this unit, you should be able to

- state, prove and apply the implicit function theorem,
- derive and use a sufficient condition for the functional dependence of two functions from $\mathbf{R}^2 \to \mathbf{R}^2$,
- define and identify locally invertible functions,
- apply the inverse function theorem.

10.2 IMPLICIT FUNCTION THEOREM

In this section we will discuss one of the important theorems in advanced calculus, namely "the implicit function theorem". We first illustrate the theorem for functions of two variables. Then we shall extend the theorem to functions of three variables.

10.2.1 Implicit Function Theorem for Two Variables

You know from Unit 7 that an equation of the form F(x, y) = 0 does not necessarily represent a unique function y = f(x).

For example, consider the equation $F(x,y) = x^2 + y^2 - 1 = 0$. Here we cannot find a single value of y for a given value of x.

However, there are functions such as

i) $y = f_1(x) = +\sqrt{1-x^2}, x \in [1, -1]$

ii) $y = f_2(x) = -\sqrt{1-x^2}, x \in [-1, 1]$

$$\text{iii) } y = f_3(x) = \begin{cases} \sqrt{1-x^2}, & x \in [-1, 0[\\ -\sqrt{1-x^2}, & x \in [0, \frac{1}{2}[\\ \sqrt{1-x^2}, & x \in [\frac{1}{2}, 1], \end{cases}$$

which satisfy the equation $F(x, f(x)) = 0$. Fig. 1 shows the graphs of these functions.

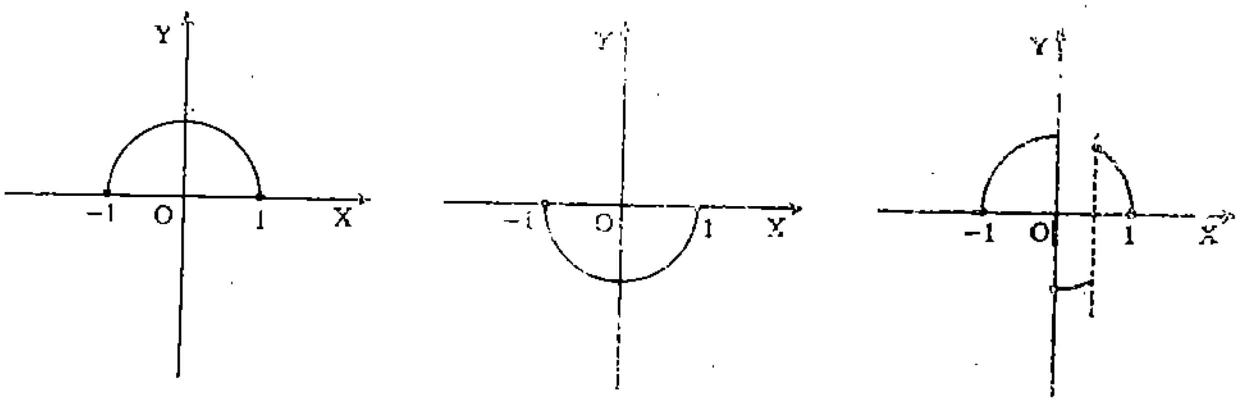


Fig. 1

In this context we can raise the following question.

"When can the equation $F(x,y) = 0$ be solved explicitly for y in terms of x, or solved explicitly for x in terms of y, yielding a unique function $y = f(x)$ or $x = g(y)$?" The implicit function theorem deals with this question. Before stating the theorem let us look at some examples.

Example 1 : Consider the equation $F(x,y) = x - y^2 = 0$.

Now, if we define $f_1 : \mathbf{R}^+ \to \mathbf{R}$, $f_1(x) = \sqrt{x}$, then f_1 satisfies the equation $F(x, f_1(x)) = 0$.

Similarly, the function $f_2 : \mathbf{R}^+ \to \mathbf{R}$, $f_2(x) = -\sqrt{x}$ also satisfies the equation $F(x, f_2(x)) = 0$.

So, corresponding to the given equation we have obtained two different functions defined on $\mathbf{R}^+$. Fig. 2 shows the graph of these functions.

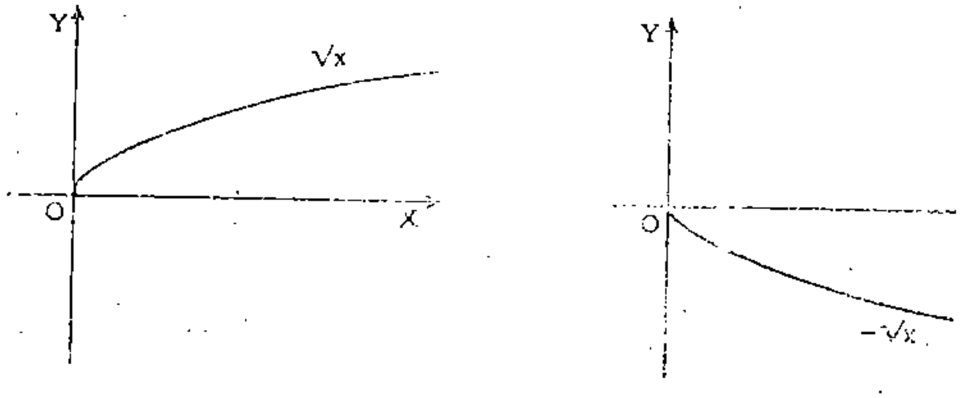


Fig. 2

Note that here both the functions f_1 and f_2 are continuous on their domains. Now, look at the three functions corresponding to the equation $F(x,y) = x^2 + y^2 - 1 = 0$, which we discussed earlier. You will agree that f_1 and f_2 are differentiable, whereas f_3 is not even continuous (see Fig. 1).

We have yet another situation in the next example.

Example 2 : Consider the equation

$$F(x, y) = x^5 + y^5 - 5x^3 y = 0.$$

We first note that we cannot solve for y to obtain it as a function of x or solve for x to obtain it as a function of y. Now let us look at the graphical representation of the function

$F(x,y)$ given in Fig. 3. We observe that there are three functions $f_1(x)$, $f_2(x)$ and $f_3(x)$ which satisfy $F(x,y) = 0$ in a neighbourhood of $x = 1$. But note that we are left with a unique function once we fix the ordinate y of the point $(1,y)$ on the curve.

Thus, we get f_1, if we fix the point $(1, 1.44)$,

f_2, if we fix the point $(1, 0.2)$,

f_3, if we fix the point $(1, -1.54)$.

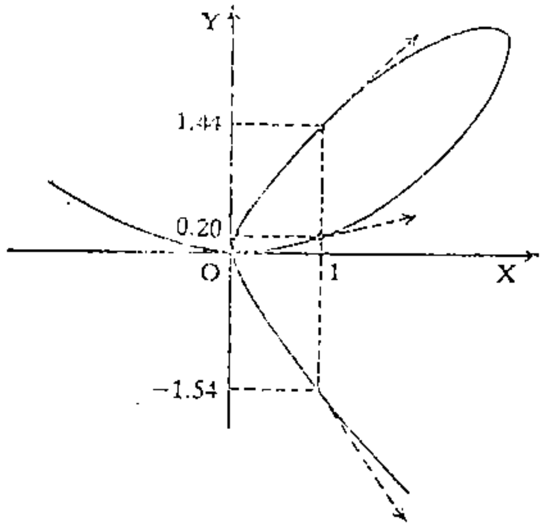


Fig. 3: Graph of $x^5 + y^5 - 5x^3y = 0$

From Fig. 3 you can see that the graphs of f_1, f_2 and f_3 are smooth curves. So, we can conclude that f_1, f_2 and f_3 are differentiable.

In these examples, given an equation $F(x,y) = 0$, we were able to find some functions f which satisfied $F(x, f(x)) = 0$. But we will be interested in knowing whether there are any functions having nice properties like continuity, differentiability, etc. More precisely, we would like to know the answer to the following question.

Suppose $F : D \to R$ is a real-valued function of two variables, where D is a non-empty subset of R^2. If $F(a, b) = 0$ for some (a, b), then does the equation $F(x, y) = 0$ yield a continuously differentiable function g defined on a neighbourhood of a, which satisfies $g(a)=b$? The implicit function theorem provides an answer to this question. It says that under some additional constraints we can give a positive answer. Here is the statement of this theorem.

Theorem 1 : (Implicit Function Theorem) Let F be a real-valued continuous function defined on some neighbourhood N of the point (a, b). If

i) $F(a, b) = 0$,

ii) $\frac{\partial F}{\partial y}$ exists and is continuous on N,

iii) $\frac{\partial F}{\partial y}(a, b) \neq 0$,

then there exists a unique function g defined on some neighbourhood N_a of a such that

i) $g(a) = b$,

ii) $F(x, g(x)) = 0$ for each $x \in N_a$, and

iii) g is continuous.

Moreover, if $\frac{\partial F}{\partial x}$ also exists and is continuous on N, then g is continuously differentiable on N_a and g' is given by

$$g'(t) = \frac{-\frac{\partial F}{\partial x}(t, g(t))}{\frac{\partial F}{\partial y}(t, g(t))}, \; t \in N_0$$

Proof : We prove the theorem in three steps. In Step 1 we prove the existence of the unique function g. in Step 2 we prove that g is continuous and in Step 3 we prove the differentiability of g.

Step 1 : We first note that the given function F is such that $\frac{\partial F}{\partial y}$ exists, is continuous and $\frac{\partial F}{\partial y}(a, b) \neq 0$. Suppose $\frac{\partial F}{\partial y}(a, b)$ is positive. Then there exits a neighbourhood N_1 of (a, b) contained in N such that $\frac{\partial F}{\partial y}$ is positive for all points in N_1 (see Theorem 6, Unit 4). Now we choose a rectangle T contained in N_1 (see Fig. 4).

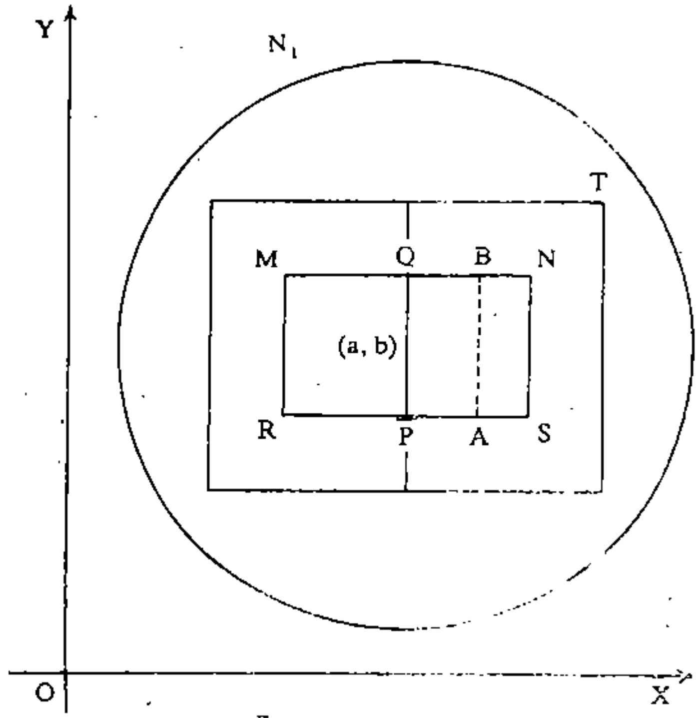


Fig. 4

Now consider the restriction $\hat{F}$ of F to the line through (a, b), parallel to the y-axis. Then $\hat{F}$ is a function of a single variable, y. Thus,

$$\hat{F}(y) = F(a, y) \text{ and } \hat{F}'(y) = \frac{\partial F}{\partial y}(a, y) > 0.$$

We know from Calculus (Unit 14) that a function of one variable increases when its derivative is positive. Therefore, we can conclude that $\hat{F}$ is an increasing function, and hence, F must be increasing on the line through (a,b), parallel to the y-axis. Further, we have

F(a,b) = 0. Therefore, F will have to be negative at some point, say P, and positive at some point, say Q, of this line. (See Fig 4).

Now let us use the fact that F is continuous. Since F is negative at P, there is a line segment RS through P and parallel to the x-axis, along which F is negative (see Fig. 4). Similarly, since F is positive at Q and is continuous, it must be positive along a line segment say MN through Q and parallel to the x-axis. (See Fig 4). Do you agree that we can choose MN and RS so that they are of equal length?

We thus have a rectangle MNRS = [c, d] × [e, f]. Then for each x_0 in the interval]c, d[, we can find a line AB in MNRS through x_0 , parallel to the y-axis. As the value of y goes from its value at A to its value at B, F goes from a negative value to a positive value. Note that the x coordinate is constant on AB. Now $F(x_0, y)$ is a continuous function of a single variable. Therefore, by the intermediate value theorem we can say that $F(x_0, y)$ must be zero at some point (x_0, y_0) on the line AB. Further, $\frac{\partial F}{\partial y}$ is positive on the rectangle T. This means that the derivative of the function $y \to F(x_0, y)$, which is equal to $\frac{\partial F}{\partial y}$ is positive on AB. This implies that the function $y \to F(x_0, y)$ is increasing along AB. Thus, F is zero only once on AB. That is, corresponding to x_0, there is a unique y_0, for which $F(x_0, y_0) = 0$.

(Note that corresponding to $x = a \in \,]c, d[$ we have the unique value b such that $F(a, b) = 0$.)

Hence, the correspondence $g : x_0 \to y_0$ is a function. The domain of g is the interval $]c, d[$. Thus, we have defined a function g on a neighbourhood $N_a = \,]c, d[$ of a such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in N_a$. Note that the function g defined on $]c, d[$ has its range contained in $]e, f[$.

Step 2 : Next we prove that g is continuous. For that we have to show that given $x_0 \in N_a = \,]c, d[$ and $\varepsilon > 0$, there is a number $\delta > 0$, such that

$$|g(x) - g(x_0)| < \varepsilon \text{ for } x \in \,]x_0 - \delta, x_0 + \delta[.$$

This can be achieved by repeating the above argument by choosing PQ properly. We choose PQ such that the length of PQ is not greater than 2ε and such that it has y_0 as its centre. Now having chosen PQ, you can see that the rest of the proof is irrespective of our choice of PQ. Thus, we get the same function g such that $F(x, g(x)) = 0$ for all $x \in \,]c, d[$.

Now choose δ such that $]x_0-\delta, x_0+\delta[$ is contained in $]c, d[$. Note that for each $x \in \,]c, d[$, $(x, g(x))$ lies on some line AB, parallel to PQ in MNRS. Hence we get that $|g(x) - g(x_0)| < \varepsilon$ for all $x \in \,]x_0 - \delta, x_0 + \delta[$.

This is true for all x_0 in $]c, d[$. Hence g is continuous throughout the interval.

Step 3 : It remains to prove that g is differentiable and that its derivative is continuous on $]c, d[$. In order to prove this we shall use the fact that $\frac{\partial F}{\partial x}$ exists and is continuous on N. To investigate the differentiability of g, we have to look at

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}.$$

The mean value theorem for functions of two variables tells us that

$$F(x+h, g(x+h)) - F(x, g(x))$$

$$= h \frac{\partial F}{\partial x}(\alpha, \beta) + [g(x+h) - g(x)] \frac{\partial F}{\partial y}(\alpha, \beta) \qquad \ldots(3)$$

for some (α, β) lying on the line segment joining $(x, g(x))$ and $(x + h, g(x + h))$.

But by the definition of g,

$$F(x, g(x)) = 0 \text{ and } F(x + h, g(x + h)) = 0 \text{ in T}.$$

Therefore, from (3) we have

$$\frac{g(x+h) - g(x)}{h} = \frac{-\frac{\partial F}{\partial x}(\alpha, \beta)}{\frac{\partial F}{\partial y}(\alpha, \beta)} \qquad \ldots(4)$$

Note that $\frac{\partial F}{\partial y}$ is non-zero on T.

We first observe that since g is continuous, $g(x + h) \to g(x)$ as $h \to 0$.

Hence $(\alpha, \beta) \to (x, g(x))$ as $h \to 0$.

Therefore, since $\frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial x}$ are continuous,

$$\lim_{h \to 0} \frac{\partial F}{\partial y}(\alpha, \beta) = \frac{\partial F}{\partial y}(x, g(x)), \text{ and}$$

$$\lim_{h \to 0} \frac{\partial F}{\partial x}(\alpha, \beta) = \frac{\partial F}{\partial x}(x, g(x)).$$

Thus, if we take the limit on both sides of (4), we get

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = \frac{-\frac{\partial F}{\partial x}(x, g(x))}{\frac{\partial F}{\partial y}(x, g(x))}.$$

Thus, g is differentiable.

Now the continuity of $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ ensures that g' is continuous.

This completes the proof of Theorem 1.

You must have noticed that Theorem 1 guarantees only the existence of a unique function g. It does not provide an explicit formula.

Now we illustrate this theorem with some examples.

Example 3 : Let us verify the implicit function theorem for the equation

$$F(x, y) = xy + x^2 = 0.$$

We first note that $F(x, y) = xy + x^2$ is a continuous function defined on $\mathbf{R}^2$. Then we find the set of points (x, y) at which

$$F(x, y) = xy + x^2 = 0.$$

$$xy + x^2 = 0 \Rightarrow x(x + y) = 0$$

i.e., either $y = -x$ or $x = 0$.

Therefore, $(x, -x)$, $x \neq 0$ and $(0, 0)$ are the points at which $F(x, y) = 0$. Let us take the point $(x, -x)$, $x \neq 0$.

Now $\frac{\partial F}{\partial y} = x$ is a continuous function in $\mathbf{R}^2$ and $\frac{\partial F}{\partial y}(x, -x) = x$ which is non-zero.

Thus, F satisfies all the conditions of Theorem 1. Therefore by Theorem 1, there exists a unique function g defined in a neighbourhood of x such that $g(x) = -x$ and g is continuous.

Moreover, since $\frac{\partial F}{\partial x} = y + 2x$ is continuous g is continuously differentiable and the derivative g' is given by

$$g'(x) = \frac{-\frac{\partial f}{\partial x}(x,y)}{\frac{\partial f}{\partial y}(x,y)}$$

$$= -\frac{y+2x}{x}, \; x \neq 0.$$

$$= -1, \text{ since } y = -x.$$

In fact, you can directly see from the equation that the function g is given by $g(x) = -x$.

Now when we consider the origin (0, 0), then $\frac{\partial F(0,0)}{\partial y} = 0 = \frac{\partial F(0,0)}{\partial x}$. Therefore, in this case, we cannot apply Theorem 1 to obtain y as a function of x or x as a function of y.

Here is another example.

Example 4 : Let us show that there exists a continuously differentiable function g defined by the equation

$$F(x, y) = x^3 + y^3 - 3xy - 4 = 0 \qquad \ldots (5)$$

in a neighbourhood of x = 2 such that g (2) = 2, and also find its derivative.

We first observe that F(2, 2) = 0 and

$$\frac{\partial F}{\partial y} = 3y^2 - 3x, \quad \frac{\partial F}{\partial x} = 3x^2 - 3y.$$

$\therefore \quad \frac{\partial F(2,2)}{\partial y} = 6 \neq 0$, and $\frac{\partial F}{\partial y}$ is continuous everywhere. Therefore, by Theorem 1, there exists a unique function g defined in a neighbourhood of x = 2 by g(x) = y, where F(x, y) = 0 and such that g(2) = 2. Moreover, since $\frac{\partial F}{\partial x}$ is continuous, by Theorem 1 again

$$g'(x) = -\frac{\dfrac{\partial F(x,y)}{\partial x}}{\dfrac{\partial F(x,y)}{\partial y}} = -\frac{x^2 - y}{y^2 - x} \text{ for all } x \text{ in a neighbourhood of } 2.$$

Why don't you try some exercises now ?

E 1) Apply Theorem 1 to $F(x, y) = x^2 - y^2 = 0$ at the points (1, 1) and (1, -1). Does the theorem apply at the point (0, 0)?

E 2) Consider the equation $F(x,y) = x^3 + y^3 - 16x^4 y - 1 = 0$. Apply Theorem 1 to F(x, y) at (1, 2) and see whether there is a function g defined by the equation in a neighbourhood of (1, 2) such that g(1)=2.

E 3) Show that the equation $2xy - \ln xy = 2$ determines a solution ϕ around the point x = 1 such that ϕ (1) = 1. Find the first derivative of the solution.

In the next sub-section we shall discuss the implicit function theorem for functions of three variables.

10.2.2 Implicit Function Theorem for Three Variables

In this sub-section we first discuss the implicit function theorem for functions of three variables in a form which directly generalises Theorem 1. Then we discuss how Jacobians play an important role in the implicit function theorem for more complicated situations.

Suppose we are concerned with the solution of an equation F(x, y, z) = 0 for one of the variables as a function of the others, say, we want to solve for z in terms of x and y. In this case we have to find a function f of two variables x and y such that

$$F(x, y, f(x, y)) = 0$$

for all x, y.

We now state a theorem, which is similar to Theorem 1, and which ensures the existence of the function f(x, y), mentioned above. As in the case of Theorem 1, this function f(x,y) also possesses continuous partial derivatives and therefore is continuously differentiable in a neighbourhood of the point under consideration. The proof runs exactly parallel to the proof of Theorem 1. We do not give it here as the technique of the proof is not used in any further discussions. However, we will illustrate the theorem with some examples.

Theorem 2 : Let F (x, y, z) be a real-valued function of three variables, which is continuously differentiable in a neighbourhood of a point $P_0 = (x_0, y_0, z_0)$ in R^3. Assume that $F(P_0) = 0$ and that $\frac{\partial F}{\partial z}(P_0) \neq 0$. Then there exists a unique function f which is continuously differentiable in a neighbourhood N of (x_0, y_0) in R^2 such that $f(x_0, y_0) = z_0$, and F(x, y, f(x, y)) = 0 for (x, y) in N.

We now look at Theorem 2 in another form. From the equation $F(x, y, z) = 0$ in Theorem 2, we construct a transformation

$$T : R^3 \to R^3,$$

given by

$$T(x, y, z) = (x, y, F(x, y, z)). \qquad \dots (6)$$

You have seen how to compute the Jacobians of some transformations in Unit 9. Now the Jacobian of the transformation T defined by (6) at a point $P_0(x_0, y_0, z_0)$ is given by

$$\begin{vmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = \frac{\partial F}{\partial z}(x, y, z)$$

This means that we can replace the condition $\frac{\partial F}{\partial z} \neq 0$ at a point P_0 in Theorem 2 by the condition that the Jacobian of T at P_0 is non-zero. In fact, in more complicated cases, where we want to find the solution of a system of equations, the non-vanishing of the Jacobian plays an important role.

Now we give a simple illustration of Theorem 2.

Example 5 : Consider a function $F : R^3 \to R$ given by

$$F(x, y, z) = x^2 + y^2 + z^2 - 1.$$

Suppose we want to find out whether the equation $F(x, y, z) = 0$ defines a unique function f in a neighbourhood of $(\frac{1}{2}, \frac{1}{2})$ such that $f(\frac{1}{2}, \frac{1}{2}) = \frac{-1}{\sqrt{2}}$.

We first note that

$$F(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} - 1 = 0.$$

Now, $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$ and $\frac{\partial F}{\partial z} = 2z$. Thus F is continuously differentiable since all its partial derivatives are continuous.

Further, $\frac{\partial F}{\partial z}(\frac{1}{2}, \frac{1}{2}, \frac{-1}{\sqrt{2}}) = -\sqrt{2} \neq 0$.

Therefore, F satisfies all the conditions of Theorem 2. Hence the equation defines a unique function f in a neighbourhood of $(\frac{1}{2}, \frac{1}{2})$, such that $f(\frac{1}{2}, \frac{1}{2}) = \frac{-1}{\sqrt{2}}$.

In the next example you can see an application of Theorem 2.

Example 6 : Let f be a continuously differentiable function of one variable such that $f(1) = 0$. Suppose we want to find the conditions under which the equation

$F(x, y, z) = f(xy) + f(yz) = 0$ can be solved for z in terms of x and y in a neighbourhood of (1, 1, 1).

Now since f is continuously differentiable, and $\frac{\partial F}{\partial x} = yf'(xy)$, $\frac{\partial F}{\partial y} = xf'(xy) + zf'(yz)$ and $\frac{\partial F}{\partial z} = yf'(yz)$, we can say that F is a continuously differentiable function.

Further $F(1, 1, 1) = 0$.

Also, we have $\frac{\partial F}{\partial z}(1, 1, 1) = f'(1)$.

Therefore, if we put the condition that $f'(1) \neq 0$, then F satisfies all conditions of Theorem 2. Thus, by Theorem 2, we can solve for z in terms of x and y in a neighbourhood of (1,1), provided $f'(1) \neq 0$.

Example 7 : Let us try to answer the question : Can the surface whose equation is $x + y + z - \sin(xyz) = 0$ be described by an equation of the form $z = f(x, y)$ in a neighbourhood of the point (0, 0), such that $f(0, 0) = 0$?

To answer this question we apply Theorem 2 to

$$F(x, y, z) = x + y + z - \sin(xyz) = 0.$$

Check that F is a continuously differentiable function such that $F(0, 0, 0) = 0$.

Also, $\frac{\partial F}{\partial z} = 1 - xy\cos(xyz)$.

Thus,

$$\frac{\partial F}{\partial z}(0, 0, 0) = 1 \neq 0.$$

Therefore by Theorem 2, there exists a neighbourhood of (0, 0) and a continuously differentiable function f defined on it such that $z = f(x, y)$ gives the same surface as $F(x,y,z) = 0$ in a neighbourhood of (0,0,0).

Here we would like to state the most general form of the implicit function theorem. A particular case of this theorem for $n = 3$, $m = 2$ was used in proving Theorem 3 in Unit 9.

Theorem 3 : Let $F_1(x_1, \ldots, x_n, u_1, \ldots, u_m), \ldots, F_m(x_1, \ldots, x_n, u_1, \ldots, u_m)$, be m functions of n+m variables defined in a neighbourhood N of the point (a, u) with $a = (a_1, \ldots, a_n)$, $u = (u_1, \ldots, u_m)$, such that

i) $F_j(a_1, \ldots, a_n, u_1, \ldots, u_m) = 0$, $1 \leq j \leq m$

ii) F_j is continuously differentiable for each j, $1 \leq j \leq m$.

iii) $\frac{\partial(F_1, \ldots, F_m)}{\partial(u_1, \ldots, u_m)} \neq 0$ at the point $(a_1, \ldots, a_n, u_1, \ldots, u_m)$.

Then there exist exactly m functions g_i of n variables such that each g_i is defined in a neighbourhood S of $(a_1, \ldots, a_n)$ and

i) $g_i(a_1, \ldots, a_n) = u_i$, $1 \leq i \leq m$

ii) $F_i(x_1, \ldots, x_n, g_1, \ldots, g_m) = 0$ for $(x_1, \ldots, x_n) \in S$ and $i = 1, 2, \ldots, m$

iii) Each g_i is continuously differentiable in S.

We will not prove this theorem here. We also do not expect you to remember the statement.

Now you can try this exercise.

E 4) For each of the following functions $F : \mathbf{R}^3 \to \mathbf{R}$, show that the equation $F(x, y, z) = 0$ defines a continuously differentiable function $z = f(x, y)$ in a neighbourhood of the given point (a,b).

a) $F(x, y, z) = x^3 + y^3 + z^3 - xyz - 2$ at (1, 1) such that $f(1, 1) = 1$

b) $F(x, y, z) = x^2 + y^2 - xy\sin z$ at (1, −1) such that $f(1, -1) = 0$.

In Unit 9 we had discussed the functional dependence of two functions. There we had also proved that if two differentiable functions f(x,y) and g(x,y) are functionally dependent on some domain $D \subset \mathbf{R}^2$, then the Jacobian

$$\frac{\partial(f, g)}{\partial(x, y)} = 0 \text{ for all } (x, y) \text{ in } D.$$

We had made a remark that the converse of this result also holds. Now we are in a position to prove it. The proof is an application of the implicit function theorem.

Theorem 4 : Suppose $u = f(x, y)$ and $v = g(x, y)$ are real-valued continuously differentiable functions defined in some open sphere S. If

$$\frac{\partial(u, v)}{\partial(x, y)} = 0$$

for all $(x, y) \in S$, then u and v are functionally dependent in S.

Proof : By definition,

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}$$

If all the entries in this determinant are identically zero in S, then by Theorem 5, Unit 9, u and v are constants. Therefore, they are functionally dependent. Now suppose there exists at least one entry which is not identically zero in S. Let us suppose that $\frac{\partial f}{\partial x} \neq 0$ at some point $(a, b) \in S$. Now consider the equation

$$F(x, y, u) = f(x, y) - u = 0$$

in three variables x, y and u. Clearly, the function $F(x, y, u)$ is defined in some neighbourhood of the point (a, b, u_0), where $u_0 = f(a, b)$. Since $\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} \neq 0$ at the point (a, b, u_0), the conditions of Theorem 2 are satisfied. Therefore, we can express x as a function of y and u, defined in some neighbourhood N of (b, u_0). If $x = \phi(y,u)$, then

$$v - g(\phi(y, u), y) = 0 \text{ or } v = G(y, u)$$

for $y \in]b - \delta^*, b+\delta^*[$ and $u \in]u_0 - \delta^*, u_0+\delta^*[$ for some $\delta^* > 0$.

We shall now show that $\frac{\partial G}{\partial y} \equiv 0$ for all $y \in]b - \delta, b+\delta[$ for some $\delta > 0$, which is enough to prove that G is a function of u alone. (see E 11 of Unit 9).

$$\frac{\partial v}{\partial x} = \frac{\partial G}{\partial u}\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial y} = \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial y}$$

But we are given that

$$0 = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial G}{\partial u}\frac{\partial u}{\partial x} & \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial G}{\partial y}$$

We know that $\frac{\partial u}{\partial x} \neq 0$ at (a, b) and is continuous at (a, b). Therefore, there exists a neighbourhood of (a,b) in which

$\frac{\partial u}{\partial x} \neq 0$. That is, there exists a $\delta > 0$ such that $\frac{\partial u}{\partial x} \neq 0$ for $x \in]a-\delta, a+\delta[$ and $y \in]b - \delta, b+\delta[$ and $0 < \delta < \delta^*$. Consequently, $\frac{\partial G}{\partial y} = 0$ for $y \in]b - \delta, b + \delta[$ and the proof is complete.

You may find this proof a little difficult to understand at the first attempt. But if you re-read it, we are sure you will be able to grasp it. The following example illustrates this theorem.

Example 8 : Show that there is a functional relationship between

$$u = 2 \ln x + \ln y, \; v = e^{x\sqrt{y}} \quad (x, y, > 0), \qquad \ldots(8)$$

and determine it.

We have

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{2}{x} & \frac{1}{y} \\ \sqrt{y}\, e^{x\sqrt{y}} & e^{x\sqrt{y}} \frac{x}{2\sqrt{y}} \end{vmatrix} = 0.$$

Therefore, by Theorem 4, there is a functional relationship between u and v.

Solving for y from the first of the Equations (8), we get $y = \frac{e^u}{x^2}$. Substituting for y in the second equation, we get

$$\ln v = x\sqrt{y} = e^{u/2}$$

i.e. $e^{u/2} - \ln v = 0.$

Now we make an important remark.

Remark 1 : i) The hypothesis in Theorem 4 that $\frac{\partial(u, v)}{\partial(x, y)} = 0$ on an open sphere is essential. For example,

if $u = x^3$, $v = y^3$, then $\frac{\partial(u, v)}{\partial(x, y)} = 0$ at (0, 0). But this Jacobian does not vanish throughout any open sphere containing (0, 0). You will also agree that u and v cannot be linked by any functional relationship.

ii) Theorem 4 can also be extended to three functions of three variables.

Now we pass onto the next section where we reconsider the problem of invertibility which we have outlined in Unit 9. But before that it is time to do an exercise.

E 5) Show that the following pairs of functions are functionally dependent.

a) $f(x, y) = e^{x-y}$, $g(x, y) = \sqrt{x^2 - 2xy + y^2 - 2x + 2y}$.

b) $u = x^2 + 2xy + y^2$

$v = 3x + 3y$

10.3 INVERSE FUNCTION THEOREM

Let us recall some known facts about functions of one variable.

Suppose f is a real-valued, continuously differentiable function defined on some open subset D of **R**. If for any point $x_0 \in D$, $f'(x_0) \neq 0$, then f' is not zero in $I =]x_0 - \delta, x_0 + \delta[\subset D$ for a suitable $\delta > 0$. In fact, $f'(x)$ has the sign of $f'(x_0)$ in I. If $f'(x_0) > 0$, then f(x) is strictly increasing in I, and if $f'(x_0) < 0$, then f(x) is strictly decreasing in I. In any case, f is one-one on I. Clearly, f(I) is an open interval containing $f(x_0)$. Thus, the function $f : I \to f(I)$ is one-one and onto and hence is invertible on I. Moreover, you may recall (Theorem 1, Sec. 4.3, Calculus) that the function $f^{-1} : f(I) \to I$ is differentiable at $f(x_0)$ and

$$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}.$$

Thus, if $f'(x_0) \neq 0$ for every $x \in D$, then the above holds for every point of D.

A similar result is true for functions of several variables and is known as "the inverse function theorem". In this section we state the theorem for functions of several variables and illustrate with examples for functions of two and three variables. We do not give the proof here as it is beyond the scope of this course. Before we state the theorem, we recall the definition of an inverse function and give some examples.

Definition 1 : A function f with domain $D \subset \mathbf{R}^n$ and range $D^* \subset \mathbf{R}^n$ is said to be **invertible** on D if there exists a function $g : D^* \to D$ such that $g(f(P)) = P$ and $f(g(Q)) = Q$ for every $P \in D$ and $Q \in D^*$.

Recall that $f : D \to D^*$ is invertible on D if and only if f is one-one (it is already onto). Moreover, the function g is uniquely determined by f and is called the **inverse of f**. It is usually denoted by f^{-1}.

We explain this definition with the following example.

Example 9 : Let D be a subset of $\mathbf{R}^2$ consisting of all pairs (r, θ) with $r > 0$ and $0 < \theta < \pi$. Define a function F on D by

$$F(r, \theta) = (f(r, \theta), g(r, \theta)),$$

where $f(r,\theta) = r\cos\theta$, $g(r, \theta) = r\sin\theta$.

We first note that the image of D, say D′, under this map is the upper half-plane consisting of all (x, y) such that $y > 0$ ($y > 0$ because $r > 0$ and $0 < \theta < \pi$). See Fig. 5.

You already know that a map F from $\mathbf{R}^2$ to $\mathbf{R}^2$ is given by the coordinate function : $F(x,y) = (f(x,y), g(x,y))$.

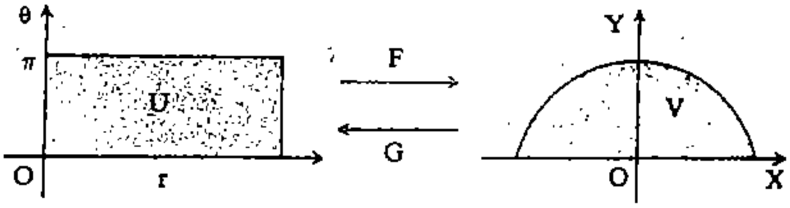


Fig. 5

Solving for r and θ we get,

$$r = \sqrt{x^2 + y^2} \text{ and } \theta = \cos^{-1}\frac{x}{r}.$$

Then the inverse map $G : D' \to D$ is given by

$$G(x, y) = \left(\sqrt{x^2 + y^2},\ \cos^{-1}\frac{x}{r}\right)$$

So far, by a neighbourhood of a point $a \in \mathbf{R}^n$ we had meant an open sphere $S(a,r) = \{ x \mid \|x - a\| < r \}$, of radius r. But from now on, we shall regard even those sets U of $\mathbf{R}^n$ as neighbourhoods of a, which contain an open sphere S (a, r) for a suitable r. This is, in fact, the universally acceptable definition of a neighbourhood in Euclidean spaces. We did not introduce this earlier, because we did not need it specifically. Thus, the closed disc $\{ (x,y) \mid (x-2)^2 + y^2 \leq 4 \}$ is also a neighbourhood of (2, 0). The space $\mathbf{R}^3$, or for that matter, any open set is a neighbourhood of each one of its points.

Now with this new interpretation of a neighbourhood of a point, we give the following definition.

Definition 2 : Let $f : D \to \mathbf{R}^n$, $D \subset \mathbf{R}^n$. We say that f is **locally invertible** at a point $p \in D$ if there exist a neighbourhood N of p contained in D and a neighbourhood N* of f(p), such that

i) $f(N) = N^*$

ii) f is 1–1 on N.

The following example would make this definition clear.

Example 10 : Consider the transformation

$F(x, y) = (2xy, x^2 - y^2)$.

It maps the whole plane $\mathbf{R}^2$ to $\mathbf{R}^2$. However, it is not 1–1 in the whole plane, since $f(1,1) = f(-1, -1) = (2, 0)$. Therefore, it is not invertible. In general, $f(p) = f(-p)$. But, if we take $D = \{(x, y) \mid x > 0\}$, then f restricted to D is 1–1.

To see this, let $f(x,y) = f(a,b)$. We will prove that x = a and y = b. Thus, we are given that

$2xy = 2ab$ and $x^2 - y^2 = a^2 - b^2$.

i.e., $x^2 - y^2 - a^2 + b^2 = 0$.

Therefore, since $x \neq 0$, $y = \frac{ab}{x}$ and on substituting the value of y in the second equation we obtain

$0 = x^2 - a^2b^2 - x^2a^2 + x^2b^2$ (because $xy = ab$).

$= (x^2 + b^2)(x^2 - a^2)$

Thus $x^2 - a^2 = 0$ or $x^2 + b^2 = 0$. But $x^2 + b^2$ cannot be zero. Therefore $x^2 = a^2$.

But $x > 0$ and $a > 0$ (on D). Therefore we get x = a. Then xy = ab gives that y = b. Thus, f maps the open half plane D into $\mathbf{R}^2$ in a one-one manner.

We shall show that

$D^* = \{(u,v) \mid v > 0 \text{ if } u = 0\}$

$= \mathbf{R}^2 -$ negative y–axis

is the range of f.

If u = 0, then y has to be zero, because u = 2xy and x > 0 in D. In that case $v = x^2 > 0$. Thus, no point on the negative y–axis can be the image of a point of D under f. Notice that for any $(x, y) \in D^*$, an open disc around (x, y) is contained in D^*. If $u \neq 0$, then $(u, v) = f(x, y)$, where

$$x = \left[\frac{v + \sqrt{u^2 + v^2}}{2}\right]^{1/2}$$

$$y = u\left[2v + 2\sqrt{u^2 + v^2}\right]^{-1/2} \quad \ldots (9)$$

We leave the verification of this to you as an exercise. (see E 6).

Now f is locally invertible at every point of D. This is because the sets D and D^* being open sets, can be regarded as neighbourhoods of each of their points. Therefore, for any point of D, both the requirements of local invertibility are satisfied. However, the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ as defined above is not locally invertible at (0, 0). The reason for this is that given any neighbourhood N of (0,0), we can find $x, y \in \mathbf{R}$ such that $(x, y) \in N$ and $(-x, -y) \in N$. Now since we know that $f(x, y) = f(-x, -y)$, we conclude that f is not 1–1 on N. We leave it as an exercise to you to check the local invertibility of f at (x,y) when $x < 0$ (see E 7).

Here is another example.

Example 11: Consider the function F given in Example 9 by $F(r, \theta) = (r\cos\theta, r\sin\theta)$ on the whole of $\mathbf{R}^2$.

You can easily see that the function is not invertible since F is not one-one. Note that $F(0, \theta) = (0, 0)$ for all θ. Let us check whether F is locally invertible. From Example 9, we can see that F is locally invertible at all points (r, θ) such that r > 0 and $0 < \theta < \pi$. But when we take the point (0,0), then any neighbourhood of this point contains points $(0, \theta)$, $\theta \neq 0$ which are mapped to (0, 0), i.e., F is not one-one in any neighbourhood of

(0, 0). Therefore F is not locally invertible at (0, 0). Later when we state Theorem 5, you will learn that this function is locally invertible at all points, (r, θ), such that $r \neq 0$.

Here are some exercises for you.

E 6) Prove the relations in (9).

E 7) Show that the function $f(x, y) = (2xy, x^2 - y^2)$ defined in Example 10 is locally invertible at (x, y), where $x < 0$.

We are now ready to state the inverse function theorem which provides us a sufficient criterion for the local invertibility of a function. After this theorem, you would realise that the above examples and exercises could have been done easily as we know how to calculate Jacobians.

Theorem 5 (Inverse Function Theorem) : Let $F_1, F_2, \ldots, F_n$ be n real-valued functions defined on an open subset D of $\mathbf{R}^n$.

Let $F = (F_1, \ldots, F_n)$ be a function from $\mathbf{R}^n$ to $\mathbf{R}^n$ with domain D. If F is continuously differentiable at a point $P_0 = (a_1, \ldots, a_n) \in D$ and if the Jacobian of F, i.e.,

$\dfrac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)}$ is non-zero at P_0, then the function F is locally invertible at P_0. Moreover, the local inverse F^{-1} of F is continuously differentiable at the point $F(P_0)$.

A vector - valued function $F = (f_1, \ldots, f_n)$ is differentiable (continuously differentiable) if $f_1, \ldots, f_n$ are differentiable (continuously differentiable).

For instance, look at Example 10. You can see that the Jacobian of F is given by $-4(x^2 + y^2)$ and the map is invertible at all points different from (0,0). We have already checked its invertibility at those points (x, y) for which $x > 0$.

Look at this example now.

Example 12 : Let us check the local invertibility of the function

$F : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$F(x, y) = (y \sin x, x + y + 1)$

at (0,1).

We apply the inverse function theorem (Theorem 5) to F. We first note that the function F is continuously differentiable, since both the functions $f(x,y) = y \sin x$ and $g(x, y) = x + y + 1$ are continuously differentiable.

The Jacobian of F is

$$JF = \begin{vmatrix} y \cos x & \sin x \\ 1 & 1 \end{vmatrix}$$

$$= y \cos x - \sin x$$

Therefore, $JF(0, 1) = 1 \neq 0$. Hence, by the inverse function theorem, F is locally invertible at (0, 1).

Now you can easily do these exercises.

E 8) Prove that the map $F : \mathbf{R}^2 \to \mathbf{R}^2$ given by $F(x, y) = (e^x \cos y, e^x \sin y)$ is not invertible on the whole of $\mathbf{R}^2$, but is locally invertible at each point of $\mathbf{R}^2$.

E 9) Determine whether the following maps are locally invertible at the given point.

a) $F(x, y) = (x^3 y + 1, x^2 + y^2)$ at (1, 2)

b) $F(x, y) = (e^{xy}, \ln x)$ at (1, 4)

c) $F(x, y) = (\sin x, \cos xy)$ at $(\pi, \frac{\pi}{2})$

d) $F(x, y, z) = (x + y + z, e^x \cos z, e^x \sin z)$ at (x, y, z).

Here is an important observation about Theorem 5.

Remark 2 : According to Theorem 5, the non-vanishing of the Jacobian at a point in the domain of definiton guarantees that the function has an inverse in a neighbourhood of the point. Now suppose that the Jacobian is non-zero at a.l points in the domain of definition of a functioj. Then by Theorem 5 we know that each point thas a neighbourchood in which the function in invertible. Now does this mean that the function is invertible in the entire domain? No. In (E8) we have already seen a function F from R^2 to R^2 for which

$$JF = \begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix} = e^{2x} \neq 0$$

for all $(x, y) \in R^2$. Thus, F is locally invertible at all points of R^2, but is not invertible, because it is not 1 – 1.

Therefore, it does not follow that the function is invertible in the domain of definition, even when the Jacobian is non-zero for all points in the domain.

Also note that a function may be locally invertible at a point, even when its Jacobian vanishes at that point. For example, consider the function $f(x, y) = (x^3, y^3)$. The Jacobian of fat (0, 0) is 0. But f is invertible in $]-1, 1[\times]-1, 1[$, the inverse being given by $x^{1/3}, y^{1/3})$.
That brings us to the end of this unit. Let us now take a quick look at the points covered in it.

10.4 SUMMARY

In this unit we have

- Stated the implicit function theorem for functions of two and three variables and proved the theorem for two variables.

Implicit function theorem for two variables :

Let F be a real-valued continuous function defined n some neighbourhood N of the point (a, b). If

(i) $F(a, b) = 0$.

(ii) $\frac{\partial F}{\partial y}$ exists and is continuous on N. and

(iii) $\frac{\partial F}{\partial y}(a, b) \neq 0$.

then there exists a unique function g defined on some neighbourhood N_a of a such that

(i) $g(a) = b$.

(ii) $F(x, g(x)) = 0$ for each $x \in N_a$, and

(iii) g is continuous

Moreover, if $\frac{\partial F}{\partial x}$ also exists and is continuous on N, then g is continuously differentiable on N_a and g' is given by

$$g'(t) = \frac{-\frac{\partial F}{\partial x}(t, g(t))}{\frac{\partial F}{\partial y}(t, g(t))}, \; t \in N_a.$$

Obtained a sufficient condition for the functional dependence of two functions:

Suppose $u = f(x, y)$ and $v = g(x, y)$ are functions from R^2 to R which are continuously differentiable in an open sphere S.

Then u and v are functionally dependent in S if

$$\frac{\partial(u, v)}{\partial(x, y)} = 0 \text{ at all points of } S$$

- Defined the local invertibility of maps from $\mathbf{R}^n$ to $\mathbf{R}^n$
- Discussed the inverse function theorem which gives a sufficient condition for local invertibility :

  Let $F = (F_1,\ldots,F_n)$ be a function from R^n to R^n with comain D. If F is continuously differentiable at a point $P_0 = (a_1,\ldots, a_n) \in D$ and if the lacobian of F, i.e.

  $\frac{\partial(F_1,\ldots,F_n)}{\partial(x_1,\ldots,x_n)}$ is non-zero at P_0, then the function F is locally invertible at P_0.

  Moreover, the local inverse F^{-1} of F is continuously differentiable at the point F (P_0).

10.5 SOLUTIONS AND ANSWERS

(E1) $F(1, 1) = 0, F(1, -1) = 0, F(0, 0) = 0$

$\frac{\partial F}{\partial y} = -2y$ exists and is continuous.

$\frac{\partial F}{\partial y} \neq 0$ at (1, 1).

$\frac{\partial F}{\partial y} = 0$ at (0, 0).

$\frac{\partial F}{\partial x} = 2x$ is also continuous.

∴ The theorem does not apply at (0, 0). According to the theorem. ∃ a continous function g defined on a neighbourhood N of (1, a) such that g (1) = 1.

$F(x, g(x)) = 0, \forall x \in N$.

In fact, you can see that g (x) = x.

$$g'(x) = \frac{-\partial F / \partial x}{\partial F / \partial y} = \frac{-2x}{-2y} = \frac{x}{y} = 1$$

Similarly, you can apply the theorem to (1, −1). In this case you will get a.g.

$g_1(1) = -1$.

$F(x, g_1(x)) = 0 \ \forall x \in N_1$.

In fact, $g_1(x) = -\sqrt{x^2}$.

$$g_1(x) = \frac{-\partial F / \partial x}{\partial F / \partial y} = \frac{x}{y} = \frac{x}{-\sqrt{x^2}} = -1.$$

(E2) $F(1, 2) = 0$.

$\frac{\partial F}{\partial x} 5x^4 - 48x^2y$

$\frac{\partial F}{\partial y} = 5y^4 - 16x^3$. $\frac{\partial F}{\partial y}(1, 2) = 64 \neq 0$.

$\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous.

Therefore, there exists a continuous function g defined on a neighbourhood N of (1, 2), such that

$g(1) = 2$, and $F(x, g(x)) = 0 \ \forall x \in N$.

Further, $g'(x) = \frac{-5x^4 + 48x^2y}{5y^4 - 16x^3}$.

(E3) Let $F(x, y) = 2xy - \ln xy - 2$

Then $F(1, 1) = 0$

$$\frac{\partial F}{\partial x} = 2y - \frac{1}{x}$$

$$\frac{\partial F}{\partial y} = 2x - \frac{1}{y}, \; \frac{\partial F}{\partial y}(1,1) = 1 \neq 0.$$

$\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are continuous in a neighbourhood N of (1,1). Hence there exists a continuous function ϕ defined on N such that $\phi(1) = 1$.

$$\phi'(x) = \frac{-(2y-1/x)}{2x-1/y} = \frac{y(1-2xy)}{x(2xy-1)} = \frac{-y}{x}$$

E 4) a) $F(1,1,1) = 0$.

$$\frac{\partial F}{\partial x} = 3x^2 - yz,$$

$$\frac{\partial F}{\partial y} = 3y^2 - xz,$$

$\frac{\partial F}{\partial z} = 3z^2 - xy$ are all continuous functions.

Further, $\frac{\partial F}{\partial z}(1,1,1) = 2 \neq 0$.

Hence, by Theorem 2, there exist a continuously differentiable function f defined in a neighbourhood of (1, 1), such that f(1, 1) = 1.

b) Similar.

E 5) a)

$$\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} e^{x-y} & -e^{x-y} \\ \frac{x-y-1}{\sqrt{(x-y)^2-2(x-y)}} & \frac{-(x-y-1)}{\sqrt{(x-y)^2-2(x-y)}} \end{vmatrix} = \begin{vmatrix} A & -A \\ B & -B \end{vmatrix}, \text{ say}$$

$$= 0$$

Hence, f and g are functionally dependent.

b) $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x+2y & 2x+2y \\ 3 & 3 \end{vmatrix} = 0.$$

Hence u and v are functionally dependent.

E 6) $u \neq 0 \Rightarrow 2xy \neq 0 \Rightarrow x \neq 0, y \neq 0$,

and $y = \frac{u}{2x}$.

$$\therefore v = x^2 - \frac{u^2}{4x^2}$$

$$\Rightarrow 4x^4 - 4x^2v - u^2 = 0$$

$$\Rightarrow x^2 = \frac{4v + \sqrt{16v^2 + 16u^2}}{8}$$

We do not consider the other root as it will mean that $x^2 < 0$.

$$\therefore x^2 = \frac{v + \sqrt{v^2 + u^2}}{2}$$

$$\therefore x = \left[\frac{v + \sqrt{v^2 + u^2}}{2}\right]^{1/2}$$

$$\Rightarrow \quad y = u\left[2v + 2\sqrt{v^2 + u^2}\right]^{-1/2}$$

E 7) Let $D_1 = \{(x,y) \mid x < 0\}$.

Then f restricted to D_1 is 1–1.

The proof is similar to that of "f restricted to D is 1–1" given in Example 10.

Thus f maps D_1 into $\mathbf{R}^2$ in a one-one manner.

Now, $D_1^* = \{(u,v) \mid v > 0 \text{ if } u = 0\}$

$= \mathbf{R}^2$ – the negative y-axis

is the range of f. The proof of this also is similar to the one given in Example 10.

Further, D_1^* is open.

$\therefore$ $f: D_1 \to D_1^*$ is 1–1 and onto, where D_1 and D_1^* are both open sets.

$\therefore$ f is locally invertible at any point (x, y), where $x < 0$.

E 8) $F(x, y) = F(x, y + 2\pi)\ \forall\ (x, y) \in \mathbf{R}^2$.

Hence F is not 1–1 in $\mathbf{R}^2$.

$\therefore$ F is not invertible on the whole of $\mathbf{R}^2$.

$$\text{Now } JF = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$= e^{2x} \neq 0$ for any $(x,y) \in \mathbf{R}^2$.

$\therefore$ By Theorem 5, F is locally invertible at each point of $\mathbf{R}^2$.

E 9) a)

$$JF = \begin{vmatrix} 3x^2y & x^3 \\ 2x & 2y \end{vmatrix} = 6x^2y^2 - 2x^4.$$

$\therefore$ $JF(1, 2) = 24 - 2 = 22 \neq 0$.

$\therefore$ F is locally invertible at (1, 2).

b)

$$JF = \begin{vmatrix} ye^{xy} & xe^{xy} \\ \frac{1}{x} & 0 \end{vmatrix} = -e^{xy}$$

$\therefore$ $JF(1, 4) = -e^4 \neq 0$.

$\therefore$ F is locally invertible at (1,4).

c) F is locally invertible at $(\pi, \pi/2)$.

d)

$$JF = \begin{vmatrix} 1 & 1 & 1 \\ e^x \cos z & 0 & -e^x \sin z \\ e^x \sin z & 0 & e^x \cos z \end{vmatrix} = -e^{2x}\sin^2 z - e^{2x}\cos^2 z$$

$= -e^{2x} \neq 0$ at any (x, y, z)

$\therefore$ The given map is locally invertible at every point.

ERRATA
MTE-07
Block 1

Page No.	Line No.	Should be
26	13	Then $x \,.\, k = x \,.\, \frac{\infty}{\infty} \ldots\ldots = k$, if $x > 0$.
30	19	$\lim_{x \to \infty} (\cos \frac{1}{x}) = \cos (\lim_{x \to \infty} \frac{1}{x}) = 1$.
31	11, 12	given ...... Then $\left\lvert \frac{1}{\ln x} \right\rvert < \varepsilon \Leftrightarrow \frac{1}{\ln x} < \varepsilon$ $\Leftrightarrow x > e^{1/\varepsilon}$
43	5	..... at $x = a$ (a may be ∞ or $-\infty$).

ERRATA
MTE-07
Block 2

Page No.	Line No.	Should be
5	32	. . . . $\{x \mid x \in R^n$, . . .
6	21	. . . . $m = L - \varepsilon$ and $M = L + \varepsilon$.
6	47	. . . . then $\mid f(x) - L \mid < \varepsilon$,
11	23	if $\lim_{x \to a} f(x) = f(a)$.
17	14	. . . . if given $\varepsilon > 0$, . . .
17	19	. . . . if given $\varepsilon > 0$, ...
19	27	. . . . . with $m_1 = 1$, $m_2 = -1$.
20	33	Let $f(x_1, x_2, \ldots, x_n)$ . . . . .
44	15	. . . . $\frac{\varepsilon}{\sqrt{8}} \Rightarrow$ . . . .
44	26	. . . . $= 0 = f(0, 0)$.
48	1	. . . . $f(x, y) = \frac{2m}{1 + m^2}$ . . . . .
48	2	. . . . . is $\frac{2m}{1 + m^2}$, and . . . . .
48	9	d) $x^2 + y^2 \geq x$ [illegible]
60	1	derivatives. The conditions . . . . .
67	6	. . . . $= h$
72	28	. . . . . point $(f(t_0), g(t_0))$ . . . .
[illegible]	23	. . . . . and $y = g(t) = \tan t$
79	8	. . . . . . and $z = \phi(x, y)$ . . . . . .
79	27	$= -6t$ . . . . .
88	32	the plane $y = 1$ . . .
92	18	$g_x(a, b) = f_{[illegible]}(g(a, b)) g_x(a, b)$
92	19	$g_y(a, b)$ [illegible] (a, b)
92	21	$g'(t_0) = g_1'(t_0) f_x(g_1(t_0), g_2(t_0)) + g_2'(t_0) f_y(g_1(t_0), g_2(t_0))$ where $g = (g_1, g_2)$.
95	9	$\frac{\tan [illegible]}{\tan A - [illegible]}$
96	26	$\frac{\partial w}{\partial z}$ . . .
99	19	$\lim_{t \to 0} \frac{\sin(t \cos \pi/5)}{t \cos \pi/5}$ $(\cos \pi/5)$
100	26	. . . . $\Rightarrow$ [illegible] $\cos \theta$, ...

NOTES

UTTAR PRADESH
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UGMM-07
Advanced Calculus

Block

4

MULTIPLE INTEGRATION

BLOCK 4 MULTIPLE INTEGRATION

This is the fourth and the last block of this course. In the earlier blocks you studied the differential calculus of functions of several variables, that is, you studied the concepts of limit, continuity, partial derivatives, differentiability and Jacobians for functions of several variables. You also saw some applications of these concepts to problems of maxima-minima, invertibility, etc.

In this block we take up the study of the other aspect of the calculus of several variables, namely, the integral calculus.

In the first unit of this block, i.e., Unit 11, we confine our attention to functions of two variables. We first deal, in some detail, with the integrability of functions over closed rectangles, a natural generalisation of definite integral studied by you in Calculus. Here we show how to evaluate double integrals with the help of repeated integrals, which are really definite integrals with respect to each of the two variables. We do define double integrals over bounded subsets, but confine our attention to very special types of regions over which double integrals can be evaluated using iterated integrals.

Unit 12 is really a repetition of what we have done in the preceding unit, the only change being that here we deal with functions of 3 variables and regions of integration are subsets of $\mathbf{R}^3$.

In Unit 13 we discuss the applications of double and triple integrals to some geometric and physical problems, like evaluation of areas, volumes, centre of gravity and moments of inertia.

In the last unit, i.e. Unit 14, we discuss line integrals, which provide another way of generalising the definite integral. Even though we could have easily discussed line integrals in space also, yet we confine our attention to the integrals in $\mathbf{R}^2$ only, because the study of line integrals in $\mathbf{R}^3$ would be incomplete without the study of surface integrals, the introduction of which requires some sophisticated ideas, beyond the scope of the present course. This has forced us to omit theorems like Stoke's theorem, even though these were mentioned in your programme guide. We conclude this unit by proving Green's theorem for a special type of regions. This theorem establishes a link between line and double integrals

We hope that you will have a chance to study surface integrals and other related topics in some future course.

Make sure to work out all the intermediate steps in the solved examples. Spend some time on the exercises before looking at the solutions. We have omitted a large number of proofs in this block, and have included only a few. Read carefully the proofs which we have given, as you may be expected to write them in your examination.

Notations and Symbols

$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$	:	Sum of a_{ij}, $1 \le i \le n$, $1 \le j \le m$.
$P = \{a = x_0 < x_1 < \ldots\ldots < x_n = b\}$	:	Partition of $[a,b]$ into n sub-intervals
$\mathcal{P}$	:	Set of all partitions (of an interval)
$\iint_D f(x,y)\, dxdy$	:	Double integral of f over a bounded set D in $\mathbf{R}^2$
$\int_a^b \left[\int_c^d f(x,y)dy \right] dx$	:	Repeated integral of f first w.r.t. y, and then w.r.t. x.
$\int\int_W\int f(x,y,z)\, dx\, dy\, dz$	:	Triple integral of f over a bounded region W in $\mathbf{R}^3$
$\int_a^b \left[\int_c^d \left\{ \int_s^t f(x,y,z)dz \right\} dy \right] dx$	:	Repeated integral of f first w.r.t. z, then w.r.t. y and finally w.r.t. x.
M_x	:	Moment about the x-axis
$(\bar{x}, \bar{y})$	:	Centre of gravity of an object in the plane
M_{xy}	:	Moment about the xy-plane
$(\bar{x}, \bar{y}, \bar{z})$	:	Centre of gravity of a solid region in space
$\int_C f(x,y)\, dx$	:	Line integral of f over the curve C w.r.t. x
$\int_C f(x,y)\, ds$	:	Line integral of f over the curve C w.r.t. the arc length.

Also see the lists in previous blocks.

UNIT 11 DOUBLE INTEGRATION

Structure

11.1 INTRODUCTION

In the earlier calculus course (Block 3 Unit 10) we introduced you to the concept of integration for real-valued functions of one variable over a closed interval. In this unit we extend the notion of integrability to real-valued functions of two variables defined over a bounded set in the plane. Such integrals are called double integrals. We first of all consider real-valued functions defined over a closed rectangle, a natural generalisation to plane of a closed interval on the real line. We also introduce two other integrals called repeated integrals, which cannot have an analogue for functions of one variable. Roughly speaking, a repeated integral is a definite integral with respect to the two variables, successively. We show that in a large number of cases repeated integrals are equal to the double integral. This enables us to compute a double integral using the techniques of integration of functions of one variable. In Section 11.3 we extend the definition of double integral to functions which are defined over bounded sets. We also discuss some properties of these integrals in this section. In the last section we consider regions which are easily described by polar coordinates and describe how to evaluate integrals over such regions.

In this unit we omit quite a few proofs as these involve ideas which are beyond the scope of this course. However, we have tried to give a large number of examples to illustrate the results stated.

In the next unit we shall take up the study of triple integrals.

Objectives

After reading this unit you should be able to

- define double integral and repeated integrals of a real-valued function of two variables over a closed rectangle,
- evaluate double integrals using repeated integrals,
- define and evaluate double integrals over some special types of regions,
- effect change of variables in double integrals,
- compute double integrals using polar co-ordinates.

11.2 DOUBLE INTEGRAL OVER A RECTANGLE

In this section, we extend the theory of integrability of functions of one variable to functions of two variables. We shall consider integrability of functions defined over a closed rectangle. Before starting the discussion for two variables, we quickly review the concept of integration of functions of one variable in the following sub-section. Recall that you have studied integration in Block 3 of Calculus.

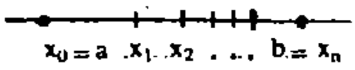
11.2.1 Preliminaries

We first recall the definition of integration of a function of one variable. Then we talk about partitions of rectangles in R^2 and the analogue of upper and lower sums in the case of a function of two variables.

To start with, let $f : [a, b] \rightarrow R$ be a bounded function.

Let $P : \{a = x_0 < x_1 < x_2 < \dots x_n = b\}$

be a partition of [a,b] into sub-intervals $[x_{i-1}, x_i]$, $1 \le i \le n$. See Fig. 1. Since f is a bounded real-valued function on [a,b], for each i, $1 \le i \le n$, there exist real numbers m_i and M_i, such that



$$m_i = \inf_{x_{i-1} \le x \le x_i} \{f(x)\}$$

$$M_i = \sup_{x_{i-1} \le x \le x_i} \{f(x)\}$$

Now we get

$$L(P,f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

and

$$U(P,f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}).$$

Note that $x_i - x_{i-1}$ is the length of the sub-interval $[x_{i-1}, x_i]$. The sums L(P,f) and U(P,f) are called **lower sum** and **upper sum** of f corresponding to the partition P. These sums are also referred to as lower and upper Riemann sums. Then we have

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a), \qquad \dots\dots (1)$$

where m and M are the infimum and supremum, respectively, of f in [a,b] (See Theorem 1, Unit 10, Calculus Block 3).

Let $\mathcal{P}$ be the set of all partitions of [a,b]. Now consider the set

$L = \{L(P,f) \mid P \in \mathcal{P}\}$ and $U = \{U(P,f) \mid P \in \mathcal{P}\}$.

Then the non-empty set L of real numbers is bounded above and the non-empty set U of real numbers is bounded below. Let

$I_L = \sup \{L(P,f) \mid P \in \mathcal{P}\}$

$I_U = \inf \{U(P,f) \mid P \in \mathcal{P}\}$

If $I_L = I_U = I$,

then we say that f is integrable over [a, b] and define

$$\int_a^b f(x)\,dx = I = \sup_{\mathcal{P}} \{L(P,f)\} = \inf_{\mathcal{P}} \{U(P,f)\}$$

I is also called the definite integral of f over [a, b].

You must have done some exercises in computing the integral by the above procedure in your Calculus course. To refresh your memory, you can try an exercise now.

E 1) Show that the function $f(x) = x$ is integrable on [a, b] and

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2).$$

(**Hint** : Use the inequality $x_{i-1} \le \frac{1}{2}(x_i + x_{i-1}) \le x_i$)

Now we adopt a similar procedure to define the integral of a real-valued function of two variables. A closed rectangle seems to be a natural generalisation of a closed interval on the real line. Here we assume that the area of a rectangle with sides of lengths a and b units is ab square units.

We begin our discussion of two variables by considering partition of a closed rectangle analogous to that of a closed interval.

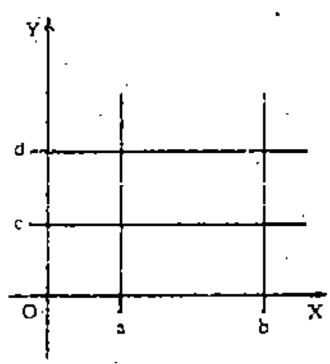


Fig. 2

Let T be a closed rectangle in R^2 formed by the lines $x = a$, $x = b$, $y = c$, $y = d$ as in Fig. 2. The rectangle T is nothing but the Cartesian product of closed intervals [a, b] and [c, d], i.e., $T : [a, b] \times [c, d]$ (see Unit 3 of Block 1).

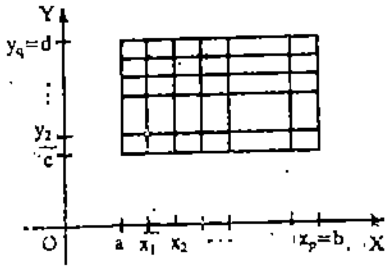
The most natural way to subdivide this rectangle is

i) to divide the interval [a, b] of the x-axis into p sub-intervals,
   $[a = x_0, x_1], [x_1, x_2], \ldots, [x_{i-1}, x_i], \ldots, [x_{p-1}, x_p = b]$,

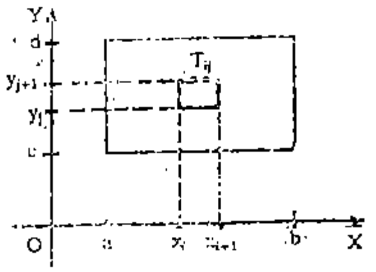
ii) to divide the interval [c, d] of the y-axis into q sub-intervals
   $[c = y_0, y_1], [y_1, y_2], \ldots, [y_{i-1}, y_i], \ldots, [y_{q-1}, y_q = d]$, and

iii) to consider the rectangles formed by the intervals $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$, $1 \leq i \leq p$, $1 \leq j \leq q$.

These three steps are illustrated in Fig. 3.



(a)



(b)

Fig. 3

Let us denote the rectangle formed by $[x_{i-1}, x_i]$ and $[y_{j-1}, y_j]$ by T_{ij}. Thus we have divided the rectangle T into p.q sub-rectangles T_{ij}s (see Fig. 3(b)). These T_{ij}s constitute a partition of T. More formally we have the following definition :

Definition 1 : Let T be a closed rectangle $[a,b] \times [c,d]$. Let $\{T_i\}_{1 \leq i \leq n}$ be a sequence of finitely many closed rectangles such that

i) $T_i \subset T$ for each i.

ii) Sides of T_i are parallel to the coordinate axes.

iii) T_i and T_j intersect on the boundary.

iv) $T = \bigcup_{i=1}^{n} T_i$.

Then the sequence (T_i), $1 \leq i \leq n$, is said to be a partition of the rectangle T.

Let $T = [a, b] \times [c, d]$ be a rectangle in $\mathbb{R}^2$. Let

$P_1 = \{x_0, x_1, \ldots, x_i, \ldots, x_p\}$ be a partition of [a, b] and

$P_2 = \{y_0, y_1, \ldots, y_j, \ldots, y_q\}$ be a partition of [c, d]. Then P_1 [illegible] P_2 give rise to a partition P (denoted by $P_1 \times P_2$) which divides T in[illegible] rectangles T_{ij}, where $T_{ij} = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}$. Conversely, [illegible] a partition P of T, which divides T into rs sub-rectangles as in Fig. 3, there exi[illegible] P_1 of [a, b] which divides [a, b] into r sub-intervals and a partition P_2 [illegible] divides [c, d] into s sub-intervals, such that $P = P_1 \times P_2$.

As in the case of one variable, we say that a partition [illegible] rectangle T is a refinement of another partition Q of T, if each sub-rectangle [illegible] is contained in some sub-rectangle of Q. We write $Q \subset P$ to denote that [illegible] refinement of Q.

We are now ready to define upper sums and low[illegible] functions of [illegible] variables.

Upper Sums and Lower Sums

We start with a bounded real-valued function [illegible] rectangle

$T : [a, b] \times [c, d]$. Let $P = P_1 \times P_2$ be a partition [illegible]

$P_1 = \{x_0, x_1, \ldots, x_p\}$

$P_2 = \{y_0, y_1, \ldots, y_q\}$

are partitions of [a, b] and [c, d], respectively.

Now let T_{ij} be the sub-rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$.

Consider the set $S_{ij} = \{f(x,y) \mid x \in [x_{i-1}, x_i], y \in [y_{j-1}, y_j]\}$

Since f is a bounded real-valued function, S_{ij} must be a non-empty bounded subset of **R**. This means it has a supremum and an infimum. We write

$M_{ij} = \sup S_{ij}$, $m_{ij} = \inf S_{ij}$.

We define the upper sum U(P,f) and the lower sum L(P,f) analogous to the one-variable case by

$$U(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} \Delta x_i \Delta y_j,$$

$$L(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} m_{ij} \Delta x_i \Delta y_j, \qquad \ldots\ldots (1)$$

where $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ and the product $\Delta x_i \Delta y_j$ is the area of the sub-rectangle T_{ij}. The right hand side of the expressions in (1) involves double summation, $\Sigma\,\Sigma$. But since both these sums are finite, the double summation poses no problem. For example, we can write

$$\sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} \Delta x_i \Delta y_j = \sum_{i=1}^{p} \left(\sum_{j=1}^{q} M_{ij} \Delta x_i \Delta y_j \right)$$

That is, we first take the sum as j varies from 1 to q, and then take the sum as i varies from 1 to p. If we reverse the order of i and j, we still get the same sum. Thus, when we expand the sums in (1), we get

$$\begin{aligned} U(P,f) = &\{M_{11} (\text{area of } T_{11}) + M_{12} (\text{area of } T_{12}) + \ldots + M_{1q} (\text{area of } T_{1q})\} \\ &+ \{M_{21} (\text{area of } T_{21}) + \ldots + M_{2q} (\text{area of } T_{2q})\} + \ldots \\ &+ \{M_{p1} (\text{area of } T_{p1}) + \ldots + M_{pq} (\text{area of } T_{pq})\}. \end{aligned}$$

Further,

$$\begin{aligned} L(P,f) = &\{m_{11} (\text{area of } T_{11}) + m_{12} (\text{area of } T_{12}) + \ldots + m_{1q} (\text{area of } T_{1q})\} \\ &+ \{m_{21} (\text{area of } T_{21}) + \ldots + m_{2q} (\text{area of } T_{2q})\} + \ldots \\ &+ \{m_{p1} (\text{area of } T_{p1}) + \ldots + m_{pq} (\text{area of } T_{pq})\}. \end{aligned}$$

Thus, to get U(P,f), we have multiplied the supremum of each S_{ij} by the area of T_{ij} and then taken the sum of all these products. Similarly, for obtaining L(P,f), we have multiplied the infimum of each S_{ij} by the area of T_{ij}. If you compare this with the one-variable case, you will see that the only difference is that Σ is replaced by $\Sigma\,\Sigma$, and the length of an interval is replaced by the area of a rectangle. Now suppose M and m denote the bounds of f in T. Then for any i, j we have

$m \Delta x_i \Delta y_j \leq m_{ij} \Delta x_i \Delta y_j \leq M_{ij} \Delta x_i \Delta y_j \leq M \Delta x_i \Delta y_j$

Now we vary i over 1,....., p and j over 1, ..., q and take double summation. We get

$$\begin{aligned} \sum_{i=1}^{p} \sum_{j=1}^{q} m \Delta x_i \Delta y_j &\leq \sum_{i=1}^{p} \sum_{j=1}^{q} m_{ij} \Delta x_i \Delta y_j \\ &\leq \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij} \Delta x_i \Delta y_j \leq \sum_{i=1}^{p} \sum_{j=1}^{q} M \Delta x_i \Delta y_j \end{aligned}$$

That is,

$$m \sum_{i=1}^{p} \sum_{j=1}^{q} \Delta x_i \Delta y_j \leq L(P,f) \leq U(P,f) \leq M \sum_{i=1}^{p} \sum_{j=1}^{q} \Delta x_i \Delta y_j$$

But $\sum_i \sum_j \Delta x_i \Delta y_j$ = area of T = A. Therefore, we have

$mA \leq L(P,f) \leq U(P,f) \leq MA.$

Now we state a theorem (without proof) which gives a relation between upper and lower sums corresponding to two partitions.

Theorem 1 : Let $f : T = [a, b] \times [c, d] \rightarrow R$ be a bounded function and let P and Q be two partitions of T. If Q is finer than P, then

$L(P,f) \leq L(Q,f) \leq U(Q,f) \leq U(P,f)$.

This means that as the partitions become finer, the upper and lower sums come closer. Now here is an example which illustrates how these sums are obtained.

Example 1 : Let us consider the function

$f(x,y) = x+y-2$

on the rectangle $T : [1,4] \times [1,3]$. Let $P = P_1 \times P_2$ be a partition of T, where $P_1 = \{1,2,3,4\}$ is a partition of [1,4] and $P_2 = \{1, \frac{3}{2}, 3\}$ is a partition of [1, 3]. Let us calculate U(P,f) and L(P,f).

We first note that the function f is bounded on T and $f(x,y) \geq 0$ for all $(x,y) \in T$, i.e., f is a non-negative bounded function. The partition $P = P_1 \times P_2$ breaks up the rectangle T into six rectangles as shown in Fig. 4.

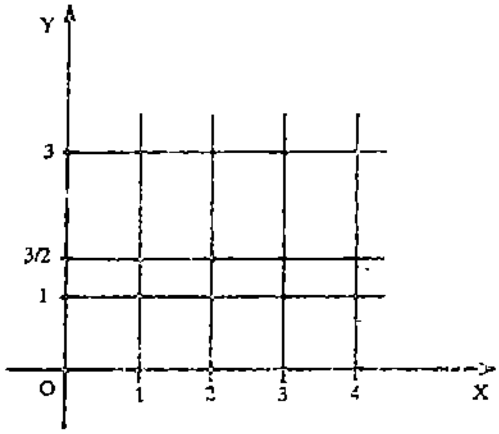


Fig. 4

First, let us calculate U(P,f). By definition

$$U(P,f) = \{M_{11} (\text{area of } T_{11}) + M_{12} (\text{area of } T_{12})\}$$
$$+ \{M_{21} (\text{area of } T_{21}) + M_{22} (\text{area of } T_{22})\}$$
$$+ \{M_{31} (\text{area of } T_{31}) + M_{32} (\text{area of } T_{32})\}$$

where $M_{ij} = \sup \{f(x,y) \mid (x,y) \in T_{ij}\}$.

Now, on each rectangle T_{ij}, the point at which f takes maximum value is (x_i, y_j), the corner farthest from the origin.

Therefore,

$M_{ij} = f(x_i,y_j) = x_i + y_j - 2.$

Thus,

$$U(P,f) = \frac{3}{2}\left(\frac{1}{2}\right) + 3\left(\frac{3}{2}\right) + \frac{5}{2}\left(\frac{1}{2}\right) + \text{[illegible]}$$

$$= \frac{87}{4}.$$

Next, we calculate L(P,f). By definition

$$L(P,f) = \{m_{11} (\text{area of } T_{11}) + m_{12} (\text{area of } T_{12})\}$$
$$+ \{m_{21} (\text{area of } T_{21}) + m_{22} (\text{area of } T_{22})\}$$
$$+ \{m_{31} (\text{area of } T_{31}) + m_{32} (\text{area of } T_{32})\},$$

where $m_{ij} = \inf \{f(x,y) \mid (x,y) \in T_{ij}\}$.

Again, you can see from Fig. 4 that in T_{ij} the point at which f takes minimum value is (x_{i-1}, y_{j-1}), the corner closest to the origin. Thus,

$m_{ij} = f(x_{i-1}, y_{j-1}) = x_{i-1} + y_{j-1} - 2.$

Therefore,

$$L(P,f) = 0\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{3}{2}\right)+1\left(\frac{1}{2}\right)+\frac{3}{2}\left(\frac{3}{2}\right)+2\left(\frac{1}{2}\right)+\frac{5}{2}\left(\frac{3}{2}\right)$$

$$= \frac{33}{4}.$$

Why don't you try some exercise on your own now?

E2) Find U(P,f) and L(P,f) for the function $f(x,y) = x+2y$ defined on $T:[0,2]\times[0,1]$. P is the partition $P_1 \times P_2$, where $P_1 = \{0, 1, \frac{3}{2}, 2\}$, $P_2 = \{0, \frac{1}{2}, 1\}$.

E3) Verify the result in Theorem 1 for the function in E 2) by taking the partition $Q = P_1' \times P_2'$, where

$P_1' = \{0, 1, \frac{3}{2}, 2\}$, $P_2' = \{0, \frac{1}{4}, \frac{1}{2}, 1\}$.

In the next sub-section we use these upper and lower sums to define the double integral of a bounded function over a rectangle. You will see that the procedure is the same as in the one-variable case.

11.2.2 Double Integrals and Repeated Integrals

In this sub-section we define the double integral of a bounded function of two variables. Then we introduce another type of integral called repeated integral, which makes the evaluation of double integral quite easy.

Let $f : T \to \mathbf{R}$ be a bounded function, where $T = [a,b] \times [c,d]$. Let A denote the area of T. Let M and m be the bounds of f in T. Let $\boldsymbol{P}$ be the set of all partitions of T. Then we have seen that to each partition P of $\boldsymbol{P}$, there correspond an upper sum U(P,f) and a lower sum L(P,f). Also, $mA \le L(P,f) \le U(p,f) \le MA$. This shows that the set $S = \{U(P,f) \mid P \in \boldsymbol{P}\}$ is a subset of **R** and is bounded below. Thus, it is possible to find the infimum of S. Similarly, the set $S' = \{L(P,f) \mid P \in \boldsymbol{P}\}$ is bounded above and therefore, we can find the supremum of S'. Let U be the infimum of S and L be the supremum of S'.

Now we are in a position to define the double integral of a bounded function.

Definition 2 : Let $f : T = [a, b] \times [c, d] \to \mathbf{R}$ be a bounded function. f is said to be **integrable** over T if L = U.

This common value is called the **double integral** of f over the rectangle T and is denoted by any one of the symbols

$$\iint_T f(x, y)\, dx\, dy \quad \text{or} \quad \int_a^b \int_c^d f(x, y)\, dx\, dy.$$

Now we take a simple example.

Example 2 : Let us check whether the function $f : T \to \mathbf{R}$, defined by $f(x) = k$, where $k > 0$ and $T = [a, b] \times [c, d]$, is integrable or not.

For this, let us take any partition $P = P_1 \times P_2$ of T, where

$P_1 : a = x_0 < x_1 < \ldots.. < x_p = b$ and

$P_2 : c = y_0 < y_1 < \ldots.. < y_q = d.$

Since f is a constant function, on each T_{ij} we have

$m_{ij} = k = M_{ij}$

$$\therefore L(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} k\, \Delta x_i\, \Delta y_j$$

$$= k \sum_{i=1}^{p} \sum_{j=1}^{q} \Delta x_i\, \Delta y_j$$

$$= k\,(b-a)\,(d-c)$$

Similarly, $U(P,f) = k(b-a)(d-c) = L(P,f)$ for every partition P.

This means that $S = S' = \{k(b-a)(d-c)\}$.

Therefore, inf S = sup S = k (b–a) (d–c).

Thus, we can say that f is integrable over T and

$$\iint_T f(x)\,dx\,dy = \iint_T k\,dx\,dy = k\,(b-a)\,(d-c).$$

In Example 2 we had put the restriction $k > 0$. But the result is true even when $k \leq 0$, and the proof is similar.

Before considering more examples of evaluating double integrals, let us look at their geometrical interpretation. As in the case of functions of one variable, we consider non-negative functions for this.

Let f be a non-negative, bounded function defined on $T : [a, b] \times [c, d]$. Let $P = P_1 \times P_2$ be a partition of T. Let us see what these sums U(P,f) and L(P,f) represent geometrically. Suppose we consider the region D between the graph of f and the rectangle T as shown in Fig. 5(a).

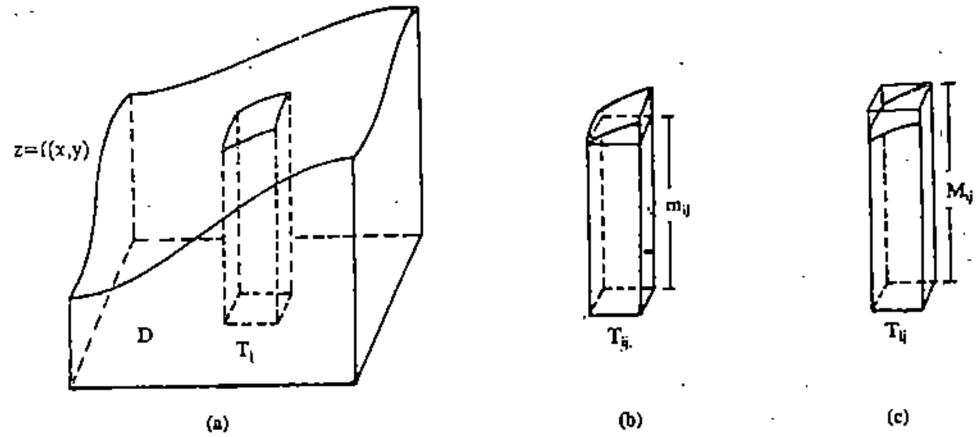


Fig. 5

Let V denote the volume of this solid region. Let P divide the rectangle T into p.q sub-rectangles T_{ij}, $1 \leq i \leq p$, $1 \leq j \leq q$

This divides the entire solid region D into parts S_{ij} as shown in Fig. 5(a). Then, for each pair (i,j), consider the rectangular parallelopipeds with base area = area of T_{ij} $= \Delta x_i\, \Delta y_j$ and heights m_{ij} and M_{ij}. You can see these in Fig. 5(b) and (c), respectively. Then $M_{ij}\Delta x_i \Delta y_j$ gives the volume of the rectangular parallelopiped with base T_{ij} and height M_{ij}, and $m_{ij}\Delta x_i \Delta y_j$ gives the volume of the rectangular parallelopiped with base T_{ij} and height m_{ij}. The rectangular parallelopiped in Fig. 5 (b) is called inner rectangular parallelopiped, and that in Fig. 5(c) is termed as outer rectangular parallelopiped.

Now, if v_{ij} denotes the volume of S_{ij}, you can see from Fig. 5(b) and (c) that

$$m_{ij}\, \Delta x_i \Delta y_j \leq v_{ij} \leq M_{ij}\, \Delta x_i \Delta y_j \qquad \ldots\ldots (2)$$

Further,

$$U(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} M_{ij}\, \Delta x_i\, \Delta y_j$$

= Sum of the volumes of outer rectangular parallelopipeds as in Fig. 5(c).

$$L(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} m_{ij}\, \Delta x_i\, \Delta y_j$$

= Sum of the volumes of inner rectangular parallelopipeds as in Fig. 5(b).

Now, if V is the volume of D, then

$$V = \sum_{i=1}^{p} \sum_{j=1}^{q} v_{ij}. \qquad \ldots\ldots (3)$$

Thus, from (2) and (3) we get

$$L(P,f) \leq V \leq U(P,f). \qquad \ldots\ldots (4)$$

(4) is true for all partitions P of T. Now, if f is integrable, then there is a unique number lying between L(P,f) and U(P,f) for all $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of all partitions of T. This unique number is the double integral of f over T (see Definition 2). This together with (4) tells us that

$$\iint_T f(x, y)\, dx\, dy = V.$$

Thus, if f is a non-negative, bounded function, then we can view the double integral of f over T as the volume of the 3-dimensional region lying above the rectangle T and bounded above by the graph of f.

Now, here is a remark, which tells us that double integral can be viewed as the limit of a sum.

Remark 1 : Suppose a function f is integrable on a closed rectangle $T = [a, b] \times [c, d]$. Consider a partition $P = P_1 \times P_2$ of T, where

$P_1 : a = x_0 < x_1 < \ldots\ldots < x_p = b,$

$P_2 : c = y_0 < y_1 < \ldots\ldots < y_q = d.$

Let $T_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, $1 \le i \le p$, $1 \le j \le q$.

Let us choose a point P_{ij} in T_{ij}. Then the sum

$\sum_{i=1}^{p} \sum_{j=1}^{q} f(P_{ij})\, \Delta x_i\, \Delta y_j$, where $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, is called the **Riemann sum** of f over T, corresponding to P. It can be proved that as the diameter of each T_{ij} tends to zero, this Riemann sum approaches $\iint_T f(x, y)\, dx\, dy$.

Thus, we can write

$$\iint_T f(x, y)\, dx\, dy = \lim_{||P|| \to 0} \sum_{i=1}^{p} \sum_{j=1}^{q} f(P_{ij})\, \Delta x_i\, \Delta y_j,$$

where $||P||$, the norm of the partition P, is the largest diameter of the rectangle in P.

In what follows, we shall often use the result in Remark 1. So make sure that you have understood it.

Now here is an example.

Example 3 : Let $T = [1, 2] \times [3, 4]$, and let $f : T \to \mathbf{R}$ be defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Let us check whether this function is integrable or not.

Let P be any partition of T into sub-rectangles T_i, $1 \le i \le n$. Now, when we take any rectangle T_i, there exists a point (x,y) in T_i with x rational, and there exists another point (x_1, y_1) in T_i with x_1 irrational. Thus, we have

m_i = infimum of f in T_i = 0, and

M_i = supremum of f in T_i = 1.

This shows that L(P,f) = 0 and U(P,f) = 1, since the area of T = 1. This is true for all partitions, Thus, $L = \sup \{L(P,f) \mid P \in \mathcal{P}\} = 0$ and $U = \inf \{U(P,f) \mid P \in \mathcal{P}\} = 1$. Therefore, the double integral of f does not exist.

Next we state a theorem, which gives a criterion for the integrability of functions of two variables.

Theorem 2 : Let T be a closed rectangle. A real-valued, bounded function $f : T \to \mathbf{R}$ is integrable over T if and only if given $\varepsilon > 0$, there exists a partition P of T such that

$U(P,f) - L(P,f) < \varepsilon$.

The proof of this theorem is exactly similar to the proof in the one-variable case (see Theorem 3, Unit 10, Calculus). To refresh your memory we give the proof of the "if part" of the theorem here. The proof of the "only if part" is left to you as an exercise. (See E 4).

Proof (if part) : Let $L = \sup \{L(P,f) \mid P \in \mathcal{P}\}$ and $U = \inf \{U(P,f) \mid P \in \mathcal{P}\}$, where $\mathcal{P}$ is the set of all partitions of T. Then

$L(P,f) \le L \le U \le U(P,f) \quad \forall P \in \mathcal{P}$,

or, $U - L \le U(P,f) - L(P,f) \quad \forall P \in \mathcal{P}$.

Now, given $\varepsilon > 0$, $\exists P \in \mathcal{P}$ such that

$U(P,f) - L(P,f) < \varepsilon$.

This implies that

$U - L < \varepsilon$.

Since this is true for all $\varepsilon > 0$, we get that

$U - L = 0$ or $U = L$.

This shows that f is integrable over T.

Now do the following exercise and complete the proof of Theorem 2.

E 4) Prove the "only if" part in Theorem 2.

With the help of Theorem 2, it is possible to identify a large class of integrable functions. We shall not worry about the proof here, but let us state the result formally now.

Theorem 3 : If a function $f : T = [a, b] \times [c, d] \to \mathbf{R}$ is continuous, then f is integrable.

So, continuity $\Rightarrow$ integrability. But the converse is not true. Here is an example which shows just this.

Example 4 : Consider the function,

$$f(x,y) = \begin{cases} 1, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Clearly, f(x,y) is not continuous at (0,0). We shall show that f(x,y) is integrable over $T : [-1,1] \times [-1,1]$. Given any $\varepsilon > 0$, find a partition P of the rectangle T such that the area of the sub-rectangle T^* containing (0,0) is less than ε. Note that for any other sub-rectangle of P, infimum of f is equal to the supremum of f (each being equal to 1). Thus,

$$U(P,f) - L(P,f) = 1.\text{ area of } T^* - 0.\text{ area of } T^*.$$
$$< \varepsilon,$$

showing that f is integrable on T.

In this example you saw a function which is discontinuous at a point in T, but still is integrable on T. In fact, even if a function is discontinuous at a finite number of points, it can still be integrable.

In Unit 5, you have seen that differentiability $\Rightarrow$ continuity. Then by Theorem 3, we have

differentiability $\Rightarrow$ continuity $\Rightarrow$ integrability.

Now look at the functions below.

i) $f(x,y) = x^2 - 2xy + 4y$, in $[0,2] \times [0,1]$

ii) $f(x,y) = \sqrt{x+y}$ in $[3,5] \times [1,4]$

Clearly, both these functions, being continuous, are integrable. At this stage, apart from the definition, we have no other means to evaluate their double integral. As in the case of functions of one variable, it is not at all easy to evaluate double integrals of most of the functions by applying directly the definition. In the one-variable case, the Fundamental Theorem of Calculus came to our rescue. Here we overcome this difficulty by reducing the calculation of a double integral to calculation of integrals of functions of one variable with the help of **repeated integrals** which we now introduce.

Let $T = [a, b] \times [c, d]$ be a rectangle in $\mathbf{R}^2$ and f(x, y) be a real-valued, bounded function on T. If we keep x fixed and allow y to vary over the interval [c, d], then we get a function $f^x : [c, d] \to \mathbf{R}$, defined by

f^x is a function of a single variable y.

$f^x(y) = f(x,y)$ for $y \in [c, d]$.

Do you agree that f^x is bounded in [c, d]?

Now suppose f^x is integrable over [c, d].

Then the integral $\int_c^d f^x(y)dy$ depends on x, and thus defines a function of x, say $F(x) = \int_c^d f^x(y)\,dy$, on [a, b]. If F(x) is integrable over [a, b], then $\int_a^b F(x)dx = I$ is called a **repeated integral** of f(x,y) on T.

Clearly,

$$I = \int_a^b \left(\int_c^d f(x,y)dy \right) dx.$$

Roughly speaking, to obtain this repeated integral of f, we first integrate f(x,y) over [c,d], regarding it as a function y (treating x as a constant), and then integrate the resulting function of x over [a, b]. Recall that you have seen a similar procedure in Unit 4, while evaluating repeated limits.

By interchanging the roles of the variables x and y, we can define the other repeated integral

$$\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy,$$

provided f(x,y) is integrable on [a,b] for each fixed y, and the function $\int_a^b f(x,y)dx$ is integrable on [c,d]. It is obvious from the definitions of the two repeated integrals that, at any given time, we are really integrating a function of one variable. Therefore, all the techniques of integration that we have learnt so far are available to us for the evaluation of repeated integrals. We shall illustrate this with the help of a few examples.

Example 5 : Consider the function $f(x,y) = 3x^2y$.

Let us find the repeated integral of f over the rectangle T = [1,2] × [–3,4], i.e., let us evaluate the integral

$$\int_1^2 \left[\int_{-3}^4 f(x,y)\,dy \right] dx.$$

To do this, we first compute the integral of $f^x(y) = 3x^2y$ with respect to y, keeping x constant over [–3,4]. We get

$$\int_{-3}^4 3x^2y\,dy = 3x^2 \int_{-3}^4 y\,dy \text{ (Since x is a constant, we can take } x^2 \text{ outside.)}$$

$$= 3x^2 \left[\frac{y^2}{2} \right]_{-3}^4$$

$$= 3x^2 \left[\frac{16-9}{2} \right]$$

$$= \frac{21x^2}{2}$$

We then integrate this w.r.t. x and get

$$\int_1^2 \frac{21x^2}{2}\,dx = \frac{21}{2} \left[\frac{x^3}{3} \right]_1^2$$

$$= \frac{21}{2} \left[\frac{8-1}{3} \right] = \frac{49}{2}.$$

Thus,

$$\int_1^2 \left[\int_{-3}^4 3x^2y\,dy \right] dx = \frac{49}{2}.$$

Check that $\int_{-3}^4 \left[\int_1^2 3x^2y\,dx \right] dy$ is also equal to $\frac{49}{2}$. So, in this case, both the repeated integrals are equal. But there is no reason to expect that the two repeated integrals will always be equal. In fact, it is possible that one repeated integral exists and the other does not exist at all. This is the situation in our next example.

Example 6 : Let T : [−1,1] × [−1,1] and let f : T → **R** be defined by

$$f(x,y) = \begin{cases} y, \text{ if } x \text{ is rational,} \\ 0, \text{ if } x \text{ is irrational.} \end{cases}$$

We will show that $\int_{-1}^{1}\left[\int_{-1}^{1} f(x,y)dy\right]dx$ exists, and the other repeated integral is not defined.

We first integrate f(x,y) w.r.t.y, keeping x constant. Then the fixed value of x can be rational or irrational. In both cases you can easily see that

$$\int_{-1}^{1} f(x,y)\,dy = \begin{cases} \int_{-1}^{1} y\,dy, \ x \text{ rational} \\ \int_{-1}^{1} 0\,dy, \ x \text{ irrational,} \end{cases}$$

and therefore, $\int_{-1}^{1} f(x,y)\,dy = 0$. Consequently,

$$\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dy\right)dx = 0$$

Now we have to show that the other repeated integral $\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)dx\right)dy$ does not exist.

For this we fix y, say, y = 1, and define the function $f_1(x)$ on [−1,1] by

$$f_1(x) = \begin{cases} 1, \text{ if } x \text{ is rational,} \\ 0, \text{ if } x \text{ is irrational.} \end{cases}$$

You have seen in Calculus (Example 4, Unit 10) that such a function is not integrable.

Therefore, $\int_{-1}^{1} f_1(x)dx$ does not exist, and hence we can't even talk of the repeated integral

$$\int_{-1}^{1}\left(\int_{-1}^{1} f(x,y)\,dx\right)dy.$$

To gain some practice in evaluating repeated integrals, you can try these exercises.

E5) Evaluate the following repeated integrals :

a) $\int_{-1}^{4}\left(\int_{1}^{2} x^2y\,dx\right)dy$

b) $\int_{0}^{1}\left(\int_{0}^{1} (x^2+y^2)\,dx\right)dy$

c) $\int_{1}^{2}\left(\int_{3}^{4} (xy + e^y)\,dy\right)dx$

E6) Check whether the repeated integrals obtained by interchanging the order of integration in E5) a) b) and c) are the same or not.

In Example 6 we have seen a function f(x,y) defined on a rectangle T, for which one of the repeated integrals exists and the other is not even defined. Is it also possible that both the repeated integrals are defined but are not equal? In fact, examples of bounded functions whose repeated integrals exist but are unequal have been found. It is not possible to give an example of such a function here. But don't worry, because in this course we shall only consider those functions whose repeated integrals are equal.

You have seen the definitions of double integral and repeated integrals. You will agree that it is much easier in practice to evaluate a repeated integral of a function than its double integral. This is because in the case of a repeated integral, we are dealing with only one variable at a time. Repeated integrals are very helpful in computing a double integral, because for a large class of functions, repeated integrals and double integral coincide. The next theorem concerns this. We shall not prove this theorem here.

Theorem 4 : Let T be the rectangle $[a,b] \times [c,d]$, and let $f : T \rightarrow \mathbf{R}$ be a continuous function. Then both the repeated integrals exist and are equal to the double integral.

This theorem says that if the function to be integrated is a continuous function, then we can easily compute the double integral by computing any of the repeated integrals.

The condition stated in Theorem 4 for the equality of two repeated integrals and the double integral is sufficient and not necessary. For example, the function in Example 4 is not continuous. But for this function the two repeated integrals exist and are equal. However, in this course we shall only deal with continuous functions. Therefore, we do not need to look for other criteria which ensure the existence and equality of repeated integrals.

Example 7 : Let us evaluate the double integral of the function $f(x,y) = \sqrt{x+y}$ over the rectangle $[3,5] \times [1,4]$.

By Theorem 3, we have

$$\iint_T \sqrt{x+y}\, dx\, dy = \int_1^4 \left(\int_3^5 \sqrt{x+y}\, dx \right) dy$$

$$= \int_1^4 \left[\frac{2}{3}(x+y)^{3/2} \right]_3^5 dy$$

$$= \frac{2}{3} \int_1^4 \left[(5+y)^{3/2} - (3+y)^{3/2} \right] dy$$

$$= \frac{2}{3} \left[\frac{2}{5} \left\{ (5+y)^{5/2} - (3+y)^{5/2} \right\}_1^4 \right]$$

$$= \frac{4}{15} \left[9^{5/2} - 6^{5/2} - 7^{5/2} + 4^{5/2} \right]$$

You can now evaluate the double integrals in the following exercise by repeate integration.

E 7) Compute the double integrals of the following functions over T.

a) $f(x,y) = x \sin(x+y)$, $T = [0,\pi] \times [0,\pi]$

(**Hint :** Use integration by parts)

b) $f(x,y) = \frac{1}{1+x+y}$, $T = [0,1] \times [0,1]$

(**Hint :** Use the formula $\int \ln x\, dx = x \ln x - x + c$ to evaluate the definit integral.)

We list below some of the properties of double integrals over rectangles. You have studied similar properties of definite integrals in Calculus (Unit 10).

Let T be a closed rectangle in $\mathbf{R}^2$, and let f and g be such that $\iint_T f(x,y)\, dx\, dy$ and $\iint_T g(x,y)\, dx\, dy$ exist. Then

1) $\iint_T c\, f(x,y)\, dx\, dy = c \iint_T f(x,y)\, dx\, dy$, where c is a constant.

2) $\iint_T (f+g)(x,y)\, dx\, dy = \iint_T f(x,y)\, dx\, dy + \iint_T g(x,y)\, dx\, dy$

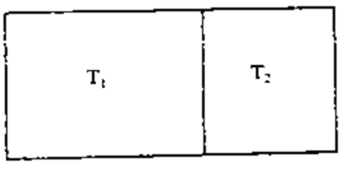


Fig. 6 : $T = T_1 \cup T_2$

3) If $f(x,y) \leq g(x,y)$ for all $(x,y) \in D$, then $\iint_T f(x,y)\, dx\, dy \leq \iint_T g(x,y)\, dx\, dy$.

4) If T is the union of two rectangles T_1 and T_2 such that T_1 and T_2 intersect only on the boundary, i.e., T_1 and T_2 are non-overlapping (see Fig. 6), then
$$\iint_T f(x,y)\, dx\, dy = \iint_{T_1} f(x,y)\, dx\, dy + \iint_{T_2} f(x,y)\, dx\, dy$$

5) $\left| \iint_T f(x,y)\, dx\, dy \right| \leq \iint_T |f(x,y)|\, dx\, dy$.

So far we have studied double integrals over rectangles. But, in practice, many times we have to evaluate double integrals over regions which are not rectangles. In the next section we will define double integrals over bounded sets in R^2.

11.3 DOUBLE INTEGRAL OVER ANY BOUNDED SET

In this section we define double integral over bounded sets in R^2. Thus, we have to first tell you what is a bounded set in a Euclidean space.

Definition 3 : A subset $X \subset R^n$ is said to be a **bounded set** if X is contained in some open sphere with centre at the origin.

For example, the set $\{(x,y) \mid x^2+y^2 \le 1\}$ is a bounded subset of $\mathbf{R}^2$, whereas the set $\{(x,y) \mid y > 0\}$ is not a bounded subset.

From Definition 3 you should be able to see that a set X is bounded if and only if X is contained in some parallelopiped or a rectangular box. Such a box is a set of the form $I_1 \times I_2 \times \ldots \times I_n$, where I_j is a closed interval $[a_j,b_j]$, $1 \le j \le n$.

Now let $f : D \to \mathbf{R}$ be a bounded function, where D is a bounded set in $\mathbf{R}^2$. Since D is bounded, there exists a closed rectangle T, which encloses D (see Fig. 7).

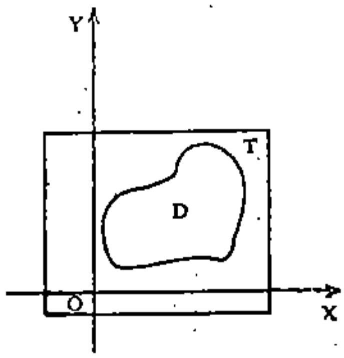


Fig. 7

We now define a function f^* on T by

$$f^*(x,y) = \begin{cases} f(x,y), \text{ if } (x,y) \in D \\ 0, \text{ if } (x,y) \notin D. \end{cases}$$

The function $f^*(x,y)$ is a real-valued, bounded function defined on a closed rectangle T and we have seen in Sec. 11.2 how to define the integral of this function over T.

We say that f(x,y) is integrable over D if $f^*(x,y)$ is integrable over T, and set

$$\int_D\int f(x,y)\,dx\,dy = \int_T\int f^*(x,y)\,dx\,dy.$$

We can easily prove that this definition is independent of the choice of T.

Suppose T_1 is another closed rectangle which encloses D.

Let

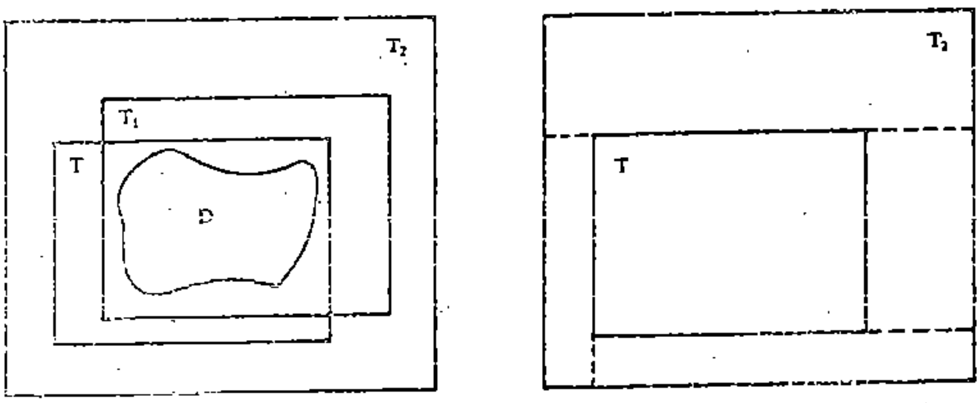
$$f_1(x,y) = \begin{cases} f(x,y), x \in D \\ 0, x \in T_1 \setminus D. \end{cases}$$

We have to show that

$$\int_T\int f^*(x,y)\,dx\,dy = \int_{T_1}\int f_1(x,y)\,dx\,dy.$$

Now let T_2 be another closed rectangle such that $T_2 \supset T$ and $T_2 \supset T_1$. See Fig. 8(a).

$$\text{Let } f_2(x,y) = \begin{cases} f(x,y), x \in D. \\ 0, x \in T_2 \setminus D. \end{cases}$$



(a) (b)

Fig. 8

Then by using the properties of double integrals listed at the end of the last section (also see Fig. 8(b)) you can see that

$$\int_T\int f^*(x,y) = \int_{T_2}\int f_2(x,y) = \int_{T_1}\int f_1(x,y)$$

You will agree that this definition gives meaning to the double integral over any bounded set in the plane. But we are not interested in any arbitrary bounded set because even good functions defined over them need not be integrable. For example, let

$D = \{(x,y) \mid 1 \leq x \leq 2, 3 \leq y \leq 4$, where x is rational$\}$.

Then D is a bounded set enclosed by the closed rectangle $[1,2] \times [3,4]$. Now if we consider the constant function $f(x,y) = 1$ defined over D, then by Example 3, we can conclude that f is not integrable over D.

In addition, the above definition is not very useful in computing the integral. Because of these reasons we restrict ourselves to some particular types of regions where the computation is fairly easy.

11.3.1 Regions of Type I and Type II

We shall now define two simple types of regions.

Definition 4 : Let ϕ_1 and ϕ_2 be two continuous, real-valued functions defined on a closed interval [a,b], such that $\phi_1(x) \leq \phi_2(x)$ for all $x \in [a,b]$.

Let $D = \{(x,y) \mid a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$.

Such a region D in the plane is called a **region of Type I.** See Fig. 9(a).

Definition 5 : A region D in the plane is called a **region of Type II,** if there are continuous, real-valued functions ψ_1 and ψ_2 defined on [c,d], such that $\psi_1(y) \leq x \leq \psi_2(y)$, and

$D = \{(x,y) \mid \psi_1(y) \leq x \leq \psi_2(y), c \leq y \leq d\}$ (see Fig. 9(b)).

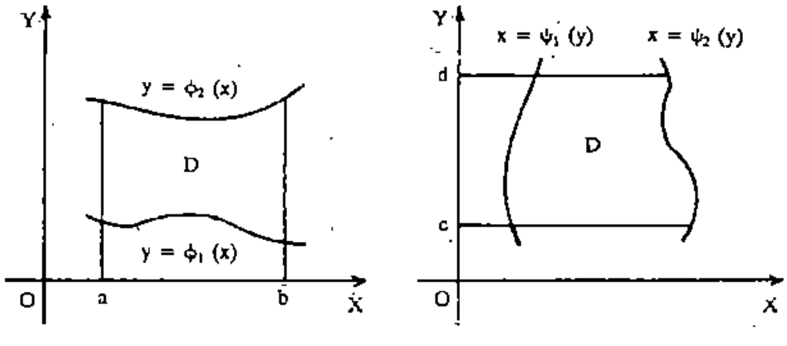


Fig. 9 : A region of (a) Type I (b) Type II in R^2

You will agree that the shape of such regions is very similar to a closed rectangle. Clearly, if $\phi_1(y) = a$ and $\phi_2(y) = b$ or $\psi_1(x) = c$ and $\psi_2(x) = d$, then both the regions mentioned above are nothing but the closed rectangle $[a,b] \times [c,d]$.

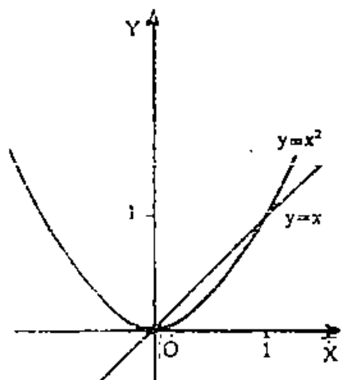


Fig. 10

Let us see some examples of such regions.

Example 8 : Let $D = \{(x,y) \mid 0 \leq x \leq 1$ and $x^2 \leq y \leq x\}$. Let us describe the region D geometrically (see Fig. 10). Instead of writing D in the set form, we also describe D as the region bounded by the straight line $y = x$ and the parabola $y = x^2$. Note that here $\psi_1(x) = x$ and $\psi_2(x) = x^2$. To find the range of values of x, we have to find the points of intersection of these two curves. They are given by $x = 0$ and $x = 1$. Thus, the range of x is, $0 \leq x \leq 1$.

Let us take one more example.

Example 9 : Suppose D is the region formed by the triangle bounded by the lines $x = 0$, $y = 0$, and $x+y = 6$.

Let us describe D geometrically (see Fig. 11).

In Fig. 11, we observe that for points (x,y) in D, y ranges from 0 to 6, and for any y, x extends from the y-axis to the line L : x+y = 6, i.e., x ranges between the lines x = 0 and x = 6 − y.

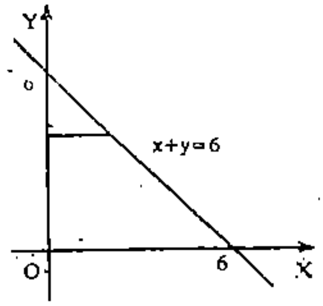


Fig. 11

Thus we can express D as

$D = \{(x,y) \mid 0 \le y \le 6, 0 \le x \le 6-y\}$

Thus, D is of Type II.

Note that here $\psi_1(y) = 0$ and $\psi_2(y) = 6-y$.

In this example we could also describe D as

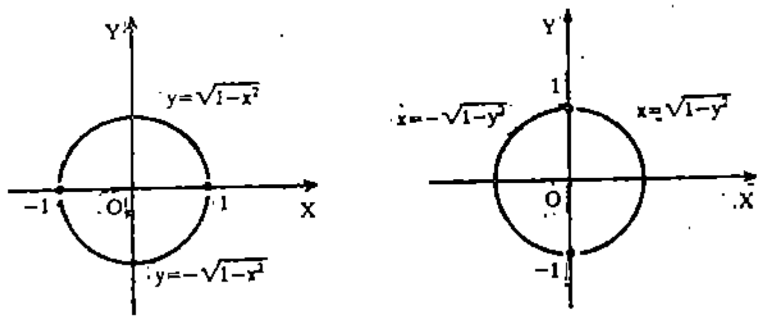
$D = \{(x,y) \mid 0 \le x \le 6, 0 \le y \le 6-y\}$.

That is, D is a region of Type I. Thus, it is possible that a region may be both of Type I and Type II at the same time.

Here is another example of this type.

Example 10 : Suppose D is the region bounded by the unit circle $x^2+y^2 = 1$.

Fig. 12 gives the geometric description of D as a region of both Type I and Type II.



(a) (b)

Fig. 12 : The unit circle as a region of (a) Type I (b) Type II

In Fig. 12 (a), we observe that x ranges between −1 and 1, and y ranges between $-\sqrt{1-x^2}$ and $\sqrt{1-x^2}$. Thus, D is of Type I. Similarly, from Fig. 12(b), we observe that y ranges between −1 and 1, and x ranges between $-\sqrt{1-y^2}$ and $\sqrt{1-y^2}$. That is, D is expressed as

$D = \{(x,y) \mid -\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, -1 \le y \le 1\}$, and therefore is of Type II.

It is also possible that a region is neither of Type I nor of Type II. For instance, the annulus D in Fig. 13 given by $D = \{(x,y) \mid 4 \le x^2+y^2 \le 9\}$ is of neither type. But D can be expressed as the union of two regions D_1 and D_2, which are of Type I (see Fig. 13).

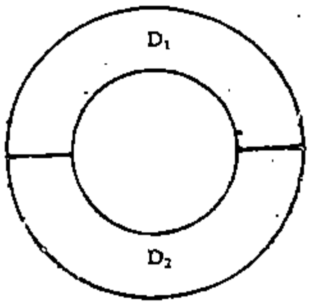


Fig. 13

From now on in this unit, we shall be dealing with double integrals over regions of Type I or Type II or over those regions which can be written as unions of these two types of regions.

Here is a remark about the terminology.

Remark 2 : If a region D is of Type I, then any line parallel to the y-axis meets D in at most two points. Such regions are also called **quadratic** or **regular** with respect to the y-axis.

Similarly, regions of Type II are called quadratic or regular w.r.t. the x-axis.

To get more practice, you can try this exercise.

E8) See whether the regions given below are of Type I, Type II, both or neither :

a) The region D bounded by $y = 0$, $x = 2$, $y = x^2$.

b) The region D lying between the circles $x^2+y^2 = a^2$ and $x^2+y^2 = b^2$, $b > a$.

c) The region D bounded by $y = x^2$ and $y = x^{1/4}$

d) The region D lying between the curves $x^2+y^2 = 1$, $x^2+y^2 = 9$, $y \geq 0$.

As in the case of double integrals over closed rectangles, it is not at all easy to evaluate $\int\int_D f(x,y)\,dx\,dy$ by using just the definition. However, if D is a region of Type I or Type II, or if D can be broken up into finitely many regions, each of which is of Type I or Type II, then we can evaluate the double integral with the help of repeated integrals. This is what we are going to discuss in the next sub-section.

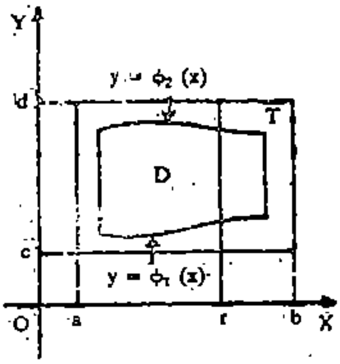


Fig. 14

11.3.2 Repeated Integrals over Regions of Type I and Type II

Let f(x,y) be a bounded function defined on a region D of Type I. Then we can write D as $D = \{(x,y) \mid a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$, where a, b, ϕ_1 and ϕ_2 are as in Definition 4. Let T be a rectangle $[a,b] \times [c,d]$, which encloses D. Let f^* be defined by

$$f^*(x,y) = \begin{cases} f(x,y), & (x,y) \in D, \\ 0; & (x,y) \in T \setminus D. \end{cases}$$

Then, for any x in the interval [a,b], we have $c \leq \phi_1(x) \leq \phi_2(x) \leq d$. Now look at Fig. 14. You will see that, for any fixed x, we can write

$$\int_c^d f^*(x,y)\,dy = \int_c^{\phi_1(x)} f^*(x,y)\,dy + \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy + \int_{\phi_2(x)}^{d} f^*(x,y)\,dy, \qquad \ldots\ldots (5)$$

by the Interval Union Property of integrals of functions of one variable, provided the integral

$\int_c^d f^*(x,y)\,dy$ exists.

Since $f^*(x,y) = 0$ whenever $c \leq y \leq \phi_1(x)$ and $\phi_2(x) \leq y \leq d$, we get

$$\int_c^{\phi_1(x)} f^*(x,y)\,dy = 0 = \int_{\phi_2(x)}^{d} f^*(x,y)\,dy$$

Therefore, from (5) we get

$$\int_c^d f^*(x,y)\,dy = \int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy$$

Thus, the repeated integral of f^* over the rectangle (if it exists) is in fact equal to the repeated integral

$$\int_c^d \left[\int_{\phi_1(x)}^{\phi_2(x)} f^*(x,y)\,dy \right] dx.$$

We have a result similar to Theorem 3, which shows that the double integral of a continuous function over a region of Type I or Type II can be computed by repeated integration. We now give the statement for regions of Type I.

Theorem 5 : Let $D = \{(x,y) \mid a \leq x \leq b,\ \phi_1(x) \leq y \leq \phi_2(x)\}$ be a region in the plane where ϕ_1 and ϕ_2 are as in Definition 4. Let $f : D \to \mathbb{R}$ be a continuous function. Then the repeated integral

$\int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy \right] dx$ exists and is equal to the double integral

$$\int\int_D f(x,y)\,dx\,dy.$$

The statement of the theorem for regions of Type II is exactly similar. See if you can write it (see E 11). After you have written it, don't forget to tally it with the answer given in Sec. 11.6.

E9) Give the statement of a result for regions of Type II, which is analogous to Theorem 5.

Now we illustrate Theorem 5 with the help of some examples.

Example 11 : Let us calculate the double integral of $f(x,y) = x+y$ over the region $D = \{(x,y) \mid 0 \leq x \leq 1, 1 \leq y \leq e^x\}$.

Clearly, D is a region of Type I.

Since the function $f(x,y) = x+y$ is continuous over D, by Theorem 5, we have

$$\int_D\int (x+y)\,dx\,dy = \int_0^1 \left[\int_0^{e^x} (x+y)\,dy \right] dx.$$

$$= \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^{e^x} dx$$

$$= \int_0^1 \left(xe^x + \frac{e^{2x}}{2} \right) dx$$

$$= \left[\left(xe^x - e^x + \frac{e^{2x}}{4} \right) \right]_0^1$$

$$= e - e + \frac{e^2}{4} + 1 - \frac{1}{4}$$

$$= \frac{e^2+3}{4}$$

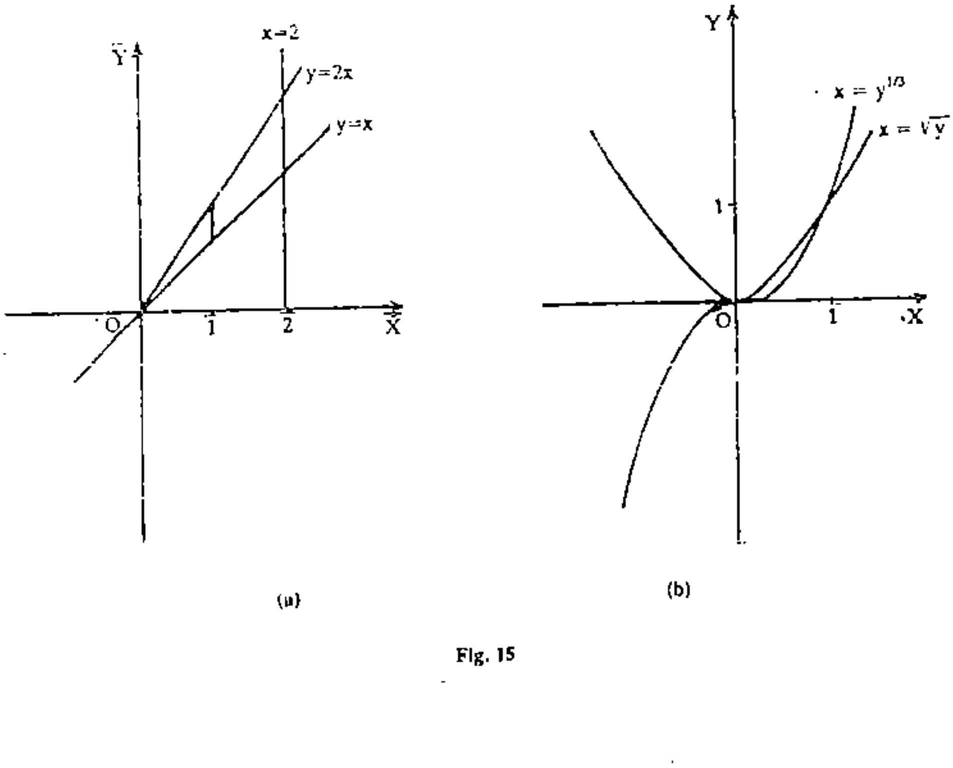
Example 12 : Let us find the integrals of the following functions over the indicated regions.

i) $f(x,y) = \sqrt{xy}$ over the region bounded by $y = x$, $y = 2x$ and $x = 2$.

ii) $f(x,y) = x^3+y^2$ over the region bounded by $x = y^{1/3}$ and $x = \sqrt{y}$.

Let us take these one by one.

i) We first describe the region D geometrically in Fig. 15(a).



(a) (b)

Fig. 15

From the figure we can observe that x ranges between 0 and 2, i.e., $0 \le x \le 2$ and y ranges between the lines $\phi_1(x) = x$ and $\phi_2(x) = 2x$. Thus,

$D = \{(x,y) \mid 0 \le x \le 2, x \le y \le 2x\}$, which is a region of Type I.

Since the function $f(x,y) = \sqrt{xy}$ is continuous in D, by Theorem 5 we have

$$\int\int_D f(x,y)\,dx\,dy = \int_0^2 \left[\int_x^{2x} \sqrt{xy}\,dy\right] dx$$

$$= \frac{2}{3}\int_0^2 \sqrt{x}\, y^{3/2} \Big]_x^{2x} dx$$

$$= \frac{2}{3}\int_0^2 \sqrt{x}\,[(2x)^{3/2} - x^{3/2}]\,dx$$

$$= \frac{2}{3}(2^{3/2} - 1)\int_0^2 x^2\,dx$$

$$= \frac{2}{3}(2\sqrt{2} - 1)\frac{x^3}{3}\Big]_0^2$$

$$= \frac{2}{3}(2\sqrt{2} - 1)\frac{8}{3}$$

$$= \frac{16}{9}(2\sqrt{2} - 1)$$

ii) The region D in this case is shown in Fig. 15(b). D can be described as

$D = \{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le y^{1/3}\}$.

Thus, D is a region of Type II. Now since $f(x,y) = x^4+y^2$ is continuous, applying the result in E 9), we get

$$\int_0^1\left[\int_{\sqrt{y}}^{y^{1/3}} (x^4+y^2)\,dx\right]dy$$

$$= \int_0^1 \left[\frac{x^5}{5} + xy^2\right]_{\sqrt{y}}^{y^{1/3}} dy$$

$$= \int_0^1 \left[\frac{1}{5}(y^{5/3} - y^{5/2}) + y^{7/3} - y^{5/2}\right] dy$$

$$= \left[\frac{1}{5}\cdot\frac{3}{8}y^{8/3} + \frac{3}{10}y^{10/3} - \frac{6}{5}\cdot\frac{2}{7}y^{7/2}\right]_0^1$$

$$= \frac{3}{40} + \frac{3}{10} - \frac{12}{35}$$

$$= \frac{9}{280}$$

In some cases it so happens that while applying Theorem 5 or its analog, we obtain an integrand where the Fundamental Theorem of Calculus is not helpful. In such a situation we try to describe the region differently. This enables us to reverse the order of integration which may be useful for calculations. You will see one such situation in the following examples.

Example 13 : Consider the integral $\int\int_D f(x,y)\,dx\,dy$,

where $f(x,y) = \begin{cases} \frac{\sin y}{y}, & y \ne 0, \\ 1, & y = 0. \end{cases}$

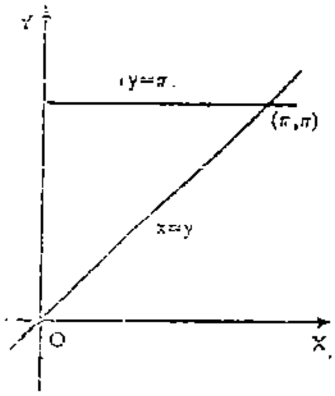


Fig. 16

and D is the region bounded by the lines $x = 0$, $y = \pi$, $x = y$ as shown in Fig. 16.

Here D, described as

$D = \{(x,y) \mid 0 \le x \le \pi, x \le y \le \pi\}$ is a region of Type I.

Then by Theorem 5, we get

$$\int\int_D f(x,y)\,dx\,dy$$

$$= \int_0^{\pi}\left(\int_x^{\pi} \frac{\sin y}{y}\,dy\right)dx.$$

But we cannot use the Fundamental Theorem of Calculus to evaluate

$$\int_x^{\pi} \frac{\sin y}{y}\,dy.$$

However, if we describe D as $D = \{(x,y)\,|\,0 \le y \le \pi,\ 0 \le x \le y\}$, then D is a region of Type II. Now applying the analog of Theorem 5 which you have obtained in E 9), you can see that

$$\int\int_D \frac{\sin y}{y}\,dx\,dy = \int_0^{\pi}\left(\int_0^{y} \frac{\sin y}{y}\,dx\right)dy$$

$$= \int_0^{\pi} \sin y\,dy = -\cos y\big]_0^{\pi} = 2.$$

You can now try some exercises on your own.

E 10) Find the value of the following repeated integrals.

a) $\int_0^{\pi}\int_0^{x} x \sin y\,dy\,dx$

b) $\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx$

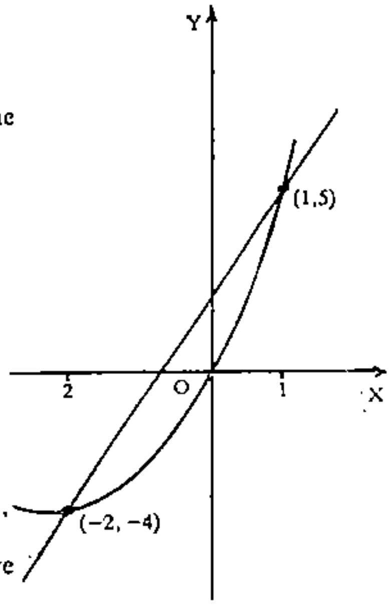
E 11) Obtain the double integral of the function $f(x,y) = e^{x^2}$ over the region bounded by the triangle formed by the x-axis, and the lines $2y = x$ and $x = 2$.

E 12) Express the following repeated integral as a double integral and describe the region of integration. Express this double integral as a repeated integral with the order of integration reversed.

$$\int_1^{2}\left(\int_{x^2}^{2x} f(x,y)\,dy\right)dx$$

E 13) Evaluate the following integral by reversing the order of integration.

$$\int_0^{2}\left(\int_0^{x^2} 3xy\,dy\right)dx$$



E 14) Write both the repeated integrals associated with the double integral $\int\int_D dx\,dy$, where D is the region shown alongside. Evaluate them and check that they are equal.

In the next section, we shall see how a change of variables affects the integrals of functions of two variables over regions of Type I and Type II. In particular, this will enable us to convert double integrals in Cartesian coordinates to double integrals in polar coordinates. This conversion becomes essential when the region under consideration is bounded by a curve like a cardioid or a circle, which can be more easily described using polar coordinates.

11.4 CHANGE OF VARIABLES

In Calculus (Unit 11) you have seen that a suitable change in the variable of integration often makes integration quite easy. You have seen that in the integral $\int f(v)\,dv$, the substitution $v = g(x)$, where the function g satisfies certain suitable conditions, leads to the formula

$\int f(v)\, dv = \int f(g(x))\, g'(x)\, dx.$

Don't you think then that we would be justified in expecting that changing variables may simplify evaluation of double integrals too? In what follows, we shall informally discuss how the transformation $x = \phi(u,v)$, $y = \psi(u,v)$ affects the double integral $\int\int_D f(u,v)\, du\, dv$, and then state the precise result without proof.

You have seen that to define double integrals over a region, we first partitioned the region into small rectangles. The areas of these rectangles were then used to define upper and lower sums. Now suppose we change the variables u and v to the variables x and y. Let us assume that the new variables x and y are related to the variables u and v by

$$x = \phi(u,v),\ y = \psi(u,v). \qquad \ldots\ldots (6)$$

These equations define a transformation of the uv-plane to the xy-plane. We shall be interested in only those transformations which map the region of integration for any given integral onto another region in a 1–1 manner. Now let us see how this transformation affects the area of a small rectangle ABCD in the uv-plane. The image of ABCD under the transformation given by (6) will be some region A'B'C'D' in the xy-plane. See Fig. 17.

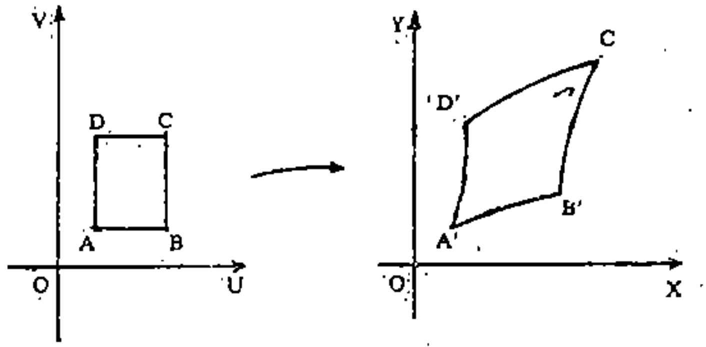


Fig. 17

Suppose the coordinates of A, B, C and D are as follows :

$A = (a,b)$, $B = (a+h, b)$
$C = (a+h, b+k)$, $D = (a, b+k)$.

The image of A, that is,
$A' = (\phi(A), \psi(A))$.

Similarly, we have

$B' = (\phi(B), \psi(B))$, $C' = (\phi(C), \psi(C))$
and $D' = (\phi(D), \psi(D))$.

Now, when the rectangle ABCD is small, i.e. when h and k are small, the figure A'B'C'D' will look like a parallelogram.

Thus, we can write

A = area of A'B'C'D' = 2. Area of $\triangle$ A'B'D'

$$= \pm \begin{vmatrix} \phi(A) & \phi(B) & \phi(D) \\ \psi(A) & \psi(B) & \psi(D) \\ 1 & 1 & 1 \end{vmatrix}$$

The area of a $\triangle PQR$, where
$P = (x_1, y_1)$
$Q = (x_2, y_2)$
$R = (x_3, y_3)$
is
$$\pm\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$
The sign is chosen so as to get a positive value.

Subtracting the first column from each of the columns, and evaluating the determinant, we get

$$A = \pm \begin{vmatrix} \phi(B)-\phi(A) & \phi(D)-\phi(A) \\ \psi(B)-\psi(A) & \psi(D)-\psi(A) \end{vmatrix}.$$

Now applying the mean value theorem to each of the entries in this 2×2 determinant we get

$$A = \pm hk \begin{vmatrix} \phi_u(\xi) & \phi_v(\eta) \\ \psi_u(\xi') & \psi_v(\eta') \end{vmatrix}$$

Here ξ and ξ' are points on the line joining A and B, and η and η' are points on the line joining A and D.

Now the right hand side of (7) is approximately equal to $|J|$ hk, where J is the Jacobian of the transformation given by (6).

Thus, the rectangular region ABCD with area hk is transformed into the region A'B'C'D' with area $|J|$ hk.

Let f(x,y) be an integrable function over D, and let $P = \{T_i\}$ be any partition of D. Then

$$L(P,f) = \sum_i hk \inf_{(x,y)\in T_i} f(x,y)$$

is approximated by

$$\sum_i |J| hk \inf_{(u,v)\in T_i'} f(\phi(u,v), \psi(u,v)), \quad \text{.....} (8)$$

where T_i' is such that it gets transformed to T_i by the transformation given by (6).

Now the sum in (8) becomes

$$\iint_{D'} f(\phi(u,v), \psi(u,v))\, du\, dv\, |J|,$$

as the norm of the partition tends to zero. Thus, it follows that the change of variables from (x,y) to (u,v) should lead to the equality

$$\iint_D f(x,y)\, dx\, dy = \iint_{D'} f(\phi(u,v), \psi(u,v))\, |J|\, du\, dv,$$

where D is the image of D' under the transformation given by (6). More precisely, we have the following result :

Theorem 6 : Let D be a bounded set in R^2, and let f be a continuous function defined on D. Let $x = \phi(u,v)$, $y = \psi(u,v)$ be a transformation from the uv-plane to the xy-plane, such that

i) there exists a region D' in the uv-plane such that D' is mapped onto D in a 1–1 manner,

ii) ϕ, ψ have continuous partial derivatives on D', and

iii) $J = \dfrac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in D'.

Then

$$\iint_D f(x,y)\, dx\, dy = \iint_{D'} f(\phi(u,v), \psi(u,v))\, |J|\, du\, dv$$

We are not going to prove this theorem here. But now we shall see how it proves useful in the evaluation of some double integrals. Since you will need to calculate many Jacobians while changing variables in double integrals, it will be useful to go back and do a quick revision of Unit 9.

Example 14 : Suppose S is a triangle in the uv-plane with vertices (0,0), (1,0) and (0,1), and R is the corresponding region in the xy-plane obtained under the transformation

$x = 2u - 3v$

$y = 5u + 7v$.

Let us evaluate $\iint_R x\, dx\, dy$.

The Jacobian of transformation in this case is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -3 \\ 5 & 7 \end{vmatrix}$$

$$= 29$$

Thus, using Theorem 6, we get

$$\iint_R x\,dx\,dy = \iint_S (2u-3v)\,(29)\,du\,dv,$$

where S is the triangular region in the uv-plane, given by

$0 \le u \le 1,\ 0 \le v \le 1-u.$

Thus,

$$\iint_R x\,dx\,dy = 29\int_0^1\int_0^{1-u} (2u-3v)\,dv\,du$$

$$= 29\int_0^1 \left[2uv-\frac{3v^2}{2}\right]_0^{1-u} du$$

$$= 29\int_0^1 \left\{2u\,(1-u)-\frac{3}{2}(1-u)^2\right\} du$$

$$= \frac{-29}{6}$$

Example 15 : Let us find the area of the region D lying in the first quadrant bounded by the curves $xy = 1$, $xy = 9$, $y = x$ and $y = 4x$ by using the transformation $x = u/v$, $y = uv$, $u > 0$, $v > 0$. The region D is shown in Fig. 18.

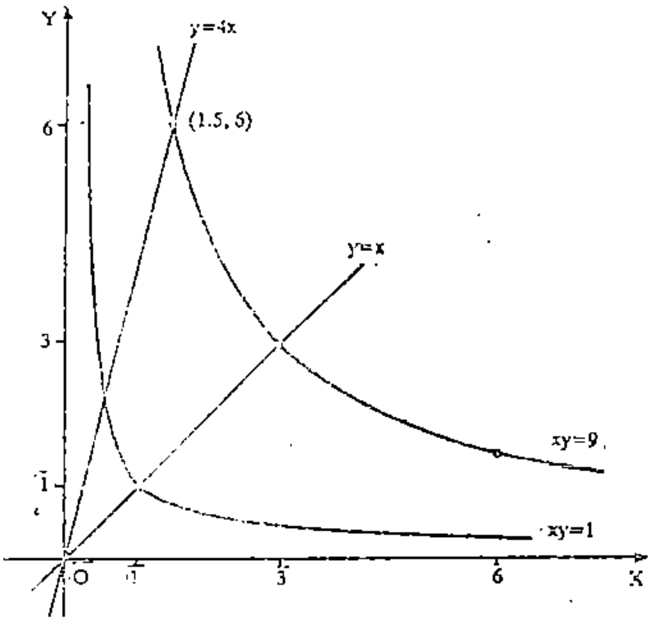


Fig. 18

Clearly, it is not at all easy to express D as a union of regions of Type I or Type II. But transformation $x = u/v$, $y = uv$ shows that D is the image of the rectangle bounded by the lines $u = 1$, $u = 3$, $v = 1$ and $v = 2$.

Therefore,

$$\iint_D dx\,dy = \int_1^2\int_1^3 |J|\,du\,dv,$$

$$\text{where } |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix}$$

Thus, the required area is equal to

$$\int_1^2 \int_1^3 \frac{2u}{v}\, du\, dv = 8 \ln 2.$$

You must have noticed that the change of variables made the whole computation very simple.

We shall now use the change of variables formula to convert double integrals in Cartesian coordinates to double integrals in polar coordinates.

Double integrals in Polar coordinates

You know that a point in plane can also be represented by polar coordinates (r,θ), and that these are connected to the Cartesian coordinates by the equations

$x = r\cos\theta \quad y = r\sin\theta.$

We can use the above relations to convert the integral $\int\int_D f(x,y)\, dx\, dy$ to an integral of the type $\int\int_{D^*} f^*(r,\theta)\,|J|\, dr d\theta$, where $|J| = \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right|$ and D^* is the region D described in polar coordinates. You know (Example 1, Unit 9) that $|J| = r$. Therefore, to evaluate a double integral in polar coordinates we have to describe the region D in terms of polar coordinates, and then express it as a region of the type

$\{(r,\theta) \mid \alpha \le \theta \le \beta,\ g_1(\theta) \le r \le g_2(\theta)\}$,

or $\{(r,\theta) \mid a \le r \le b,\ h_1(r) \le \theta \le h_2(r)\}$,

or a union of such regions so that we can evaluate the double integral with the help of iterated integrals. Of course we have to check that the conditions stated in Theorem 6 are satisfied.

In the first two examples below you will see how to describe a region in polar coordinates. Then we compute a few integrals with the help of polar coordinates. Note that if the region D is already described in polar coordinates, then, we consider the integral

$$\int\int_D f(r,\theta)\, r\, d\theta\, dr$$

as the integral of f over D in polar coordinates, and not the integral $\int\int_D f(r,\theta)\, d\theta\, dr$, because it is the first one which is equal to the double integral of f over D when expressed in Cartesian coordinates.

Let us look at some examples :

Example 16 : Let D be the region enclosed by a circle of radius a and centre at the origin (Fig. 19). Let us describe D in terms of polar coordinates.

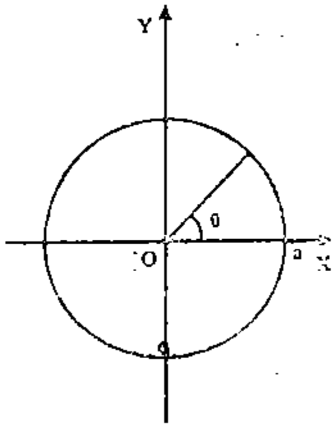
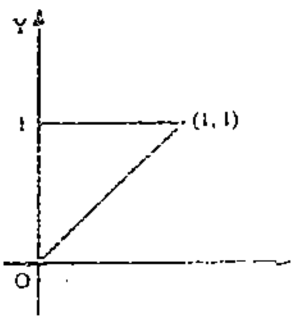


Fig. 19

We note that in D, θ varies from 0 to 2π. Holding θ fixed, we notice that on the ray of angle θ, r varies from 0 to a. Thus D has the description :

$0 \le \theta \le 2\pi,\ 0 \le r \le a$

Example 17 : Let D be a triangular region with vertices (in Cartesian coordinates) at (0,0), (1,1), (0,1). We shall now describe D in polar coordinates. From Fig. 20 we can see that θ varies between π/4 and π/2. Further, for a given θ, r varies between 0 and $\frac{1}{\sin\theta}$, i.e.,



$\frac{\pi}{4} \le \theta \le \frac{\pi}{2},\ 0 \le r \le \frac{1}{\sin\theta}$

Alternatively, we observe that r varies from 0 to $\sqrt{2}$. For fixed r, when $0 \le r \le 1$, θ clearly ranges from π/4 to π/2, while for $1 \le r \le \sqrt{2}$ θ ranges from π/4 to a value β such that $r\sin\beta = 1$, i.e.,

$0 \le r \le 1,\ \pi/4 \le \theta < \pi/2$

$1 \le r \le \sqrt{2},\ \pi/4 \le \theta \le \sin^{-1}(1/4)$

We now give some examples to illustrate the evaluation of double integrals in polar coordinates.

Example 18 : Suppose D is the quarter ring with radii, r = 1 and r = 2. Let us evaluate the integral

$$\int_D\int (3x+8y^2)\,dy\,dx$$

We first note that we can write D as

$D = \{(r,\theta) : 0 \le \theta \le \frac{\pi}{2}, 1 \le r \le 2\}$.

In polar co-ordinates the function f can be written as

$f^*(r,\theta) = 3r\cos\theta + 8r^2\sin^2\theta$

Then we have

$$\int_D\int (3x+8y^2)\,dy\,dx = \int_0^{\pi/2}\int_1^2 (3r\cos\theta + 8r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_1^2 (3r^2\cos\theta + 8r^3\sin^2\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} [r^3\cos\theta + 2r^4\sin^2\theta]_1^2\,d\theta$$

$$= \int_0^{\pi/2} [7\cos\theta + 30\sin^2\theta]\,d\theta$$

$$= \int_0^{\pi/2} (7\cos\theta + 15 - 15\cos 2\theta)\,d\theta$$

$$= [7\sin\theta + 15\theta - \frac{15}{2}\sin 2\theta]_0^{\pi/2}$$

$$= 7 + \frac{15}{2}\pi$$

Example 19 : Let D be the region between the polar graphs of $r = \theta$ and $r = 2\theta$ for $0 \le \theta \le 3\pi$. Let us calculate the double integral

$$\int_D\int (x^2+y^2)\,dy\,dx.$$

We note that the function $f(x,y) = x^2+y^2$ is continuous on D and $f^*(r,\theta) = r^2\cos^2\theta + r^2\sin^2\theta = r^2$. Therefore, we have

$$\int_D\int (x^2+y^2)\,dy\,dx = \int_0^{3\pi}\int_0^{2\theta} r^2 .\, r\,dr\,d\theta$$

$$= \int_0^{3\pi}\int_0^{2\theta} r^3\,dr\,d\theta$$

$$= \int_0^{3\pi} \left[\frac{r^4}{4}\right]_0^{2\theta} d\theta$$

$$= \frac{15}{4}\int_0^{3\pi} \theta^4\,d\theta$$

$$= \frac{15}{4}\left[\frac{\theta^5}{5}\right]_0^{3\pi}$$

$$= \frac{363}{2}\pi^5$$

Example 20 : Let us find the integral of $f(x,y) = y$ over the region D which is inside the cardioid $r = 2(1+\cos\theta)$ and outside the circle $r = 2$.

Let us look at the region D given in Fig 21.

Then D is the set of points (x,y) whose polar co-ordinates satisfy $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $2 \le r \le 2(1+\cos\theta)$.

Since the function $f(x,y) = y$ is continuous on D, we have

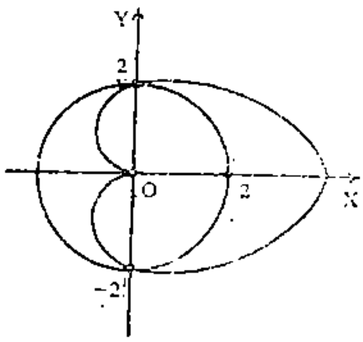


Fig. 21

$$\iint_D y\,dy\,dx = \int_0^{\pi/2}\int_2^{2(1+\cos\theta)} (r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^3}{3}\sin\theta\right]_2^{2(1+\cos\theta)} d\theta$$

$$= \frac{8}{3}\left[\int_0^{\pi/2} (1+\cos\theta)^3\sin\theta - \sin\theta\right]d\theta$$

$$= \frac{8}{3}\left[-\frac{(1+\cos\theta)^4}{4} + \cos\theta\right]_0^{\pi/2}$$

$$= \frac{22}{3}$$

Sometimes we have to use the change of variables formula twice to evaluate a given integral, as our next example shows.

Example 21 : Let us evaluate $\iint_R x^2\,dxdy$, where R is the region given by $\frac{x^2}{4} + \frac{y^2}{9} \le 1$, by using the change of variables : $x = 2u$, $y = 3v$.

This change of variables transforms the region R (which is bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$) in the xy-plane to a circular region S in the uv-plane given by $u^2 + v^2 \le 1$.

The function $f(x,y) = x^2 = (2u)^2$, and $\frac{\partial(x,y)}{\partial(u,v)} = 6$.

Therefore, $\iint_R x^2\,dxdy = \iint_S (2u)^2\, 6\,dudv$.

Now using (r,θ) co-ordinates in the uv-plane, i.e., $u = r\cos\theta$, $v = r\sin\theta$, we have

$$\iint_R x^2\,dxdy = 6\int_0^{2\pi}\int_0^1 (4r^2\cos^2\theta)\, r\,dr\,d\theta$$
$$= 6\pi.$$

Here are some exercises for you.

E 15) Find the integral of $e^{x^2+y^2}$ over the region consisting of points (x,y) such that $x^2 + y^2 \le 1$.

E 16) Find the integral of the function $f(x,y) = y$ over the region bounded in polar coordinates by $r = 1 - \cos\theta$.

E 17) Evaluate the following integrals by making the indicated change of variables.

a) $\iint_D (a^2 - x^2 - y^2)\,dxdy$, where D is the semi-circular disc $x^2+y^2 \le ax$ in the first quadrant.

Transformation : $x = r\cos\theta$, $y = r\sin\theta$

(Hint : the transformed region is $\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le a\cos\theta\}$)

b) $\iint_D \frac{x^2+2y^2}{xy}\,dxdy$, where D is bounded by $x = y^2$, $x = y^2 + 2$, $y = \frac{1}{x}$, $y = \frac{3}{x}$.

Transformation : $u = xy$, $v = x - y^2$.

E18) Evaluate the following integrals by making a suitable change of variables.

a) $\int\int_R \frac{x+y}{1+x-y}\, dxdy$, where R is bounded by $x-y=0$, $x+y=0$, $x-y=4$, $x+y=4$.

b) $\int\int_R e^x\, dx\, dy$, where R is bounded by

$y = 3x + 1$, $y = 3x - 3$, $y = -x + 1$, $y = -x + 5$.

With this we come to the end of this unit. In the next unit we shall discuss triple integrals. Let us now recall the points covered in this unit.

11.5 SUMMARY

In this unit we have

1) defined double integral of a function defined over a rectangle.
2) described repeated integrals using which double integrals are easily calculated.

   Thus, under suitable conditions

   $$\int\int_D f(x,y)\, dxdy = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy,$$

   where $D = [a,b] \times [c,d]$.
3) extended the definition of double integrals to bounded sets of R^2 and also evaluated the integrals over some regions of Type I and Type II, using repeated integrals.

   For example,

   $$\int\int_D f(x,y)\, dxdy = \int_c^d \left(\int_{g_1(x)}^{g_2(x)} f(x,y)dy \right) dx,$$

   where f is a continuous function defined over a bounded region D of Type I described by

   $D = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$.
4) described the change of variables formula :

   $$\int\int_D f(x,y)\, dxdy = \int\int_{D'} f(\phi(u,v), \psi(u,v))\, |J|\, dudv$$
5) evaluated double integrals in terms of polar coordinates.

11.6 SOLUTIONS AND ANSWERS

E 1) Since the function $f(x) = x$ is continuous on $[a,b]$, it is integrable over $[a,b]$ (see Calculus Block 3, Unit 10, Theorem 5).

Then $I_L = \sup \{L(P,f)\} = \inf. \{U(P,f)\} = I_U = \int_a^b f(x)\, dx$. We will show that $I_L = I_U = \frac{1}{2}(b^2 - a^2)$.

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be an arbitrary partition of $[a,b]$. On each sub-interval $[x_{i-1}, x_i]$, the function $f(x) = x$ has a maximum $M_i = x_i$ and a minimum $m_i = x_{i-1}$. Therefore, $U(P,f) = \Sigma M_i \Delta x_i = \Sigma x_i (x_i - x_{i-1})$

and

$L(P,f) = \Sigma x_{i-1} (x_i - x_{i-1})$

Now for each index i, $x_{i-1} \leq \frac{1}{2}(x_i + x_{i-1}) \leq x_i$

Therefore, we get

$U(P,f) \geq \Sigma \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1})$

$U(P,f) \geq \Sigma \frac{1}{2}(x_n^2 - x_0^2)$,

$U(P,f) \geq \frac{1}{2}(b^2 - a^2)$

∴ This is true for all partitions $P \in \mathcal{P}$ of $[a,b]$. Thus $I_U = \inf \{U(P,f)\} \geq \frac{1}{2}(b^2 - a^2)$

Similarly, we can show that

$I_L \leq \frac{1}{2}(b^2 - a^2)$. Thus

$I_L \leq \frac{1}{2}(b^2 - a^2) \leq I_U$

But $I_L = I_U$.

Hence, $\int_a^b f(x)\,dx = \frac{1}{2}(b^2 - a^2)$

E2) $U(P,f) = M_{11}$ (area of T_{11}) + M_{12} (area of T_{12}) + M_{21} (area of T_{21}) + M_{22} (area of T_{22}) + M_{31} (area of T_{31}) + M_{32} (area of T_{32})

Area of $T_{11} : [0,1] \times [0, \frac{1}{2}] = \frac{1}{2}$

Area of $T_{12} : [0,1] \times [\frac{1}{2}, 1] = \frac{1}{2}$

Area of $T_{21} : [1, \frac{3}{2}] \times [0, \frac{1}{2}] = \frac{1}{4}$

Area of $T_{22} : [1, \frac{3}{2}] \times [\frac{1}{2}, 1] = \frac{1}{4}$

Area of $T_{31} : [\frac{3}{2}, 2] \times [0, \frac{1}{2}] = \frac{1}{4}$

Area of $T_{32} : [\frac{3}{2}, 2] \times [\frac{1}{2}, 1] = \frac{1}{4}$

Now to calculate M_{ij}'s and m_{ij}'s we note that $f(x,y) = x+2y$ is an increasing function on $[0,2] \times [0,1]$. Therefore,

$M_{ij} = \sup \{f(x,y) \mid (x,y) \in T_{ij}\}$
$= f(x_i, y_j)$
$= x_i + 2y_j$

Thus $U(P,f) = 2 \times \frac{1}{2} + 3 \times \frac{1}{2} + \frac{5}{2} \times \frac{1}{4} + \frac{7}{2} \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4}$

$= 5\frac{3}{4}$

Similarly you can calculate $L(P,f)$ by noting that

$m_{ij} = x_{i-1} + 2y_{j-1}$

Then $L(P,f) = 2\frac{1}{4}$

E3) Here $x_0 = 0, x_1 = 1, x_2 = \frac{3}{2}, x_3 = 2; y_0 = 0, y_1 = \frac{1}{4}, y_2 = \frac{1}{2}, y_3 = 1$. Then

$L(Q,f) = \sum_{j=1}^{3} \sum_{i=1}^{3} m_{ij}$ (area of T_{ij})

Area of $T_{11} = \frac{1}{4}$

Area of $T_{12} = \frac{1}{4}$

Area of $T_{13} = \frac{1}{2}$

Area of $T_{21} = \frac{1}{8}$

Area of $T_{22} = \frac{1}{8}$

Area of $T_{23} = \frac{1}{4}$

Area of $T_{31} = \frac{1}{8}$

Area of $T_{32} = \frac{1}{8}$

Area of $T_{33} = \frac{1}{4}$

We know from E 2) that $m_{ij} = x_{i-1} + 2y_{j-1}$
and $M_{ij} = x_i + 2y_j$
Then $L(Q,f) = \Sigma\, m_{ij}$ (area of T_{ij}) = 2.5
and
$U(Q,f) = \Sigma\, M_{ij}$ (area of T_{ij}) = 5.5
Thus, we have $L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$

E 4) Suppose f is integrable over T. Then

$$\sup \{L(P,f) \mid P \in \boldsymbol{P}\} = \inf. \{U(P,f) \mid P \in \boldsymbol{P}\} = \iint_T f(x,y)\, dx\, dy$$

Then for every $\varepsilon > 0$ there exist partitions P′ and P″ of T such that

$$0 \le \iint_T f(x,y)\, dx\, dy - L(P',f) < \frac{\varepsilon}{2}$$

and

$$0 \le U(P'',f) - \iint_T f(x,y)\, dx\, dy < \frac{\varepsilon}{2}$$

Let P be a partition which is finer than P′ and P″. Then by Theorem 1, we have

$$U(P,f) - L(P,f) = U(P,f) - \iint_T f(x,y)\, dx\, dy + \iint_T f(x,y)\, dx\, dy - L(P,f)$$

$$\le U(P'',f) - \iint_T f(x,y)\, dx\, dy + \iint_T f(x,y)\, dx\, dy - L(P',f)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$< \varepsilon$$

E 5) a) To evaluate the repeated integral $\int_{-3}^{4}\left[\int_1^2 3x^2y\, dx\right] dy$,

we first calculate $\int_1^2 3x^2y\, dx$.

$$\int_1^2 3x^2y\, dx = 3y\left[\frac{x^3}{3}\right]_1^2 = 7y$$

$$\text{Then} \int_{-3}^{4}\left[\int_1^2 3x^2y\, dx\right] dy = \int_{-3}^{4} 7y\, dy = 7\left[\frac{y^2}{2}\right]_{-3}^{4} = \frac{49}{2}$$

b) $\int_0^1\left[\int_0^1 (x^2+y^2)\, dx\right] dy = \frac{2}{3}$

c) $\int_3^4 (xy + e^y)\, dy = \left[x\frac{y^2}{2} + e^y\right]_3^4$

$$= \frac{7}{2}x + e^4 - e^3$$

$$\int_1^2\left[\int_3^4 (xy + e^y)\, dy\right] dx = \int_1^2 \left(\frac{7}{2}x + e^4 - e^3\right) dx$$

$$= \left[\frac{7}{2} \cdot \frac{x^2}{2} + (e^4 - e^3)\, x \right]_1^2$$

$$= \frac{21}{4} + (e^4 - e^3)$$

E6) Example 5 shows that if we interchange the order of integration in E 5 (a), then the repeated integral obtained is the same. If you look at the integral in E 5 (b) carefully, you will be able to say directly that both the integrals are the same.

Suppose we interchange the order of integration in E 5 (c), then we get

$$\int_3^4 \left[\int_1^2 (xy + e^y)\, dx \right] dy = \int_3^4 \left[\frac{x^2 y}{2} + e^y x \right]_1^2 dy$$

$$= \int_3^4 \left[\frac{3y}{2} + e^y \right] dy$$

$$= \frac{21}{4} + e^4 - e^3$$

This is the answer we have got in E 5 (c). Thus both the repeated integrals are the same.

E7) a) Since the function $f(x,y) = x \sin(x+y)$ is continuous over $[0,\pi] \times [0, \frac{\pi}{2}]$, then by Theorem 3, we have

$$\int\int_T f(x,y)\, dx\, dy = \int_0^{\pi} \left[\int_0^{\pi/2} x \sin(x+y)\, dy \right] dx$$

$$= \int_0^{\pi} [-x \cos(x+y)]_0^{\pi/2}\, dx$$

$$= -\left[\int_0^{\pi} x \cos\left(\frac{\pi}{2} + x\right) dx - \int_0^{\pi} x \cos x\, dx \right]$$

$$= -\left[\int_0^{\pi} -x \sin x\, dx - \int_0^{\pi} x \cos x\, dx \right]$$

$$= \int_0^{\pi} x \cos x\, dx + \int_0^{\pi} x \sin x\, dx$$

$$\int_0^{\pi} x \cos x\, dx = x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x\, dx$$

$$= -[-\cos x]_0^{\pi} = -2$$

$$\int_0^{\pi} x \sin x\, dx = -x \cos x \Big|_0^{\pi} - \int_0^{\pi} (-\cos x)\, dx$$

$$= -\pi \cos(\pi) + \sin x \Big|_0^{\pi}$$

$$= \pi$$

Hence $\int_0^{\pi} \left[\int_0^{\pi/2} x \sin(x+y)\, dy \right] dx = \pi - 2$

b) $$\int\int_T \frac{1}{1+x+y}\, dx\, dy = \int_0^1 \left[\int_0^1 \frac{1}{1+x+y}\, dx \right] dy$$

$$= \int_0^1 \ln(1+x+y)\Big|_0^1\, dy$$

$$= \int_0^1 \ln(2+y)\, dy - \int_0^1 \ln(1+y)\, dy$$

To evaluate these integrals we make use of the substitution method.

Put u = 2+y, v = 1+y.
Then

$$\int_0^1 \ln(2+y)\,dy = \int_2^3 \ln u\,du = u\ln u - u\Big|_2^3$$
$$= 3\ln 3 - 3 - [2\ln 2 - 2]$$
$$= \ln\left(\frac{27}{4}\right) - 1$$

Similarly, $\int_0^1 \ln(1+x)\,dx = \int_1^2 \ln u\,du = \ln(4) - 1$

Then $\int\int_T \frac{1}{1+x+y}\,dy\,dx = \ln\left(\frac{27}{16}\right)$

E8) a) In D, x ranges between 0 and 2, and y ranges between 0 and x^2. Thus D expressed as

$D = \{(x,y) \mid 0 \le x \le 2, 0 \le y \le x^2\}$ is of Type I.

Similarly, we can express D also as a Type II region, by

$D = \{(x,y) \mid \sqrt{y} \le x \le 2, 0 \le y \le 4\}$

Thus D is of both Type I and Type II.

b) D is neither of Type I or Type II.

c) $D = \{(x,y) \mid 0 \le x \le 1, x^{1/4} \le y \le x^2\}$. This shows that D is of Type I.

D can also be expressed as

$D = \{(x,y) \mid \sqrt{y} \le x \le y^4, 0 \le y \le 1\}$ which is of Type II.

d) Here x ranges from 1 to 3 and also from −1 to −3. Correspondingly we get that y ranges from $\sqrt{1-x^2}$ to $\sqrt{9-x^2}$

Thus, D is of Type I but not of Type II.

E9) Let $D = \{(x,y) \mid \psi_1(y) \le x \le \psi_2(y), c \le y \le d\}$ be a region in the plane where ψ_1 and ψ_2 are as in Definition 4. Let $f : D \to \mathbf{R}$ be a continuous function.

Then the repeated integral

$$\int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy$$

exists and is equal to the double integral

$$\int\int_D f(x,y)\,dx\,dy.$$

E 10) a) By Theorem 1, $\int_0^\pi \int_0^x x\sin y\,dy\,dx = \int_0^\pi \left[\int_0^x x\sin y\,dy\right] dx$

Now $\int_0^x x\sin y\,dy = x\,[-\cos y]_0^x$

$$= -x(\cos x - 1) = x - x\cos x.$$

$$\int_0^\pi \left[\int_0^x x\sin y\,dy\right] dx = \int_0^\pi (x - x\cos x)\,dx$$
$$= \int_0^\pi x\,dx - \int_0^\pi x\cos x\,dx$$
$$= \frac{x^2}{2}\Big|_0^\pi - \left(x\sin x \Big|_0^\pi - \int_0^\pi \sin x\,dx\right)$$

$$= \frac{\pi^2}{2} - [\cos x]_0^{\pi}$$

$$= \frac{\pi^2}{2} - [\cos\pi - \cos 0] = 2 + \frac{\pi^2}{2}$$

b) $$\int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\int_0^{\sin x} y\,dy\right]dx$$

$$= \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

$$= \int_0^{\pi}\frac{\sin^2 x}{2}\,dx$$

$$= \int_0^{\pi}\frac{1-\cos 2x}{4}\,dx$$

$$= \frac{1}{4}\int_0^{\pi}(1-\cos 2x)\,dx$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

$$= \frac{\pi}{4}$$

E 11) Let D be the region given. Then D can be expressed as

$D = \{ (x,y) \mid 0 \le x \le 2, 0 \le y \le \frac{1}{2}x \}$. That is D is of Type I. Therefore, by Theorem 5,

$$\int\int_D e^{x^2}\,dy\,dx = \int_0^2\left[\int_0^{x/2} e^{x^2}\,dy\right]dx$$

$$= \int_0^2 [e^{x^2} y]_0^{x/2}\,dx$$

$$= \int_0^2 \frac{e^{x^2}x}{2}\,dx$$

$$= \frac{1}{2}\int_0^2 x\,e^{x^2}\,dx$$

$$= \frac{1}{2}\times\frac{1}{2}\int_0^4 e^u\,du. \quad \text{Put } u = x^2, \frac{du}{dx} = 2x$$

$$= \frac{1}{4}[e^u]_0^4$$

$$= \frac{1}{4}(e^4 - 1)$$

E 12) The integral can be expressed as the double integral $\int\int_D f(x,y)\,dx\,dy$,

where $D = \{(x,y) \mid 0 \le x \le 2, x^2 \le y \le 2x\}$.
Then D is the region bounded by the line $y = 2x$ and the parabola $y = x^2$. In D we observe that for a fixed y, x ranges from $\frac{y}{2}$ to $\sqrt{y}$ and y ranges from 0 to 4. Thus we can express D also as

$$D = \{ (x,y) : \frac{y}{2} \le x \le \sqrt{y}, 0 \le y \le 4 \}$$

Therefore, by the analog of Theorem 5, we get

$$\int\int_D f(x,y)\,dx\,dy = \int_0^4\left[\int_{y/2}^{\sqrt{y}} f(x,y)\,dx\right]dy.$$

E 13) The region $D = \{(x,y) : 0 \le x \le 2, 0 \le y \le x^2\}$ is of Type I. When we interchange the order of integration, the evaluation is possible only if we express D as a Type II region. In D, y ranges from 0 to 4 and for a fixed y, x ranges from $\sqrt{y}$ to $\sqrt{2}$. Thus

$D = \{(x,y) \mid 0 \le y \le 4, \sqrt{y} \le x \le 2\}$

which is of Type II. Thus we have

$$\int_0^2 \left[\int_0^{x^2} 3xy\, dy \right] dx = \int_0^4 \left[\int_{\sqrt{y}}^{2} 3xy\, dx \right] dy$$

$$= \int_0^4 3y \left(2 - \frac{y}{2}\right) dy$$

$$= \int_0^4 \left(6y - \frac{3y^2}{2}\right) dy$$

$$= \left[3y^2 - \frac{y^3}{2} \right]_0^4$$

$$= 16$$

E 14) From the figure we observe that x ranges from -2 to 1 and y ranges from $x^2 + 4x$ to $3x + 2$. Thus

$D = \{(x,y) \mid -2 \le x \le 1, x^2 + 4x \le y \le 3x + 2\}$

is of Type I.

The repeated integral corresponding to this region is

$$\int_{-2}^{1} \left[\int_{x^2+4}^{3x+2} dy \right] dx = \int_{-2}^{1} (x^2 + 4x - 3x - 2)\, dx$$

$$= -\frac{27}{6}$$

Similarly we observe from the figure that y ranges from -4 to 5 and x ranges from the point x_1 to $\frac{y-2}{3}$ where x_1 is such that

$x_1^2 + 4x_1 - y = 0$

i.e., $y + 4 = (x_1 + 2)^2$

or $x_1 = \sqrt{y + 4} - 2$

Thus we can express D as

$D = \{(x,y) \mid \sqrt{y + 4} - 2 \le x \le \frac{y-2}{3}, -4 \le y \le 5\}$

The repeated integral corresponding to this region is

$$\int_{-4}^{5} \left[\int_{\sqrt{y+4}-2}^{\frac{y-2}{3}} dx \right] dy = \int_{-4}^{5} \left[\frac{y-2}{3} - (\sqrt{y+4} - 2) \right] dy$$

$$= \frac{y^2}{6} + \frac{4}{3} y - \frac{2}{3} (y+4)^{3/2} \Big]_{-4}^{5}$$

$$= -\frac{27}{6}$$

Thus both the repeated integrals are the same.

E 15) Since the region is a disc D with radius 1, the integral can be more easily evaluated by polar coordinates. The region D described in polar coordinates is the set D^* such that

$D^* = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$.

Moreover, the function $f(x,y) = e^{x^2+y^2}$ is continuous on D and $f^*(r,\theta) = e^{r^2}$. Therefore we have

$$\iint_D f(x,y)\,dx\,dy = \int_0^{2\pi}\left[\int_0^1 e^{r^2} r\,dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\int_0^1 \frac{e^u}{2}du\right]d\theta,\ u = r^2,\ du = 2r\,dr$$

$$= \frac{1}{2}\int_0^{2\pi}(e-1)\,d\theta$$

$$= \frac{e-1}{2}\,\theta\ \Big|_0^{2\pi} = (e-1)\,\pi$$

E 16) Here the region D is described in polar coordinates by
$D = \{(r,\theta) \mid 0 \le r \le 1 + \cos\theta,\ 0 \le \theta \le \pi\}$
The function f(x,y) is continuous on D and
$f^*(r,\theta) = r\sin\theta$. Therefore, we have

$$\iint_D f(x,y)\,dx\,dy = \int_0^{\pi}\left[\int_0^{1+\cos\theta}(r\sin\theta)\,r\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\int_0^{1+\cos\theta}\sin\theta\, r^2\,dr\right]d\theta$$

$$= \int_0^{\pi}\sin\theta\ \frac{r^3}{3}\Big|_0^{1+\cos\theta}d\theta$$

$$= \int_0^{\pi}\sin\theta\ \frac{(1+\cos\theta)^3}{3}\,d\theta$$

To evaluate this integral we make the substitution $u = 1+\cos\theta$, and obtain

$$\iint_D f(x,y)\,dx\,dy = -\frac{1}{12}(1+\cos\theta)\Big|_0^{\pi} = \frac{1}{6}$$

E 17) a) $$\iint_D (a^2-x^2-y^2)\,dxdy = \int_0^{\pi/2}\int_0^{a\cos\theta}(a^2-r^2)\,r\,drd\theta$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta}a^2 r\,drd\theta - \int_0^{\pi/2}\int_0^{a\cos\theta}r^3\,drd\theta$$

$$= a^2\int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{a\cos\theta}d\theta - \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^{a\cos\theta}$$

$$= a^2\int_0^{\pi/2}\frac{a^2\cos^2\theta}{2}\,d\theta - \frac{a^4}{4}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= \frac{a^4}{2}\left[\int_0^{\pi/2}\cos^2\theta\,d\theta - \frac{1}{2}\int_0^{\pi/2}\cos^4\theta\,d\theta\right]$$

Now, $$\int_0^{\pi/2}\cos^2\theta\,d\theta = \int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}$$

By reduction formula $\int_0^{\pi/2}\cos^4\theta\,d\theta = \frac{3}{4}\frac{\pi}{4} = \frac{3\pi}{16}$

Thus, $$\iint_D (a^2-x^2-y^2)\,dx\,dy = \frac{a^4}{2}\left[\frac{\pi}{4}-\frac{3\pi}{32}\right]$$

$$= \frac{5\pi a^4}{64}$$

b). The transformation is $u = xy$, $v = x-y^2$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & 1 \\ x & -2y \end{vmatrix} = -2y^2 - x$$

$$\left| \frac{\partial(u,v)}{\partial(x,y)} \right| = 2y^2 + x$$

Then $\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{x + 2y^2}$

W is the image of the rectangle $[1,3] \times [0,2]$.

Here we don't have to solve for x and y in terms u and v because

$$\int\int_D \frac{x + 2y^2}{xy} \, dx\, dy = \int_0^2\int_1^3 \frac{x + 2y^2}{xy} \cdot \frac{1}{x + 2y^2} \, du\, dv$$

$$= \int_0^2\int_1^3 \frac{1}{u} \, du\, dv$$

$$= \int_0^2 [\ln u]_1^3 \, dv$$

$$= 2 \ln 3.$$

E 18) a) Put $u = x-y$, $v = x+y$. Then the region D is the image of the rectangle $D^* : [0,4] \times [0,4]$ in the uv–plane.

Now, $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = -2$. Then

$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{2}$$

Therefore,

$$\int\int_D f(x,y) \, dx\, dy = \int_0^4 \left[\int_0^4 \frac{v}{1+u} \frac{1}{2} du \right] dv$$

$$= \frac{1}{2} \int_0^4 v \ln(1+u) \Big|_0^4 dv$$

$$= \frac{1}{2} \ln 5 \frac{v^2}{2} \Big|_0^4$$

$$= 4 \ln 5$$

b) Put $u = 3x-y$, $v = x+y$. The region D is the image of the rectangle $D^* : [-1,3] \times [1,5]$.

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 4.$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4}$$

Solving for x, from $u = 3x-y$ and $v = x+y$, we get $u+v = 4x$. $\therefore x = \frac{1}{4}(u+v)$. Therefore,

$$\int\int_D f(x,y) \, dx\, dy = \int_{-1}^3 \left[\int_1^5 e^{(u+v)/4} \frac{1}{4} dv \right] du$$

$$= \frac{1}{4} \int_{-1}^3 \left[e^{u/4} \times 4 e^{v/4} \right]_1^5 du$$

$$= \int_{-1}^3 (e^{5/4} - e^{1/4}) e^{u/4} \, du$$

$$= 4 (e^{5/4} - e^{1/4}) (e^{3/4} - e^{-1/4}).$$

UNIT 12 TRIPLE INTEGRATION

Structure

12.1 INTRODUCTION

In the last unit (Unit 11) you have seen how the concept of integration can be extended to real-valued functions of two variables. We can very easily modify the method described there to apply to functions over $\mathbf{R}^n$, for any $n > 2$. But in this course we will not consider the integration of functions of more than three variables. In this unit, we'll talk about triple integration. You will soon see that this unit proceeds exactly like the previous one.

Objectives

After reading this unit you should be able to

- define the triple integral of a real-valued function of three variables over a rectangular box,
- evaluate triple integrals using repeated integrals,
- describe regions which are analogues of Type I and Type II regions in the plane and evaluate triple integrals over such regions,
- effect change of variables in triple integrals,
- evaluate triple integrals using cylindrical and spherical coordinates.

12.2 INTEGRAL OVER A REGION IN SPACE

As in the last unit, we will first define triple integral i.e. integral of a real-valued function of three variables over a closed rectangular box or a closed parallelopiped. You know that a closed rectangular box in the 3-dimensional Cartesian coordinate space is a natural generalisation in space of a closed interval on the real-line or a closed rectangle in the plane. The faces of such a closed box (referred to simply as 'box' in the sequel) are given by the planes $x = a$, $x = b$, $y = c$, $y = d$, $z = s$ and $z = t$. After somewhat detailed discussion about triple integrals over a box we define triple integral over bounded sets in $\mathbf{R}^3$.

So, let us first define triple integrals over a box.

12.2.1 Integral Over a Rectangular Box

In Sec. 11.2, while defining the double integral, we had first defined it over a rectangle. The only thing that we have assumed there was that the area of a rectangle of length l units and breadth b units is lb sq. units. Here we are going to assume that the volume of a rectangular box of dimensions l, b and h units, is lbh cu. units.

By now, you must be having a clear idea about how we are going to proceed from here. Suppose f is a bounded real-valued function of three variables defined on the box $B:[a,b] \times [c,d] \times [s,t]$
We will

- partition B,
- form lower and upper sums,
- take the supremum and infimum of these sums, and then
- define integrability.

So let us partition B first.

If P_1, P_2 and P_3 are partitions of [a,b], [c,d] and [s,t], respectively, then $P_1 \times P_2 \times P_3$ defines a partition of B. See Fig. 1.

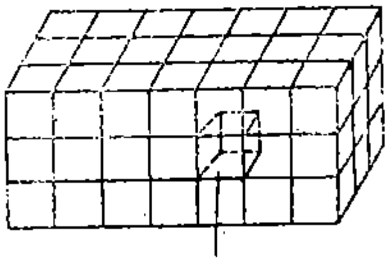


Fig. 1

If P_1 divides [a,b] in p sub-intervals, P_2 divides [c,d] in q sub-intervals and P_3 divides [s,t] in r sub-intervals, then the partition $P_1 \times P_2 \times P_3$ divides B into p.q.r small boxes B_{ijk}. Conversely, if P is any partition of B into boxes whose faces are parallel to the coordinates planes, then there exist partitions P_1, P_2, P_3 of [a,b], [c,d], [s,t], respectively, such that $P = P_1 \times P_2 \times P_3$. We shall consider only such partitions of B.

Now, if $P_1 = \{x_0, x_1, \ldots\ldots, x_p\}$,

$P_2 = \{y_0, y_1, \ldots\ldots, y_q\}$ and

$P_3 = \{z_0, z_1, \ldots\ldots, z_r\}$, then

$[x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ denotes a typical box B_{ijk} of B (See Fig. 1). The volume of B_{ijk}, denoted by V_{ijk} is

$V_{ijk} = (x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})$.

Thus, we have that V, the volume of B, is given by

$$V = (b-a)(d-c)(t-s) = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} V_{ijk} \qquad \ldots\ldots (1)$$

The right hand side of (1) involves triple summation over i, j, k. But that should pose no problem. We can write a triple summation

$\sum_i \sum_j \sum_k a_{ijk}$ as $\sum_i \left[\sum_j \left[\sum_k a_{ijk} \right] \right]$

and evaluate it. It may be noted that the order of summation is immaterial.

Now corresponding to a sub-box B_{ijk}, consider the set
$S_{ijk} = \{f(x,y,z) \mid x \in [x_{i-1}, x_i], y \in [y_{j-1}, y_j], z \in [z_{k-1}, z_k]\}$. Since f is a bounded function, this S_{ijk} is a bounded set in **R**. This means we can talk about the greatest lower bound and the least upper bound of S_{ijk}. Suppose

$M_{ijk} = \sup S_{ijk}$ and $m_{ijk} = \inf S_{ijk}$.

Then we can define the lower sum L(P,f) as

$$L(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} m_{ijk} \Delta x_i \Delta y_j \Delta z_k.$$

Here $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ and $\Delta z_k = z_k - z_{k-1}$.

Similarly, the upper sum U(P,f) can be defined as :

$$U(P,f) = \sum_{i=1}^{p} \sum_{j=1}^{q} \sum_{k=1}^{r} M_{ijk} \Delta x_i \Delta y_j \Delta z_k.$$

In this case, the product $\Delta x_i \Delta y_j \Delta z_k$ is nothing but the volume of B_{ijk}. Thus, L(P,f) can be obtained by first taking the product of the infimum of S_{ijk} and the volume of B_{ijk}, and then taking the sum of all such products. Similarly, to obtain U(P,f) we multiply the supremum of f on S_{ijk} by the volume of B_{ijk}, and then take the sum of these products.

In the case of non-negative functions of a single variable, you know that L(P,f) and U(P,f) are the total areas of inscribed and circumscribed rectangles. For non-negative-valued functions of two variables we have seen that L(P,f) and U(P,f) give the total volumes of the inscribed and circumscribed parallelopipeds. What can we say about functions of three variables? No similar interpretation is possible in this case, as we cannot geometrically visualise a 4-dimensional Euclidean space. You may recall that we have mentioned in Unit 3 that it is not possible to draw the graph of a function defined on $\mathbf{R}^3$.

Now, as in the case of functions of one or two variables, we can make the following two statements about the upper and lower sums of a function f defined on a box B in $\mathbf{R}^3$.

1) $L(P,f) \le U(P,f)$, and

2) $L(P,f) \le L(Q,f) \le U(Q,f) \le U(P,f)$,

where P and Q are partitions of B, and Q is finer than P. In particular,

$m\,V \le L(P,f) \le U(P,f) \le M\,V$,

where m and M are the lower and upper bounds of f on B.

A partition Q is said to be finer than a partition P, ($Q \supset P$) if each sub-box of Q is contained in a sub-box of P.

Now, let us denote the set of all lower sums by $\boldsymbol{u}$, i.e.,

$\boldsymbol{u} = \{L(P,f) \mid P \in \boldsymbol{P}\}$. We further write

$\boldsymbol{u}' = \{U(P,f) \mid P \in \boldsymbol{P}\}$.

Now $\boldsymbol{u}$ is a set of real numbers which is bounded above by MV and $\boldsymbol{u}'$ is a set of real numbers which is bounded below by mV. So, we can take the supremum of $\boldsymbol{u}$ and the infimum of $\boldsymbol{u}'$. If $\sup \boldsymbol{u} = \inf \boldsymbol{u}'$, then we say that f is **triple integrable**, and the common value is called the **triple integral** of f over B. We denote this integral by

$$\iiint_B f(x,y,z)\,dxdydz, \text{ or by } \int_a^b\int_c^d\int_p^q f(x,y,z)\,dxdydz.$$

Thus, $\iiint_B f(x,y,z)\,dxdydz = \text{lub}\ \boldsymbol{u} = \text{glb}\ \boldsymbol{u}'$

Now we can make a similar remark as Remark 1 of Unit 11, which tells us that we can view triple integrals as the limit of a sum.

Remark 1: Suppose f is a bounded function defined over a box $B:[a,b] \times [c,d] \times [s,t]$. We partition B into pqr sub-boxes, B_{ijk}, by

$a = x_0 < x_1 < \ldots\ldots < x_p = b$

$c = y_0 < y_1 < \ldots\ldots < y_q = d$

$s = z_0 < z_1 < \ldots\ldots < z_r = t$

Let $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, $\Delta z_k = z_k - z_{k-1}$ and P_{ijk} be any point in the sub-box B_{ijk}. Then the sum

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} f(P_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

is called the **Riemann sum of f over B.**

For any sub-box B_{ijk}, let $\Delta(B_{ijk}) = \max\{\Delta x_i, \Delta y_j, \Delta z_k\}$, and let $\Delta(P) = \max_{i,j,k} \Delta(B_{ijk})$

When $\Delta(P) \to 0$, then the Riemann sum of f approaches the triple integral of f. That is,

$$\iiint_B f(x,y,z)\,dxdydz = \lim_{\Delta(P)\to 0} \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} f(P_{ijk})\,\Delta x_i\,\Delta y_j\,\Delta z_k$$

We often make use of this result in the following units.

The statements of Theorem 2 and Theorem 3 of Unit 11 are also true for functions of three variables defined on closed boxes. Thus, the statement that

differentiability $\Rightarrow$ continuity $\Rightarrow$ integrability

is also true for real-valued functions of three variables. Moreover, continuity is only a sufficient condition for integrability and not a necessary condition. We can show this easily by using Theorem 2 and modifying Example 4 of Unit 11. We leave this to you for checking as an exercise (See E 1).

E 1) Show that the function f defined by

$$f(x,y,z) = \begin{cases} 1, & (x,y,z) \ne (0,0,0) \\ 0, & (x,y,z) = (0,0,0) \end{cases}$$

is integrable over the box $B = [-1,1] \times [-1,1] \times [-1,1]$

For the evaluation of triple integrals we make use of the repeated integral defined below.

Let f(x,y,z) be a real-valued function defined on a box B:[a,b]×[c,d]×[s,t]. Then for fixed (x,y) belonging to the rectangle T = [a,b] × [c,d], the function

$f_1(z) = f(x,y,z)$

is a function of one variable defined on [s,t]. If the function f_1 is integrable on [s,t] then we get a function

$$g(x,y) = \int_s^t f_1(z)dz = \int_s^t f(x,y,z)\,dz$$

defined on [a,b] × [c,d]. If the repeated integral

$$\int_a^b \left[\int_c^d g(x,y)dy \right] dx$$

exists over [a,b] × [c,d], then we say that the repeated integral

$$\int_a^b \left[\int_c^d \left[\int_s^t f(x,y,z)\, dz \right] dy \right] dx \qquad \text{..... (2)}$$

of f(x,y,z) over B exists. By interchanging the roles of x,y,z, we get the following five more repeated integrals.

$$\int_s^t \left[\int_c^d \left[\int_a^b f(x,y,z)\, dx \right] dy \right] dz \qquad \text{..... (3)}$$

$$\int_a^b \left[\int_s^t \left[\int_c^d f(x,y,z)\, dy \right] dz \right] dx \qquad \text{..... (4)}$$

$$\int_c^d \left[\int_s^t \left[\int_a^b f(x,y,z)\, dx \right] dz \right] dy \qquad \text{..... (5)}$$

$$\int_c^d \left[\int_a^b \left[\int_s^t f(x,y,z)\, dz \right] dx \right] dy \qquad \text{..... (6)}$$

$$\int_s^t \left[\int_a^b \left[\int_c^d f(x,y,z)\, dy \right] dx \right] dz \qquad \text{..... (7)}$$

In (2) we first integrate f(x,y,z) w.r.t.z, treating x and y as constants. Then the resulting function is a function of x and y, which we integrate w.r.t.y while treating x as a constant. The resulting function of x is then integrated w.r.t.x; of course provided all the functions involved are integrable.

In the case of functions of two variables you have seen that the repeated integrals do not always exist. And even if they exist, they may not always be equal. A similar situation exists for functions of three variables. The six repeated integrals may not always exist, and even if they exist, they may be all different. But we won't worry about such cases. We'll only be interested in those functions for which all the repeated integrals exist and are equal. The following theorem identifies one class of functions with this property. It also establishes a link between the triple and repeated integrals of these functions.

Theorem 1 : Let B = [a,b] × [c,d] × [s,t] be a rectangular box in $\mathbf{R}^3$ and let f : B → R be a continuous function. Then the triple integral $\iiint_B f\, dxdydz$ and the six repeated integrals exist and are all equal.

We would like to remind you as before, that the condition stated in the above theorem is only sufficient and not necessary. We will illustrate the usefulness of the above theorem through some examples now.

Example 1 : Let us integrate the function f(x,y,z) = z on the rectangular box B = [0,3] × [0,4] × [0,2].

By Theorem 1 we can write

$$\iiint_B f(x,y,z)\, dxdydz = \int_0^3 \left[\int_0^4 \left[\int_0^2 z\, dz \right] dy \right] dx$$

$$= \int_0^3 \left(\int_0^4 2\, dy \right) dx$$

$$= \int_0^3 8\, dx$$

$$= 24.$$

Now we take an example which is slightly more complicated.

Example 2 : Let B be the box $[0,2] \times [-\frac{1}{2}, 0] \times [0, \frac{1}{3}]$ and consider the function $f : B \to \mathbf{R}$.

$f(x,y,z) = (x+2y+3z)^2$.

Let us evaluate the triple integral of f over B. Since f is a polynomial function, it is continuous on B. Therefore, by Theorem 1, we can use any of the six repeated integrals to evaluate the triple integral of f over B. Let us consider

$$\int_0^2 \left(\int_0^{1/3} \left(\int_{-1/2}^0 (x+2y+3z)^2 dy \right) dz \right) dx$$

So we have to first integrate w.r.t. y, then w.r.t. z and finally w.r.t. x.

Now

$$\int_{-1/2}^0 (x+2y+3z)^2\, dy = \frac{(x+2y+3z)^3}{6} \Bigg]_{-1/2}^0 = \frac{1}{6}\left[(x+3z)^3 - (x+3z-1)^3\right]$$

Next

$$\int_0^{1/3} \left(\int_{-1/2}^0 (x+2y+3z)^2 dy \right) dz = \frac{1}{6} \int_0^{1/3} \left[(x+3z)^3 - (x+3z-1)^3\right] dz$$

$$= \frac{1}{18}\left[\frac{(x+3z)^4}{4} - \frac{(x+3z-1)^4}{4} \right]_0^{1/3}$$

$$= \frac{1}{72}\left[(x+1)^4 - 2x^4 + (x-1)^4\right]$$

Thus,

$$\int_0^2 \left(\int_0^{1/3} \left(\int_{-1/2}^0 (x+2y+3z)^2 dy \right) dz \right) dx$$

$$= \frac{1}{72} \int_0^2 \left[(x+1)^4 - 2x^4 + (x-1)^4\right] dx$$

$$= \frac{1}{72}\left[\frac{(x+1)^5}{5} - \frac{2x^5}{5} + \frac{(x-1)^5}{5} \right]_0^2$$

$$= \frac{1}{2}.$$

See if you can solve this exercise now.

E 2) Integrate :

a) $f(x,y,z) = x^2 + y^2 + z^2$ over $[0,1] \times [2,4] \times [1,3]$.

b) $f(x,y,z) = \sin(x+y+z)$ over $[0,\pi] \times [0,\pi] \times [0,\pi]$.

c) $f(x,y,z) = z\, e^{x+y}$ over $[0,1] \times [0,1] \times [0,1]$.

By now the concept of integration over rectangular boxes would have become quite clear to you. In the next sub-section we shall discuss integration over other regions in space.

12.2.2 Integral Over Bounded Regions

Now we'll consider integration of functions defined over bounded sets of $\mathbf{R}^3$.

Let $f : W \to \mathbf{R}$, where W is a bounded set in $\mathbf{R}^3$. Since W is bounded, we can enclose it in a rectangular box, say B. Fig. 2 shows one such bounded set enclosed in a box.

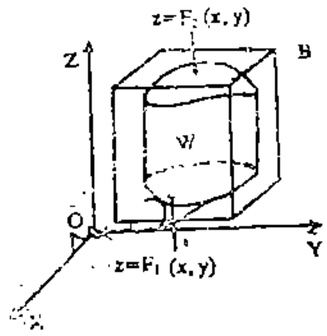


Fig. 2

Now we define a new function F on B by

$$F(x,y,z) = \begin{cases} f(x,y,z), \text{ if } (x,y,z) \in W \\ 0, \text{ if } (x,y,z) \in B \setminus W. \end{cases}$$

That is, F agrees with f on W and vanishes outside W. Now, if F is integrable on the rectangular box B, then we say that **f is integrable on the bounded set W**, and

$$\iiint_W f(x,y,z)\, dxdydz = \iiint_B F(x,y,z)\, dxdydz$$

Proceeding exactly as in Sec. 11.3, we can see that this definition also is independent of the choice of B.

Even though we have managed to define triple integration over all bounded sets, we are sure to have serious problems in actually evaluating them. But we would like to assure you, that for a special type of regions we can tackle the evaluation of triple integrals fairly easily.

For these regions it is possible to reduce the triple integrals to repeated integrals, with suitable limits of integration. We shall now describe these regions in the following definition.

Definition 1 : Suppose D is a region in $\mathbf{R}^2$ which is either of Type I or of Type II in $\mathbf{R}^2$. Let $\gamma_1(x,y)$ and $\gamma_2(x,y)$ be two continuous functions defined on D such that $\gamma_1(x,y) \le \gamma_2(x,y)$. Let

$W = \{(x,y,z) \mid (x,y) \in D \text{ and } \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$.

Then W is said to be of **Type I in $\mathbf{R}^3$**.

Note that there are two possible descriptions of regions of Type I in $\mathbf{R}^3$:

i) $W = \{(x,y,z) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x),\ \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$, ... (8)

where ϕ_1 and ϕ_2 are two continuous functions defined on a closed interval [a,b], i.e. D is of Type I in $\mathbf{R}^2$.

ii) $W = \{(x,y,z) \mid \psi_1(y) \le x \le \psi_2(y),\ c \le y \le d,\ \gamma_1(x,y) \le z \le \gamma_2(x,y)\}$, ... (9)

where ψ_1 and ψ_2 are two continuous functions defined on the closed interval [c,d], i.e., D is of Type II in $\mathbf{R}^2$.

Also see Figs. 3(a) and 3(b).

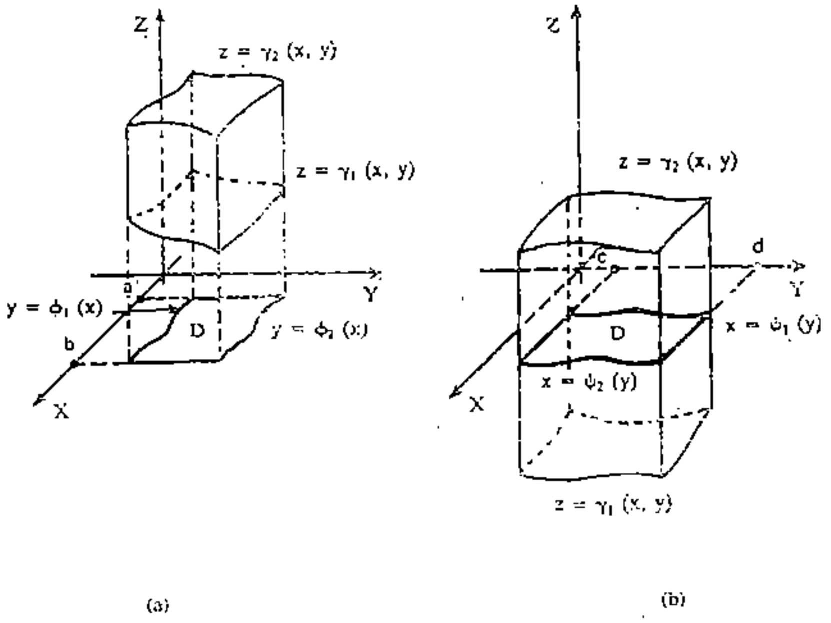


Fig. 3 : Regions of Type I in $\mathbf{R}^3$

A region W is of **Type II in $\mathbf{R}^3$** if it can be expressed in the form (8) or (9) with the roles of x and z interchanged. A region W is of **Type III** if it can be expressed in the form (8) or (9) with y and z interchanged. The following figure (Fig. 4) gives a clear picture of regions of Type I, Type II and Type III.

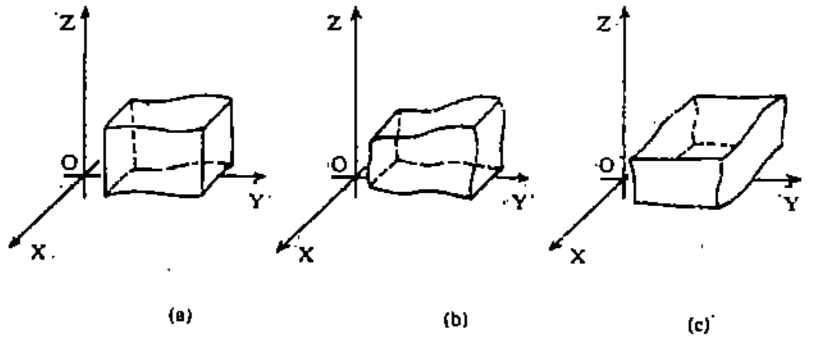


Fig. 4 : Region of (a) Type I (b) Type II (c) Type III.

You will agree that these regions in R^3 are generalisations of regions of Type I and Type II in the plane. Notice that a given region may be of two or even three types at once.

Let us see some examples of these regions.

Example 3 : Let us show that the ball $x^2 + y^2 + z^2 \leq 4$ in R^3 is a region of all three types

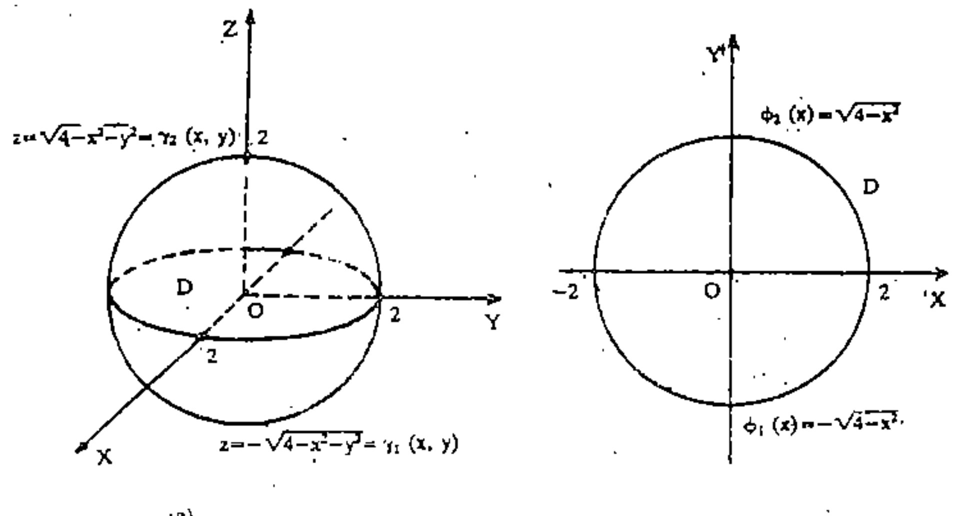


Fig. 5

Let us first describe the region as Type I. To do that we first note that z ranges from $-\sqrt{4-x^2-y^2}$ to $\sqrt{4-x^2-y^2}$. In Fig. 5 (a) you can see the top and bottom hemispheres given by $z = \sqrt{4-x^2-y^2}$ and $z = -\sqrt{4-x^2-y^2}$, respectively. So comparing with (8), we can say that

$\gamma_1(x,y) = -\sqrt{4-x^2-y^2}$ and $\gamma_2(x,y) = \sqrt{4-x^2-y^2}$.

What can we say about y? From Fig. 5(b) you will be able to say that y varies from $-\sqrt{4-x^2}$ to $\sqrt{4-x^2}$.

Thus, we can write $\phi_1(x) = -\sqrt{4-x^2}$ and $\phi_2(x) = \sqrt{4-x^2}$. We also note that x varies from -2 to 2.

Thus, we write W as

$$-2 \leq x \leq 2$$
$$-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$$
$$-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$$

We can write W as a region of Type II or Type III by interchanging the roles of x,y,z.

Example 4 : Suppose W is the region bounded by the planes $x = 0$, $y = 0$, $z = 5$ and the surface $z = x^2 + y^2$.

Let us express this region as a region of Type I.

You can see a sketch of this region in Fig. 6.

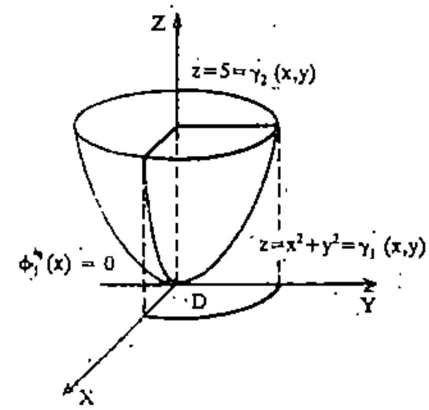


Fig. 6

Examine Fig. 6 carefully, and see if you agree with the following observations.

x varies from 0 to $\sqrt{5}$.

y varies from 0 to $\sqrt{5-x^2}$.

z varies from $x^2 + y^2$ to 5.

Thus, W is a region of Type I with

$\phi_1(x) = 0, \phi_2(x) = \sqrt{5-x^2}$

$\gamma_1(x,y) = x^2 + y^2, \gamma_2(x,y) = 5.$

We can express this region as a region of Type II also. This is what we ask you to do in E 3.

Here are some exercises for you.

E3) Express the region W in Example 4 as a region of Type II.

E4) Describe the region W given by a hemisphere $z = \sqrt{a^2-x^2-y^2}$, $z \geq 0$, as a region of Type I.

As we have said earlier, we are interested in these regions because the integrals over these regions can be written as repeated integrals. Now, if f is a continuous function defined on a region W which is of Type I, II or III, then the following theorem gives us a method for evaluating the integral of f over W.

Theorem 2 : Let $f : W \to R$ be a continuous function, where W is a region of Type I in R^3. Then

$\int\int\limits_W\int f(x,y,z)\, dxdydz$ exists and is equal to

$$\int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz \right) dy \right) dx, \text{ if W is described by (8), and is equal to}$$

$$\int_c^d \left(\int_{\phi_1(y)}^{\phi_2(y)} \left(\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz \right) dx \right) dy, \text{ if W is described by (9).}$$

Similar formulas hold if W is of Types II or III. We can obtain these by interchanging the roles of x,y,z. We leave it as an exercise to you (see E 5).

After doing the exercise, don't forget to tally your answer with the one given in Sec. 12.5.

E 5). Suppose f is a continuous function defined on a region W in R^3. Write the formula for the triple integral of f, if W is of

a) Type II
b) Type III.

We now use Theorem 2 to evaluate the triple integral in the next example. Henceforth we will consider only those functions which are continuous on the relevant regions.

Example 5 : Let us integrate the function $f(x,y,z) = y+z$ over the region W which is a hemisphere $z = \sqrt{a^2-x^2-y^2}$, $z > 0$.

If you have done E 3, then you know that the region W is of Type i. We write W as

$$-a \le x \le a$$
$$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$$
$$0 \le z \le \sqrt{a^2-x^2-y^2}$$

Thus, by Theorem 2, we write

$$\iiint\limits_W (y+z)\,dxdydz = \int_{-a}^{a}\left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\int_0^{\sqrt{a^2-x^2-y^2}}(y+z)\,dz\right)dy\right)dx$$

$$= \int_{-a}^{a}\left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(y\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2-y^2}{2}\right)dy\right)dx$$

$$= \frac{2}{3}\int_{-a}^{a}(a^2-x^2)^{3/2}\,dx$$

$$= \frac{\pi a^4}{4}.$$

We are sure you will be able to check the evaluation of the integrals in these steps.

Here is another example. It involves integration over a section of a paraboloid.

Example 6 : Suppose W is the region given in Example 4, bounded by $x = 0$, $y = 0$, $z = 5$ and the surface $z = x^2 + y^2$.

Fig. 6 shows this region. Let us compute $\iiint\limits_W x\,dx\,dy\,dz$.

In Example 4, we observed that W can be written as

$$0 \le x \le \sqrt{5}$$
$$0 \le y \le \sqrt{5-x^2}$$
$$x^2 + y^2 \le z \le 5$$

Therefore, by Theorem 2, we write

$$\iiint\limits_W x\,dxdydz = \int_0^{\sqrt{5}}\left(\int_0^{\sqrt{5-x^2}}\left(\int_{x^2+y^2}^{5} x\,dz\right)dy\right)dx$$

$$= \int_0^{\sqrt{5}}\left(\int_0^{\sqrt{5-x^2}} x(5-x^2-y^2)\,dy\right)dx$$

Now,

$$\int_0^{\sqrt{5-x^2}} x(5-x^2-y^2)dy = x(5-x^2)y - \frac{xy^3}{3}\Big]_0^{\sqrt{5-x^2}}$$

$$= x(5-x^2)^{3/2} - \frac{x}{3}(5-x^2)^{3/2}$$

$$= \frac{2x}{3}(5-x^2)^{3/2}$$

Therefore,

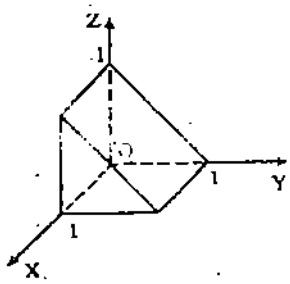
$$\iiint_W x\,dx\,dy\,dz = \frac{2}{3}\int_0^{\sqrt{5}} x\,(5-x^2)^{3/2}\,dx$$

$$= -\frac{1}{3}\left.\frac{(2-x^5)^{5/2}}{5/2}\right]_0^{\sqrt{5}}$$

$$= \frac{10}{3}\sqrt{5}.$$

Thus, you see, the most important step in integration functions over such regions is to decide the limits of integration of the variables. And to be able to do this you will have to first get a clear idea of the graph of the region.

You can try your hand at these exercises now.



E6) Integrate $f(x,y,z) = 2x+z-3$ over the cylindrical region given by $x^2 + y^2 = 1$, $0 \leq z \leq 1$.

E7) Integrate the function $f(x,y,z) = xyz$ over the region bounded by the planes, $x = 0$, $z = 0$, $x = 1$, $y = 0$ and $y + z = 1$. This region is shown alongside.

As in the case of double integrals, change of variables may prove very useful for the evaluation of some triple integrals. In the next section we discuss the change of variables formula for triple integrals.

12.3 CHANGE OF VARIABLES IN TRIPLE INTEGRALS

In this section we shall state a theorem analogous to Theorem 6 of Unit 11. This theorem gives us a formula to express the triple integral in one coordinate system as the triple integral in some other coordinate system. Then we apply this theorem to change a triple integral from Cartesian coordinates to cylindrical or spherical coordinates.

Let us begin with the statement of the following theorem. In the statement we don't expect you to bother about the definition of a regular region in $\mathbf{R}^3$. All the regions that would enter our discussion here would satisfy the requirements of the theorem.

Theorem 3 : Let W be a regular region in $\mathbf{R}^3$ and let f be a continuous real-valued function defined on W. Let $x = g_1(u,v,w)$, $y = g_2(u,v,w)$, $z = g_3(u,v,w)$ describe a transformation from the uvw–space to the xyz–space such that

i) there exists a region W′ in the uvw-space such that W′ is mapped onto W in a 1–1 manner,

ii) g_1, g_2, g_3 have continuous partial derivatives on W′, and

iii) $J = \dfrac{\partial(g_1,g_2,g_3)}{\partial(u,v,w)} \neq 0$ in W′.

Recall from Unit 9, the definition of a transformation from one space to another and the formula to compute the Jacobian J of the transformation.

Then

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W'} f(g_1(u,v,w), g_2(u,v,w), g_3(u,v,w))\,|J|\,du\,dv\,dw.$$

Here is an example to explain the utility of this theorem.

Example 7 : Let us use Theorem 3 to evaluate

$\iiint_W \sqrt{x+y+z}\,dx\,dy\,dz$, where W is described by

$0 \leq x + y + z \leq 9$, $1 \leq x + 2y \leq 4$, $2 \leq y - 3z \leq 6$.

We shall use a transformation which will convert W into a simpler region. So let us set

$u = x+y+z$, $v = x+2y$ and $w = y-3z$ ..... (10)

The Jacobian of the transformation is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -3 \end{vmatrix} = -2 \neq 0.$$ Thus, the inverse of this transformation exists (see Unit 9).

We now consider the inverse of the transformation given by (10).

Under the inverse transformation then, W is the image of a rectangular box bounded by the surfaces,

$0 \le u \le 9,\ 1 \le v \le 4,\ 2 \le w \le 6$ in the uvw-space.

Also, the Jacobian of its inverse transformation will be $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2} \neq 0$ (see Theorem 4, Unit 9). Thus, the inverse transformation satisfies all the requirements of Theorem 3.

Therefore, by Theorem 3 we can write

$$\iiint_W \sqrt{x+y+z}\ dx\,dy\,dz = \iiint_{W'} \sqrt{u}\left(\frac{1}{2}\right) du\,dv\,dw$$

$$= \frac{1}{2}\int_0^9\left[\int_1^4\left[\int_2^6 \sqrt{u}\ dw\right]dv\right]du$$

$$= 6\int_0^9 \sqrt{u}\ du$$

$$= 4u^{3/2}\Big]_0^9$$

$$= 108.$$

Now you can try your hand at these exercises.

E 9) Evaluate the following integral by making the indicated change of variables :

$$\iiint_W \frac{x+y-z}{1+(y+2z)^2}\ dx\,dy\,dz,$$ where W is described by

$0 \le x+y-z \le 2,\ 0 \le x-y+z \le 3,\ 0 \le y+2z \le 4.$

Transformation : $u = x+y-z,\ v = x-y+z,\ w = y+2z.$

E 10) Evaluate the following integral by making a suitable change of variables :

$$\iiint_W [4x^2 - 4y^2 - 4(y-z)^3]\ dx\,dy\,dz,$$

where W is described by

$-1 \le x-y \le 1,\ 1 \le x+y \le 3,\ -2 \le y-z \le 0.$

In many cases we have to evaluate triple integrals over regions which have cylindrical or spherical symmetry. In the next two sub-sections you will see how conversion to cylindrical or spherical coordinates simplifies the evaluation of triple integrals. Let us first consider the cylindrical coordinate system.

12.3.1 Triple Integrals in Cylindrical Coordinates

The cylindrical coordinates of a point P in space are (r, θ, z), where z is its distance from the xy-plane and r and θ are the polar coordinates of its projection in the xy-plane. Also see Fig. 7. That is, cylindrical coordinates consist of polar coordinates in the xy-plane together with the z-coordinate. Thus, we have $x = r\cos\theta,\ y = r\sin\theta,\ z = z$. Further, r is always taken to be non-negative, and θ varies between 0 and 2π.

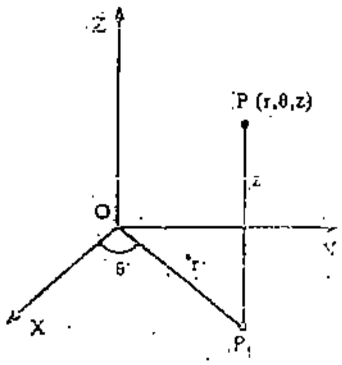


Fig. 7

Now we apply Theorem 3 to the transformation $x = r\cos\theta,\ y = r\sin\theta,\ z = z$.

Here $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r \neq 0$. Then by Theorem 3,

$$\iiint_W f(x,y,z)\ dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta,\ r\sin\theta, z)\ r\,dr\,d\theta\,dz, \qquad \text{..... (11)}$$

where f and W are as in Theorem 3 and [illegible] is the region W described in cylindrical coordinates. The integral on the right hand side of (11) is called the triple integral in cylindrical coordinates.

Note that if W is a region described in cylindrical coordinates, then triple integral over W is $\int\int\int_W f(r,\theta,z)\, r\, dr\, d\theta\, dz$, and not $\int\int\int_W f(r,\theta,z)\, dr\, d\theta\, dz$.

Suppose now that the region W^* in Equation (11) is of Type I or Type II or Type III. (See Definition 1 and replace x,y,z by r,θ,z.) Let us say W^* is of Type I, described by

$$\gamma_1(\theta,z) \le r \le \gamma_2(\theta,z)$$
$$\phi_1(z) \le \theta \le \phi_2(z) \text{ and}$$
$$z_1 \le z \le z_2.$$

Then we can write

$$\int\int\int_W f(x,y,z)\, dx\, dy\, dz = \int\int\int_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

$$= \int_{z_1}^{z_2} \left(\int_{\phi_1(z)}^{\phi_2(z)} \left(\int_{\gamma_1(\theta,z)}^{\gamma_2(\theta,z)} f(r\cos\theta, r\sin\theta, z)\, r\, dr \right) d\theta \right) dz \quad \ldots\ldots (12)$$

If W^* is of Type II or Type III, correspondingly we get other repeated integrals. We shall express the region in such a form that we can easily find the limits of integration.

Now we are giving some examples which will help you understand the evaluation of triple integrals in cylindrical coordinates.

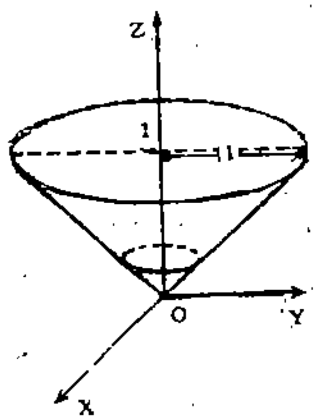


Fig. 8

Example 8 : Consider the function $f(r,\theta,z) = zr^2\cos^2\theta$ defined on the region W described by

$0 \le \theta \le 2\pi, 0 \le r \le z, 0 \le z \le 1$.

In Fig. 8 you can see this region. Let us evaluate the triple integral of f over W.

$$\int\int\int_W f(r,\theta,z)\, f\, dr\, d\theta\, dz = \int_0^1 \left[\int_0^{2\pi} \left(\int_0^z [illegible] \cos^2\theta\, r\, dr \right) d\theta \right] dz$$

$$= \int_0^1 \left[\int_0^{2\pi} \frac{[illegible]}{4}\, d\theta \right] dz$$

$$= \int_0^1 [illegible]$$

$$= \frac{\pi}{20}$$

$$\int_0^{2\pi} \cos^2\theta d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \pi$$

Example 9 : Let us evaluate [illegible] the function $f(r,\theta,z) = 1$ over the region W, described by $0 \le \theta \le 2\pi$, [illegible] $-\sqrt{[illegible]} \le z \le \sqrt{a^2 - r^2}$, using cylindrical coordinates.

$$\int\int\int_W 1 r\, dr\, d\theta\, dz = \int_0^{2\pi} \left[\int_0^a \left(\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} [illegible] \right) dr \right] d\theta$$

$$= \int_0^{2\pi} \left[\int_0^a 2r \sqrt{[illegible]}\, dr \right] d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} a^3\, d\theta$$

$$= \frac{4\pi a^3}{3}$$

$$\int_0^a 2r\sqrt{a^2 - r^2}\, dr$$

$$= \left[\frac{-2(a^2 - r^2)^{3/2}}{3}\right]_0^a$$

$$= \frac{2a^3}{3}$$

Now we give an example of a function which is defined over a region described in terms of Cartesian coordinates. In this case you will find that if you describe the region in cylindrical coordinates and then apply the formula of triple integration in the cylindrical coordinate system, then the evaluation becomes very simple.

Example 10 : Let us integrate the function $f(x,y,z) = (x^2+y^2)z^2$ over the cylindrical region W given by $x^2 + y^2 \leq 1, -1 \leq z \leq 1$. If we write $x = r\cos\theta, y = r\sin\theta, z = z$, then

$f(x,y,z) = (x^2+y^2)z^2 = (r^2\cos^2\theta+r^2\sin^2\theta)z^2 = r^2z^2 = f^*(r,\theta,z)$.

Now, $\int\int\limits_W\int f(x,y,z)\,dx\,dy\,dz = \int\int\limits_{W^*}\int f^*(r,\theta,z)\,r\,dr\,d\theta\,dz.$

You know that we can describe the unit ball $x^2 + y^2 \leq 1$ in polar coordinates (r,θ) by $0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$.

Thus, we can describe W in r, θ, z as

$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, -1 \leq z \leq 1.$

Hence,

$$\int\int\limits_{W^*}\int f^*(r,\theta,z)\,r\,dr\,d\theta\,dz = \int_{-1}^{1}\left[\int_0^{2\pi}\left\{\int_0^1 (z^2r^2)\,r\,dr\right\}d\theta\right]dz$$

$$= \int_{-1}^{1}\left[\int_0^{2\pi}\frac{z^2}{4}\,d\theta\right]dz$$

$$= \int_{-1}^{1}\frac{\pi z^2}{2}\,dz$$

$$= \frac{\pi}{3}.$$

If we had not changed over to cylindrical coordinates, we would have had to evaluate

$$\int_{-1}^{1}\left[\int_{-1}^{1}\left\{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)z^2\,dy\right\}dx\right]dz.$$

Try to carry out the integration. You will soon realise that it is not a very easy task! In this example the conversion from Cartesian to cylindrical coordinates was useful since the region was a cylinder! If you have understood Examples 8, 9 and 10, you should be able to solve these exercises.

E 10) Integrate the following functions over the indicated regions :

a) $f(r,\theta,z) = \cos\theta$, W: $0 \leq \theta \leq \pi/2, 0 \leq r \leq 1+\sin\theta, r \leq z \leq 2$.

b) $f(r,\theta,z) = r^2$, W is the region bounded by $z = r^2, 0 \leq \theta \leq \frac{\pi}{2}$ and $z = 1$.

E 11) Evaluate the integral of $f(x,y,z) = z+3x-2$ over the cylinder bounded by $x^2+y^2=1, z = 0$ and $z = 1$ by changing to cylindrical coordinates.

In the next section we will look at the evaluation of triple integrals in spherical coordinates.

12.3.2 Triple Integrals in Spherical Coordinates

Until now in this unit we have considered the evaluation of triple integrals in the Cartesian and the cylindrical coordinate systems. There is yet another very useful way of determining the position of a point in space – the spherical coordinate system. Consider a point P in space. Let P_1 be its projection in the xy-plane. See Fig. 9. Then P can be uniquely determined by r, θ and φ, where r is the distance |OP| from the origin to P, θ is the polar angle associated with the projection P_1 of P on the xy-plane and φ is the angle between the positive z-axis and the line segment OP. Further, note that $r \geq 0, 0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.

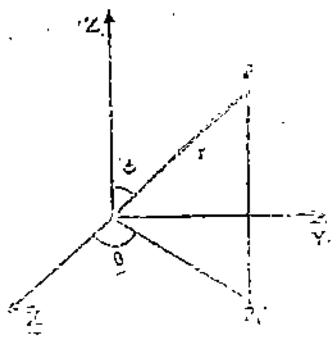


Fig. 9

These coordinates are called spherical coordinates because r = constant represents a sphere.

The Cartesian and the spherical coordinates of a point P are related by

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

Now we apply Theorem 3 to the transformation $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$. Recall that you have seen this transformation in Unit 9, Example 2. There we calculated the Jacobian of the transformation as

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\phi.$$

Thus, we have

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^{**}} f^*(r,\theta,\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi, \qquad \ldots (13)$$

where f, W are as in Theorem 3, $f^*(r,\theta,\phi) = f(r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$, and W^{**} is the region corresponding to W in the (r,θ,ϕ) – space. The integral on the right hand side of (13) is called a **triple integral in spherical coordinates.**

Further, suppose that W is of Type I, II or III, defined in Sub-sec. 12.2.2 with (x,y,z) replaced by (r,θ,ϕ), say, W is described as

$$\alpha \le \theta \le \beta,$$
$$h_1(\theta) \le \phi \le h_2(\theta)$$
$$\psi_1(\theta,\phi) \le r \le \psi_2(\theta,\phi),$$

and f is continuous on W. Then we evaluate $\iiint_W f\,dx\,dy\,dz$ by

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{\gamma_1(\theta,\phi)}^{\gamma_2(\theta,\phi)} f(r,\theta,\phi)\,r^2\sin\phi\,dr\,d\phi\,d\theta.$$

Depending on the description of W, we may use alternative forms of repeated integrals.

Before we give a few examples, we would like to remind you of one thing. As in the case of cylindrical coordinates, if W is a region in spherical coordinates, then triple integral in spherical coordinates is

$$\iiint_W f(r,\theta,\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi, \text{ and not } \iiint_W f(r,\theta,\phi)\,dr\,d\theta\,d\phi.$$

We now give a few examples of triple integration in spherical coordinates.

Example 11 : Suppose W is the region described by

$0 \le r \le 1, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi$

Let us integrate the function

$f(r,\theta,\phi) = e^{r^3}$

over W.

$$\iiint_W f(r,\theta,\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\pi}\int_0^1 e^{r^3} r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} \sin\phi\left\{\frac{1}{3}\int_0^1 e^{r^3}3r^2\,dr\right\}d\phi\right]d\theta$$

$$= \frac{(e-1)}{3}\int_0^{2\pi}\left\{\int_0^{\pi}\sin\phi\,d\phi\right\}d\theta$$

$$= \frac{(e-1)}{3}\int_0^{2\pi} 2\,d\theta$$

$$= \frac{2}{3}(e-1).\,2\pi$$

$$= \frac{4\pi}{3}(e-1).$$

Note that in the above example W is the unit ball $x^2+y^2+z^2<1$ expressed in (r,θ,ϕ) – coordinates.

In the next example you will find a situation where conversion from Cartesian to spherical coordinates makes things easy.

Example 11: Consider the function

$f(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}$. Let us integrate it over the solid region W bounded by the two spheres :

$x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 + z^2 = b^2$, $a > b > 0$.

Converting to spherical coordinates, we get

$f(x,y,z) = \dfrac{1}{r^3} = f^*(r,\theta,\phi)$.

We can describe the region W as :

$b < r < a,\ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \pi$.

Thus,

$$\iiint\limits_W f(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi}\left[\int_0^{\pi}\left\{\int_b^a \frac{r^2\sin\phi}{r^3}\,dr\right\}d\phi\right]d\theta.$$

$$= \int_0^{2\pi}\left[\int_0^{\pi} \sin\phi \left\{\int_b^a \frac{1}{r}\,dr\right\}d\phi\right]d\theta.$$

You can check that this repeated integral is equal to

$4\pi \ln \frac{a}{b}$.

You can do the following exercises now.

E 12) Integrate $f(r,\theta,\phi) = r^2\cos^2\phi$ over the quarter sphere given by $r \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq \pi/2$.

E 13) Use spherical coordinates to evaluate $\iiint\limits_W \sin(x^2+y^2+z^2)^{3/2}\,dx\,dy\,dz$, where W is the region bounded by the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 9$.

With this we come to the end of this unit. Let us now summarize the points covered in it.

2.6 SUMMARY

In this unit we have

1) discussed triple integrals over a rectangular box in R^3.
2) seen the evaluation of triple integrals over a box using repeated integrals. Thus, if f is a continuous function defined over a box $B = [a,b] \times [c,d] \times [s,t]$, then

$$\iiint\limits_B f(x,y,z)\,dx\,dy\,dz = \int_s^t\left(\int_c^d\left(\int_a^b f(x,y,z)\,dx\right)dy\right)dz.$$

3) defined triple integrals over bounded sets in R^3.
4) seen that the triple integrals over special types of bounded regions in R^3 can be evaluated using repeated integrals.
5) stated the change of variables formula for triple integrals.
6) explained the evaluation of triple integrals in cylindrical coordinates using change of variables formula. Thus,

$$\iiint\limits_W f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{W^*} f^*(r,\theta,z)\,r\,dr\,d\theta\,dz,$$

where

$f^*(r,\theta,z) = f(r\cos\theta, r\sin\theta, z)$ and W^* is the region corresponding to W in the (r,θ,z)-space.

7) explained the evaluation of triple integrals in spherical coordinates. Thus,

$$\iiint\limits_W f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{W^{**}} f^{**}(r,\theta,\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi,$$

where W^{**} is the region W described in spherical coordinates and
$f^*(r,\theta,\phi) = f(r\cos\theta\sin\phi,\ r\sin\theta\sin\phi,\ r\cos\phi)$.

12.5 SOLUTIONS AND ANSWERS

E1) Given $\varepsilon > 0$, find a partition P of B such that the volume of the sub-box B*
$0 \le z \le 5$
For any other sub-box in P, f is a constant function. Therefore, the infimum of f is the same as the supremum of f, which is equal to 1.

Thus,

$U(P,f) - L(P,f) = 1.\text{ volume of } B^* - 0 \text{ volume of } B^* < \varepsilon.$

$\therefore$ f is integrable on B.

E2) a) f is a polynomial function.

$\therefore$ The required integral $= \int_0^1 \left[\int_2^4 \left\{ \int_1^3 (x^2+y^2+z^2)\,dz \right\} dy \right] dx$

$$= \int_0^1 \left[\int_2^4 \left\{ (x^2+y^2)z + \frac{z^3}{3} \right\}_1^3 dy \right] dx$$

$$= \int_0^1 \left[\int_2^4 \left\{ 2x^2+2y^2+\frac{26}{3} \right\} dy \right] dx$$

$$= \int_0^1 \left[2x^2y + \frac{2y^3}{3} + \frac{26y}{3} \right]_2^4 dx$$

$$= \int_0^1 \left(4x^2 + \frac{164}{3} \right) dx$$

$$= \frac{4x^3}{3} + \frac{164x}{3} \Big]_0^1$$

$$= 56$$

b) $\int_0^{\pi} \left[\int_0^{\pi} \left\{ \int_0^{\pi} \sin(x+y+z)\,dz \right\} dy \right] dx = -8.$

c) $\frac{(e-1)^2}{2}$

E3) $W: \{(x,y,z) \mid 0 \le z \le 5,\ 0 \le y \le \sqrt{z},\ 0 \le x \le \sqrt{z-y^2}\}$

So, W is a region of Type II with

$0 \le z \le 5$

$\phi_1(z) = 0 \le y \le \phi_2(z) = \sqrt{z}$

$\gamma_1(z,y) = 0 \le x \le \gamma_2(z,y) = \sqrt{z-y^2}$

E4) W can be expressed by

$-a \le x \le a$

$-\sqrt{a^2-x^2} \le y \le \sqrt{a^2-x^2}$

$0 \le z \le \sqrt{a^2-x^2-y^2}$

E5) a) If W is of Type II, then W can be described as

$a \le z \le b$

$\phi_1(z) \le y \le \phi_2(z)$

$\gamma_1(z,y) \le x \le \gamma_2(z,y)$ where

ϕ_1, ϕ_2, γ_1 and γ_2 are continuous functions.

$$\therefore \iiint_W f\,dx\,dy\,dz = \int_a^b \left[\int_{\phi_1(z)}^{\phi_2(z)} \left\{ \int_{\gamma_1(z,y)}^{\gamma_2(z,y)} f(x,y,z)\,dx \right\} dy \right] dz.$$

b) If W is of Type III, then W can be described as :

$a \le x \le b$

$\phi_1(x) \le z \le \phi_2(x)$

$\psi_1(x,z) \le y \le \psi_2(x,z)$,

where $\phi_1, \phi_2, \psi_1, \psi_2$ are continuous.

$$\therefore \iiint_W f\,dv = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} \left\{ \int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x,y,z)\,dx \right\} dy \right) dz.$$

E6) $W : x^2+y^2 = 1, 0 \le z \le 1$

$$\iiint_W f\,dv = \int_0^1 \left[\int_0^1 \left\{ \int_0^{\sqrt{1-y^2}} (2x+z-3)\,dx \right\} dy \right] dz.$$

$$= \int_0^1 \left[\int_0^1 \left[x^2+zx-3x\right]_0^{\sqrt{1-y^2}} dy \right] dz.$$

$$= \int_0^1 \left[\int_0^1 \left\{ 1-y^2 + (z-3)\sqrt{1-y^2} \right\} dy \right] dz$$

$$= \int_0^1 \left[y - \frac{y^3}{3} + (z-3)\left\{ \frac{y}{2}\sqrt{1-y^2} + \frac{1}{2}\sin^{-1} y \right\} \right]_0^1 dz$$

$$= \int_0^1 \left[\frac{2}{3} + (z-3).\ \frac{\pi}{4} \right] dz$$

$$= \left[\frac{2}{3} z + \left(\frac{z^2}{2} - 3z \right) \frac{\pi}{4} \right]_0^1$$

$$= \frac{2}{3} - \frac{5\pi}{8}$$

E7) $\int_0^1 \left[\int_0^1 \left\{ \int_0^{1-y} xyz\,dz \right\} dy \right] dx$

$$= \frac{1}{48}.$$

E8) $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -6$

$$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{1}{6}.$$

By applying Theorem 3, we get

$$\iiint_W \frac{x+y-z}{1+(y+2z)^2}\,dx\,dy\,dz$$

$$= \iiint_{W'} \frac{u}{1+w^2} \cdot \frac{1}{6}\,du\,dv\,dw,$$

where W' is bounded by

$0 \le u \le 2,\ 0 \le v \le 3,\ 0 \le w \le 4.$

$$= \frac{1}{6}\int_0^2 \left[\int_0^3 \left\{\int_0^4 \frac{u}{1+w^2}\,dw\right\} dv\right] du$$

$$= \frac{1}{6}\int_0^2 \left[u\int_0^3 \left\{\int_0^4 \frac{1}{1+w^2}\,dw\right\} dv\right] du$$

$$= \frac{1}{6}\int_0^2 \left[u\int_0^3 \left[\tan^{-1}w\right]_0^4 dv\right] du$$

$$= \frac{\tan^{-1}4}{6}\int_0^2 3u\,du$$

$$= \tan^{-1}4.$$

E9) Let $u = x-y,\ v = x+y,\ w = y-z.$

Then $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = -2.$

$$\therefore |J| = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{2}.$$

$$\therefore \iiint_W [4x^2 - 4y^2 - 4(y-z)^3]\,dx\,dy\,dz$$

$$= \iiint_W [4(x+y)(x-y) - 4(y-z)^3]\,dx\,dy\,dz$$

$$= 4\iiint_{W'} (uv - w^3)\,\frac{1}{2}\,du\,dv\,dw, \text{ where}$$

W' is given by $-1 \le u \le 1,\ 1 \le v \le 3,\ -2 \le w \le 0.$

$$= 2\int_{-1}^1 \left[\int_1^3 \left\{\int_{-2}^0 (uv - w^3)\,dw\right\} dv\right] du$$

$$= -32.$$

E10) a) $\displaystyle\iiint_W f(r,\theta,z)\, r\,dr\,d\theta\,dz$

$$= \int_0^{\pi/2}\left[\int_0^{1+\sin\theta}\left\{\int_r^2 r\cos\theta\,dz\right\} dr\right] d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^{1+\sin\theta} r\cos\theta\ [\text{illegible}]\right] d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[r^2 - \frac{r^3}{3}\right]_0^{[\text{illegible}]} d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[(1+\sin\theta)^2 - \frac{1}{3}(1+\sin\theta)^3\right] d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left(\frac{2}{3} - [\text{illegible}] + \sin\theta\right) d\theta$$

$$= \left[\frac{2\sin\theta}{3} - \frac{\sin^{[\text{illegible}]}\theta}{[\text{illegible}]}\ [\text{illegible}]\ \frac{[\text{illegible}]}{4}\right]_0^{\pi/2} = \frac{15}{12}$$

b) Let W be the region.

$$\iiint_W f\,dv = \int_0^1\left[\int_0^{\pi/2}\left\{\int_{r^2}^{1} r^2 . r dz\right\}d\theta\right]dr$$

$$= \frac{\pi}{2}\int_0^1 (r^3 - r^5)\,dr$$

$$= \frac{\pi}{24}.$$

E11) Let W be the cylinder.

$$\iiint_W f\,dv = \int_0^1\left[\int_0^1\left\{\int_0^{2\pi}(z+3r\sin\theta-2)r\,d\theta\right\}dz\right]dr$$

$$= \int_0^1\left[\int_0^1\left\{(z-2)\,r\theta - 3r^2\cos\theta\Big|_0^{2\pi} dz\right]dr$$

$$\int_0^1\left\{2\pi r\int_0^1(z-2)\,dz\right\}dr$$

$$= 2\pi\int_0^1 r.\left[\frac{z^2}{2}-2z\right]_0^1 dr$$

$$= -3\pi\int_0^1 r\,dr$$

$$= -\frac{3\pi}{2}$$

E12) $$\int_0^1\left[\int_0^{\pi}\left\{\int_0^{\pi/2} f(r,\theta,\phi)\,r^2\sin\theta\,d\phi\right\}d\theta\right]dr$$

$$\int_0^1\left[\int_0^{\pi}\left\{\int_0^{\pi/2} r^4\cos^2\phi\sin\phi\,d\phi\right\}d\theta\right]dr$$

$$= \frac{\pi}{15}.$$

E13) $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$.

$f(x,y,z) = \sin(x^2+y^2+z^2)^{3/2} = \sin r^3 = f^*(r,\theta,\phi)$

The region W is described as :

$1 \le r \le 3$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

$$\therefore \text{The given integral} = \int_1^3\left[\int_0^{2\pi}\left\{\int_0^{\pi} \sin r^3 r^2\sin\phi\,d\phi\right\}d\theta\right]dr$$

$$= \frac{4\pi}{3}[\cos 1 - \cos 27].$$

UNIT 13 APPLICATIONS OF INTEGRALS

Structure

13.1 INTRODUCTION

You have seen in Calculus (Units 15 and 16) that the definite integral of a function of a single variable has got many applications. It enables us to calculate the area under a curve, the length of an arc of a curve, the volume of a cone and other solids of revolution. In this unit we shall discuss some applications of double and triple integrals. In Sec. 13.2 you will see that double integrals can be used to calculate the area of a region in the xy-plane and the volume of a region in space lying above a closed bounded region in the plane. Apart from these, double integral has got some physical applications also. For instance, it is useful in finding the mass of a plate with variable density, or the moment of inertia. We shall also discuss some applications of triple integrals. You will see that most of the applications of double integral can be carried over directly from double to triple integrals.

Objectives

After reading this unit you should be able to

- find the area of a planar region, the volume of a solid region lying under a surface and the surface area of a given surface by using double integrals,
- calculate the mass, moments, centre of gravity and moment of inertia of planar regions,
- compute the moments, centre of gravity and moments of inertia of solid regions,
- find the volume and mass of a solid region in space using triple integrals.

13.2 APPLICATIONS OF DOUBLE INTEGRALS

This section deals with some applications of double integrals. Let us start with a simple application.

13.2.1 Area of a Planar Region and Volume of a Solid

In this sub-section we will see how double integrals are used to find the area of a planar region and the volume of a solid.

In Unit 11 we have seen that if a function f(x,y) is non-negative on a rectangle D, then the double integral of f over D represents the volume of a three-dimensional solid with D as the base and bounded by the surface z = f(x,y) (See Fig. 5(a) Unit 11). Now we shall show that in the same way, the double integral of a non-negative function over a region of Type I or Type II also can be viewed as the volume of a certain three-dimensional region.

Suppose D is a region described by

$$D = \{ (x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x) \}.$$

Let f be a real-valued function defined on D such that $f(x,y) \ge 0$ on D. Let D be enclosed in the rectangle $T = [a,b] \times [c,d]$. Then we set :

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in T \setminus D. \end{cases}$$

Suppose F is integrable on T, or equivalently, suppose f is integrable on D. We now partition T into pq sub-rectangles with the help of a partition,

$P = P_1 \times P_2$, where

$P_1 = \{ a = x_0, x_1, \ldots\ldots, x_p = b \}$ and

$P_2 = \{ c = y_0, y_1, \ldots\ldots, y_q = d \}$.

Let $\Delta x_i = x_i - x_{i-1}$, $1 \le i \le p$ and $\Delta y_j = y_j - y_{j-1}$, $1 \le j \le q$.

Then the norm of P is $\|P\| = \max\limits_{\substack{1\le i\le p\\ 1\le j\le q}} (\Delta x_i, \Delta y_j)$.

Let P_{ij} be any point in the sub-rectangle T_{ij}. Then by Remark 1 in Unit 11,

$$\lim_{\|P\|\to 0} \sum_{i=1}^{p} \sum_{j=1}^{q} F(P_{ij})\, \Delta x_i\, \Delta y_j = \iint_T F(x,y)\, dx\, dy$$

This means,

$$\lim_{\|P\|\to 0} \sum_i \sum_j f(P_{ij})\, \Delta x_i\, \Delta y_j = \iint f(x,y)\, dx\, dy.$$

On the other hand, consider the three-dimensional region L shown in Fig. 1, which lies beneath the graph of $z = f(x,y)$, and above the region D.

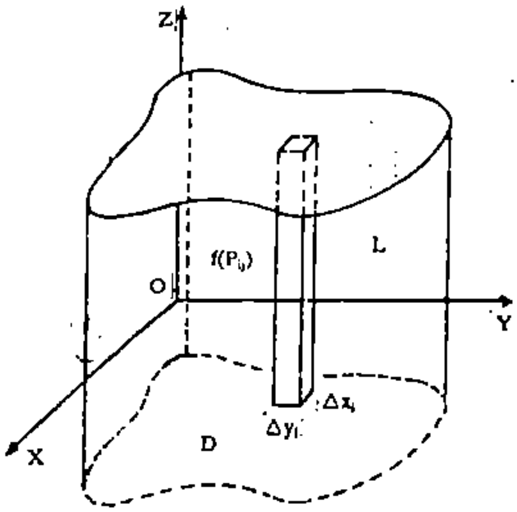


Fig. 1

We construct small rectangular boxes whose bases are the pq rectangles T_{ij} with base area equal to $\Delta x_i\, \Delta y_j$, and whose heights are $f(P_{ij})$. The sum of the volumes of these boxes is equal to

$$V(P) = \sum_i \sum_j f(P_{ij})\, \Delta x_i\, \Delta y_j.$$

It is reasonable to expect that V(P) will get closer and closer to the volume of L as P becomes finer and finer. Indeed, it is true that

$$\lim_{\|P\|\to 0} \sum_i \sum_j f(P_{ij})\, \Delta x_i\, \Delta y_j = \text{Volume of L.}$$

Thus, if a real-valued function $f(x,y) \ge 0$ on a closed bounded region D and is integrable on D, then the volume of the solid region under the graph of the surface $z = f(x,y)$ over D is equal to the double integral of $f(x,y)$ over D.

When $f(x,y) = 1$ on D, then $\iint_D dxdy$ represents the area of the region D (when it exists). This does not mean that the area of a region D is equal to the volume of the solid of height 1 and base D. Actually both the quantities have the same numerical value, but their units are different.

Thus, we can make use of double integrals to evaluate areas of some regions and to evaluate volumes of certain solids. We illustrate this discussion with a few examples.

Example 1 : Let us find the area of the region bounded by the curve $y = x^2$ and the line $y = x+2$ shown in Fig. 2.

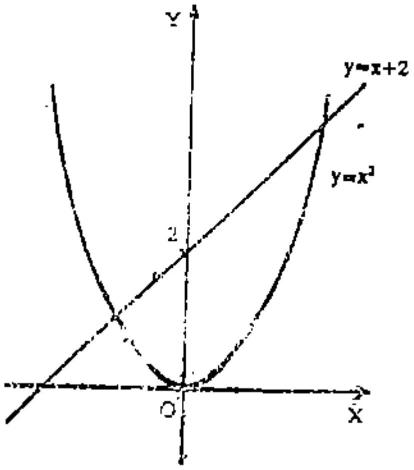


Fig. 2

From the figure we can see that x varies between −2 and 1, and y varies between x^2 and x+2. Remember, to get the range of x, we solve the equations $y = x^2$ and $y = x+2$ simultaneously for x. Thus,

$$D = \{ (x, y) \mid -2 \leq x \leq 1,\ x^2 \leq y \leq x+2 \}$$

Then

$$\text{the area of } D = \iint_D dydx$$

$$= \int_{-2}^{1} \left(\int_{x^2}^{x+2} dy \right) dx$$

$$= \int_{-2}^{1} \left[y \right]_{x^2}^{x+2} dx$$

$$= \int_{-2}^{1} (x+2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-2}^{1}$$

$$= \frac{3}{2}.$$

If the region D is described by polar co-ordinates, we calculate the area of the region by the formula,

$$\text{Area of } D = \iint_D r\, dr\, d\theta$$

Let us see an example where the region D is described by polar co-ordinates.

Example 2 : Let us find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

We know that in the first quadrant, θ lies between 0 and $\frac{\pi}{2}$ and r lies between 0 and $1 + \sin\theta$. Therefore,

$$D = \{ (r, \theta) \mid 0 \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 1+\sin\theta \}$$

In Fig. 3 you can see a sketch of D.

Now

$$\text{the area of D} = \int_0^{\pi/2}\int_0^{1+\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{1+\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (1 + \sin\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(\frac{3}{2} + 2\sin\theta - \frac{\cos 2\theta}{2}\right) d\theta$$

$$= 1 + \frac{3\pi}{8}$$

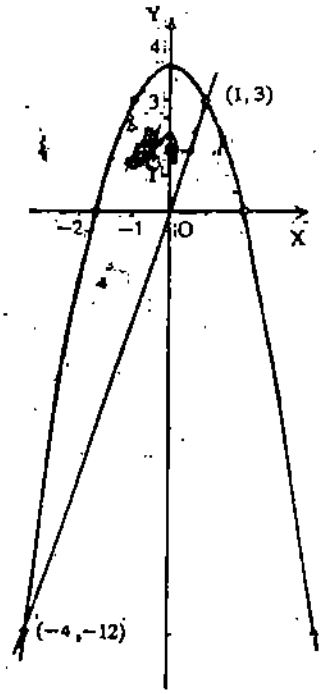


Fig. 4

In the next two examples we compute the volume of a region in space using double integral.

Example 3 : Suppose we want to find the volume of a solid whose base is the region in the xy-plane, bounded by the parabola $y = 4-x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x+4$.

Let D be the given region. The line $y = 3x$ and the parabola $y = 4-x^2$ intersect in the points (1, 3) and (−4, −12). This shows that on D, we have $-4 \le x \le 1$ and $3x \le y \le 4-x^2$. Also see Fig. 4. Thus,

$$D = \{ (x, y) \mid -4 \le x \le 1,\ 3x \le y \le 4-x^2\}.$$

Also, let $f(x,y) = x+4$. Then $f(x,y) \ge 0$ for all (x, y) in D. Therefore we have

$$\text{volume of the solid} = \iint_D (x + 4)\,dy\,dx$$

$$= \int_{-4}^{1}\int_{3x}^{4-x^2} (x+4)\,dy\,dx$$

$$= \int_{-4}^{1} (x + 4)\,y\Big]_{3x}^{4-x^2} dx$$

$$= \int_{-4}^{1} (x + 4)(4 - x^2 - 3x)\,dx$$

$$= 52\frac{1}{12}$$

Example 4 : Let us compute the volume within the cylinder $x^2 + y^2 = 9$ between the planes $y + z = 4$ and $z = 0$.

Here the region D is bounded by the circle $x^2 + y^2 = 9$. Thus, we can write D in polar coordinates as

$$D = \{ (r, \theta) \mid 0\le r\le 3,\ 0\le\theta\le 2\pi\}.$$ See Fig. 5.

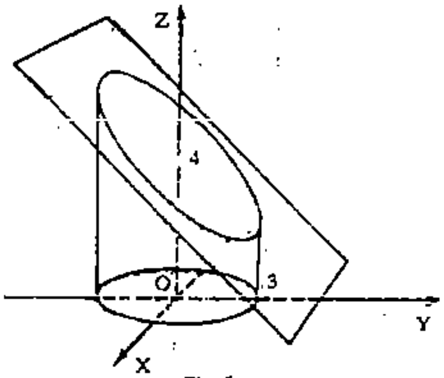


Fig. 5

The required volume lies under the plane $y + z = 4$. Now $y+z=4 \Rightarrow z=4-y$. So let $f(x,y) = 4 - y$. Then $f^*(r, \theta) = 4 - r\sin\theta$.

Therefore, the volume

$$V = \int_0^3 \int_0^{2\pi} (4-r\sin\theta)\, r\, d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} 4r\, d\theta\, dr - \int_0^3 \int_0^{2\pi} r^2\sin\theta\, d\theta\, dr$$

$$= 4\int_0^3 \int_0^{2\pi} r\, d\theta\, dr - 0$$

$$= 36\pi.$$

Why don't you try some exercises now?

E 1) Find the area of the region bounded by

a) the y-axis and the lines $y = 4$, $y = 2x$

b) the x-axis, the curve $y = e^x$, and the lines $x = 0$, $x = 1$.

E 2) Sketch the regions given in the following integrals and compute their area

a) $\int_0^1 \int_y^{\sqrt{y}} dxdy$

b) $\int_0^3 \int_{-x}^{x(2-x)} dxdy$

E 3) Find the area cut off from the first quadrant by the curve $r = (2 - \sin 2\theta)^{1/2}$

E 4) Find the volume of the solid whose base is a triangle in the xy-plane bounded by the x-axis, the line $y=x$, and the line $x=1$, while the top of the solid is in the plane,

$z = f(x,y) = 3-x-y$.

E 5) Find the volume of the solid whose base is in the xy-plane and is bounded by the circle $x^2+y^2 = a^2$, while the top of the solid is bounded by the paraboloid $az = x^2+y^2$ (**Hint**: Use polar co-ordinates).

In the next sub-section you will see another important application of double integrals.

13.2.2 Surface Area

In Calculus (Unit 16) you have seen that we can use definite integrals to determine the area of surfaces of revolution. In this sub-section you will see that we can find the area of general curved surfaces using double integrals.

Here we shall consider those curved surfaces, which are given by a graph $z = f(x,y)$, where $f(x,y)$ is a function of two variables defined over a region D of Type I or Type II, and f has continuous partial derivatives w.r.t. x and y at each point of D. In Fig. 6 we have shown one such surface.

The area S of such a surface is given by the formula

$$S = \int_D \int \sqrt{1+f_x^2(x,y) + f_y^2(x,y)}\, dxdy$$

$$= \int_D \int \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy. \qquad \ldots\ldots\ldots(1)$$

Here we are not going to give a rigorous proof of this formula, but we give a brief discussion, which suggests that this formula is plausible.

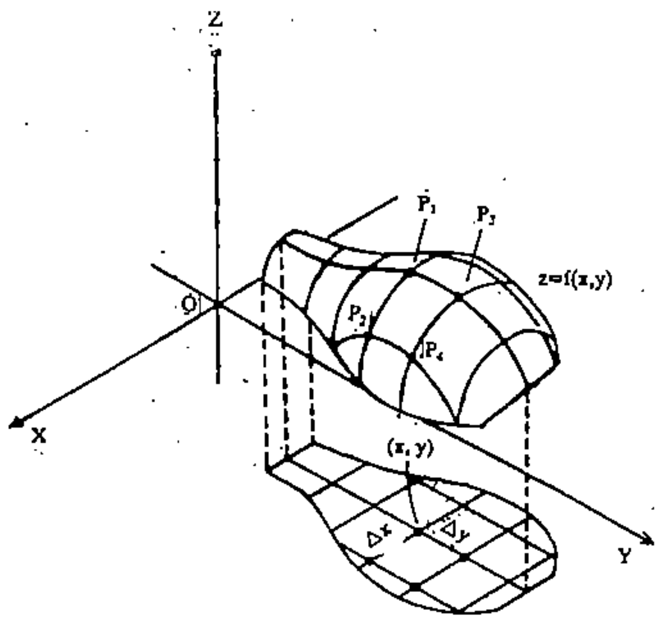


Fig. 6

As we did in the last sub-section, we partition D into small rectangles which are of the form $[x, x+\Delta x] \times [y, y+\Delta y]$ (see Fig. 6). When Δx and Δy are small, this rectangle will be a projection of a figure, which is approximately a parallelogram with vertices,

$$P_1 = (x, y, f(x,y)),$$
$$P_2 = (x+\Delta x, y, f(x+\Delta x, y)),$$
$$P_3 = (x, y+\Delta y, f(x,y+\Delta y)),$$
$$P_4 = (x+\Delta x, y+\Delta y, f(x+\Delta x, y+\Delta y)).$$

Now the area ΔA of the parallelogram $P_1P_2P_3P_4$ = 2 area of $\triangle P_1P_2P_3$.

Since

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

and

$$f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x,y+\Delta y) - f(x,y)}{\Delta y},$$

we write $f(x+\Delta x, y) \simeq f_x(x,y)\Delta x + f(x,y)$

and $f(x,y+\Delta y) \simeq f_y(x,y)\Delta y + f(x,y)$.

Therefore, ΔA is approximately equal to

2 area of $\triangle P_1P_2^*P_3^*$ with $P_1 = (x,y\ f(x,y))$

$$P_2^* = (x+\Delta x, y, f_x(x,y)\Delta x + f(x,y))$$
$$P_3^* = (x, y+\Delta y, f_y(x,y)\Delta y + f(x,y))$$

Now to calculate the area of $\triangle P_1P_2^*P_3^*$, we make use of a simple technique in analytical solid geometry. Let P_x denote the projection of the $\triangle P_1P_2^*P_3^*$ on the xy-plane. This means that the vertices of P_x are $(x,y,0)$, $(x+\Delta x, y, 0)$ and $(x,y+\Delta y, 0)$. Similarly, let P_y and P_z denote the projections of the $\triangle P_1P_2^*P_3^*$ on the yz-plane and the zx-plane, respectively. Also, we denote by A_x, A_y, A_z the areas of P_x, P_y and P_z, respectively. Then a result in analytical solid geometry says that

$$\text{Area of } \triangle P_1P_2^*P_3^* = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Note that P_x, P_y and P_z are triangles in the co-ordinate plane, whose vertices are known. Therefore, we know how to calculate the areas of these triangles. Let us calculate the areas A_x, A_y and A_z one by one.

We first consider P_x. The triangle P_x has vertices (x,y), $(x+\Delta x, y)$ and $(x, y+\Delta y)$. Therefore,

$$\text{Area of } P_x = A_x = \frac{1}{2}\begin{vmatrix} x & y & 1 \\ x+\Delta x & y & 1 \\ x & y+\Delta y & 1 \end{vmatrix}$$

$$= x(-\Delta y) - y(\Delta x) + (x+\Delta x)(y+\Delta y) - xy$$

$$= \frac{1}{2}\Delta x \Delta y.$$

Now to calculate the area of P_y, note that P_y has vertices $(x, f(x,y))$, $((x+\Delta x), f_x(x,y)\Delta x + f(x,y))$ and $(x, f_y(x,y)\Delta y + f(x,y))$.
Therefore,

$$\text{Area of } P_y = A_y = \frac{1}{2}\begin{vmatrix} x & f(x,y) & 1 \\ x+\Delta x & f_x(x,y)\Delta x + f(x,y) & 1 \\ x & f_y(x,y)\Delta y + f(x,y) & 1 \end{vmatrix}$$

$$= \frac{1}{2} f_y(x,y)\Delta x \Delta y.$$

Similarly, the area of P_z, which has vertices $(y, f(x,y))$, $(y, f_x(x,y)\Delta x + f(x,y))$ and $(y+\Delta y, f_y(x,y)\Delta y + f(x,y))$, is $A_z = \frac{1}{2} f_x(x,y)\ \Delta x \Delta y$.

Then the area of $\Delta P_1P_2P_3 = \frac{1}{2}\sqrt{\Delta x^2\Delta y^2 + f_y^2\Delta x^2\Delta y^2 + f_x^2\Delta x^2\Delta y^2}$

$$= \frac{1}{2}\Delta x \Delta y \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}$$

Therefore,

Area of the parallelogram $P_1P_2P_3P_4 = \Delta x\, \Delta y \sqrt{1 + f_x^2(x,y) + f_y^2(x,y)}$.

In this discussion we have assumed that the parallelogram $P_1P_2P_3P_4$ is not degenerate.

So, the total area S is approximately equal to

$\sum \Delta x \Delta y \sqrt{1 + f_x^2 + f_y^2}$ where the sum is taken over all parallelograms corresponding to a given partition of D. Thus, it is reasonable to define the area of the surface under consideration to be equal to $\int\int_D \sqrt{1 + f_x^2 + f_y^2}\, dxdy$.

We now illustrate this formula with some examples.

Example 5: Suppose we want to find the surface area of the part of the sphere $x^2+y^2+z^2 = 1$, lying above the ellipse $x^2+4y^2 = 1$.

We first note that here the surface is given by the equation $x^2+y^2+z^2=1$. Solving for z from this equation we get that the surface is the upper hemisphere given by the function

$$f(x,y) = z = \sqrt{1-x^2-y^2}.$$

The partial derivatives of f are

$$f_x(x,y) = \frac{-x}{\sqrt{1-x^2-y^2}} \text{ and } f_y(x,y) = \frac{-y}{\sqrt{1-x^2-y^2}}.$$

We denote by D, the ellipse $x^2 + 4y^2 \leq 1$. Then D is a closed bounded region described by

$$D = \{ (x,y) \mid -1 \leq x \leq 1, -\frac{1}{2}\sqrt{1-x^2} \leq y \leq \frac{1}{2}\sqrt{1-x^2} \}$$

See Fig 7.

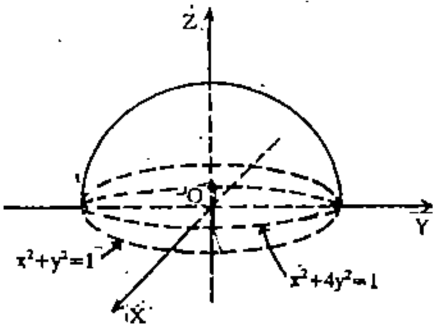


Fig. 7

Then the function f is a continuous function defined on D, and has continuous partial derivatives on D. Therefore, by Formula (1), the surface area

$$S = \iint_D \sqrt{1+f_x^2 + f_y^2}\, dxdy$$

$$= \iint_D \sqrt{1+\frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}}\, dxdy$$

$$= \iint_D \frac{1}{\sqrt{1-x^2-y^2}}\, dxdy$$

$$= \int_{-1}^{1} \left[\int_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} \frac{dy}{\sqrt{1-x^2-y^2}}\right] dx$$

$$= \int_{-1}^{1} \left[\sin^{-1} \frac{y}{\sqrt{1-x^2}}\right]_{-\sqrt{1-x^2}/2}^{\sqrt{1-x^2}/2} dx$$

$$= 2\int_{-1}^{1} \sin^{-1}\left(\frac{1}{2}\right) dx$$

$$= 4\sin^{-1}\left(\frac{1}{2}\right)$$

Example 6: Let us find the surface area, S, of the portion of the paraboloid

$$z = 9 - x^2 - y^2$$

that lies above the xy-plane.

The given surface lies over the region D in the xy-plane bounded by the circle $x^2 + y^2 = 9$. See Fig. 8.

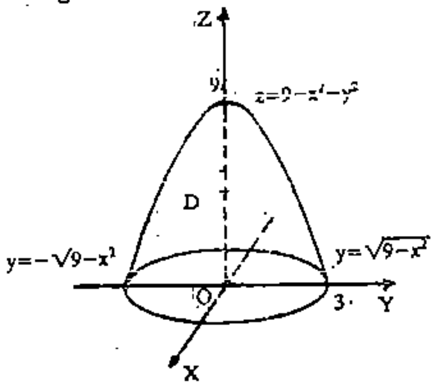


Fig. 8

Let $f(x,y) = 9 - x^2 - y^2$. Then

$$f_x(x,y) = -2x \text{ and } f_y(x,y) = -2y.$$

Therefore, by Formula (1),

$$S = \iint_D \sqrt{4x^2+4y^2+1}\, dx\, dy$$

Here the region D is a disc. Therefore, we make use of polar co-ordinates to evaluate this integral. In polar co-ordinates we can describe the region D as

$$D = \{ (r, \theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 3 \}.$$

Therefore

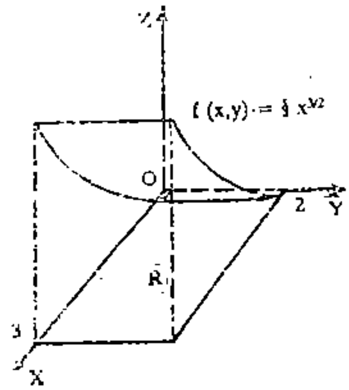
$$S = \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\, r dr\, d\theta.$$

$$= \frac{1}{12}\int_0^{2\pi} [(4r^2+1)^{3/2}]_0^3\, d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} (37^{3/2}-1) d\theta = \frac{1}{6}\pi (37^{3/2}-1)$$

Here are some exercises for you.

E 6) Let R be the rectangular region bounded by the lines $x = 0$, $x = 3$, $y = 0$ and $y = 2$, and let $f(x,y) = \frac{2}{3}x^{3/2}$. Find the surface area S of the portion of the graph of f that lies over R (see the figure alongside).



E 7) Find the area of the surface $z = x^2 + y^2$ below the plane $z = 9$.

In the next sub-section we shall discuss some physical applications of double integrals.

13.2.3 Mass and Moments

Here we explain how double integrals are useful in evaluating two physical quantities related to an object, namely, mass and moment of inertia.

We first consider the mass. Let us take a flat sheet that is so thin, that we may consider it to be two-dimensional. See Fig. 9(a). Suppose that the sheet is made of non-homogeneous material, i.e., the density of the sheet is non-uniform. We want to find an expression for the mass of this sheet using double integrals.

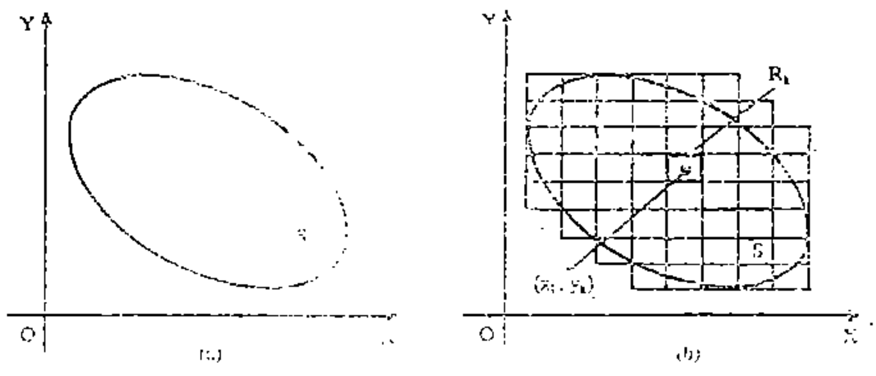


Fig. 9

Let S be the region that the sheet occupies in the xy-plane and let the density at a point (x,y) be $\delta(x,y)$. Since the density of the sheet varies from point to point, we consider it as a function of the points in the sheet. Now partition S into small rectangles, $R_1, R_2, \ldots\ldots, R_n$, as shown in Fig. 9(b). Let us pick a point (x_k, y_k) on R_k. Then the mass of R_k is approximately $\delta(x_k, y_k)$. area of R_k.

= area × density

Thus, the total mass of the sheet will be approximately

$$\sum_{k=1}^{n} \delta(x_k,y_k)\ A\ (R_k)$$

The actual mass is obtained by taking the limit of the above expression as the diameter of R_k tends to zero. That is,

$$m = \lim_{n} \sum_{k=1}^{n} \delta(x_k,y_k)\ A\ (R_k)$$

But, by Remark 1 in Unit 11, this limit is the double integral of the function $\delta(x,y)$ over D. Therefore,

$$m = \iint_D \delta(x,y)\ dxdy$$

Remark 1: Instead of a thin sheet if we take a flat plate with uniform thickness h, then the mass of the plate will be

$$m = h \iint_D \delta(x,y)\ dxdy$$

This follows easily because the plate can be considered as made up of h thin sheets patched together.

Let us look at an example now.

Example 7 : Suppose we want to find the total mass of a thin sheet with density $\delta(x,y) = xy$, which is bounded by the x-axis, the line $x=8$ and the curve $y = x^{2/3}$.

Here the region $D = \{ (x,y) \mid 0 \le x \le 8,\ 0 \le y \le x^{2/3}\}$

The total mass is given by the formula

$$m = \iint_D \delta(x,y)\ dxdy$$

$$= \int_0^8 \left(\int_0^{x^{2/3}} xydy \right) dx$$

$$= \int_0^8 \left[\frac{xy^2}{2} \right]_0^{x^{2/3}} dx$$

$$= \frac{1}{2} \int_0^8 x^{7/3}\ dx$$

$$= \frac{1}{2} \cdot \frac{3}{10} \left[x^{10/3} \right]_0^8$$

$$= \frac{768}{5} = 153.6$$

Next we will consider another interesting phenomenon in physics, namely, moments.

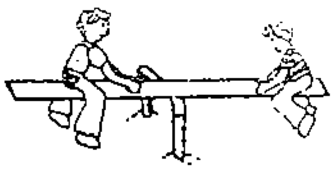
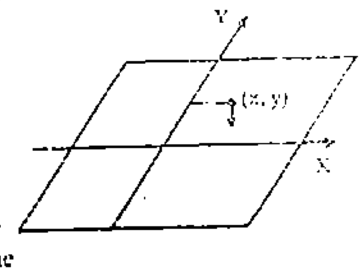


Fig. 10

Let us imagine two children playing on a see-saw (Fig. 10). It is a simple fact that a heavier child has more effect on the rotation of the sea-saw than does a lighter child. You must have also observed that a lighter child can balance a heavier one by moving farther away from the axis of rotation. This leads us to define a quantity called "moment", which measures the tendency of a mass to produce a rotation. Let us first consider the idealised situation in which an object of a positive mass m is concentrated at a point (x,y) in the plane. Such an object is called a **point mass**. The moment of the point mass **about the y-axis** is defined to be mx. We can think of mx as a measure of the tendency of the point mass to rotate about the y-axis (see Fig. 11). The larger the x or m is, the larger the magnitude of the moment. Thus, our definition of the moment is consistent with the observation that it is easier to rotate a see-saw about its axis if the person is heavier or farther away from the axis.



Next, suppose there are several point masses, $m_1, m_2, \ldots, m_n$, located at the respective points, $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ in the plane.

We define the moment M_y of the collection of point masses about the y-axis to be the sum of the moments of the individual point masses about the y-axis. Thus,

$$M_y = m_1x_1 + m_2x_2 + \ldots\ldots + m_nx_n = \sum_{k=1}^{n} m_k x_k \qquad \ldots\ldots(2)$$

We can think of M_y as a measure of the tendency of the collection of masses to produce a rotation about the y-axis. If $M_y = 0$, then there will be no tendency for rotation. In this case the collection of point masses is said to be in equilibrium position.

Analogously, we can define the moment M_x of the point masses $m_1, m_2, \ldots\ldots, m_n$ about the x-axis by setting

$$M_x = m_1y_1 + m_2y_2 + \ldots\ldots + m_ny_n \qquad \ldots\ldots(3)$$

We say that the point masses are in equilibrium with respect to rotation about the x-axis, if $M_x = 0$.

Now let $m = m_1 + m_2 + \ldots\ldots + m_n$ be the combined mass of the point masses just considered. Let us find a point $(\bar{x}, \bar{y})$ with the property that if we place a point mass with mass m at $(\bar{x}, \bar{y})$, then its moments about the x and y axes will be M_x and M_y, respectively. But according to our definition, the moment of the single point mass m about the y-axis is $m\bar{x}$, and its moment about the x-axis is $m\bar{y}$. Hence by (2) and (3) we get that

$$m\bar{x} = M_y = \sum_{k=1}^{n} m_k x_k$$

and

$$m\bar{y} = M_x = \sum_{k=1}^{n} m_k y_k$$

Thus,

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{m} = \frac{M_y}{m} \qquad (4)$$

and

$$\bar{y} = \frac{\sum_{k=1}^{n} m_k y_k}{m} = \frac{M_x}{m} \qquad (5)$$

The point $(\bar{x}, \bar{y})$ is called the centre of gravity, or centroid, of the given collection of point masses.

Now we find expressions for moments and centre of gravity using double integrals. The procedure is the same as that for mass, volume and surface area.

Consider a thin sheet of variable density $\delta(x,y)$ covering a region S in the xy-plane as in Fig. 9(a).

Partition this as in Fig. 9(b). We assume, as an approximation, that the mass of each R_k is concentrated at $(\bar{x}_k, \bar{y}_k)$, $k = 1, 2 \ldots\ldots, m$. Now the mass of $R_k = \delta(\bar{x}_k, \bar{y}_k) A(R_k)$ and so, the moments of R_k are

$$M_y^k = \delta(\bar{x}_k, \bar{y}_k) A(R_k) \bar{x}_k$$

and

$$M_x^k = \delta(\bar{x}_k, \bar{y}_k) A(R_k) \bar{y}_k$$

Then the sum of the moments of masses occupying all the spaces R_k, $k = 1, 2 \ldots, n$, about the y-axis is given by

$$\sum_{k=1}^{m} \bar{x}_k \, \delta(\bar{x}_k, \bar{y}_k) \, A(R_k)$$

This will be an approximation for the moment M_y of the total mass of the thin sheet. Thus, we get that

$$M_y = \iint_D x \, \delta(x,y) \, dx \, dy.$$

Similarly we get that

$$M_x = \iint_D y \, \delta(x,y) \, dx \, dy.$$

If $\bar{x}$ and $\bar{y}$ denote the centre of gravity of the thin sheet, from (4) and (5) we have

$$\bar{x} = \frac{M_y}{m} \text{ and } \bar{y} = \frac{M_x}{m},$$

where m is the total mass. We have already seen that the total mass m is expressed in terms of a double integral by

$$m = \iint_D \delta(x,y) \, dx \, dy.$$

Therefore,

$$\bar{x} = \frac{\iint_D x \, \delta(x,y) \, dx \, dy}{\iint_D \delta(x,y) \, dx \, dy}$$

and

$$\bar{y} = \frac{\iint_D y \, \delta(x,y) \, dx \, dy}{\iint_D \delta(x,y) \, dx \, dy}$$

The derivations of these formulas may appear a little difficult to you. But you don't have to worry. We expect you only to remember these formulas and to apply them. Here are some examples.

Example 8 : Let us find the centre of gravity of the thin sheet given in Example 7.

In Example 7, we calculate the mass m of the sheet as

$$m = \frac{768}{5}.$$

The moments M_x and M_y w.r.t. the x and y axes are given by

$$M_x = \int_D \int y \, \delta(x,y) \, dxdy$$

and

$$M_y = \int_D \int x \, \delta(x,y) \, dxdy$$

Therefore,

$$M_x = \int_0^8 [\int_0^{x^{2/3}} xy^2 \, dy] \, dx$$

$$= \frac{1}{3} \int_0^8 \left[xy^3 \right]_0^{x^{2/3}} dx$$

$$= \frac{1}{3} \int_0^8 x^3 \, dx$$

$$= \frac{1024}{3}$$

Similarly

$$M_y = \int_0^8 [\int_0^{x^{2/3}} x^2 y \, dy] \, dx$$

$$= \frac{1}{2} \int_0^8 x^{10/3} dx = \frac{12288}{13}$$

Thus,

$$\bar{x} = \frac{M_y}{m} = \frac{12288}{13} \cdot \frac{5}{768} = \frac{80}{13}, \text{ and}$$

$$\bar{y} = \frac{M_x}{m} = \frac{1024}{3} \cdot \frac{5}{768} = \frac{20}{9}.$$

That is, the centre of gravity is at the point $\left[\frac{80}{13}, \frac{20}{9}\right]$.

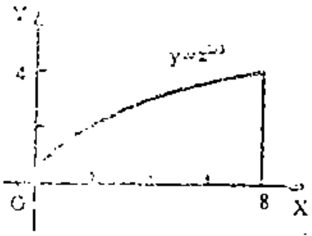


Fig. 12

In Fig. 12 you can see that the point $(\bar{x}, \bar{y})$ is in the upper right portion of D. But that is to be expected, since a thin sheet with density $\delta(x,y) = xy$ gets heavier as the distances from the x-and y-axes increase.

Now you can try your hand at these exercises.

E 8) Find the moments and the centre of gravity of a thin sheet in the shape of a quarter-circle of radius 2, whose density at a point is k times the distance from the centre to that point $(k > 0)$.
(**Hint** : Use polar coordinates for evaluating the double integral.)

E 9) Find the centre of gravity of a mass in the shape of the rectangle $[0,1] \times [0,1]$, if the density at (x,y) is e^{x+y}

E 10) Find the y-coordinate of the centre of gravity of the following objects :

a) The part of the disc of radius 1, lying in the first quadrant with uniform density.

b) A thin plate bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$ with density $\delta(x,y) = y^2$

Before we end this section, we will discuss another physical concept called 'moment of inertia'. Those of you, who have studied Physics must have learnt that moment of inertia is used in studying the rotation of a mass around a line. If a particle of mass m is at a distance r from a line L, then the expression r^2m is called the 'moment of inertia' of that particle about L. For a system of n particles in a plane with masses $m_1, m_2, \ldots\ldots, m_n$ at distances $r_1, r_2, \ldots\ldots, r_n$ from a line L, the moment of inertia of the system about L is defined as

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots\ldots + m_n r_n^2$$

We can express moment of inertia also in terms of double integrals. Suppose we have an object with varying density $\delta(x,y)$, covering a region D in the plane. We want to find an expression for the moment of inertia of this object using double integrals. By now you must be quite familiar with the process: partition the regions into slices, calculate the moment of inertia of a sample slice, form the summation. This summation approximates the moment of inertia when we take the limit as the norm of the partition tends to [illegible]. This gives us the following formulas.

The moments of inertia about the x- and y-axes are, respectively, given by

$$I_x = \iint_D y^2 \, \delta(x,y) \, dA$$

and

$$I_y = \iint_D x^2 \, \delta(x,y) \, dA.$$

We shall illustrate these formulas with an example.

Example 9 : Let us calculate the moment of inertia of the object given in Example 7.

The density of the object is $\delta(x,y) = xy$.

Therefore, the moment of inertia about the x-axis is

$$I_x = \int_S\int y^2\,\delta(x,y)\,dxdy = \int_0^8\int_0^{x^{2/3}} xy^3\,dydx$$

$$= \frac{1}{4}\int_0^8 x^{11/3}\,dx = \frac{6144}{7}$$

Similarly, you can calculate I_y. We are leaving it to you as an exercise (see E11).

E 11) Compute I_y for the object given in Example 7.

E 12) Find the moment of inertia I_x and I_y of a thin plate with density $\delta(x,y) = y$, bounded by the curves $y = x^2$ and $x = 2$.

13.3 APPLICATIONS OF TRIPLE INTEGRALS

Some of the applications you have seen in Sec 13.2, carry over directly from double to triple integrals. The only difference here is that we consider a region in space and a mass lying over this region will have density $\delta(x,y,z)$, a function of three variables.

We are just listing the formulas by which we can compute the volume and mass. You can see that the formulas are exactly similar with double integrals replaced by triple integrals.

$$\text{Volume} = \int\int_W\int dxdydz,$$

$$\text{Mass} = \int\int_W\int \delta(x,y,z)\,dxdydz,$$

where W is the three-dimensional region which is occupied by an object with density $\delta(x,y,z)$.

Let us see some examples where we can apply this formula.

Example 10 : Suppose we want to find the mass of an object which is in the form of a cube $[1, 2] \times [1, 2] \times [1,2]$. Suppose the density at any point (x,y,z) on the cube is given by $\delta(x,y,z) = (1+x)e^zy$. Then the mass m of the object is given by

$$m = \int_1^2\int_1^2\int_1^2 (1+x)e^zy\,dxdydz$$

$$= \int_1^2\int_1^2\left[\left(x+\frac{x^2}{2}\right)e^zy\right]_1^2 dydz$$

$$= \int_1^2\int_1^2 \frac{5}{2}e^zy\,dydz$$

$$= \int_1^2 \frac{15}{4}e^z\,dz$$

$$= \frac{15}{4}(e^2 - e)$$

Example 11 : Suppose we want to find the volume of the region W in space which lies inside the surface $z = \frac{1}{9}(x^2+y^2)$, and below the plane $z = 1$.

We first note that in W, z varies from $\frac{1}{9}(x^2+y^2)$ to 1. Now if we put

$\psi_1(x,y) = \frac{1}{9}(x^2+y^2)$ and $\psi_2(x,y) = 1$, then

$\psi_1(x,y) \leq z \leq \psi_2(x,y)$

From Fig. 13 you will be able to say that y varies from $-\sqrt{9-x^2}$ to $\sqrt{9-x^2}$, and x ranges from -3 to 3. Therefore, we can write W as

$$W = \{ (x, y, z) \mid -3 \le x \le 3, -\sqrt{9-x^2} \le y \le \sqrt{9-x^2}, \frac{1}{9}(x^2+y^2) \le z \le 1 \}$$

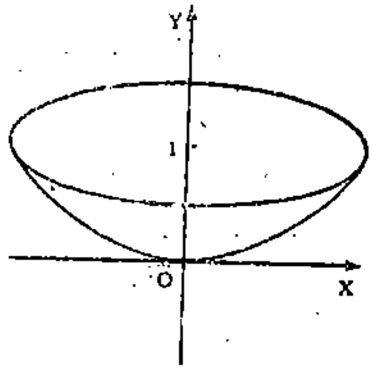


Fig. 13

Then the volume is given by

$$V = \iiint_W dxdydz$$

$$= \int_{-3}^{3}\left[\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(\int_{(x^2+y^2)/9}^{1} dz \right) dx \right] dy$$

$$= \int_{-3}^{3} \left[\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \left(1 - \frac{x^2}{9} - \frac{y^2}{9}\right) dy \right] dx$$

$$= \int_{-3}^{3} \left[\left(1 - \frac{x^2}{9}\right) y - \frac{y^3}{27} \right]_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} \left[\left(1 - \frac{x^2}{9}\right) 2\sqrt{9-x^2} - \frac{2}{27}(9-x^2)^{3/2} \right] dx$$

$$= \int_{-3}^{3} \frac{4}{27} (9-x^2)^{3/2}\, dx$$

$$= \frac{9\pi}{2}.$$

Now here are some exercises for you.

E 13) Find the volume of the region lying in the first octant which is common to the two cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

E 14) Find the mass of the solid in the first octant which is bounded above by the surface $z = 1 - x^2$ and on the right side by $x = y^2$; the density function is $\delta(x,y,z) = xy$.

As in the case of double integrals, we can find expressions for moments and centre of gravity of objects occupying solid regions using triple integrals. Let us first see what is meant by moments in this case. Suppose we have a point mass m located at (x,y,z). We define its moments M_{xy}, M_{xz}, M_{yz} about the coordinate planes as follows :

moment about the xy-plane $= M_{xy} = zm$,
moment about the yz-plane $= M_{yz} = xm$,
moment about the zx-plane $= M_{zx} = ym$.

Note that in this case moments are defined w.r.t. the coordinate planes whereas in the two-variable case moments are defined w.r.t. the coordinate axes. Also note that z, x and y are the distances from the point (x,y,z) to the xy-plane, yz-plane and zx-plane, respectively.

Now we find expressions for the moments of an object occupying a solid region W and having density $\delta(x,y,z)$, using triple integrals. We first consider the moment about the xy-plane. We circumscribe W with a rectangular box W'. Then we partition W' into small rectangular boxes of which $W_1, W_2, \ldots, W_n$ are entirely contained in W (see Fig. 1 Unit 12).

Let Δx_k, Δy_k, Δz_k be the dimensions of W_k. Then the volume V_k of W_k is $V_k = \Delta x_k \Delta y_k \Delta z_k$.

For each k between 1 and n, choose any point (x_k,y_k,z_k) in W_k. Then the mass of W_k will be approximately

$$m_k = \delta(x_k,y_k,z_k)\, V_k$$

Now the distance from any point in W_k to the xy-plane is approximately equal to z_k. Therefore, by definition, M^k_{xy} will be

$$M^k_{xy} = \text{mass} \times \text{the distance from } W_k \text{ to the xy-plane}$$
$$= \delta(x_k,y_k,z_k)\, V_k z_k$$

Then $\sum_{k=1}^{n} \delta(x_k,y_k,z_k)\, V_k z_k$ approximates the moment of W' w.r.t. the xy-plane. This approximation gets better and better as we take finer and finer partitions. Thus, we get

$$M_{xy} = \iiint_W z\, \delta(x,y,z)\, dxdydz$$

Similarly, we get

$$M_{zx} = \iiint_W y\, \delta(x,y,z)\, dxdydz \text{ and}$$

$$M_{yz} = \iiint_W x\, \delta(x,y,z)\, dxdydz$$

Now we give the expression for the centre of gravity of an object using the moments without giving any details. Let $(\bar{x},\bar{y},\bar{z})$ denote the centre of gravity of the object. Then

$$\bar{x} = \frac{M_{yz}}{m} = \frac{\iiint_W x\, \delta(x,y,z)\, dxdydz}{\iiint_W \delta(x,y,z)\, dxdydz}$$

$$\bar{y} = \frac{M_{xz}}{m} = \frac{\iiint_W y\, \delta(x,y,z)\, dxdydz}{\iiint_W \delta(x,y,z)\, dxdydz}$$

$$\bar{z} = \frac{M_{xy}}{m} = \frac{\iiint_W z\, \delta(x,y,z)\, dxdydz}{\iiint_W \delta(x,y,z)\, dxdydz}$$

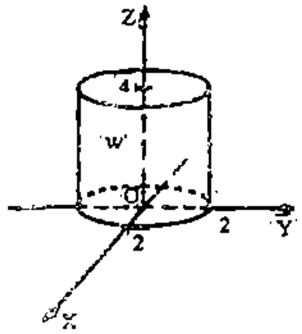


Fig. 14

Example 12 : Let us compute the moments, M_{xy}, M_{xz}, M_{yz} and the centre of gravity of an object occupying a solid region W bounded by the circular cylinder shown in Fig. 14, and which has mass density δ given by

$$\delta(x,y,z) = 20-z^2.$$

Let us first compute the moment M_{xy}. By definition,

$$M_{xy} = \int\int_W\int z\,\delta(x,y,z)\,dxdydz$$

To evaluate this integral we describe the region W in cylindrical coordinates by

$$W = \{ (r, \theta, z) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ 0 \le z \le 4\}.$$

Thus,

$$M_{xy} = \int_0^2 \left[\int_0^{2\pi} \left\{ \int_0^4 z(20-z^2)\,dz \right\} d\theta \right] r dr$$

$$= \int_0^2\int_0^{2\pi} \left[\int_0^4 (20z-z^3)\,dz\right] r\,drd\theta$$

$$= \int_0^2\int_0^{2\pi} \left[10z^2 - \frac{z^4}{4}\right]_0^4 r\,drd\theta$$

$$= \int_0^2\int_0^{2\pi} 96\,r\,dr\,d\theta$$

$$= 384\,\pi.$$

Next we calculate

$$M_{xz} = \int\int_W\int y\,\delta(x,y,z)\,dxdydz$$

$$= \int_0^2\int_0^{2\pi}\int_0^4 r\sin\theta\,(20-z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^2\int_0^{2\pi} r^2\sin\theta\left[20z - \frac{z^3}{3}\right]_0^4 drd\theta$$

$$= \int_0^2\int_0^{2\pi} \frac{176}{3} r^2\sin\theta\,d\theta dr$$

$$= \int_0^2 \frac{176}{3} r^2\left[-\cos\theta\right]_0^{2\pi} dr$$

$$= \int_0^2 \frac{176}{3} r^2 (\cos 0 - \cos 2\pi)\,dr$$

$$= 0$$

Similarly, we can check that $M_{yz} = 0$.

To compute the centre of gravity, we have to first calculate the mass of the object. Thus,

$$m = \int\int_W\int \delta(x,y,z)\,dxdydz$$

$$= \int_0^2\int_0^{2\pi}\int_0^4 (20-z^2)\,rdrd\theta dz$$

$$= \int_0^2\int_0^{2\pi}\left[20z - \frac{z^3}{3}\right]_0^4 rdrd\theta$$

$$= \int_0^2 \int_0^{2\pi} \frac{176}{3} \, r dr d\theta$$

$$= \frac{176}{3} \int_0^2 r \left[\theta \right]_0^{2\pi} dr$$

$$= \frac{352}{3} \pi \left[\frac{r^2}{2} \right]_0^2$$

$$= \frac{704}{3} \pi.$$

Therefore, the centre of gravity is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} = 0, \quad \bar{y} = \frac{M_{xz}}{m} = 0, \quad \bar{z} = \frac{M_{xy}}{m} = \frac{384\pi}{704} \cdot \frac{3}{\pi} = \frac{18}{11},$$

i.e., $(0, 0, \frac{18}{11})$.

Triple integral also allows us to find the **moment of inertia** of objects about the x,y and z-axes. We now give the formula :

I_x = moment of inertia about the x-axis $= \iiint_W (y^2+z^2) \, \delta(x,y,z) \, dxdydz$

I_y = moment of inertia about the y-axis $= \iiint_W (x^2+z^2) \, \delta(x,y,z) \, dxdydz$

I_z = moment of inertia about the z-axis $= \iiint_W (x^2+y^2) \, \delta(x,y,z) \, dxdydz$,

where W is the solid region which is occupied by an object and $\delta(x,y,z)$ is the density of the object.
Here is an example.

Example 13 : Let us find the moment of inertia of a solid sphere W of uniform density and radius a about the z-axis.

Here density is given to be a constant, say k, i.e., $\delta(x,y,z) = k$ for all (x,y,z). According to the definition the moment of inertia about the z-axis is

$$I_z = \iiint_W (x^2+y^2) \, \delta(x,y,z) \, dxdydz$$

$$= \iiint_W k \, (x^2+y^2) \, dxdydz$$

You will agree that the region of integration W in this case can be easily described by spherical coordinates. You have seen in Unit 12 that in this situation, spherical coordinates are more useful in evaluating the integral. In these coordinates the region W can be described as

$$W = \{ (r, \theta, \phi) \mid 0 \le r \le a, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi \}$$

Further, $x^2+y^2 = r^2\sin^2\phi$, and hence,

$$I_z = \int_0^a \int_0^{\pi} \int_0^{2\pi} k \, [r^2\sin^2\phi] \, r^2\sin\phi \, d\theta d\phi dr$$

$$= k \int_0^a \int_0^{\pi} \int_0^{2\pi} r^4\sin^3\phi \, d\theta d\phi dr$$

$$= 2\pi \int_0^a \int_0^{\pi} r^4\sin^3\phi \left[\theta \right]_0^{2\pi} d\phi dr$$

But $\int_0^{\pi} \sin^3 \phi \, d\phi = \int_0^{\pi} \sin\phi \, (1-\cos^2\phi) \, d\phi$

$$= -\left[\cos\phi - \frac{\cos^3\phi}{3} \right]_0^{\pi}$$

$$= \frac{4}{3}$$

Thus,

$$I_z = \int_0^a 2\pi k \cdot \frac{4}{3} \cdot r^4 \, dr$$

$$= \frac{8\pi k}{3} \left[\frac{r^5}{5} \right]_0^a$$

$$= \frac{8\pi k a^5}{15}$$

Are you ready for some exercises now?

E 15) Find the moment of inertia about the x-axis of a solid region with uniform density cut from the sphere $x^2+y^2+z^2 = 4a^2$ by the cylinder $x^2+y^2 = a^2$. (**Hint** : Use cylindrical coordinates. Then $z^2 = 4a^2 - (x^2 + y^2) = 4a^2 - r^2$.)

E 16) Find the moments and the centre of gravity of a solid occupying a portion of the sphere $x^2+y^2+z^2 = a^2$, which lies between the planes $\theta = \frac{\pi}{4}$ and $\theta = \frac{-\pi}{4}$, given that $\delta(x,y,z) = 1$.

13.4 SUMMARY

In this unit we have seen how to find

1) the area of some planar regions using double integrals :

$$\text{Area of D} = \int_D \int dx dy$$

2) the volume of a solid region lying under the graph of a function f(x,y) and above a region D in the xy-plane using double integrals :

$$V = \int_D \int f(x,y) \, dx dy$$

3) the surface area of a general curved surface using double integral

$$S = \int_D \int \sqrt{1+f_x^2+f_y^2} \, dx \, dy$$

4) the mass, moments, centre of gravity of objects like thin sheets using double integrals.

$$m = \int_D \int \delta(x,y) \, dx dy$$

$$M_x = \int_D \int y \, \delta(x,y) \, dx dy$$

$$M_y = \int_D \int x \, \delta(x,y) \, dx dy$$

Centre of gravity $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{M_y}{m}, \quad \bar{y} = \frac{M_x}{m}.$$

5) the moment of inertia of a thin sheet using double integrals.
6) the volume of a region in space.
7) the mass, moments, centre of gravity and moment of inertia of an object occupying a solid region in space using triple integrals

13.5 SOLUTIONS AND ANSWERS

E 1) a) Let D be the region bounded by the y-axis and the lines y=4, y=2x. We can write D as a Type II region by

$$D = \{ (x,y) \mid 0 \le x \le \frac{y}{2}, 0 \le y \le 4\}.$$

$$\text{The area of D } = \int_D \int dx\,dy = \int_0^4 \left[\int_0^{y/2} dx \right] dy$$

$$= \int_0^4 \frac{y}{2}\,dy$$

$$= \frac{y^2}{4}\Big]_0^4$$

$$= 4$$

b) The region $D = \{ (x,y) \mid 0 \le x \le 1, 0 \le y \le e^x\}$

$$\text{Area of D } = \int_D \int dx\,dy$$

$$= \int_0^1 \left[\int_0^{e^x} dy \right] dx$$

$$= \int_0^1 e^x\,dx$$

$$= e-1.$$

E 2) a) Let $D = \{ (x,y) \mid y \le x \le \sqrt{y}, 0 \le y \le 1\}$. D is of Type II. In fact, D is the region bounded by the parabola $y=x^2$ and the line $y=x$.

$$\text{Then} \int_D \int dx\,dy = \int_0^1 \left[\int_y^{\sqrt{y}} dx \right] dy$$

$$= \int_0^1 (\sqrt{y} - y)\,dy$$

$$= \left[\frac{2}{3} y^{3/2} - \frac{y^2}{2} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6}$$

b) $D = \{(x,y) \mid 0 \le x \le 3, -x \le y \le x(2-x) \}$.
D is of Type I. D is the region bounded by line $y = -x$ and the curve $y = -x^2+2x$.

$$\text{Area of D } = \int_0^3 \left[\int_{-x}^{x(2-x)} dy \right] dx$$

$$= \int_0^3 (-x^2+2x+x)\,dx$$

$$= \left[-\frac{x^3}{3} + 3\frac{x^2}{2} \right]_0^3$$

$$= \frac{9}{2}.$$

E 3) Here the region can be described in polar coordinates by

$D = \{ (r,\theta) \mid 0 \le r \le \sqrt{2-\sin 2\theta},\ 0 \le \theta \le \frac{\pi}{2} \}$

Thus, area of D $= \int_0^{\pi/2} \left[\int_0^{\sqrt{2-\sin 2\theta}} r\, dr \right] d\theta$

$$= \int_0^{\pi/2} \left[\frac{r^2}{2}\right]_0^{\sqrt{2-\sin 2\theta}} d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} (2-\sin 2\theta)\, d\theta$$

$$= \frac{1}{2} \left[2\theta + \frac{\cos 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi}{2} - \frac{1}{2}$$

E 4) The region $D = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le x \}$.

Thus, volume $= \int\int_D (3-x-y)\, dx\, dy$

$$= \int_0^1 \left[\int_0^x (3-x-y)\, dy \right] dx$$

$$= \int_0^1 \left(3y - xy - \frac{y^2}{2}\right) \Big]_0^x dx$$

$$= 1.$$

E 5) Let $f(x,y) = \frac{x^2+y^2}{a}$

Since the base region is a circle, we use polar coordinates. Then $D = \{ (r,\theta) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi\}$ and $f^*(r,\theta) = \frac{r^2}{a}$.

Thus,

$$\text{volume} = \int_0^{2\pi} \left[\int_0^a \frac{r^2}{a} \cdot r\, dr \right] d\theta = \frac{\pi a^3}{2}$$

E 6) The surface area $S = \int\int_D \sqrt{f_x^2 + f_y^2 + 1}\ dx\, dy$.

Let $f(x,y) = \frac{2}{3} x^{3/2}$. Then $f_x(x,y) = x^{1/2}$, $f_y(x,y) = 0$.

$$S = \int\int_D \sqrt{x+1}\ dxdy$$

$$= \int_0^3 \left[\int_0^2 \sqrt{x+1}\ dy \right] dx$$

$$= 2\left[\frac{2}{3} (x+1)^{3/2} \right]_0^3$$

$$= \frac{28}{3}.$$

E 7) Here the region D is the disc $x^2+y^2 \le 9$.

$f(x,y) = x^2+y^2$

$f_x(x,y) = 2x,\ f_y(x,y) = 2y.$

The region D can be more easily described in polar coordinates by $D = \{(r,\theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$. Therefore, we use polar coordinates to evaluate the integral. Thus

$$\text{Surface area } S = \int\int_D \sqrt{4(x^2+y^2)+1}\ dx\,dy.$$

$$= \int_0^{2\pi}\int_0^3 \sqrt{4r^2+1}\ r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{8} \times \frac{2}{3}\left[(4r^2+1)^{3/2}\right]_0^3 d\theta$$

$$= \frac{\pi}{6}\left(37^{3/2}-1\right)$$

E 8) The distance from (0,0) to a point (x,y) is $\sqrt{x^2+y^2}$.

Then $\delta(x,y) = k\sqrt{x^2+y^2}$, where k is a constant such that $k > 0$.

We first calculate the mass m given by

$$m = \int\int_D \delta(x,y)\ dxdy,$$

where D is the quarter circle of radius 2. We write D in polar coordinates as,
$D = \{(r,\theta) \mid 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$
and $\delta^*(r,\theta) = kr$.

$$\text{Thus, } m = \int_0^{\pi/2}\int_0^2 kr\cdot r\ dr\,d\theta$$

$$= \int_0^{\pi/2} k\left[\frac{r^3}{3}\right]_0^2 d\theta$$

$$= \frac{8k}{3}\int_0^{\pi/2} d\theta$$

$$= \frac{4k\pi}{3}$$

Now the moment M_x about x-axis is

$$M_x = \int\int_D y\,k\sqrt{x^2+y^2}\ dx\,dy$$

$$= k\int_0^{\pi/2}\int_0^2 r\sin\theta\ r^2 dr\,d\theta$$

$$= k\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^2 \sin\theta\ d\theta$$

$$= k\cdot\frac{16}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= 4k$$

Similarly, the moment M_y about y-axis is

$M_y = 4k.$

Then the centre of gravity is the point $(\bar{x}, \bar{y})$,

$$\text{where } \bar{x} = \frac{M_y}{m} = \frac{3}{\pi} = \bar{y}.$$

E 9) $\delta(x,y) = e^{x+y}$

The mass $m = \int_0^1 \int_0^1 e^{x+y}\, dx\, dy$

$$= \int_0^1 e^y \left[e^x \right]_0^1 dy$$

$$= (e-1)\, e^y \Big]_0^1 = (e-1)^2 .$$

$$\bar{x} = \frac{M_y}{m} = \frac{\int_0^1 \left[\int_0^1 x\, e^{x+y}\, dx \right] dy}{(e-1)^2} = \frac{(e-1)}{(e-1)^2} = \frac{1}{(e-1)}$$

$$\bar{y} = \frac{1}{(e-1)}.$$

E 10) a) We assume that the density $\delta(x,y) = k$ for all (x,y) where k is a constant, $k > 0$.

Here $D = \{ (r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{2} \}$

The mass $m = \int_0^{\pi/2} \int_0^1 k\, r\, dr d\theta$

$$= k \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^1 d\theta$$

$$= \frac{k\pi}{4}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int\int_D y\, \delta(x,y)\, dx\, dy}{\frac{k\pi}{4}}$$

$$= \int_0^{\pi/2} \int_0^1 r \sin\theta\, k\, r\, dr\, d\theta \Big/ \frac{k\pi}{4}$$

$$= k \int_0^{\pi/2} \sin\theta \left[\frac{r^3}{3} \right]_0^1 d\theta \Big/ \frac{k\pi}{4}$$

$$= \frac{k}{3} \left[-\cos\theta \right]_0^{\pi/2} \Big/ \frac{k\pi}{4}$$

$$= \frac{4}{3\pi}$$

b) $D = \{ (x,y) \mid 0 \le x \le 1,\ 0 \le y \le e^{-x} \}$

$\delta(x,y) = y^2.$

$$m = \int_0^1 \left[\int_0^{e^{-x}} y^2\, dy \right] dx$$

$$= \int_0^1 \left[\frac{y^3}{3} \right]_0^{e^{-x}} dx$$

E 16) $W = \{ (r,\theta,z) \mid 0 \le r \le a, -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, -\sqrt{a^2-r^2} \le z \le \sqrt{a^2-r^2} \}$

$$\text{mass} = \int_0^a \int_{-\pi/4}^{\pi/4} \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r dz' d\theta\, dr$$

$$= \int_0^a \int_{-\pi/4}^{\pi/4} 2r \sqrt{a^2-r^2}\, d\theta\, dr$$

$$= 2r \int_0^a \sqrt{a^2-r^2}\, 2r \frac{\pi}{4}\, dr$$

$$= \left. \frac{-\pi(a^2-r^2)^{3/2}}{3/2} \right]_0^a$$

$$= \frac{\pi a^3}{3}$$

$$M_{xy} = \int_{-\pi/4}^{\pi/4} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} z\, r\, dz\, dr\, d\theta = 0$$

$$M_{xz} = 0$$

$$M_{yz} = \int_{-\pi/4}^{\pi/4} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} (r \cos\theta)\, r\, dz\, dr\, d\theta$$

$$= \int_0^a \left[\int_{-\pi/4}^{\pi/4} 2\, r^2 \sqrt{a^2-r^2} \cos\theta\, dr \right] d\theta$$

$$= 2\sqrt{2} \int_0^a r^2 \sqrt{a^2-r^2}\, dr$$

$$= \frac{\pi\sqrt{2}\, a^4}{8}$$

$$\bar{x} = \frac{\pi\sqrt{2}\, a^4}{8} \frac{3}{\pi a^3} = \frac{3\sqrt{2}\, a}{8}$$

$$\text{centroid} = \left[\frac{3\sqrt{2}\, a}{8}, 0, 0 \right]$$

UNIT 14 LINE INTEGRALS IN R^2

Structure

14.1 INTRODUCTION

In the first two units of this block you have seen one way of generalising the concept of the definite integral of a function of one variable. In this unit we shall introduce a totally different generalisation of the integral $\int_a^b f(x)dx$. Here we shall replace the interval [a,b] by a curve C in the plane and define three types of integrals: $\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$, and $\int_C f(x,y)\,ds$, where s denotes the arc length. All these integrals are called line integrals or curve integrals. We shall also evolve some methods to evaluate these line integrals. Finally, we prove Green's theorem, which gives a connection between line integrals and double integrals.

Objectives

After reading this unit, you should be able to

- define the line integrals, $\int_C f(x,y)\,dx$, $\int_C f(x,y)\,dy$, and $\int_C f(x,y)\,ds$, where C is a curve in the plane,
- evaluate the above mentioned line integrals,
- state and prove Green's theorem connecting line integrals with double integrals.

14.2 LINE INTEGRALS

In this section we are going to talk about curvilinear integrals or integrals over curves. In order to define curvilinear integrals we need to know what we mean by a curve in a plane.

Let us start with the following definition.

Definition 1: Let ψ_1 and ψ_2 be two real-valued continuous functions defined on the closed interval [a,b]. Then the set C defined by

$$C = \{ (\psi_1(t), \psi_2(t)) \mid a \le t \le b\}$$

is said to be a **curve** in R^2. The point P $(\psi_1(a), \psi_2(a))$ is called the **initial point** of C and the point Q $(\psi_1(b), \psi_2(b))$ is called the **end point** of the curve C. A curve C is called a closed curve if its initial and end points coincide. A curve C is called a simple curve if for $t_1, t_2 \in]a,b[$, $t_1 \ne t_2$, the points $(\psi_1(t_1), \psi_2(t_1))$ and $(\psi_1(t_2), \psi_2(t_2))$ corresponding to t_1 and t_2, respectively, are distinct. This means that the curve does not cross itself, see Fig. 1(a). In Fig. 1(b) you can see a curve which is not simple.

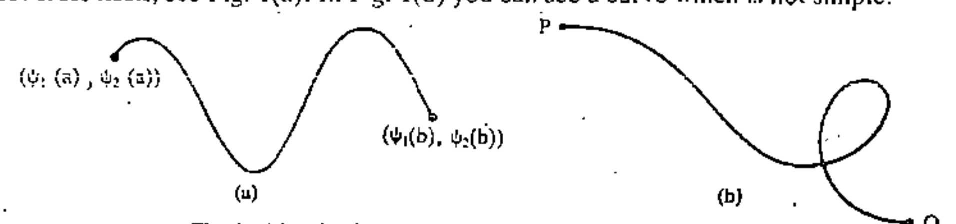


Fig. 1 : (a) a simple curve (b) a curve which is not simple.

$$= \frac{1}{3} \int_0^1 e^{-3x}\, dx$$

$$= \frac{1}{3} \left. \frac{e^{-3x}}{-3} \right]_0^1$$

$$= \frac{1-e^{-3}}{9}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\int_0^1 \left[\int_0^{e^{-x}} y^3\, dy \right] dx}{m}$$

$$= \frac{\int_0^1 \left[\frac{y^4}{4} \right]_0^{e^{-x}} dx}{m}$$

$$= \frac{\int_0^1 e^{-4x}\, dx}{m}$$

$$= -\frac{1}{16} \frac{\left[e^{-4x} \right]_0^1}{m}$$

$$= -\frac{1}{16} \frac{(e^{-4}-1)}{\frac{1}{9}(1-e^{-3})}$$

$$= \frac{9}{16} \frac{(1-e^{-4})}{(1-e^{-3})}$$

E 11) $I_y = \int_D \int x^2 \delta(x,y)\, dxdy$

Here $D = \{(x,y) \mid 0 \le x \le 8,\ 0 \le y \le x^{2/3}\}$ and $\delta(x,y) = xy$.

Then

$$I_y = \int_0^8 \left[\int_0^{x^{2/3}} x^3\, y\, dy \right] dx$$

$$= \int_0^8 x^3 \left[\frac{y^2}{2} \right]_0^{x^{2/3}} dx$$

$$= \frac{1}{2} \int_0^8 x^{13/3}\, dx$$

$$= \frac{1}{2} \frac{x^{16/3}}{16/3} = \frac{3}{32}\, 8^{16/3} = 6144$$

E 12) $D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x^2\}$

$$I_x = \int_0^2 \left[\int_0^{x^2} y^3\, dy \right] dx = \int_0^2 \left[\frac{y^4}{4} \right]_0^{x^2} = \frac{1}{4} \int_0^2 x^8 dx$$

$$= \frac{2^9}{36}$$

$$I_y = \frac{2^5}{10}$$

E 13) The region W can be described as

$W = \{ (x,y,z) \mid 0 \le x \le a,\ 0 \le y \le \sqrt{a^2-x^2},\ 0 \le z \le \sqrt{a^2-x^2} \}$

Thus the volume V is

$$V = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} dx\, dy\, dz$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\ dy\, dx$$

$$= \int_0^a (a^2-x^2)\ dx$$

$$= \frac{2a^3}{2}$$

E 14) The mass of the solid $= \int\int_W\int \delta(x,y,z)\ dx\, dy\, dz$

$W = \{ (x,y,z) \mid 0 \le x \le \sqrt{4-z},\ 0 \le y \le \sqrt{x},\ 0 \le z \le 4 \}$

$\delta(x,y,z) = xy$

$$\text{Mass} = \int_0^4 \left[\int_0^{\sqrt{4-z}} \left[\left\{ \int_0^{\sqrt{x}} xy\ dy \right\} dx \right] dz \right.$$

$$= \int_0^4 \left[\int_0^{\sqrt{4-z}} \frac{x^2}{2}\ dx \right] dz$$

$$= \int_0^4 \left[\frac{x^3}{6} \right]_0^{\sqrt{4-z}} dz$$

$$= \frac{1}{6} \int_0^4 (4-z)^{3/2}\ dz$$

$$= \frac{32}{15}.$$

E 15) Suppose $\delta(x,y,z) = k$.

The region W can be described in cylindrical coordinates by

$W = \{ (r,\theta,z) \mid 0 \le r \le a,\ 0 \le \theta \le 2\pi,\ -\sqrt{4a^2-r^2} \le z \le \sqrt{4a^2-r^2} \}$

$$I_x = k \int\int_W\int (y^2+z^2)\ dx\, dy\, dz$$

$$= k \int_0^a \int_0^{2\pi} \int_{-\sqrt{4a^2-r^2}}^{\sqrt{4a^2-r^2}} (r^2\sin^2\theta + z^2)\ r\ dz\, d\theta\, dr$$

$$= k \int_0^a \int_0^{2\pi} \left[2\, r^2\sin^2\theta \cdot \sqrt{4a^2-r^2} + \frac{2}{3} (4a^2-r^2)^{3/2} \right] r\ d\theta\, dr$$

$$= k \int_0^a \left[r^2 \sqrt{4a^2-r^2} + \frac{2}{3} (4a^2-r^2)^{3/2} \right] 2\pi r\ dr$$

$$= \frac{2\pi a^5 k}{15} (128 - 51\sqrt{3})$$

A curve C is said to be smooth if ψ_1 and ψ_2 have continuous derivatives at all points of [a,b], and $\psi_1'(t)$, $\psi_2'(t)$, are not simultaneously zero on [a,b] (i.e., $[\psi_1'(t)^2 + \psi_2'(t)^2] > 0$ for all $t \in [a,b]$). See Fig. 1(a). We can explain a smooth curve like this : Suppose we imagine that an object is moving along the curve given in Fig. 1(a) so that its position at time t is given by $(\psi_1(t), \psi_2(t))$. Then that object would suffer no sudden change of direction, (continuity of $\psi_1'(t)$ and $\psi_2'(t)$ ensures this) and would not stop or double back ("$\psi_1'(t)$ and $\psi_2'(t)$ not simultaneously zero" ensures this).

Now for the purpose of integration, it is useful to assign an orientation or a direction to a curve C. Look at Fig. 1(a). We can consider C as directed either from P to Q or from Q to P. That means the curve C has two directions, one in which t increases from a to b, and the other in which t decreases from b to a. In what follows, by C we shall denote the curve C described by the parameter t moving from a to b, and by −C, the curve described by the parameter t moving from b to a.

Note that, the initial point of C = the end point of −C and the end point of C = the initial point of −C. When a curve is described by increasing t, it is said to be positively oriented. Throughout our discussion, we shall deal with oriented, smooth and simple curves only.

Now suppose that C is an oriented, smooth and simple curve given by

$$C = \{ (x(t), y(t)) \mid a \le t \le b\},$$

where x(t) and y(t) are continuous functions over [a,b]. Let f(x,y) be a bounded real-valued function defined over C. We shall define $\int_C f(x,y)\, dx$.

As in the case of double integrals, the first step to define an integral is to partition C. To do this let us first consider a partition P of [a,b], given by

$a = t_0 < t_1 < t_2 < \dots\dots < t_n = b$.

For each i, $0 \le i \le n$, let P_i denote the point $(x(t_i), y(t_i))$ on C. Let C_i denote the arc of C joining the points P_{i-1} and P_i. See Fig. 2.

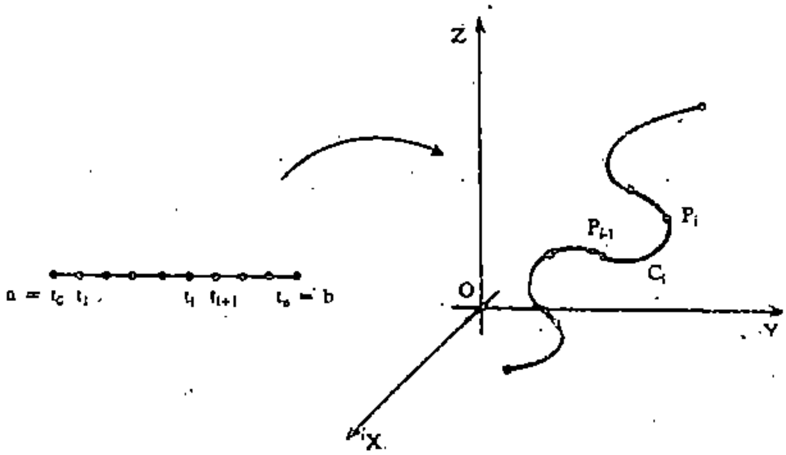


Fig. 2

Set

$$\Delta t_i = t_i - t_{i-1}$$
$$\Delta x_i = x(t_i) - x(t_{i-1})$$
$$M_i = \sup \{f(x,y) \mid (x,y) \in C_i\}$$
$$m_i = \inf \{f(x,y) \mid (x,y) \in C_i\}$$

Define upper sum and lower sums by

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad \dots\dots\dots\dots(1)$$

$$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i \qquad \ldots\ldots(2)$$

Let $l = \sup_{P} \{L(P,f)\}$ and $u = \inf_{P} \{U(P,f)\}$, where P is the set of all partitions of [a,b].

If $l=u$, then we say that the line integral $\int_C f(x,y)\, dx$ exists and

$$\int_C f(x,y)\, dx = \text{the common value of } l \text{ and } u.$$

Remark 1 : As in the case of double integrals, we can write the line integral $\int_C f(x,y)\, dx$ also as the limit of a sum. Let (ξ_i, η_i) be any point on the arc C_i. Consider the sum

$$S(P,f) = \sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta x_i \qquad \ldots\ldots(3)$$

Then the line integral $\int_C f(x,y)\, dx$ exists if and only if

$$\lim_{\mu(P)\to 0} S(P,f) = \lim_{\mu(P)\to 0} \sum_{i=1}^{n} f(\xi_i,\eta_i)\,\Delta x_i$$

exists, where $\mu(P)$ denotes the norm of the partition P. Moreover,

$$\lim_{\mu(P)\to 0} S(P,f) = \int_C f(x,y)dx.$$

Next we define another line integral $\int_C f(x,y)dy$.

Let Δx_i be replaced by Δy_i in Equations (1) and (2) above, and let

$$U^*(P,f) = \sum_{i=1}^{n} M_i \Delta y_i$$

$$L^*(P,f) = \sum_{i=1}^{n} m_i \Delta y_i$$

Suppose that

$$l^* = \sup_{P} L^*(P,f), \text{ and } u^* = \inf_{P} U^*(P,f).$$

If $l^* = u^*$, we say that the line integral $\int_C f(x,y)dy$ exists and

$$\int_C f(x,y)dy = \text{the common value of } l^* \text{ and } u^*$$

As we remarked before, $\int_C f(x,y)dy$ exists if and only if

$$\lim_{\mu(P)\to 0} S^*(P,f) = \lim_{\mu(P)\to 0} \sum_{i=1}^{n} f(\xi_i,\eta_i)\,\Delta y_i \text{ exists and}$$

$$\int_C f(x,y)dy = \lim_{\mu(P)\to 0} S^*(P,f)$$

Now we shall define a third line integral $\int_C f(x,y)ds$, where s denotes the arc length of the curve C. To define this we make use of the formula for the arc length of a curve, which you have seen in Calculus (Sub-sec. 16.2.2. Block 4). If a curve C is given in the parametric form $(x(t),y(t))$, $t \in [a,b]$, then the arc length from the point $(x(a),y(a))$ to an arbitrary point $(x(t),y(t))$ $t \in [a,b]$ is given by

$$s = \int_a^t \sqrt{x'(t)^2+y'(t)^2}\, dt.$$

So we can think of s as a function of t. Then we have $\frac{ds}{dt} = \sqrt{x'(t)^2+y'(t)^2}$ by the Fundamental Theorem of Calculus (Theorem 7, Unit 10, Calculus).

Let Δs_i = arc length of $C_i = s(t_i)-s(t_{i-1})$, and let

$$U_s(P,f) = \sum_{i=1}^{n} M_i \Delta s_i$$

$$L_s(P,f) = \sum_{i=1}^{n} m_i \Delta s_i$$

If $l_s = \sup_{P\in \mathcal{P}} L_s(P,f) = \inf_{P\in \mathcal{P}} U_s(P,f) = u_s$ then we say that $\int_C f(x,y)\, ds$ exists and

$\int_C f(x,y)\, ds$ = the common value of l_s and u_s.

Moreover

$$\int_C f(x,y)ds = \lim_{\mu(P)\to 0} \sum_{i=1}^{n} f(\xi_i, \eta_i)\, \Delta s_i.$$

So, we have defined three types of line integrals. Now here are some important remarks.

Remark 2 : Let $f(x,y) \geq 0$ on a curve C. Then $\int_C f(x,y)ds$ represents the area of the curved vertical curtain, and $U(P,f)$ is the sum of the areas of the rectangles in Figure 3. This sum clearly approximates the area of the curtain better and better as P gets finer and finer.

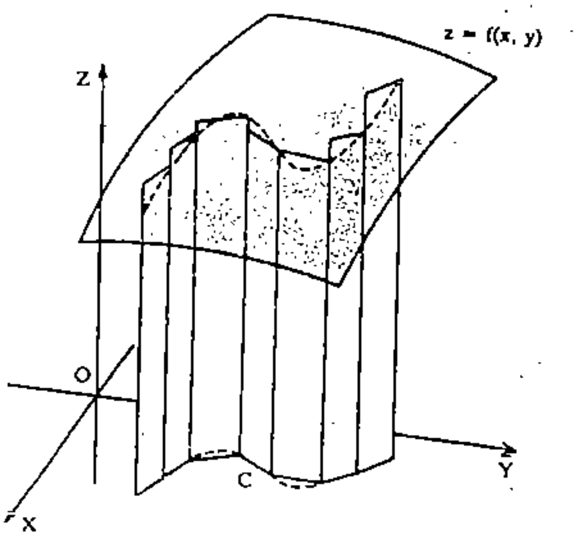


Fig. 3

Remark 3 : Let $f(x,y)$ and $g(x,y)$ be two real valued functions defined on a curve C. If both the line integrals $\int_C f(x,y)dx$ and $\int_C g(x,y)dy$ exist, then we shall denote the sum $\int_C f(x,y)dx + \int_C g(x,y)dy$ by $\int_C f(x,y)dx + g(x,y)dy$. Those of you who are familiar with vector calculus would recognise that

$$\int_C f(x,y)dx+g(x,y)\,dy = \int_C (f\hat{i}+g\hat{j})\cdot(dx\hat{i}+dy\hat{j}),$$

where $\hat{i}$, $\hat{j}$ are the unit vectors along the two coordinate axes.

Now we give a theorem which identifies a large class of functions for which all the three types of line integrals exist.

Theorem 1 : If f(x,y) is continuous on a simple smooth curve C, then the line integrals

$$\int_C f(x,y)dx, \quad \int_C f(x,y)dy \text{ and } \int_C f(x,y)ds \text{ exist.}$$

As usual, we don't expect you to know the proof of the above theorem.

Until now we have defined different types of line integrals. We have also mentioned an important class of functions for which the line integrals exist. But all this does not help us in the actual evaluation of the line integrals. We shall now show that for a large number of functions the line integrals can be expressed as ordinary definite integrals of real-valued functions of a single variable. Thus, all the techniques, you have learnt in Calculus are available to you for evaluating line integrals.

Theorem 2 : Let f(x,y) be a real-valued continuous function defined on a simple, smooth, positively oriented curve C given by

$$C = \{(x(t),y(t)) \mid a \le t \le b\}$$

Then

i) $\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt$

ii) $\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt$

iii) $\int_C f(x,y)ds = \int_a^b f(x(t),y(t))s'(t)dt$

We know $s'(t) = \sqrt{x'(t)^2 + y'(t)^2}$

Therefore (iii) can be written as $\int_C f(x,y)ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$

We shall now use Formulas (i), (ii) and (iii), mentioned in Theorem 2, to evaluate certain line integrals.

Example 1 : Let us evaluate $\int_C (x^2+y^2)dy$, where C is the curve given by

$x(t) = at^2$, $y(t) = 2at$, $0 \le t \le 1$.

You can check that the function $f(x,y) = x^2+y^2$ and the curve C satisfy the requirements of Theorem 2. Therefore, using Formula (ii) in Theorem 2, we get

$$\int_C (x^2+y^2)dy = \int_0^1 (a^2t^4+4a^2t^2)\,2a\,dt$$

$$= 2a^3\left[\frac{t^5}{5}+\frac{4t^3}{3}\right]_0^1$$

$$= 2a^3\left[\frac{1}{5}+\frac{4}{3}\right] = \frac{46}{15}a^3$$

The parametric equation of the curve in Example 1 actually represents the arc of the parabola joining the points (0,0) and (a,2a).

Example 2 : Let us evaluate the line integral

$$\int_C x^2dx+xy\ dy,$$

where C is the curve defined by x=cost, y = sint, joining the points (1,0) and (0,1). Using Remark 3, we write

$$\int_C x^2dx+xy\ dy = \int_C x^2dx + \int_C xy\ dy.$$

Then we make use of Formulas (i) and (ii) in Theorem 2 to evaluate both the integrals on the right hand side of the above expression. Therefore, we get

$$\int_C x^2dx+xy\ dy = \int_0^{\pi/2} \cos^2 t(-\sin t)dt + \int_0^{\pi/2} \sin t \cos t \cos t\ dt$$

$$= 0$$

The next example illustrates the use of Formula (iii).

Example 3 : Suppose C is the curve given by the parametric equations $x=t$, $y=t^2$, $1 \le t \le 2$. Let us find $\int_C x\ ds$.

You can check that the function f(x,y) = x and the curve C satisfy the requirements of Theorem 2. Therefore, by Formula (iii) in Theorem 2, we get

$$\int_C x\ ds = \int_1^2 t\sqrt{1+(2t)^2}\ dt, \text{ since } x'(t)=1 \text{ and } y'(t) = 2t.$$

$$= \int_1^2 t\sqrt{1+4t^2}\ dt$$

$$= \frac{1}{8}\int_4^{16} \sqrt{1+\theta}\ d\theta, \text{ where } \theta = 4t^2$$

$$= \frac{1}{12}(1+\theta)^{3/2}\Big]_4^{16}$$

$$= \frac{1}{12}\left[17^{3/2}-5^{3/2}\right]$$

Now you can try to evaluate some line integrals on your own.

E 1) Evaluate

a) $\int_C xy^{2/5}ds$; $C = \{(x,y) \mid x = \frac{t}{2}, y=t^{5/2}, 0 \le t \le 1\}$

b) $\int_C (\sin x + \cos y)\ ds$, where C is the line segment from (0,0) to $(\pi,2\pi)$.

(Hint : You will have to find a parametric form of the equation of this line segment.)

Don't forget to check your answer with the one given in Sec. 14.6.

Line integrals prove useful in solving certain physical problems.

Suppose we have a thin wire shaped in the form of a circle $x(t) = a \cos t$, $y(t) = a \sin t$, $0 \le t \le 2\pi$.

Suppose that the density of the wire at a point (x,y) is given by $\delta(x,y)$. Can we find the mass of the wire?

In this case you can see that if Δs is the length of a small portion of the wire, then the mass of this small portion is approximately equal to $\delta(x,y)\Delta s$.

This means that in this case we can interpret the upper and lower sums as approximations to the mass of the wire. Therefore, we have that

$$\text{the mass of the wire} = \int_C \delta(x,y)\,ds$$

Here is an example.

Example 4 : Let us find the mass of a thin wire shaped in the form of a circle, $x(t) = a\cos t$, $y(t) = a\sin t$, $0 \le t \le 2\pi$, having density $\delta(x,y) = 5y^2$.

Let us denote by C, the curve given by $x(t) = a\cos t$, $y(t) = a\sin t$, $0 \le t \le 2\pi$. Then

$$\text{the mass of the wire} = \int_C \delta(x,y)\,ds$$

$$= \int_0^{2\pi} 5\,a^2\sin^2 t\,\sqrt{a^2(\cos^2 t+\sin^2 t)}\,dt$$

$$= 5a^3\int_0^{2\pi}\sin^2 t\,dt$$

$$= 5a^3\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt = 5\pi a^3.$$

Apart from this application, there are many other applications of line integrals. One of the most important among them is the calculation of the work done by a force. Let f(x,y) and g(x,y) be two real-valued continuous functions defined on a curve C such that $\int_C f(x,y)dx$ and $\int_C g(x,y)dy$ exist. Then we can interpret the line integral $\int_C f(x,y)dx + g(x,y)dy$ as the work done by a force $F = (f(x,y), g(x,y))$ in moving a particle along the curve C. You will be able to understand this after going through the next example. Before giving the example we shall mention one thing. For defining the line integral $\int_C f(x,y)ds$, we had assumed that C is smooth. But we can extend the definition to curves which are not smooth, but are piecewise smooth. A **piecewise smooth** curve C is that curve, which consists of several smooth curves, $C_1, C_2, \ldots, C_n$ joined to each other. You can see an example of a piecewise smooth curve in Fig. 4. For such a curve we define

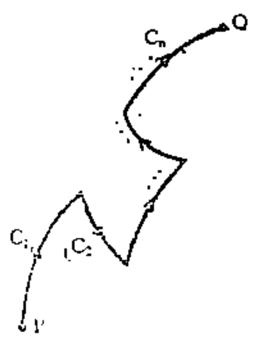


Fig. 4

$$\int_C f(x,y)dx = \sum_{i=1}^{n}\int_{C_i} f(x,y)dx$$

$$\int_C f(x,y)dy = \sum_{i=1}^{n}\int_{C_i} f(x,y)dy$$

$$\int_C f(x,y)\,ds = \sum_{i=1}^{n}\int_{C_i} f(x,y)ds$$

In the following example we consider a piecewise smooth curve.

Example 5 : Let us find the work done by a force $F = (x^2y, xy^2)$ in moving a particle from (0,0) to (1,1) along the line segment from (0,0) to (1,0) followed by the line segment from (1,0) to (1,1).

To find the work done we have to evaluate the line integral

$$\int_C x^2y\,dx+xy^2dy,$$

where $C = C_1+C_2$, C_1 is the line segment joining (0,0) to (1,0) and C_2 is the line segment joining (1,0) to (1,1) see Fig. 5. Thus,

$$\int_C x^2ydx+xy^2dy = \int_{C_1} x^2ydx+xy^2dy+\int_{C_2} x^2ydx+xy^2dy.$$

On C_1, $y = 0$, $dy = 0$.

Thus,

$$\int_{C_1} x^2ydx+xy^2dy=0$$

On C_2, $x=1$ and hence $dx=0$. Thus,

$$\int_{C_2} x^2y\,dx+xy^2dy = \int_0^1 xy^2dy = \frac{1}{3}.$$

Hence,

$$\int_C x^2y\,dx+xy^2dy = \frac{1}{3}.$$

You can try these exercises now.

E 2) Find the mass of a wire bent in the shape of the curve $y = x^2$ between (−2,4) and (2,4), if its density is given by $\delta(x,y) = k\,|x|$.

(**Hint** : Here you will have to evaluate the line integral separately over the curve from (−2,4) to (0,0) and from (0,0) to (2,4).)

E 3) Calculate the work done by the force **F** given in Example 5 in moving a particle from (0,0) to (1,1) along

a) the parabola $y = x^2$,

b) the line $y = x$.

E 4) Calculate the work done by the force $F = (xy, y^2)$ in moving a particle along a piecewise smooth curve $C = C_1UC_2UC_3UC_4$, where

$C_1 = \{ (t,0) \mid 0 \le t \le 1\}$
$C_2 = \{ (1,t) \mid 0 \le t \le 1\}$
$C_3 = \{ (1-t, 1) \mid 0 \le t \le 1\}$
$C_4 = \{ (0,1-t) \mid 0 \le t \le 1\}$

Before we conclude this section we mention a few simple facts about line integrals which easily follow from the definition and can be very useful in some computations :

1) If C_1 and C_2 are two curves such that the end point of C_1 = the initial point of C_2, then

$$\int_{C_1} f(x,y)dx + \int_{C_2} f(x,y)dx = \int_{C_3} f(x,y)dx,$$

where C_3 is the union of two paths from the initial point of C_1 to the end point of C_2.

2) $\int_C \{f(x,y)+g(x,y)\}\,dx = \int_C f(x,y)dx + \int_C g(x,y)dx$

3) $\int_C \alpha\, f(x,y)dx = \alpha\int_C f(x,y)dx$ for any real number α.

4) If $|f(x,y)| \leq M$ for (x,y) on C, then

$$\left| \int_C f(x,y)dx \right| \leq ML,$$

where L = the arc length of C.
Similar results hold for the other two line integrals too.

5) $$\int_C f(x,y)dx = -\int_{-C} f(x,y)dx$$

$$\int_C f(x,y)dy = -\int_{-C} f(x,y)dy$$

$$\int_C f(x,y)ds = -\int_{-C} f(x,y)ds$$

In E 3) we had asked you to find the work done by a force in moving a particle along different paths connecting two given points. You must have got different values for the work done. This is a quite common occurrence. The line integral of a function usually depends on the curve over which it is integrated. But there is a special category of functions whose line integral is independent of the path chosen to connect two given points. We are going to consider such functions in the next section.

14.3 INDEPENDENCE OF PATH

In this section we shall mainly consider line integrals of the form

$\int_C (Mdx+Ndy)$, where M and N are real-valued functions of the two variables x, y.

In the last section we have seen that such line integrals arise while computing the work done by a force.

Now here is a definition.

Definition 2 : Let D be a domain in $\mathbb{R}^2$. If for any two points A and B in D, the line integral $\int_C (Mdx+Ndy)$ has the same value for every positively oriented path C in D joining A and B, then we say that

$\int_C (Mdx + Ndy)$ **is independent of path in D.**

Now we present a theorem, which identifies a class of functions F = (M,N) such that $\int_C (Mdx+Ndy)$ is independent of path. This theorem is also called the **Fundamental Theorem for Line Integrals.** You will be able to note its obvious similarity with the Fundamental Theorem of Calculus which you have studied earlier (See Theorem 7, Unit 10, Calculus).

Theorem 3 : Suppose F = (M,N) is such that $M = f_x$, $N = f_y$ for a continuously differentiable function, f, on a domain D. Suppose C is a simple, smooth, positively oriented curve given by

$\{x = x(t), y = y(t), a \leq t \leq b\}$ lying in D. Then

$$\int_C (Mdx+Ndy) = f(B) - f(A), \text{ where}$$

A = (x(a), y(a)) and B = (x(b), y(b)), are the initial and terminal points of C, respectively.

Proof : Now

$$\int_C (Mdx + Ndy) = \int_C f_x dx + \int_C f_y\, dy$$

$$= \int_a^b f_x \frac{dx}{dt}\, dt + \int_a^b f_y \frac{dy}{dt}\, dt \text{ (By Theorem 2)}$$

$$= \int_a^b \left[f_x \frac{dx}{dt} + f_y \frac{dy}{dt} \right] dt$$

$$= \int_a^b \frac{df}{dt}\, dt \quad \text{by chain rule (Theorem 2, Unit 7)}$$

$$= f(x(t), y(t))\Big]_a^b$$

$$= f(x(b), y(b)) - f(x(a), y(a))$$

$$= f(B) - f(A).$$

Those of you who are familiar with vector calculus would recognize that

$$\int_C Mdx+Ndy = \int_C \nabla f \cdot \mathbf{dr}, \text{ where } \nabla f = f_x \hat{i} + f_y \hat{j}$$

and $\mathbf{dr} = dx\hat{i} + dy\hat{j}$

So Theorem 3 tells us that if f is continuously differentiable, then the line integral

$$\int_C \nabla f \cdot \mathbf{dr} = f(B) - f(A).$$

Thus, the value of this line integral depends only on the end points of the curve C. Therefore, if we have another simple, smooth, positively oriented curve C_1 joining A and B, we will have

$$\int_C \nabla f \cdot \mathbf{dr} = \int_{C_1} \nabla f \cdot \mathbf{dr}.$$

In other words, $\int_C \nabla f \cdot \mathbf{dr} = \int_C (Mdx+Ndy)$ is independent of path in D.

The converse of Theorem 3 is also true. In fact, we have the following theorem, one part of which is nothing but Theorem 3.

Theorem 4 : Let F = (M, N) be continuous on a domain D. Then the line integral

$\int_C (Mdx+Ndy)$ is independent of path if and only if $M = f_x$ and $N = f_y$

for some continuously differentiable real-valued function f on D.

We are not going to prove this theorem here.

Now if $\int_C (Mdx+Ndy)$ is independent of path in a domain D, then what can we say about $\int_C (Mdx+Ndy)$, where C is any closed curve in D? To answer this, let us consider the closed curve C shown in Fig. 6. Take any two points A and B on C. Call the portion of C from A to B as C_1, and that from B to A as C_2.

Then
$$\int_C (Mdx+Ndy) = \int_{C_1} (Mdx+Ndy) + \int_{C_2} (Mdx+Ndy)$$

$$= \int_{C_1} (Mdx+Ndy) - \int_{-C_2} (Mdx+Ndy)$$

Now since C_1 and $-C_2$ have the same initial and terminal points, and $\int_C Mdx+Ndy$ is independent of path, we have

$$\int_{C_1} (Mdx+Ndy) = \int_{-C_2} (Mdx+Ndy).$$

This shows that

$$\int_C (Mdx + Ndy) = 0.$$

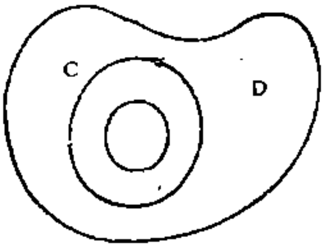
If a force F is such that $F = (f_x, f_y)$ for some continuously differentiable function f defined in a domain D, then F is called a **conservative force.** Thus, the above discussion together with Theorem 4 shows that if F is a conservative force, then the work done by F in moving a particle along a closed path is 0.

Let F be a conservative force given by $F = (f_x, f_y)$. Then we have

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(f_x) = f_{xy} = f_{yx} \text{ since f is continuously differentiable.}$$

$$= \frac{\partial N}{\partial x}$$

Conversely, we can also use this condition to check whether a force is conservative or not.



But to be able to do this, we have to put an additional constraint on the domain D: We assume that D is simply connected. Now what is a simply connected domain? Roughly speaking, a domain D is simply connected if it has no holes in it. For example the domain shown in Fig. 7 is not simply connected. This means we avoid the type of domain shown in Fig. 7. Here is the precise definition:

Definition 3 : A domain D in $\mathbf{R}^2$ is said to be **simply connected** if every smooth closed curve in D, which does not intersect itself, is the boundary of a region contained entirely in D.

The interior of a rectangle, the region bounded by the lines $y=a$ and $y=b$, $a>b$ are examples of simply connected domains.

Now we state a theorem (without proof) which gives a method of identifying conservative forces.

Theorem 5 : Suppose $F = (M,N)$, where M, N are continuously differentiable on a simply connected domain D. F is conservative, i.e., $F = \nabla f = (f_x, f_y)$, if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This theorem together with Theorem 3 is very useful in evaluating the line integral $\int_C Mdx+Ndy$. We shall illustrate this in the following example.

Example 6 : Let us evaluate $\int_C (y \sin xy \, dx + x \sin xy \, dy)$, where C is the line segment from (1,2) to (3,4).

Here $M = y \sin xy$ and $N = x \sin xy$.

$$\therefore \frac{\partial M}{\partial y} = xy \cos xy + \sin xy = \frac{\partial N}{\partial x}.$$

This shows that $F = (M,N)$ is conservative (Theorem 5). Therefore, by Theorem 3

$$\int_C (y\sin xy dx + x \sin xy dy) = f(3,4) - f(1,2),$$

where f is such that $F = \nabla f$.

Therefore, to evaluate the line integral we have to find f.

Now we know that

$f_x = M = y \sin xy$ and

$f_y = N = x \sin xy$ ....(3)

$\therefore f = \int f_x dx = \int y \sin xy dx = -\cos xy + \phi(y)$, ...(4)

where ϕ is some function of y.

Differentiating both sides of Equation (4) w.r.t. y, we get

$f_y = x \sin xy + \phi'(y)$.

If you compare this with (3), you will see that $\phi'(y) = 0$.

$\therefore f(x,y) = -\cos xy +$ a constant.

This gives us

$\int_C (y \sin xy\, dx + x \sin xy\, dy) = \cos 2 - \cos 12$.

See if you can do these exercises now.

E 5) Apply Theorem 5 to decide which of the following functions are conservative.

a) $F = (y+\cos x,\ x-1)$

b) $F = (2ye^x,\ y^2e^x)$

E 6) If possible, find a function f such that

$F = (4x^3+9x^2y^2,\ 6x^3y+6y^5) = \nabla f$.

E 7) Show that the following line integral is independent of path, and evaluate it

$\int_{(-1,2)}^{(3,1)} (y^2+2xy)dx + (x^2+2xy)dy$.

By now you have become quite familiar with line integrals. In the next section we shall discuss an important theorem. This theorem brings out the connection between the line integral on a closed curve C and the double integral over D, where D is the region enclosed by C, provided D and C satisfy certain requirements.

14.4 GREEN'S THEOREM

Green's theorem is one of the major theorems of calculus, and has wide application in physics. In fact, it arose during the study of gravitational and electric potential. It is named after the English mathematician and physicist, George Green (1793–1841), who discovered it around 1828.

Here we shall prove Green's theorem for a special type of regions and then indicate, how it can be extended to other more general regions.

You may recall that we have described different types of regions (Type I, Type II, both Type I and Type II) in Unit 11 and discussed double integration over these regions. Now we shall state and prove Green's Theorem for regions which are of both Type I and Type II.

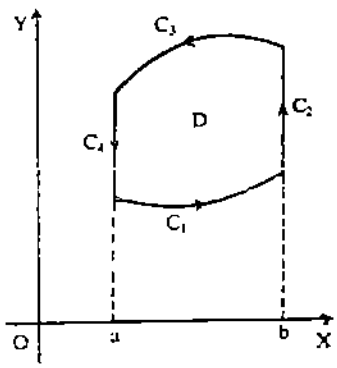


Fig. 8

Theorem 7 (Green's Theorem) : Let D be a region which is of both Type I and Type II. Let C denote its boundary, oriented in the anti-clockwise sense, i.e., when we move along C, the region D lies to our left.

Let P and Q have continuous partial derivatives in D and on C. Then

$$\int_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy.$$

Proof : Since D is of Type I, it is like the region shown in Fig. 8. Let

$D = \{ (x,y) \mid a \le x \le b,\ \phi_1(x) \le y \le \phi_2(x) \}$

Note the orientation of C.

C consists of four arcs, C_1, C_2, C_3 and C_4.

Out of these, C_2 or C_4 (or both) could be degenerate.

Now $\int_C Pdx = \int_{C_1} Pdx + \int_{C_2} Pdx + \int_{C_3} Pdx + \int_{C_4} Pdx.$

You can see that C_2 and C_4 are parallel to the y-axis. Therefore, x is a constant on both these curves and thereby $dx = 0$. This means

$$\int_{C_2} Pdx = \int_{C_4} Pdx = 0.$$

Thus,

$$\int_C Pdx = \int_{C_1} Pdx + \int_{C_3} Pdx$$

Now C_1 is the curve given by $\phi_1(x)$, $a \le x \le b$ and C_3 is the curve given by $\phi_2(x)$, $b \le x \le a$.

$$\int_C P\,dx = \int_a^b P(x,\phi_1(x))\,dx + \int_b^a P(x,\phi_2(x))\,dx.$$

$$= -\int_a^b \{P(x,\phi_2(x)) - P(x,\phi_1(x))\}\,dx$$

$$= -\int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,dy \right] dx$$

$$= -\iint_D \frac{\partial P}{\partial y}\,dxdy \qquad \text{.........................(5)}$$

Now D is also of Type II. Therefore, following exactly similar steps we can prove that

$$\int_C Qdy = \iint_D \frac{\partial Q}{\partial x}\,dxdy \qquad \text{......................(6)}$$

Combining (5) and (6) we get

$$\int_C (Pdx + Qdy) = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dxdy, \qquad \text{....................(7)}$$

where D is a region of Type I and Type II.

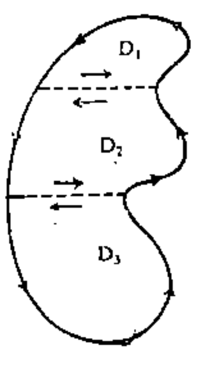


Fig. 9

We can easily extend this to those regions, which can be expressed as the union of a number of regions which are of both of Type I and Type II. For example, consider the region in Fig. 9. We can write it as the union of D_1, D_2 and D_3, and each of these is a region of both Type I and Type II. Here we have to be careful about the orientation given to the common boundaries. Remember, that the orientation should be such that when you walk along the curve in the positive direction, the enclosed region should always be to your left (see Fig. 10). You will see that if we add the line integrals on the boundaries of D_1, D_2 and D_3, the line integrals on

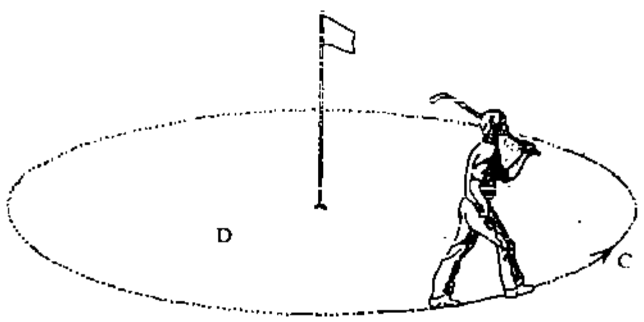


Fig. 10

the common boundaries of these regions will cancel out. Thus, we will be left with the line integral on the boundary of D.

Here we would also like to mention that Green's Theorem also holds for regions with one or more holes. Again we have to be careful about fixing the orientations of the various curves involved. See Fig. 11. Check that when you walk along any curve in the positive direction, the region of our interest is to your left. Sometimes line integrals can be easily evaluated by using Green's theorem. Here is an example to illustrate this.

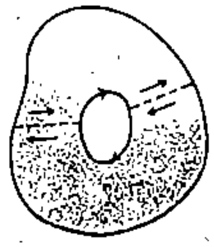


Fig. 11

Example 7 : Suppose C is the boundary of the square shown in Fig. 12. Let us evaluate

$$\int_C (x^2y\,dx + 2x^5y\,dy).$$

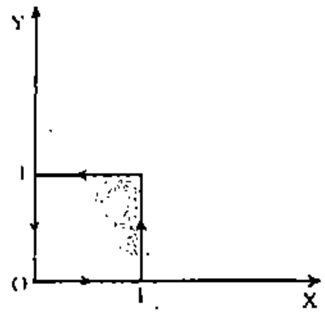


Fig. 12

By Green's theorem we get

$$\int_C (x^2y\,dx + 2x^5y\,dy) = \int_D\int \left[\frac{\partial}{\partial x}(2x^5y) - \frac{\partial}{\partial y}(x^2y)\right] dxdy$$

$$= \int_D\int (10x^4y - x^2)\,dxdy$$

$$= \int_0^1 \left[\int_0^1 (10x^4y - x^2)\,dx\right]dy$$

$$= \int_0^1 \left[2x^5y - \frac{x^3}{3}\right]_0^1 dy$$

$$= \int_0^1 \left(2y - \frac{1}{3}\right) dy$$

$$= \frac{2}{3}.$$

Green's theorem enables us to find the area of those regions for which Green's theorem holds. Let D be a region for which Green's theorem holds and let C denote its boundary.

If we take $P = -y$ and $Q = x$ in (7), we get

$$\int_C (Pdx + Ody) = \int_C (xdy - ydx) = \int_D\int \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dxdy$$

$$= 2\int_D\int dxdy.$$

Therefore, $\int_D\int dx\,dy = \frac{1}{2}\int_C (xdy - ydx)$ ....

Now the left hand side of (8) gives the area A of the region D. So we get

$A = \frac{1}{2}\int_C (xdy - ydx)$, ...

where A is the area of the region enclosed by C.

Thus, using Green's theorem, we could express the area of those regions for which Green's theorem holds, as a line integral over the boundary.

We now use Formula (9) to find the area enclosed by an ellipse.

Example 8 : Let C be the ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The parametric equations of C can be written as :

$$x = a\cos t,\ y = b\sin t,\ 0 \le t \le 2\pi.$$

Therefore, the area enclosed by this ellipse is

$$A = \frac{1}{2} \int_C (xdy - ydx)$$

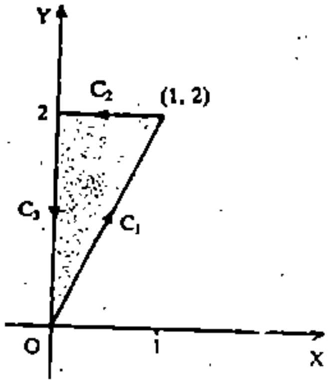
$$= \frac{1}{2} \int_0^{2\pi} (x\frac{dy}{dt}\,dt - y\frac{dx}{dt}\,dt)$$

$$= \frac{1}{2} \int_0^{2\pi} (a \cos t \,.\, b \cos t + b \sin t \,.\, a \sin t)\, dt$$

$$= \frac{1}{2} ab \int_0^{2\pi} dt$$

$$= ab\pi.$$

See if you can do the following exercises now.



E 8) Evaluate $\int_C (4x^2ydx + 2ydy)$, where C is the boundary of the triangle shown alongside, by

a) the direct method

b) using Green's theorem

E 9) Use Green's theorem to evaluate

$\int_C (x^3 + 2y)\, dx + (4x-3y^2)\, dy$, where C is the ellipse :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

E 10) Find the area of the astroid : $x = a \cos^3 t$, $y = a \sin^3 t$, $0 \le t \le 2\pi$ by using Formula (9).

This brings us to the end of this unit, this block and also of this course. Throughout this course we have discussed various theorems and proved some of them. Though our main emphasis was on the application of these theorems, their proofs are also important. So we suggest that you carefully study the proofs of the theorems given in this course. You may be asked to reproduce some of those in your term-end examination.

Let us now summarise the points covered in this unit.

14.5 SUMMARY

In this unit we have

1) defined three line integrals $\int_C f dx$, $\int_C f dy$, $\int_C f ds$.

2) described the evaluation of line integrals using definite integrals. If f is a real-valued continuous function defined on a positively oriented, simple, smooth curve

$C = \{(x(t), y(t)) \mid t \in [a,b]\}$, then

$$\int_C f(x,y)\, dx = \int_a^b f(x(t),y(t))x'(t)dt$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t),y(t))y'(t)dt$$

and

$$\int_C f(x,y)\, ds = \int_a^b f(x(t),y(t))\ \sqrt{x'(t)^2+y'(t)^2}\ dt$$

3) derived expressions for the mass of a thin wire and work done by a force in terms of line integrals.
4) defined conservative forces and developed a criterion to identify conservative forces :
   F = (M,N) is conservative if $M = f_x$ and $N = f_y$ for some continuously differentiable function, f, defined in a domain D. F = (M,N) is conservative on a simply connected domain D iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
5) stated and proved Green's theorem for regions which are of both Type I and Type II :
   $\int_C (Pdx + Qdy) = \int\int_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dxdy$, where D is a region of both Type I and Type II, C is the boundary of D, oriented anticlockwise, P and Q have continuous partial derivatives in D and on C.

14.6 SOLUTIONS AND ANSWERS

E 1) a) $\int_C xy^{2/5}\, ds = \int_0^1 \frac{t}{2} (t^{5/2})^{2/5} \sqrt{\frac{1}{4} + \frac{25}{4} t^3}\ dt$

$$= \frac{1}{4} \int_0^1 t^2 \sqrt{1+25t^3}\ dt$$

$$= \frac{1}{300} \int_0^1 75\, t^2 \sqrt{1+25t^3}\ dt$$

$$= \frac{2}{3} \frac{1}{300} (1+25t^3)^{3/2} \Big]_0^1$$

$$= \frac{1}{450} (26^{3/2}-1).$$

b) The line segment from (0,0) to $(\pi, 2\pi)$ is given by
   $x=t,\ y=2t,\ 0 \le t \le \pi$.

$\therefore \int_C (\sin x + \cos y)\, ds = \int_0^\pi (\sin t+\cos 2t)\ \sqrt{1+4}\ dt$

$$= \sqrt{5} \int_0^\pi (\sin t+\cos 2t)\ dt$$

$$= 2\sqrt{5}.$$

E 2) $x=t,\ y=t^2,\ -2 \le t \le 2$ describes the curve $y = x^2,\ -2 \le x \le 2$

$\therefore$ Mass $= \int_{-2}^{2} k\,|x(t)|\ \sqrt{[x'(t)]^2+[y'(t)]^2}\ dt$

$$= -\int_{-2}^{0} kt\ \sqrt{1+4t^2}\ dt + \int_0^2 k\,t\ \sqrt{1+4t^2}\ dt$$

$$= \frac{k}{6} (17\sqrt{17} - 1)$$

E 3) a) C : $x=t$, $y=t^2$, $0 \le t \le 1$.

$$\therefore \text{ Work done } = \int_C x^2y\,dx + xy^2dy$$

$$= \int_0^1 t^4\,dt + 2\int_0^1 t^6dt$$

$$= \frac{1}{5} + \frac{2}{7}$$

$$= \frac{17}{35}.$$

b) C : $x=t$, $y=t$, $0 \le t \le 1$

$$\text{Work done } = \int_0^1 t^3\,dt + 2\int_0^1 t^3\,dt$$

$$= \frac{1}{2}.$$

E 4) $\int_C xy\,dx+y^2dy = \int_{C_1} xy\,dx + y^2\,dy + \int_{C_2} xy\,dx + y^2\,dy.$

$$+ \int_{C_3} xy\,dx + y^2\,dy + \int_{C_4} xy\,dx + y^2\,dy.$$

Now $\int_{C_1} xy\,dx + y^2\,dy = 0.$

$\int_{C_2} xy\,dx + y^2\,dy = \int_0^1 t^2dt$, since $x = 1$, and hence $dx = 0$.

$$= \frac{1}{3}$$

$\int_{C_3} xy\,dx + y^2\,dy = -\int_0^1 (1-t)\,dt$, since $dx = -dt$ and $dy = 0$.

$$= -\frac{1}{2}.$$

$\int_{C_4} xy\,dx + y^2\,dy = -\int_0^1 (1-t)^2dt$, since $dx=0$ and $dy= -dt$

$$= -\frac{1}{3}$$

$\therefore \int_C xy\,dx + y^2dy = \frac{-1}{2}.$

E 5) a) $M = y+\cos x$, $N = x-1$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ F is conservative.

b) F is not conservative.

E 6) $\frac{\partial M}{\partial y} = 18x^2y$, $\frac{\partial N}{\partial x} = 18x^2y$

$\therefore$ F is conservative and it is possible to find f such that

$F = (M,N) = (f_x,f_y)$.

Now $f_x = \frac{\partial f}{\partial x} = 4x^3+9x^2y^2.$

$\therefore f = \int (4x^3+9x^2y^2)\, dx$

$= x^4+3x^3y^2+\phi(y)$, where ϕ is some function of y.

$\therefore f_y = 6x^3y + \phi'(y) = 6x^3y+6y^5$

$\therefore \phi'(y) = 6y^5$.

$\therefore \phi(y) = y^6$. (We don't have to consider the constant of integration)

$\therefore f = x^4+3x^3y^2+y^6$ is such that

$F = \nabla f$.

E 7) $M = y^2+2xy$, $N = x^2+2xy$.

$\frac{\partial M}{\partial y} = 2y+2x = \frac{\partial N}{\partial x}$

$\therefore$ There exists f such that $(M,N) = (f_x,f_y)$

Now $f_x = y^2+2xy \Rightarrow f = \int (y^2+2xy)dx$

$= y^2x+x^2y+\phi(y)$.

$\Rightarrow f_y = 2xy+x^2+\phi'(y) = x^2+2xy$.

$\Rightarrow \phi'(y) = 0. \Rightarrow \phi(y) =$ some constant.

$\therefore f = y^2x+x^2y$ (We can ignore the constant.) is such that $(M,N) = \nabla f$.

$\therefore$ The given line integral is independent of path.

$$\therefore \int_{(-1,2)}^{(3,1)} (y^2+2xy)\, dx+(x^2+2xy)\, dy = f(3,1) - f(-1,2)$$

$$= 12+2$$

$$= 14$$

E 8) a) See the figure alongside E 8).

On C_1, $y=2x$.

$\therefore \int_{C_1} 4x^2y dx+2y\, dy = \int_0^1 (8x^3+8x)\, dx = 6.$

On C_2, $dy = 0$.

$\therefore \int_{C_2} 4x^2y\, dx+2y\, dy = \frac{8}{3}$.

$dx = 0$ on C_3 $\therefore \int_{C_3} 4x^2ydx+2ydy = -4$.

$\therefore \int_C 4x^2ydx + 2ydy = \frac{-2}{3}$.

b) By Green's theorem,

$$\int_C 4x^2y\, dx + 2y\, dy = \int_D\int (0-4x^2)\, dx\, dy,$$

where D is the shaded triangular region : $0 \le x \le 1$, $2x \le y \le 2$.

$$\text{Now } \int_D\int -4x^2y\, dx\, dy = \int_0^1 \left[\int_{2x}^{2} -4x^2 dy \right] dx$$

$$= \frac{-2}{3}.$$

E 9) Here $P = x^2+2y$, $Q = 4x-3y^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 4-2 = 2.$

$$\therefore \int_C (x^3+2y)\,dx + (4x-3y^2)\,dy = \int_D\int 2\,dx\,dy,$$

where D is the region enclosed by the ellipse

$\therefore$ The given integral = 2 . area enclosed by C

$= 2\pi ab.$

E 10) $A = \frac{1}{2}\int_C x\,dy - y\,dx.$

$$= \frac{1}{2}\int_0^{2\pi} a\cos^3 t(3a\sin^2 t\cos t)dt + a\sin^3 t(3a\cos^2 t\sin t)\,dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \sin^2 t\cos^2 t\,dt$$

$$= \frac{3\pi a^2}{2}.$$

VIDEO PROGRAMME : DOUBLE INTEGRATION

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INTRODUCTION

In this note we will briefly describe the contents of this video programme. We hope that through this visual medium you will get a better understanding of lower and upper product sums leading to the concept of double integration. During the programme we ask you to note down some exercises and to do them after seeing the programme. We will list those exercises here, and also give the answers.

PROGRAMME SUMMARY

This programme is based on Unit 11 in Block 4. We start by recalling lower and upper product sums and the definition of the definite integral of a function from $[a,b] \to \mathbf{R}$. Then we talk about these concepts for a real-valued function of two variables defined on a rectangle. The need to extend the definition of double integral to non-rectangular regions is then discussed. We also give some practical applications of double integrals.

During the programme we have suggested that you try the following exercises.

EXERCISES

E1) Express the following quantities as double integrals.

a) A designer designs a perfume bottle. It has a rectangular base 4cm × 3 cm. Its cross-section is a parabola given by $f(x,y) = -y^2 + 4y$. How much perfume can this bottle contain?

b) The production function of a factory is given by $p(x,y) = 500x^2y^8$, where x is the number of persons employed and y is the amount of rupees spent (in thousands). What will be the average production if $10 \le x \le 50$ and $20 \le y \le 40$.

E2) Find the quantities required in E1) by using repeated integrals.

ANSWERS

E1) a) $\int_0^4 \int_0^3 (-y^2 + 4y)\, dydx$

b) Total production $= \int_{10}^{50} \int_{20}^{40} 500x^2y^8 dydx$

c) Average production $= \dfrac{\text{Total production}}{(50 - 10)(40 - 20)}$

E2) a) $\int_0^4 \int_0^3 (-y^2 + 4y)dydx = \int_0^4 [\int_0^3 (-y^2 + 4y)dy]dx$

$$= \int_0^4 \left[-\frac{y^3}{3} + 2y^2 \right]_0^3 dx$$

$$= \int_0^4 9dx$$

$$= 36$$

$$\text{Total production} = \int_{10}^{50}\left[\int_{20}^{40} 500x^2y^8\,dy\right]dx$$

$$= 500\int_{10}^{50} x^2 \left.\frac{y^9}{9}\right]_{20}^{40} dx$$

$$= \frac{500}{9}\,(40^9 - 20^9)\int_{10}^{50} x^2\,dx$$

$$= \frac{500}{27}\,(40^9 - 20^9)\,(50^3 - 10^3)$$

$$\text{Average production} = \frac{5}{216}\,(40^9 - 20^9)\,(50^3 - 10^3)$$

E R TA

MTE-07 Block-3

Page No.	Line No.	Should be
9	34	$f(x_0+h) = \sum_{r=0}^{n} f^r(x_0)\frac{h^r}{r!} + \ldots$
10	36	$\ldots\ldots e^{\|x\|} \frac{1}{n+1!}$
12	24	$T_m(x,y) = \sum_{i,j=0}^{i+j\leq n} \frac{1}{i!\,j!}\left[\frac{\partial^{i+j}f}{\partial x^i \partial y^j} (x_0,y_0)\right]$
15	29	...... of one variable
31	4	$\ldots = \frac{x}{\sqrt{x^2+y^2}+y}$
31	3	$\ldots = \frac{\frac{1}{2}x}{1+\frac{y}{\sqrt{x^2+y^2}}}$
31	27	$\ldots + 2\frac{\partial^2 f(x_0,y_0)}{\partial x\partial y}(x-x_0)(y-y_0)$
32	34	$\ldots = \sum_{m=0}^{2}\frac{f^m(1)}{m!}(x-1)^m$
34	12	$P_3(x,y) = P_2(x,y) + (x-1)^3 + (y-1)^3$ and $P_m(x,y) = P_3(x,y),\ m\geq 3$
38	8	$f(x,y) = 2\cos(x+y) + e^{xy}$
40	4	The point is (1,1).
44	23	$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \cos\theta\sin\phi \ldots\ldots \\ \ldots\ldots \\ \ldots\ldots \end{vmatrix}$
45	18	$g(x,y) = y+x^2$ at $(0,0)$
48	—	All the matrix signs, [ ] should be replaced by determinant signs, | |.
48	10	$J_F(g(x))J_g(x) = \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{vmatrix}$
54	27	$\ldots = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix} = 0$

	58	15	Replace the matrix sign [ ] by the determinant sign | |.
	63	3	$= \dfrac{f(1+y^2)^2}{(f^2+1)(1+y^2)^2}$
	63	6	$= \begin{vmatrix} -y/x^2 & 1/x \\ 1/y & -x/y^2 \end{vmatrix}$
	64	40	..... $x \in [-1, 1]$
	70	17	neighbourhood of 1.
	76	11	$0 = x^4 - a^2b^2$ ....
	79	28	.... N of 1,
	80	4	..... defined on a neighbourhood of 1 such that ....

NOTES

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