Uttar Pradesh Rajarshi Tandon
Open University

UGMM-101
Differential Calculus

DIFFERENTIAL CALCULUS

RIL-145

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BLOCK

1

Set, Relation, Function and its Property

UNIT-1

SET AND RELATION

Structure

1.1. INTRODUCTION

The notations and terminology of set theory which was originated in the year 1895 by the German mathematician G. Cantor. In our daily life, we often use phrases of words such as a bunch of keys, a pack of cards, a class of students, a team of players, etc. The words bunch, pack, class and team all denote a collection of several discrete objects. Also, the dictionary meaning of set is a group or a collection of distinct, definite and distinguishable objects selected by means of some rules or description.

In this unit we will introduce set and various examples of sets. Then we will discuss types and some operations on sets. We will also introduce Venn diagrams, a pictorial way of describing sets. Cartesian product of two sets, relation, equivalence relation, order relation, equivalence class, partition of a set. Knowledge of the material covered in this unit is necessary for studying any mathematics course, so please study this unit carefully.

Objectives

After studying this unit you should be able to:

- Use the notation of set theory;
- Find the union, intersection, difference, complement, and Cartesian product of sets;
- Identify a set, represent sets by the listing method, property method and Venn diagrams;
- Prove set identities, and apply De Morgan's laws;
- Recall the basic properties of relations;
- Derive other properties with the help of the basic ones;
- Identify various types of relations;
- Understand the relationship between equivalence classes and partition;

1.2 Set Theory

It was first of all used by **George Cantor**. According to him, '**A set is any collection into a whole of definite and distinct objects of our intuition or thought**'. However, Cantor's definition faced controversies due to the forms like 'definite' and 'collection into a whole'. Later on, a single word 'distinguishable' used to make the definition acceptable. '**A set is any collection of distinct and distinguishable objects around us'**. By the form 'distinct', we mean that no object is repeated and some lack the term 'distinguishable' we mean that whether that object is in our collection or not. The objects belonging to a set are called as elements or members of that set. For example, say A is a set of stationary used by any student i.e.

A = {Pen, Pencil, Eraser, Sharpener, Paper}

A set is represented by using all its elements between bracket {} and by separating them from each other by commas (if there are more than one element). As we have seen sets are denoted by capital letters of English alphabet while the elements are divided in general, but small letters. If x is an element of a set A, we write $x \in A$ (read as 'x belongs to A'). If x is not an element of A, we write $x \notin A$ (read as x does not belong to A). Examples:

(i) Let A = {4, 2, 8, 2, 6}. The elements of this collection are distinguishable but not distinct, hence A is not a set. Since 2 is repeated in A.

(ii) Let B ={*a*, *e* ,*i*, *o*, *u}* i.e. *B* is set of vowels in English. Here elements of B are distinguishable as well as distinct. Hence *B* is a set.

Two Forms of Representation of a Set

1. ‘Set-builders from’ representation of set, and
2. ‘Tabular form’ or ‘Roaster form’ representation of set.

In ‘**set-builder form’** of representation of set, we write between the braces { } a variable *x* which stands for each of the elements of the set, then we state the properties possessed by *x*. We denote this property of *p(x)* by a symbol: or (read as ‘such that’)

$A = \{ x: p(x)\}$

$A = \{x : x$ is Capital of a State}

A = { *x: x* is a natural number and $2 < x < 11$}

‘**Tabular Form’** or ‘**Roaster Form’**, the elements of a set listed one by one within bracket { } and one separated by each other by commas.

B = {Lucknow, Patna, Bhopal, Itanagar, Shillong}, B = { 3, 4, 5, 6, 7, 8, 9, 10}

1.3 Types of Set

II. **Finite Set:** A set is finite if it contains finite number of different elements. For examples,

a) The set of months in a year.

b) The set of days in a week.

c) The set of rivers in U.P.

d) The set of students in a class.

e) The set of vowels in English alphabets.

f) $A = \{1, 2, 4, 6\}$ is a finite set because it has four elements.

g) *B* = a null set ϕ, is also a finite set because it has zero number of elements.

III. **Infinite Set:** A set having infinite number of elements i.e. a set where counting of elements is impossible, is called an infinite set. For examples,

a) *A* = { *x : x* is the set of all points in the Euclidean planes}.

b) B = {The set of all straight lines in a given plane}.

c) $A = \{ x : x$ is a set of all natural numbers}.

IV. **Null set (or Empty Set or Void Set) :** A set having no element is called as empty set or void set. It is denoted by ϕ or { }. For examples,

a) $A = \{x: x$ is an even number not divisible by 2 }.

b) $A = \{x: x^2+4=0, x \text{ is real}\}$.

c) $\varphi = set\ of\ all\ those\ x\ which\ are\ not\ equal\ to\ x\ itself$ i.e. = $\{x: x \neq x\}$.

V. **Singleton Set:** *A* set having single element is called a singleton set. For examples,

a) $A = \{x : x$ is present Prime Minister of India}.

b) **N** = {2 }.

c) $A = \{x: 4 < x < 6, x \text{ is an integer}\}$.

VI. **Pair Set:** A set having two elements is called a pair set.

Examples: {1, 2}, {0, 3}, {4, 9} etc.

VII. **Equality of sets:** Two sets A and B are said to be equal if every element of A is an element of B and also every element of B is an element of A. The equality of two sets A and B is donated by A =B. Symbolically,

$$A = B \text{ if and only if } x \in A \Leftrightarrow x \in B$$

Examples: A = {4, 3, 2, 1} and B = { 1, 3, 2, 4}

Then A =B, because both have same and equal numbers.

VIII. **Subsets and Supersets:** Let A and B be two non-empty sets. The set A is a subset of B if and only if every element of A is an element of B. In other words, the set A is a subset of B if $x \in A \Rightarrow x \in B$. Symbolically, this relationship is written as

$$A \subseteq B \text{ if } x \in A \Rightarrow x \in B$$

which is read as 'A is a subset of B'or'A is contained in B'. If $A \subseteq B$, then B is called the superset of A and we write B $\subseteq A$ which is read as 'B is a superset of A' or B contains A'.

If the set A is not a subset of the B, that is, if at least one element of A does not belong to B and we write, $A \not\subset B$. In other words, if $x \in A \Leftrightarrow x \notin B$ which is read as 'A is not a subset of B'.

Properties of subsets

a) If the set A is a subset of the Set B, then the set B is called superset of the set A.

b) If the set A is subset of the Set B and the Set B is a subset of the set A, then the sets A and B are said to be equal , i.e., $A \subseteq B$. and $B \subseteq A \Rightarrow A=B$.

c) If the set A is a subset of the B and the set B is a subset of C, then A is a subset of C, i.e., $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$.

Example: Let A = {4, 5, 6, 9} and B = { 4, 5, 7, 8, 6}then we write $A \nsubseteq B$,

another example $A = \{1,2,3\}$, B = {2,3,1}$\Rightarrow A \subseteq B$ also $B \subseteq A$.

Here $A \subseteq B$ can also be expressed equivalently be writing $B \supseteq A$, read as B is a superset of A. So, a set A is said to be superset of another set B, if set A contains all the element of Set B.

IX. ***Proper Subset:*** Set A is said to be a proper subset of a set B if

(a) Every element of set A is an element of set B, and

(b) Set B has at least one element which is not an element of set A.

This is expressed by writing $A \subset B$ and read as A is a proper subset of B , if A is not a proper subset of B then we write it as $A \not\subset B$.

Examples

(i) Let $A = \{4,5,6\}$ and $B = \{4, 5, 7, 8, 6\}$ So, $A \subset B$

(ii) Let $A = \{1,2,3\}$, and $B = \{3, 2, 9\}$ So, $A \not\subset B$.

X. **Comparability of Sets:** Two sets A and B are said to be comparable if either one of these happens.

(i) $A \subset B$

(ii) $B \subset A$

(iii) $A = B$

Similarly if neither of these above three exist i.e. $A \not\subset B$, $B \not\subset A$ and $A \neq B$, then A and B are said to be incomparable.

Example A = (1, 2, 3}, and $B = \{1,2\}$**.** Hence set A & B are comparable.

But $A = \{1, 2, 3\}$ and $B = \{2,3,6,7\}$ are incomparable

XI. **Universal Set:** Any set which is super set of all the sets under consideration is known as the universal set and is either denoted by Ω or S or $\cup$. It is to note that universal set can be chosen arbitrarily for discussion, but once chosen, it’s is fixed for the discussion.

Example: Let $A = \{1,2,3\}$ $B = \{3,4,6,9\}$ and $C = \{0,1\}$

We can take $S = \{0,1,2,3,4,5,6,7,8,9\}$ as Universal Set for these sets A, B and C.

XII. **Power Set:** The set or family of all the subsets of a given set A is said to be the power set of A and is expressed by $P(A)$.Mathematically, $P(A)= \{B : B \subseteq A\}$ So, $B \in P(A) \Rightarrow B \subseteq A$

Example: If, $A = \{1\}$ then $P(A) = \{\phi, \{1\}\}$ If, $A = \{1,2\}$, then $P(A)$ $\{\phi, \{1\}, \{2\}, \{1,2\}\}$ Similarly if $A = \{1, 2, 3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$ So, trends show that if A has n elements then $P(A)$ has 2^n elements.

XIII. **Complements of Set:** The complement of a set A, also known as 'absolute complement' of A is the sets of all those elements of the universal sets which are not element of A. it is denoted by $\boldsymbol{A^c}$ **or** $\boldsymbol{A^1}$**.** Infact $\mathbf{A^1}$ **or** $\mathbf{A^c = U - A}$**.** Symbolically $A^1 = \{x : x \in U$ and $x \notin A\}$.

Example: *Let* $U = \{1,2,3,4,5,6,7,8,9\}$ and $A = \{2,3,5,6,7\}$, then $A^1 = U - A = \{1,4,8,9\}$

1.4 Operations on sets

We will discuss mainly three operations on sets i.e. Union of sets, Intersection of sets and Differences of Sets.

Union of Sets: The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ and read as 'A union B'. Symbolically**,** $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Example: Let, $A = \{4,5,6\}$, and $B = \{2,1,3,8\}$ then $A \cup B = \{1,2,3,4,5,6,8\}$.

Properties of Union of Sets:

(a) The Union of Sets is commutative, i.e. A and B are any two sets, then $\mathbf{A \cup B = B \cup A}$.

(b) The Union of Sets is associative ,i.e. A ,B and C are any three sets, then $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$**.**

(c) The Union of Sets is idempotent i.e., if A is any set, then

$\mathbf{A \cup A = A}$**.**

(d) $\boldsymbol{A \cup \phi = A}$**.** where φ is the null set.

(e) $\boldsymbol{A \cup U = U}$**.**

Intersection of sets*:* The intersection of two sets A and B is the set of all the elements, which are common in A and B. This set is denoted by $A \cap B$

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and read as 'A intersection of B'. i.e. Symbolically $A \cap B = \{x : x \in A$ and $x \in B\}$

Example: Let, $A = \{1,2,3\}$, and $B = \{2,1,5,6\}$ then $A \cap B = \{1,2\}$.

Properties of Intersection of Sets:

(a) The Intersection of Sets is commutative, i.e. A and B are any two sets, then $\mathbf{A \cap B = B \cap A}$.

(b) The Intersection of Sets is associative ,i.e. A ,B and C are any three sets, then $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$.

(c) The Intersection $\cap$ of Sets is idempotent i.e., if A is any set, then

$\mathbf{A \cap A = A}$.

(d) $A \cap \phi = \phi$. where φ is the null set.

(e) $A \cap U = A$.

Difference of Sets: The difference of two sets A and B, is the set of all those elements of A which are not elements of B. Sometimes, we call difference of sets as the relative components of B in A. It is denoted by $A - B$. i.e. Symbolically,

$A - B = \{x : x \in A$ and $x \notin B\}$ similarly $B - A = \{x : x \in B$ and $x \notin A\}$

Example: if $A = \{4,5,6,7,8,9\}$, and $B = \{3,5,2,7\}$ then $A - B = \{4,6,8,9\}$ and $B - A = \{3,2\}$ It is mention that $A - B \neq B - A$ So, difference of two sets is not commutative.

Properties of Difference of Sets:

(a) $\mathbf{A - A = \phi}$.

(b) $\mathbf{A - \phi = A}$.

(c) $\mathbf{(A - B) \cap B = \phi}$.

(d) $\mathbf{(A - B) \cup A = A}$.

(e) **A – B, B – A, and** $A \cap B$ are mutually disjoint.

Symmetric Difference : They symmetric difference of two sets A and B is the set of all those elements which are in A but not in B, or which are in B but not in A. It is denoted by $A \Delta B$. Symbolically $A \Delta B = (A - B) \cup (B - A)$.

It is to note that $A \Delta B = B \Delta A$, i.e. symmetric difference is commutative in nature.

Examples : Let $A = \{1,2,3,4,5\}$ and $B = \{3,5,6,7\}$ then $A - B = \{1,2,4\}$ and $B - A = \{6,7\}$, $\therefore A \Delta B = (A - B) \cup (B - A) = \{1,2,4,6,7\}$

1.5 Laws Relating Operations

These two laws are known as associative law of union and intersection. This law holds even for three sets i.e.

(i) $\mathbf{A \cup (B \cup C) = (A \cup B) \cup C}$

(ii) $\mathbf{A \cap (B \cap C) = (A \cap B) \cap C}$.

Theorem 1: For any three sets A, B and C, the following distributive laws hold:

a. $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

b. $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

i.e. union and intersection are distributive over intersection and union respectively.

1.6 De Morgan's Law

For any two sets A and B the following laws known as De Morgan's Law.

1. $\boldsymbol{(A \cup B)' = A' \cap B'}$**, and**

2. $\boldsymbol{(A \cap B)' = A' \cup B'}$.

Proof: **(1)** If $x \in (A \cup B)' \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B \Rightarrow x \in A'$ and $x \in B' \Rightarrow x \in A' \cap B' \Rightarrow x \in (A \cup B)' \Rightarrow x \in A' \cap B'$ So, $(A \cup B)' = A' \cap B'$

(2) Say $x \in (A \cap B)' \Rightarrow x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B \Rightarrow x \in A'$ or $x \in B'$ So $x \in (A \cap B)' = A' \cup B'$ Hence $(A \cap B)' = A' \cup B'$

Some more results on operations on sets

Theorem 2: If A and B are any two sets, then

(a) $\mathbf{(A - B)} = \mathrm{A} \Leftrightarrow \boldsymbol{A \cap B = \phi}$.

(b) $\boldsymbol{(A - B) \cup B = A \cup B}$.

Check your progress

(1) **(i)** Represent the set $A = \{a, e, i, o, u\}$ in set builder form.

(ii) Represent the set $B = \{x$: x is a letter in the word 'STATISTICS'} in tabular form.

(iii) Represent the set $A = \{x : x$ is an odd integer and $3 \leq x < 13\}$ in tabular form.

(2) Are the following sets equal?

$A = \{ x : x$ is a letter in the word 'wolf'}

$B = \{x : x$ is a letter in the word 'follow'}

$C = \{ x : x$ is a letter in the word How}

(3) Find the proper subset of following sets

(i) ϕ

(ii) $\{1,2,3\}$

(iii) $\{0,2,3,4\}$

(4) Find the power sets of the following sets

(i) $\{0\}$

(ii) $\{1\ (2,3)\}$

(iii) $\{4,1,8\}$

(5) If $A = \{2,3,4,5,6\}$, $B=\{3,4,5,6,7\}$, $C= \{4,5,6,7,8\}$, then find

(i) $(A \cup B) \cap (A \cup C)$

(ii) $(A \cap B) \cup (A \cap C)$

(iii) $(A - B)$ *and* $(B - C)$

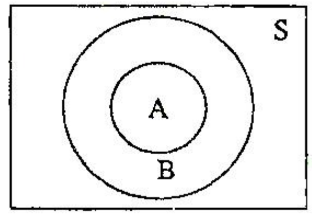
1.7 Venn Diagram

Here we will learn the operations on sets and its applications with the help of pictorial representation of the sets. The diagram formed by these sets is said to be the Venn Diagram of the statement.

A set is represented by circles or a closed geometrical figure inside the universal set. The Universal Set S, is represented by a rectangular region.

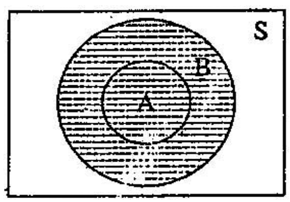
First of all we will represent the set or a statement regarding sets with the help of Venn Diagram. The shaded area represents the set written.

1.7(a) Subset:

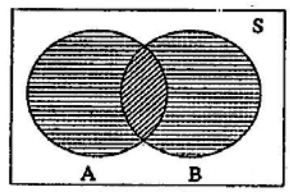


$A \subset B$

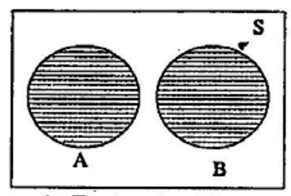
1.7 (b) Union of sets: Let $A \cup B = B$. Here, whole area represented by B represents $A \cup B$.



$A \cup B$ when $A \subset B$

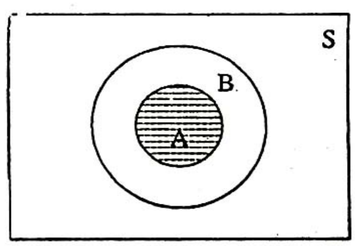


$A \cup B$ when neither $A \subset B$ nor $B \subset A$

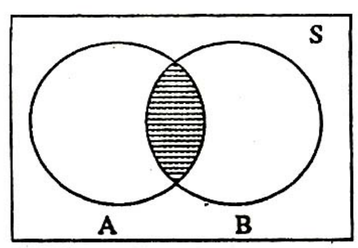


$A \cup B$ when A and B are disjoint events

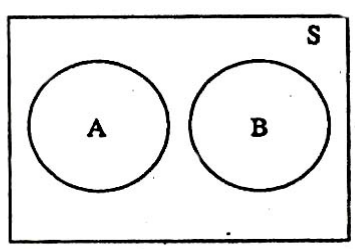
1.7 (c) Intersection of Sets: ($A \cap B$): $A \cap B$ represents the common area of A and B.



$A \cap B$ when $A \subset B$ and $A \cap B = A$

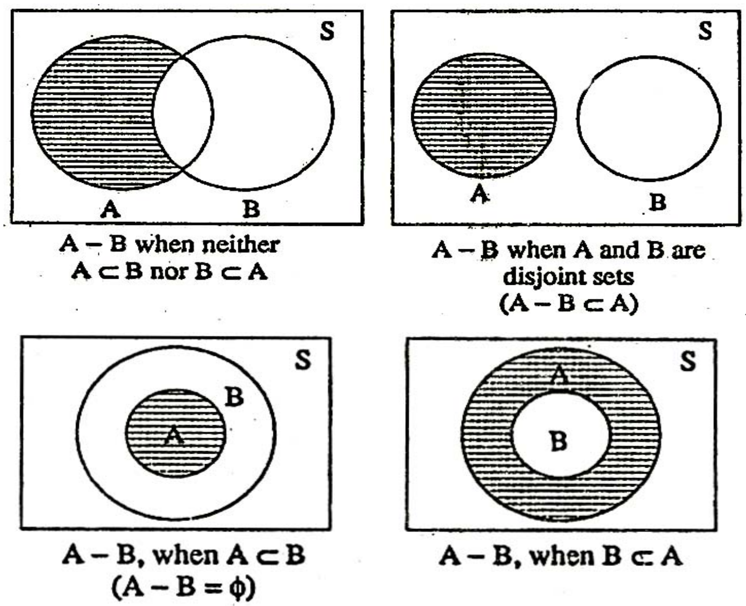


$A \cap B$ when neither $A \subset B$ nor $B \subset A$

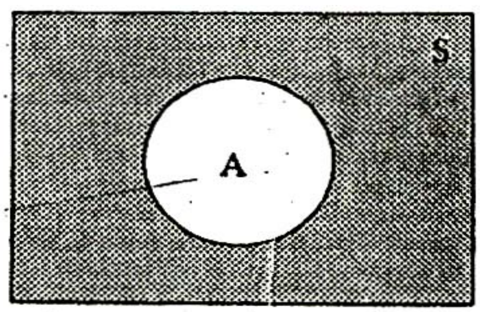


$A \cap B = \phi$

1.7 (d) Difference of sets: (A – B) represents the area of A that is not in B.



1.7(e)I Complement of Sets (A′): A′ or A^o is the set of those elements of Universal Set S which are not in A.



A′

From the above Venn-Diagram, the following results are clearly true $n(A) = n(A - B) + n(A \cap B)$

(a) $n(B) = n(B - A) + n(A \cap B)$

(b) $n(A) \cup (B) = n(A - B) + n(B - A) + n(A \cap B)$

Then result, $n(A \cap B) = n(A) + n(B) - n(A \cap B)$ can be generalized as,

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

1. $n(A \cap B') = n(A) - n(A \cap B)$

2. $n(B \cap A') = n(B) - n(A \cap B)$

3. $n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C)$

4. $n(A \cap C \cap B') = n(A \cap C) - n(A \cap B \cap C)$
5. $n(B \cap C \cap A') = n(B \cap C) - n(A \cap B \cap C)$
6. $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$
7. $n(B \cap A' \cap C') = n(B) - n(B \cap A) - n(B \cap C) + n(A \cap B \cap C)$
8. $n(C \cap A' \cap B') = n(C) - n(C \cap A) - n(C \cap B) + n(A \cap B \cap C)$

Example:- In a college there are 100 students, out of them 60 study English, 50 study Hindi, and 40 study Bengali and 40 study both English and Hindi, 35 study Hindi and Bengali, and 20 study Bengali and English and 15 study all the subjects . Is this record accurate?

Solution:- Let E → English, H →Hindi and B →Bengali.

Then $(E \cup H \cup B) = 100, n(E) = 60, n(H) = 50, n(B) = 40,$

$n(E \cap H) = 40, \; n(H \cap B) = 35, \; n(B \cap E) = 20, \; n(E \cap H \cap B) = 15$

we have

$n(E \cup H \cup B) = n\mathrm{I} + n(H) + n(B) - n(E \cap H) - n(H \cap B) - n(B \cap E) + n(E \cap H \cap B).$

$100 = 60 + 50 + 40 - 40 - 35 - 20 + 15$

$= 165 - 95$

$\neq 70$

Therefore data are not correct.

Example:- In a college 20 play Football, 15 play Hockey and 10 play both Football and Hockey. How many play only Football? Or only Hockey?.

$n(F \cap H') = n(F) - n(F \cap H)$

$= 20 - 10 = 10$

$n(H \cap F') = n(H) - n(H \cap F)$

$= 15 - 10 = 5$

1.8 Cartesian product of two sets

Ordered pair: An ordered pair consisting of two elements, say a and b in which one of them is designated as the first element and the other as the second element. An ordered pair is usually denoted by (a, b).

The element is a called the first coordinate (or first member) and the element b is called the second coordinate (or second member) of the ordered pair (a, b). There can be ordered pairs which have the same first and the second elements such as (1,1),(a, a), (2, 2), etc.

Two ordered pair (a, b) and (c, d) are said to be equal if and only if a=c and b = d.

Let us understand it by an example; the ordered pairs (1, 2) and (2, 1) though consist of the same elements 1 and 2, are different because they represent different points in the co-ordinate plane.

Cartesian product: The Cartesian product of two sets A and B is the set of all those pairs whose first co-ordinate is an element of A and the second co-ordinate is an element of B. The set is denoted by $A \times B$ and is read as 'A cross B or product set of A and B'. i.e.

$A \times B = \{(x, y): x \in A \text{ and } y \in B\}$

Example: let $A = \{1, 2, 3\}$, and $B = \{3, 5\}$

$A \times B = \{1, 2, 3\} \times \{3, 5\} = \{(1,3), (1,5), (2,3), (2,5), (3.3), (3.5)\}$

And $B \times A = \{(3, 1), (3, 2), (3,3), (5,1), (5,2), (5,3)\}$

So, it is clear that $A \times B \neq B \times A$

Similarly, we can define the Cartesian product for n set $A_1, A_2, \ldots, A_n$

$A_1 \times A_2 \times A_3, \times \ldots\ldots\ldots A_n = \{(x_1, x_2, x_3, \ldots, x_n)\} : x_1 \in A, \ x_2 \in A_2, \ x_3 \in A_3 \ldots, \text{ and } x_n \in A_n\}$

The element $(x_1, x_{1.}, \ldots x_n)$ is called as an n-tuple of $x_1, x_2, \ldots x_n$.

Note:- 1. $A \times \varphi = \varphi \times A = \varphi$.

$$A \times A = \{(x, y): x, y \in A\}$$

$$\{\varphi\} \neq \varphi$$

$$\{\{\varphi\}\} \neq \{\varphi\} \neq \varphi$$

3. Let A={1,2,3} and B= $\{\varphi\}$
4. Then $A \times B = \{(1, \varphi), (2, \varphi), (3, \varphi)\}$

Example: If A = {a, b}, and B = {b, c, d} then, find A×B and B ×A and also show that it is not commutative.

Example: If $A = \{1, 2,3\}$, $B = \{3, 4, 5\}$and C = {1, 3, 5} then, find A×B and B ×A, $A \times (B \cup C)$, $A \times (B \cap C)$.

1.9 Relations

Definition :- Let X and Y be two sets, then a relation R from X to Y, that is between $x \in X$ and $y \in Y$ is defined to be a subset R of $X \times Y$, that is $R \subseteq X \times Y$.

If $(x, y) \in R$, we say that x does stand in relation R to y or briefly as xRy. In case $(x, y) \notin R$ we say xRy (that is x is not R related to y). Similarly we may define a relation R between two elements of the same set X or a relation R in X by $R \subseteq X \times X$. If $(x_1, x_2) \in R$, then $x_1 R x_2$.

Let X be the set of all women and Y the set of all men. Then the relation 'is wife of' between women (element of X) and men (element of Y) will give us a set of ordered pairs R=(x, y) : $x \in X$, $y \in Y$, and x is wife of y}.

The ordered pairs (Kamla Nehru, Jawahar Lal Nehru), (Kasturba Gandhi, Mahatma Gandhi) are elements of R. It is clear that $R \subseteq X \times Y$.

A relation is binary if it is between two elements. Thus 'is wife of' is a binary relation involving two persons, viz Kamla Nehru is the wife of Jawahar Lal Nehru). Conversely if we are given the set R of ordered pairs (x, y) which correspond to the relation 'is wife of' man y and when not, we are only to find if (x, y) does or does not belong to R. Hence we find if we know the relation we know the set R and if we know the set R we know the relation. Thus we are led to the following definition.

A relation is binary operation between two sets. Thus 'is wife of' is a binary relation involving two persons, viz Kamla Nehru is the wife of Jawahar Lal Nehru).

Example 1: Let S be a set. Let R be a relation in $p(S)$, $R \subseteq p(S) \times p(S)$ given by

$R=\{(A, B) : A, B \in p(S)$ and $A \subseteq B\}$, Now $(A, B) \in R \Rightarrow A \subseteq B$. Or $ARB \Rightarrow A \subseteq B$.

Example 2: Let X be a set and let Δ is called the *relation of equality or diagonal relation in X and we write* x Δy iff $x = y$.

Example 3: If R = $X \times X - \Delta$. Then $(x, y) \in R \Rightarrow (x, y) \in X \times X$, $(x, y) \notin \Delta$ i.e. xRy iff $x \neq y$

R is called the relation of inequality in X. Thus we can say that the relation R of inequality in a set X is the complement of the diagonal relation Δ in $X \times X$.

Example 4: Let R be a relation in the set Z of integers given by R={x, y) : x< y, x , y∈ Z} where '<' has the usual meaning in Z. Since 3<4, therefore $(3, 4) \in R$ or $3R4$. But $(4,3) \notin R$, since 4> 3.

- Let A and B be two finite sets having m and n elements respectively. Find the number of distinct relations that can be defined from A to B. The number of distinct relations from A to B is the total number of subsets of $A \times B$. Since $A \times B$ has mn elements so total number of subsets of $A \times B$ is 2^{mn} . Hence total number of possible distinct relations from A to B 2^{mn}.

Definition:- Let R be a relation between sets X, Y, that is $R \subseteq X \times Y$. Then the domain and the range of R written as *dom R*, range R are defined by :

Dom R = $\{x \in X$: for some $y \in Y$, $(x, y) \in R$ or $x\,R\,y\}$, range $R = \{y \in Y$: for some $x \in X$, $(x, y) \in R$ or $xRy\}$.

If R is the relation 'is wife of' between the set X of women and the set Y of men, then *dom R*= set of wife, *range R* = set of husbands.

Binary Relations in a Set

A binary relation R is said to be defined in a set A then $R \subseteq A \times \text{A}$. if for any ordered pair $(x, y) \in A \times A$, it is meaningful to say that xRy is true or false. In other words, $R = \{(x, y) \in A \times A$: xRy is true.

That is, a relation R in a set A is a subset of $A \times A$. So, the binary relation is a relation between two sets, these sets may be different or may be identical, For the sake of convenience a binary relation will be written as a relation.

1.10 Domain and Range of a Relation

The domain D of the relation R is defined as the set of elements of first set of the ordered pairs which belongs to R, i.e., $D = \{(x, y) \in R$, for $x \in A\}$.

The range E of the relation R is define as the set of all elements of the second set of the ordered pairs which belong to R, i.e., $E = \{y : (x, y) \in R$, for $y \in B\}$. Obviously, $D \subseteq A$ and $E \subseteq B$.

Example: Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$. Every subset of $A \times B$ is a relation from A to B. So, if $R=\{(2, a), (4, a), (4, c)\}$, then the domain of R is the set $\{2,4\}$ and the range of R is the set $\{a,c\}$

Remark:- Total number of Distinct Relation from a set A to a set B

Let the number of elements of A and B be m and n respectively. Then the number of elements of $A \times B$ is mn. Therefore, the number of elements of the power set of $A \times B$ is $\mathbf{2^{mn}}$. Thus, $A \times B$ has $\mathbf{2^{mn}}$ different subsets. Now every subset of $A \times B$ is a relation from A to B. Hence the number of different relations A to B is $\mathbf{2^{mn}}$.

Relations as Sets of Ordered Pairs

Let R^* be any subset of $A\times B$. We can define a relation R where xRy ready '$(x,y)\in R^*$'. The solution set of this relation R is the original set R*. Thus, to every relation R there corresponds a unique solution set $R^* \subseteq A\times B$ and to every subset of R^* of $A\times B$ there corresponds a relation R for which R^* is its solution set.

1.11 Types of Relation in a set

We consider some special types of relations in a set.

1. **Reflexive relation**:- Let R be a relation in a set A that is R is subset of A cross A then R is called a reflexive relation if each element of the set A is related to itself. i.e

   $$(x, y)\in R, \forall x\in A \text{ or } xRx, \forall x\in A \text{ .}$$

   Example Let A= $\{1,2,3\}$ and consider a relation R in A such that

   $R_1 = \{(1,1), (2,2), (3,3)\}$ is reflexive relation (same element)

   But $R_2 = \{(1,1), (2,2)\}$ is not reflexive because $(3,3) \neq R_2$

   $R_3 = \{(1,1), (2,2), (3,3), (1,2)\}$ is reflexive .

2. **Symmetric relation**:- Let R be a relation in a set A that is $R \subseteq A\times A$. then R is said to be symmetric relation if

   $(x, y)\in R, \Rightarrow (y, x) \in R,$

   Or $xRy \Rightarrow yRx$.

   Example Let A= $\{1,2,3\}$ and consider a relation R in A such that

   $R_1 = \{(1,2), (2,1), (3,3)\}$ then R_1 is symmetric relation.

   But $R_2 = \{(1,1), (3,3)(1,2)\}$ is not symmetric and $R_3 = \{(1,1)\}$ is symmetric.

3. **Transitive relation:-** Let R be a relation in a set A that is $R \subseteq A\times A$. then R is said to be a transitive relation if

   $(x, y)\in R, (y, z) \in R,$ then $(x , z) \in R,$

   Or xRy and yRz then xRz .

   Example Let $A = \{1,2,3\}$ and consider a relation R in A such that

   $R_1 = \{(1,2), (2,3), (1,3)\}$ then R_1 is transitive relation because

   $(1, 2)$ and $(2 , 3) \Rightarrow (1 , 3) \in R$

   But $R_2 = \{(1,2), (2,3), (3,1), (1,3)\}$ is not transitive because 1R2, $2R3 \Rightarrow 1R3 \in R$, but $(2,3), (3,1) \in R$ *but* $(2,1) \neq R_3$.

4. **Identity Relation:** A relation R in a set A is said to be identity relation, if $I_A=\{(x,x) : x\in A\}$. Generally it is denoted by I_A.

Example : Let $A = \{1,2,3\}$ then $R=A\times A=\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2),(3,3)\}$ is a universal relation in A.

5. **Void (empty) Relation:** A relation R in a set A is said to be a void relation if R is a null set, i.e., if $R=\phi$.

***Example*:** Let $A = \{2,3,7\}$ and let R be defined as 'aRb if and only if $2a = b$ ' then we observe that $R=\phi \subset A \times A$ is a void relation.

6. **Antisymmetric Relation:** Let A be any set. A relation R on set A is said to be an antisymmetric relation iff $(a , b) \in R$ *and* $(b , a) \in R \Rightarrow a = b$ *for* $a, b \in$ A.

Example The identity relation on a set A is an antisymmetric relation.

7. **Inverse Relation:** Let R be a relation from the set A to the set B, then the inverse relation R^{-1} from the set B to the set A is defined by $R^{-1} =\{(b, a) : (a, b)\in R\}$.

In other words, the inverse relation R^{-1} consists of those ordered pairs which when reversed belong to R. Thus every relation R from the set A to the set B has an inverse relation R^{-1} from B to A.

Example 1: Let $A = \{1,2,3\}$, $B=\{a,b\}$ and $R=\{(1,a), (1, b), (3,a), (2, b)\}$ be a relation from A to B.The inverse relation of R is $R^{-1} = \{(a,1), (b, 1), (a, 3), (b,2)\}$

Example 2: Let $A= \{2,3,4\}$, $B=\{2,3,4\}$ and $R=\{x,y) : |x – y| = 1\}$ be a relation from A to B. That is, $R = \{3,2), (2,3), (4,3), (3, 4)\}$. The inverse relation of R is $R^{-1}= \{(3,2), (2, 3), (4, 3), (3, 4)\}$. It may be noted that $R=R^{-1}$.

Note: Every relation has an inverse relation. If R be a relation from A to B, then

R^{-1} is a relation from B to A and $(R^{-1})^{-1}=R$.

Theorem: If R be a relation from A to B, then the domain of R is the range of

R^{-1} and the range of R is the domain of R^{-1}.

Proof: Let $y \in$domain of R^{-1}. Then there exist $x\in A$ and $y\in B$, $(y, x)\in R^{-1}$. But $(y, x)\in R^{-1}\Rightarrow(x, y)\in R. \Rightarrow y\in$ range of R.

Therefore, $y\in$ domain $R^{-1}\Rightarrow y\in$ range of R. Hence domain of $R^{-1}\subseteq$ range of R. In a similar way we can prove that range of $R\subseteq$ domain of R^{-1}. Therefore, domain of R^{-1}= range of R. In a similar manner it can be shown that domain of R=range of R^{-1}.

Example: Let $A = \{1,2,3\}$. We consider several relations on A.

(4) Let R_1be the relation defined by $m < n$, that is, mR_1n if and only if $m < n$.

(ii) Let R_2 be the relation defined by mR_2n if and only if $|m - n| \leq 1$.

(5) Define R_3 by $m \equiv n \pmod 3$, so that mR_3n if and only if $m \equiv n \pmod 3$.

(6) Let E be the 'equality relation' on A, that is, mEn if and only if $m=n$.

Example **:** Let $A = \{1,2,3,4,5\}$ and $B=\{a, b, c\}$ and let $R= \{(1, a), (2,a), (2, c), (3, a,), (3, b), (4, a), (4, b), (4, c), (5, b)\}$.

1.12 Composition of Relation

Let R_1 be a relation from the set A to the set B and R_2 *be a* relation from the set B to the set C. That is $R_1 \subseteq A\times B$ and $R_2 \subseteq B\times C$. *The composite of the two relations* R_1 *and* R_2 *denoted by* R_2oR_1 *is a relation from the set A to C*, that is $R_2\, oR_1 \subseteq A\times C$ defined by : $R_2oR_1= \{(a, c) \in A\times C$: for some $b\in B$, $(a, b) \in R_1$ and $(b, c) \in R_2\}$. $a(R_2oR_1)c \Rightarrow$ for some $b\in B$, aR_1b and bR_2c.

Example1: Let X = Set of all women, Y= Set of all men, Z=Set of all human beings.

Let R_1 be a relation from X to Y given by $R_1 =\{(x, y) : x\in X, y\in Y$ and x is wife of $y\}$

And let R_2 be a relation from Y to Z given by $R_2= \{(y, z): y\in Y, z\in Z$ and y is father of $z\}$. Therefore $R_2oR_1= \{(x, z) \in X\times Z$: for some $y\in Y$ $(x, y)\in R_1$ and $(y, z)\in R_2\}$. Here R_2oR_1 is the relation 'is mother of' provided a man can have only wife.

Example2: If R_1 be a relation form the set X to the set Y, R_2 a relation from the set Y to the set Z and R_3 is a relation from the set Z to the set W. Then $R_3o\ (R_2oR_1) = (R_3oR_2)oR_1$, that is *composition of relation is associative.*

Now $R_2\, o\, R_1 \subseteq X \times Z$ and $R_2 \subseteq Z \times W$. Therefore $R_2\, o\, (R_2OR_1) \subseteq X\times W$, that is, a relation from X to W. Similarly $(R_2oR_1)oR_1 \subseteq X\times W$; that is, a relation from X to W. Now $(x, w) \in R_2o\, (R_2oR_1) \Leftrightarrow \exists z\in Z|(x,z)\in R_1$ and $(y, z)\in R_2$ & $(z, w)\in R_2 \Leftrightarrow \exists z \in Z, y \in Z, y \in Y\ (x, y) \in R_1$ and $(y, z) \in R_2$ & $(z, w) \in R_2$

Example3: (Since $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$) $\Leftrightarrow \exists y \in Y \mid (x, y)\in R_1$ and $(y, w)\in R_3oR_2$. $\Leftrightarrow (x, w)\in(R_3oR_2)oR_1$. Therefore $R_3o(R_2oR_1)=(R_3oR_2)oR_1$.

Definition :- Inverse of a relation: *If R be a relation from a set X to a set Y then* R^{-1} *is a relation from Y to X defined by* $R^{-1}=\{(y, x) \in Y\times X : (x, y)\in R\}$. Thus$(x, y)\in R \Leftrightarrow (y, x)\in R^{-1}$ or $xRY \Leftrightarrow yR^{-1}x$.

Check your progress

Example: A relation which is reflective but not symmetric and not transitive.

Solution: Let A= {1,2,3} and R is reflective in A as

$R_1 = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$ then

1. R is reflexive relation
2. R is not symmetric because (1 , 2))∈R but (2 , 1)≠ R
3. R is not transitive because $(1,2), (2,3) \in$ R but $(1,3) \notin R$

Example: A relation which is symmetric but not reflective and not transitive.

Solution: Let A= {1,2,3} and R= $\{(1,2), (2,1)\}$ then

R is symmetric.

R is not reflexive relation.

R is not transitive because $(1,2), (2,1) \in$ R but $(1,1) \notin R$.

Example: A relation which is reflective and symmetric but not transitive.

Solution: Let A={1,2,3} and R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)}

then, R is reflexive and symmetric relation.

R is not transitive because $(1,2), (2,3) \in$ R but $(1,3) \notin R$.

Example: A relation which is symmetric and transitive but not reflective.

Solution: Let A= {1,2,3} and R= {(1,1)} then

R is not reflexive relation since { (2,2), (3,3)} $\notin R$

R is symmetric and transitive

Example: A relation which is reflective, symmetric and transitive.

Solution: Let A= {1,2,3} and R= $\{(1,1), (2,2), (3,3)\}$

Example: A relation which is reflective and transitive but not symmetric.

Solution : Let A= {1,2,3} and R= $\{(1,1), (2,2), (3,3), (1,2)\}$

Example: Prove that $(R^{-1})^{-1}=R$.

Solution: Let $R\subseteq X\times Y$. then $R^{-1}\subseteq Y\times Y$. Therefore $(R^{-})^{-1}\subseteq X\times Y$.

Now $(x, y)\in R\Leftrightarrow(y, x)\in R^{-1} \Leftrightarrow(x, y)\in(R^{-1})$-1 Hence $R=(R^{-1})^{-1}$.

(1.2) Prove that $(R_2oR_1)^{-1}=R_1^{-1}oR_2^{-1}$.

Solution: Let $R_1\subseteq X\times Y$, $R_2\subseteq Y\times Z$. then $R_2oR_1\subseteq X\times Z$.

Hence $(R_2oR_1)^{-1}\subseteq Z\times X$. Now $R_1^{-1}oR_2^{-1}\subseteq Z\times X$ (prove)

Now $(z, x)\in(R_2oR_1)^{-1}\Leftrightarrow (x, z)\in R_2oR_1 \Leftrightarrow (x, y)\in R_1$ and $(y, z)\in R_2$ for some $y\in Y$

$\Leftrightarrow(y, x)\in R_1^{-1}$ and $(z, y)\in R_2^{-1}$ for some $y\in Y \Leftrightarrow(z, y)\in R_2^{-1}$ and $(y, x)\in R_1^{-1}$ for some $y\in Y \Leftrightarrow(z, x)\in R_1^{-1} oR_2^{-1}$.Hence $(R_2oR_1)^{-1}=R_1^{-1} oR_2^{-1}$.

Reversal Rule*: From the above we get the inverse of the composite of two relations is the composite of their inverse in the reverse order.*

1.13 Equivalence relation in a set

Definition:- *A relation R in a set S is called an equivalence relation if*

(α) R is reflexive, that is $\forall x\in S$, xRx or $(x,x)\in R$ that is, $\Delta\subseteq R$;

(β) R is symmetric, that is, $x, y\in S$, $xRy\Rightarrow yRx$ or $(x, y)\in R\Leftrightarrow(y, x)\in R$ i.e. $R^{-1}=R$.

(γ) R is transitive, that is, x, y,$z\in S$, $[xRy, yRz]\Rightarrow xRz$ Or $(x, y)\in R$, $(y, z)\in R\Rightarrow(x, z)\in R$., i.e. $RoR\subseteq R$.

Example 1: Prove that if R is an equivalence relation then $R^{-1}$is also an equivalence relation.

Solution: *Reflexive: Since R is reflexive* $\Rightarrow (x,x) \in R$, $\forall x\in R$

$\Rightarrow (x,x) \in R^{-1}$, $\forall x\in R$

Therefore R^{-1} is reflexcive.

Symmetric: Let $(x,y) \in R^{-1}$,

$\Rightarrow (y,x) \in R$

$\Rightarrow (x,y) \in R$ *because R is symmetric*

$\Rightarrow (y,x) \in R^{-1}$

Therefore R^{-1} is symmetric.

Transitive: Let (x, y) and $(y, z) \in R^{-1}$,

$$\Rightarrow (y, x) \text{and } (z, y) \in R$$

$$\Rightarrow (z, y) \text{and } (y, x) \in R$$

$\Rightarrow (z, x) \in R$ *since R is transitive.*

$\Rightarrow (x, z) \in R^{-1}$

Therefore R^{-1} *is transitive*

Hence R^{-1} *is equivalence relation.*

Example: Define a relation R in the set of integer Zsuch that aRb iff $a \equiv b$ (mod m) (read a is congruent to b or m divides a-b) where m is a positive integers. Is R is an equivalence relation?

1. *For Reflexive R is reflexive if aRa* $\forall a \in \mathbf{Z}$

i.e if m/a-a i.e. m/0

therefore R is reflexcive.

2. *For symmetric Let aRb*

$\Rightarrow$ m divides a – b

$\Rightarrow$ m divides b- a

$\Rightarrow$ bRa

Therefore R is symmetric.

3. For Transitive Let aRb and bRc

$\Rightarrow$ m divides a – b and m divides b – c

$\Rightarrow$ m divides (a – b) + (b – c)

$\Rightarrow$ m divides a – c

$\Rightarrow$ aRc

Therefore R is transitive .

Hence R is an equivalence.

Example 2: The diagonal or the equality relation Δ in a set *S* is an equivalence relation in *S*. For if $x, y \in S$ the xΔy iff $x=y$. Thus

(α) $x \Delta x \ \forall x \in S$ (reflexivity)

(β) $x \Delta y \Rightarrow x = y \Rightarrow y = x \Rightarrow y \ \Delta x$ (Symmetry)

(γ) for $x, y, z \in S$, $[x\Delta y, y\Delta z] \Rightarrow [x=y, y=z \Rightarrow x == \Rightarrow x\Delta z$. Hence $[x\Delta y$, and $y\Delta z] \Rightarrow \Delta$ (transitivity).

Example 3: Let N be the set of natural numbers. Consider the relation R in $N\times N$ given by $(a, b)\ R(c, d)$ if $a+d=b+c$, where $a, b, c\ d \in N$ and + denotes addition of natural numbers, R is an equivalence relation in $N\times N$.

(α) $(a, b)R(a, b)$ since $a+b=b+a$ (Reflexivity)

(β) $(a, b)R(c, d) \Rightarrow a+d=b+c \Rightarrow c+b=d+a \Rightarrow (c,d)R(a, b)$ (Symmetry)

(γ) $[(a, b)R(c, d), (c, d)R(e, f)] \Rightarrow [a+d=b+c, c+f=f=d+e]$

$\Rightarrow (a+d+c+f=b+c+d+e) \Rightarrow a+f=b+e$ (By cancellation laws in N) $\Rightarrow (a, b)R(e, f)$ (transitivity)

Example 4: Let a relation R in the set N of natural numbers be defined by: If m, $n \in N$, then mRn if m and n are both odd. Then R is not reflexive, since 2 is not related to 2. Thus $(x, x) \notin R\ \forall x \in N$. But R is symmetric and transitive as can be verified.

Example 5: Let X be a set. Consider the relation R in $p(x)$ given by : for $A, B \in p(X)$. ARB if $A \subseteq B$. Now R is reflexive, since $A \subseteq A$, $\forall A \in p(X)$ R is transitive, since $[A \subseteq B, B \subseteq C] \Rightarrow A \subseteq C$ where $A, B, C \in p(X)$. But R is not symmetric, since $A \subseteq B \neq \Rightarrow B \subseteq A$.

Example 6: Let S be the set of all lines L in three dimensional space. Consider the relation R in S given by; for $L_1, L_2 \in S$, $L_1 R L_2$ *if* L_1 is coplanar with L_2. Now R is reflexive, since L_1 is coplanar with L_1, R is symmetric, since L_1 coplanar with $L_2 \Rightarrow L_2$ coplanar with L_1. But R is not transitive, since (L_1 coplanar with L_2 and L_2 coplanar with L_3)$\neq\Rightarrow$ L_1 coplanar with L_3.

Example 7: (a) Let X $=\{x, x_2, x_3, x_4\}$. Define the following relations in X :

$R_1 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_2, x_3), (x_3, x_2)\}$

$R_2 = \{ x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_2, x_3), (x_2, x_4)\}$

$R_3 = \{(x_1, x_1), (x_2, x_2), (x_3, x_3),(x_4, x_4), (x_2, x_3), (x_3, x_2), (x_3, x_4), (x_4, x_3)\}$

R_1 is symmetric, transitive but not reflexive since$(x_4, x_4) \notin R_1$

R_2 is reflexive, transitive but not symmetric since $x_2 R_2 x_4$ but $(x_4, x_2) \notin R_2$

R_3 is reflexive, symmetric but not transitive since $x_2 R_3 x_3$ and $x_3 R_3 x_4$ but $(x_2, x_4) \notin R_3$.

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Note: Examples prove that the three properties of an equivalence relation viz. reflexive, symmetric and transitive are independent of each other, i.e. no one of them can be deduced from the other two.

Example8: Let A be the set of all people on the earth. Let us define a relation R in A, such that xRy if and only if 'x is father of y', Examine R is (i) reflexive, (ii) symmetric, and (iii) transitive. We have

(7) For $x \in A$, xRx does not holds, because, x is not the father of x. That is R is not reflexive.

(ii) Let xRy, i.e., x is father of y, which does not imply that y is father of x. Thus yRx does not hold. Hence R is not symmetric.

(8) Let xRy and yRz hold. i.e., x is father of y and y is father of z, but x is not father of z, i.e., xRz does not holds. Hence R is not transitive.

Example 9: Let A be the set of all people on the earth. A relation R is defined on the set A by aRb if and only if a loves b' for a, $b \in A$. Examine R is (i) reflexive, (ii) symmetric, and (iii) transitive. Here,

(9) R is reflexive, because, every people loves himself. That is, aRa holds.

(ii) R is not symmetric, because, if a loves b then b not necessarily loves, i.e., aRb does not always imply bRa. Thus, R is not symmetric.

(10) R is not transitive, because, if a loves b and b loves c then a not necessarily loves c, i.e., if aRb and bRc but not necessarily aRc. Thus R is not transitive. Hence R is reflexive but not symmetric and transitive.

Example 10: Let N be the set of all natural numbers. Define a relation R in N by 'xRy if and only if $x + y = 10$'. Examine R is (i) reflective, (ii) symmetric, and (iii) transitive. Here,

(11) Since $3 + 3 \neq 10$ i.e., $3R3$ does not hold. Therefore R is not reflexive.

(ii) If $a + b = 10$ then $b + a = 10$, i.e., if aRb hold then bRa holds. Hence R is symmetric.

(12) We have, 2+8=10 and 8+2=10 but $2+2 \neq 10$, i.e. $2R8$ and 8R2 holds but $2R2$ does not hold. Hence R is not transitive therefore R is not reflexive and transitive but symmetric.

Example 11: Let I be the set of all integers and R be a relation defined on I such that 'xRy if and only if $x > y$'. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Here,

(13) R is not reflexive, because, $x > x$ is not true, i.e., xRx is not true.

(ii) R is not symmetric also, because, if $x > y$ then $y \ngtr x$. i.e., R is not symmetric

(14) *R is transitive because* if xRy and yRz holds then xRz hold. Therefore R is not reflexive and symmetric but transitive.

Example 12: Let A be the set of all straight lines in 3-space. A relation R is defined on A by 'lRm if and only if l lies on the plane of m' for $l. m \in A$. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Here,

(15) Let $l \in A$. then l is coplanar with itself. Therefore lRl holds for all $l \in A$. Hence R is reflexive.

(ii) Let $l, m \in A$ and lRm hold. Then l lies on the plane of m. Therefore m lies on the plane of l. Therefore, $lRm \Rightarrow mRl$. Thus R is symmetric.

(16) Le $l,m,n, \in A$ and lRm and mRn both hold. The l lies on the plane of m and m lies on the plane of n. This does not always imply that l lies on the plane of n. e.g., if l is a straight line on the $x - y$ plane and m be another straight line parallel *to y* axis and n be a line on the y – z plane then lRm and mRn hold but lRn does not hold because l and n lie on $x - y$ plane and $y - z$ plane respectively. Thus R is not transitive. Hence R is reflexive and symmetric but not transitive.

Example 13: Let A be a family of sets and let R be the relation in A defined by 'A is a subset of B'. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Then R is

(17) Reflexive, because, $A \subseteq A$ is true.

(ii) Not symmetric, because if $A \subseteq B$ then B is not necessarily a subset of A.

(18) Transitive, because, if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$., i.e., if ARB and BRC hold then ARC holds. Thus R is reflexive and transitive but not symmetric.

Example 14: A relation R is defined on the set I, the set of integers, by 'aRb if and only if $ab > 0$' for $a \neq 0$, b$\neq 0 \in I$. Examine R is (i) reflexive, (ii) symmetric and (iii) transitive. Here,

(19) Let $a \in R$. Then $a.a.>0$ holds. Therefore aRa holds for all $a \in I$. Thus R is reflexive.

(ii) Let $a, b \in I$ and aRb holds. If $ab > 0$ then $ba > 0$. Therefore, $aRb \Rightarrow bRa$. Thus R is symmetric.

(20) Let $a, b, c \in I$ and aRb, bRc hold. Then $ab > 0$ and be > 0. Therefore, $(ab)(bc)>0$. This implies $ac > 0$ since $b^2>0$. So aRb and $bRc \Rightarrow$. Thus R is transitive. Hence R is reflecsive, symmetric and transitive, hence R is an equivalence relation.

( ) Let R be a relation in a set S which is symmetric and transitive. Then $aRb \Rightarrow bRa$ (by symmetry) $[aRb$ and $bRa] \Rightarrow aRa$ (by Transitivity).

From this it may not be concluded tht fefelxivity follows from symmetry and transitivity. The fallacy involved in the above argument is : for $a \in S$, to prove aRa, we have started with $aRb \Rightarrow bRa$. Now it might happen that $\exists$ no element $b \in S$ such that aRb.

Check your progress

1. Examine whether each of the following relations is an equivalence relation in the accompanying set –

(i) The geometric notion of similarity in the set of all triangles in the Euclidean plane. [Ans: It is an equivalence relation]

(ii) The relation of divisibility of a positive integer by another, the relation being defined in the set of all positive integers as follows: a is divisible by b if $\exists$ a positive integer c such that $a=bc$.

[Ans: The relation is reflexive, transitive but not symmetric.]

2. R is a relation in Z defined by: if $x, y, \in Z$, then xR_1 if $10+xy > 0$. Prove that R is reflexive, symmetric but not transitive.
Hint: $-2R3$ and $3R6$ but $(-2, 6) \notin R$

1.14 Partition of a Set

Let X be a set. A collection C of disjoint non-empty subsets of X whose union is X is called a partition of X. For example, let $X = \{a, b, c, d, e, f\}$. Then a partition of X is $[\{a\}, \{b, c, d\}, \{e, f\}]$, since intersection of any two subset of this collection is ϕ and their union is X. There may be other partitions of X. An equivalence relation in a set S may be denoted b ~. Then 'x~a' will be read as 'x is equivalent to a'.

Example: Let $A = \{a, b, c\}$. Then $A_1 = \{a\}$. $A_2 = \{b, c\}$ are the partition of A.

Example: Find all the partition of X = $\{a, b, c, d\}$.

Definition Equivalence Class: If ~ is an equivalence relation in a set S and $a \in S$, the set $\{x \in S : x \sim a\}$ is called an equivalence class of determined by a and will be denoted by $\bar{a}$. If the equivalence relation ~ is denoted by R, then the equivalence class of S determined by a may be denoted by Ra.

Theorem: If ~ is an equivalence relation in a set S, and $a, b \in S$, then

(21) $\bar{a}, \bar{b}$ are not empty.

(ii) if $b \sim a$, then $\bar{a} = \bar{b}$.

Proof *:* Since $a \sim a$ by reflexive property, $a \in \bar{a}$, hence $\bar{a}$ is a not empty. similarly $\bar{b}$ is not empty.

(22) Now $x \in \bar{a} \Rightarrow x \sim a$. $b \sim a \Rightarrow \sim b$ (by symmetry). Hence we get $x \sim a$, and $a \sim b$. Therefore $x \sim b$ (by transitivity)

Consequently $x \in \bar{b}$ thus $x \in \bar{a} \Rightarrow x \in \bar{b}$. Therefore $\bar{a} \subseteq \bar{b}$. Similarly $\bar{b} \subseteq \bar{a}$. Hence $\bar{a} = \bar{b}$.

Theorem:- Any equivalence relation in a set S partition S into equivalence classes. Conversely any partition of S into non-empty subsets, induces an equivalence relation in S, for which these subsets are the equivalence classes.

(23) Given an equivalence ~ in S. We are to prove that the collection of equivalence classes is a partition of S. Let $\bar{x}_{1,}\ \bar{x}_{2,}\ \bar{x}_i$, etc. be the equivalence classes where $x_i \in S$. We are to prove $\cup \bar{x} = S$.

Now $x \in \cup\ \bar{x}_i \Rightarrow x \in \bar{x}_i$, for some $\bar{x}_i$. $\Rightarrow x \in S$ [since $\bar{x}_i \subseteq S$]. Hence $\bigcup_i \bar{x}_I \subseteq S$

Again $x \in S \Rightarrow x \in \bar{x}_I \Rightarrow x_i \in \bigcup_i \bar{x}_I$ Therefore $\bigcup_i \bar{x}_i = S$. Now we prove that any two equivalence class $\bar{x}, \bar{y}$ where x, y$\in$S are disjoint or identical. Let $\bar{x} \cap \bar{y} \neq \phi$ $z \in \bar{x} \cap \bar{y}$, then $z \in \bar{x}$ and $z \in \bar{y}$. Now $z \in \bar{x} \Rightarrow z \sim x \Rightarrow x \sim z$ (by symmetry) $z \in \bar{y} \Rightarrow z \sim y$. Hence $z \in \bar{x} \cap \bar{y} \Rightarrow [x \sim z, z \sim y]$. $\Rightarrow x \sim y$ (by transitivity) $\Rightarrow \bar{x} = \bar{y}$.Thus $\bar{x} \cap \bar{y} \neq \phi \Rightarrow \bar{x} = \bar{y}$. Hence $\bar{x} \neq \bar{y} \Rightarrow \bar{x} \cap \bar{y} = \phi$.This completes the proof of the first part of the theorem.

(24) Let the collection $C = \{Ai\}$ be a partition of S. Then $S = \cup A_i$ and A_i's are mutually disjoint non-empty subsets of S. Now $x \in S \Rightarrow x \in Ai$ for exactly one i.

We define a relation R in S by : for $x, y \in S$. xRy if x and y are element of the same subset Ai. It can be proved that R is an equivalence relation is S and the subsets Ai are the equivalence clauses.

1.15 Quotient set of a set S

Definition: The set of equivalence classes obtained from an equivalence relation in a set S is called the quotient set of S which is denoted by $\bar{s}$ or by $S|\sim$, or by $S|R$ when the equivalence relation is denoted by R.

(1). Let S be the set of all points in the x.y plane. We define a relation R in S by: For a, b∈S, aRb if the line through the point a parallel to the X-axis passes through the point b. It can easily be proved that R is an equivalence relation in S. Now the equivalence class $\bar{a}$ determined by the point a is the line throughthe point a parallel to the x-axis and the quotient set.

$\bar{S}$= set of all straight lines in the x-y plane parallel to the x-axis.

(25) The diagonal relation or the relation of equality in a set S is an equivalence relation. If a∈S, then

$\bar{a}=\{a\}$. i.e. each equivalence class is a singleton and $\bar{S}$=set of all singletons.

(2) If S is a set, then $R=S\times S$ is an equivalence relation in S and the only equivalence class is the set S. $\bar{S}=\{S\}$.

(3) If X be the set of points in a plane and R is a relation on X defined by $A, B \in X$, ARB if A and B are equidistant from the origin. prove that R is an equivalence relation. Describe the equivalence classes. The equivalence class R_A=Set of points on the circle with centre as origin O and radius OA.

Hence the quotient set $X|R$ is the set of circles on the plane with centre as O

1.16 Order relation

Definition: A relation, R in a set A is called a partial order or partial ordering relation if and only if it following three conditions

(1) R is reflexive i.e. $xRx\ \forall x \in A$

(2) R is anti symmetric i.e. xRy and yRx *iff* $x = y$, where $x, y, \in A$

(3) R is transitive i.e. for $x, y, z \in A$. $[xRy, yRz] \Rightarrow xRz$.

If in addition $\forall x, y \in A$, either xRy or yRx, then R is called a linear order or total order relation. A set with a partial order relation is called a partially ordered set and a set with a total order relation is called a totally ordered set or a chain.

Note. 1: Generally the partial order relation is denoted by the symbol $\leq$ and is read as 'less than or equal to'.

In the set Z_+ of positive integers, the relation given by for m, $n \in Z_+$, $m \nleq n$ if m divides n, is a partial order relation not a total order relation. For (1) $m \leq m \; \forall m \in Z$, since m divides m.

(2) $m \leq n$ and $n \leq m \Rightarrow m$ divides n and n divides m => m = n.

(3) $[m \leq n, n \leq 1]$ k $\Rightarrow m$ divides n, n divides k$\Rightarrow$ m divides k$\Rightarrow$ $m \leq$ k.Thus the relation is a partial order relation.

But it is not a total order relation, since for $m, n \in Z_+$ it may happen that neither m divides n nor n divides m i.e. neither $m \leq n$ nor $n \leq m$.

Example: In the set R of real numbers, the relation $\leq$ having its usual meaning in R is a total order relation. The proof is left as an exercise.

Example: If S be a set, then the relation in $p(S)$ given by : for $A, B \in p(S)$. $A \subseteq B$, is a partial order relation but not a total order relation. The proof is left as an exercise.

Definition Let $(S, \leq)$ be a partially ordered set. If $x \leq y$ and $x \neq y$, then x is said to be strictly smaller than or strictly predecessor of y. We also say that y is strictly greater than or strictly successor of y. denote it by $x < y$.

An element $a \in S$ is said to be a **least** or first (respectively **greatest** or last) element S if $a \leq$(respectively $x \leq A) \forall x \in S)$. An element $a \in S$ is called **minimal** (respectively **maximal**) element of S is $x \leq a$ (respectively $a \leq x$) implies $a = x$ where $x \in S$.

Check your progress

(1) $(N, \leq)$, (the relation $\leq$ having its usual meaning) is a partially ordered set. 2 is strictly smaller than 5 or $2 < 5$. 1 is the least or first element of N. since, $1 \leq m \; \forall \; m \in N$, There is no greatest or last element of N. 1 is the only minimal element since if $x \in N$, Then $x \leq 1 \Rightarrow x = 1$.

(2) Consider the set S = {1, 2, 3, 4, 12}. Let $\leq$ be defined by $a \leq b$ if a divides b. Then 2 is strictly smaller than 4 or $2 < 4$. 12 is strictly greatest than 4 or $4 < 12$. Since I divides each of the number 1, 2, 3, 4, 12 so $1 \leq x \; \forall x \in S$, hence 1 is the least element of S. Again since $x \leq 12 \forall x \in S$ i.e. each element of S divides 12, so 12 is the greatest or last element of S. Here also I is the only minimal element, since $x \in S$, then $x \le 1$ i.e. x divides 1 implies $x = 1$.

(3) Let S be a set. Then $(\mathcal{P}(S), \le)$ where $\le$ is the set inclusion relation $\subseteq$, is a partially ordered set. Then ϕ is the least element, since $\phi \subseteq A \, \forall A \in \mathcal{P}(S)$, and S is the greatest element since $A \subseteq S \; \forall A \in \mathcal{P}(S)$. Every singleton is a minimal element. For if $a \in S$, $\{a\} \in \mathcal{P}(S)$ and if $X \in \mathcal{P}(S)$, then $X \subseteq \{a\} \Rightarrow X = \{a\}$.

Definition) Infimum and Supermum: Let $(S, \le)$ be a partially ordered set and A a subset of S. An element $a \in S$ is said to be a lower bound (respectively upper bound) of A if $a \le x$ (respectively $x \le a$) $\forall x \in A$.

In case A has a lower bound, we say that A is bounded below or bounded on the left. When A has an upper bound we say that A is bounded above or bounded on the right. Let $L(\neq \phi)$ be the set of all lower bounds of A, then greatest element of L if it exists is called the greatest lower bound (*g l b*) or infimum of A. Similarly if $U(\neq \phi)$ be the set of all upper bounds of A, then the least element of U if it exists is called the least upper bounded (*l.u.b.*) or supremum of A

Example: Consider the partially ordered set $(N, \le)$, where $m \le n$ if m divides n. Consider the subset $A = \{12, 18\}$. 2 is a lower bound of A since 2 divides both 12 and 18. i.e. $2 \le 12$ and $2 \le 18$. The set of al lower bounds of A viz $L = \{1,2,3,6\}$ and 6 is the greatest element of L. Hence *g.l.b.* or infimum of $A = 6$. It is called the greatest common divisor (*g.c.d*) of A. Now 36, 72, 108 etc. are upper bounds of A since x divides 36 or 72 or 108 $\forall x \in A$ thus $x \le 36$ or 72 or 108 $\forall x \in A$. Now the set of upper bounds of A viz $\{36, 72, 108, \ldots\}$, the least element of 36. Hence the *l.u.b* or supremum of $A = 36$. It is also called the L.C.M. of 12 and 18.

Example: Set S be a non-empty set which is not a singleton, consider the set $Y = P(\phi, S\}$ partially ordered by the inclusion relation. Now Y has no least or no greatest element. Each singleton as in Ex. (5.6) is the minimal element.

Let $A \subseteq Y$, $G = \cap \{X_\alpha : X_\alpha \in A\}$. If $G \neq \phi$, then G is *g.l.b* of A. Similarly $L = U \{X_\alpha : X\alpha \in A\}$ is the *l.u.b* of A and exists if $L \neq A$.

Theorem : The least (respectively greatest) element of a partially set $(s, \le)$, if it exists, is unique.

Proof. If possible let l and l' be two least element of S. Since l is the least element, so $l \le x \; \forall x \in S$ hence $l \le l'$ since $l' \in S$. Similarly taking l' as lest element $l' \le l$. Hence $\underline{l} \le \underline{l'}$ and $\underline{l'} \le \underline{l}$. Therefore by anti-symmetry $l = l'$. Similar proof can be given for the greatest element.

Remark : In contrast to the above theorem, maximal and minimal elements of a partially ordered set *X* need not be unique. In example (5.6) or (5.8)* we have shown that every singleton is a minimal element. Sometimes minimal element can also be a maximal element. For example consider the partially ordered set $\{X\ \Delta\}$ where Δ is the diagonal relation. Every element of X is a minimal as well as a maximal element of *X*. For let $a \in X$. Then $x\Delta a \Rightarrow x=a$, $a\Delta x \Rightarrow x \Rightarrow a$.

Definition : A partially ordered set $(S, \leq)$ is said to be well ordered if every non empty subset of *S* has a least element.

Theorem : A well ordered set $(S, \leq)$ is always totally ordered or linearly ordered or a chain.

Proof: Let *x*, *y* be any two element of *S*. Consider the subset $\{x, y\}$ of *S*, which is non empty and hence has a least element either *x* or *y*, then $x \leq y$ or $y \leq x$. Hence every two element of *S* are comparable and so S is totally ordered. We now state two important statements without proof.

Well ordering principle: Every set can be well ordered.

Zorn's Lemma: Let *S* be a non empty partially ordered set in which every chain i.e. every totally ordered subset has an upper bound, then S contains a maximal

Totally Ordered Sets: Two elements *a* and *b* are said to be not comparable if $a \not\leq b$ and $b \not\leq a$, that is, if neither element precedes the other. A total order in a set *A* is a partial order in A with the additional property that $a < b$, $a=b$ or $b < a$ for any two elements a and b belonging to *A*. A set *A* together with a specific total order in *A* is called a totally ordered set.

Example : Let *R* be a relation in the set of natural numbers *N* defined by '*x* is a multiple of *y*', then *R* is a partial order in *N*. 6 and 2, 15 and 3, 20 and 20 are all comparable but 3 and 5, 7 and 10 are not comparable. So *N* is not a totally ordered set.

Example : Let *A* and *B* be totally ordered sets. Then Cartesian product $A \times B$ can be totally ordered as follows: $(a, b) < (a', b')$ if $a < a'$ or if $a = a'$ and $b < \underline{b}'$. This order is called the lexicographical order of A× B, since it is similar to the way words are arranged in a dictionary.

Theorem: Every subset of a well-ordered set is well-ordered.

1.17 Summary

In this Unit, we have studied the types of sets, union and intersection of sets. Cartesian product of sets i.e. A x B, A x A, definition of relation as a subset of A x B and as subset of A x A. Types of relation in a set A i.e. reflexive, symmetric and transitive relation, equivalence relation and equivalence classes are also studied. Domain and range of a relation is also described. Partition of a set and partition theorem, composition of two relations R and S, inverse of relation R, and its properties are studied. Quotient, set, order relation, partially ordered sets and totally ordered set, infimum and supremum of a set A is described.

1.18 Terminal Questions

1. Define a relation R in NxN where N is the set of natural numbers such that (a, b) R (c, d) *iff* a + d = b + c. Prove that the relation is an equivalence relation.

2. How many relations can be defined in a set containing 10 elements? If A = {1, 2, 3} then write down the smallest and biggest reflexive relations in the set A.

3. Define a relation R in NxN such that R={(x,y) such that 2x + y =10}. find the relation R and its inverse R^{-1}. (Answer: R = {(1,8), (2,6), (3,4), (4,2)}

4. Give examples of the following relations:

(i) Reflexive but not symmetric & not transitive

(ii) Symmetric but not reflexive & not transitive

(iii) Transitive but not reflexive & not symmetric

5. If R1 and R2 be two equivalence relation then prove that R1 ∩ R2 is an equivalence relation but $R_1 \cup R_2$ need not be an equivalence relation.

6. Define a relation in a plane such that any two points of the plane are related if they are equidistant from the origin. Is R an equivalence relation?

UNIT-2

FUNCTIONS

Structure

2.1 INTRODUCTION

As we know the notion of a function is one of the most fundamental concepts in mathematics and is used knowingly or unknowingly to our day to day life at every moment. Computer Science and Mathematics is an area where a number of applications of functions can be seen. We thought it would be a good idea to acquaint with some basic results about functions. Perhaps, we are already familiar with these results. But, a quick look through the pages will help us in refreshing our memory, and we will be ready to tackle the course. We will find a number of examples of various types of functions, and also we are introduced to whole numbers, integers, rational and irrational numbers leading to the notion of real numbers. The integers and rational numbers arise naturally from the ideas of arithmetic. The real numbers essentially arise from geometry.

Greeks in 500 BC discovered irrational numbers a consequence of Pythagoras theorem. Actually this discovery shook their understanding of numbers to its foundations. They also realized that several of their geometric proofs were no longer valid. The Greek mathematician Eudoxus considered this problem and mathematicians remained unsettled by irrational numbers. Geometrically, rational numbers when represented by points on the line, do not cover every point of the line.

The modern understanding of real numbers began to develop only during the 19th century. The mathematicians were forced to invent a set of numbers which is bigger than that of rational numbers and which satisfy the equation of the type $x^n = 2$ for all n.

A set of axioms for the real numbers was developed in a middle part of the 19th century. These particular axioms have proven their worth without doubt.

Objectives

After reading this unit you should be able to:

- Describe a function in its different forms
- Derive other properties with the help of the basic ones
- Define a function and examine whether a given function is one – one/onto
- Recall the basic and other properties of real numbers.
- Recognize the different types of intervals.
- Define the function and recognize its types and inverse functions.
- Define and determine even and odd functions.
- Define and test the period of the given function.

2.2. Functions or (Mapping)

A function is a kind of relation between various objects with certain conditions.

OR

A function is a rule which maps a number to another unique number. In other words, if we start with an input, and we apply the function, we get an output.

For example: 1) The volume V of a cube is a function of its side x.

2) The velocity v of a moving body at any time t is a function of its initial velocity v_0 and time t.

Mathematically, a function is defined as follows;

1) For the sets A and B, a function from A to B is denoted by $f : A \to B$, is a correspondence which assigns to every element $x \in A$, a unique element $f(x) \in B$. The value of the function f at an element x in A is denoted by $f(x)$, which is an element in B.

2) For a function $f : A \to B$, the set A is called the domain of f and the subset $f(A) = \{f(x) : x \in A\}$ of B (i.e., set of images of f) is called the range of f.

3) If $B \subseteq \Re$ then f is said to be real valued. If $A \subseteq \Re$, then domain of f is the set of all $x \in \Re$ for which $f(x) \in \Re$.

Function is also known as **mapping**.

Alternatively

Let A and B are two sets. A function f from A to B is a rule that assigns every element $x \in A$ to a unique $y \in B$. It is written as $f : A \to B$ and $y = f(x)$.

$$\forall\ x \in A\ \exists\ y \in B, \text{ such that } y = f(x) \quad and \quad \forall\ x_1, x_2 \in A,\ f(x_1) \neq f(x_2) \Rightarrow x_1 \neq x_2.$$

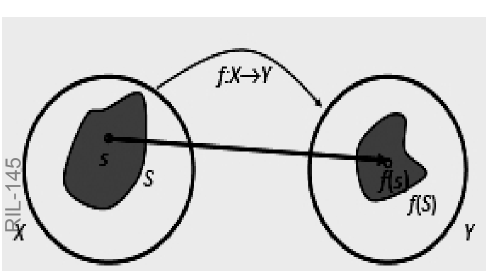
one element of A can be mapped to more than one element of B.

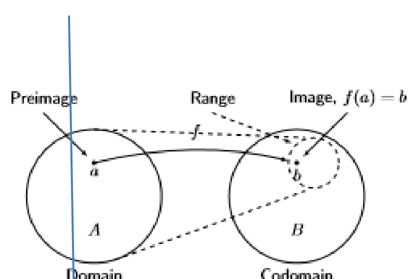
A is called domain and B is called co-domain. y is image of x under f and x is pre-image of y under f. Range is subset of B with pre-images.

Equivalently,

Let $f : X \to Y$, where X and Y are two sets, and consider the subset $S \subset X$. The image of the subset S is the subset of Y that consists of the images of the elements of S: $f(S) = \{f(s),\ s \in S\}$

Arrow Diagram of Function Or Mapping





A function, $f : A \to B$.

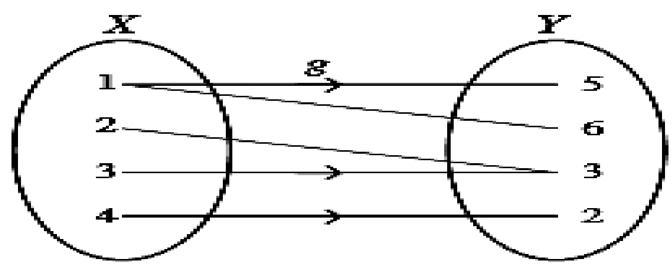
Note:

1. If we consider $f: R \to R$ is a function then f is called a real valued function of a real variable.
2. $f: R^n \to R$ ($n > 1$) is called a real valued function of a vector variable.
3. $f: R \to R^m$ ($m > 1$) is called a vector valued function of a real variable.
4. $f: R^n \to R^m$ is called a vector valued function of a vector variable.

Example 1: Domain $X = \{1, 2, 3, 4\}$ and Co-domain $Y = \{2, 3, 5, 6\}$ also Range is $Y = \{5, 6, 3, 2\}$

Solution: $g: X \to Y$ is not a function because the element 1 in set X is assigned to two elements 5 and 6 in Y

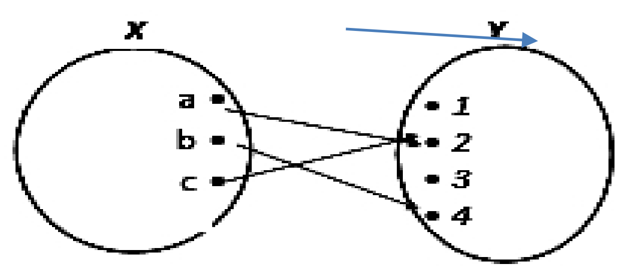
Arrow diagram of the given mapping is



Example 2: Domain $X = \{a, b, c\}$ and Codomain $Y = \{1, 2, 3, 4\}$ also Range is $Y = \{2, 4\}$

Solution: $f: X \to Y$ is a function because every element in set X is assigned to exactly one element in Y.

Arrow diagram of the given mapping is



Example 3: If $A = \{2, 4, 6, 9\}$ and $B = \{4, 6, 18, 27, 54\}$, $a \in A, b \in B$, find the set of ordered pairs such that 'a' is factor of 'b' and $a < b$.

RIL-145

Solution: We have to find a set of ordered pairs (a, b) such that 'a' is factor of 'b' and $a < b$. Since 2 is a factor of 4 and $2 < 4$. So $(2, 4)$ is one such ordered pair.

Similarly, $(2, 6), (2,18), (2, 54), (6, 18)$, *etc* are other such ordered pairs.

Thus, the required set of ordered pairs is; $\{(2, 4), (2, 6), (2,18), (2, 54), (6, 18), (6, 54), (9, 18), (9, 27), (9, 54)\}$.

Example 4: Find the domain and range of the function $f : Z \to Z$ defined by $f(x) = x^2$ where **Z** is set of integers.

Solution: Domain and codomain of the given function $f : Z \to Z$ is Z. Its range is $\{0, 1, 4, 9, 16, ...\}$.

<u>The Functions Vs Non Functions</u>

Consider $X = \{a, b, c\}$ and $Y = \{1, 2, 3, 4\}$. Then we can observe the following,

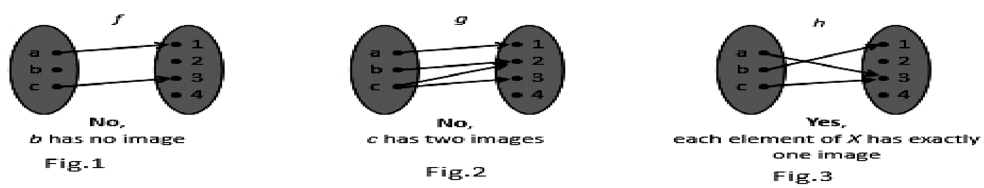


Fig.1 and **Fig.2** f is not a function but in fig.3, f is a function.

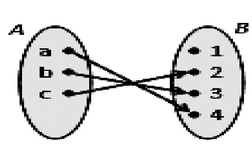
2.2.1 One To One Function (or Injective Function)

A function $f : X \to Y$ is one-to-one (or injective), iff $f(x) = f(y)$ implies $x = y$ for all x and y in the domain X of f. In other words "All elements in the domain of f have different images".

Equivalently; $\forall x, y \in X \; [f(x) = f(y) \Rightarrow x = y]$.

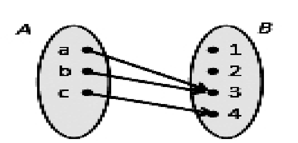
If $f : X \to Y$ is subjective then Range = Co-domain

Example 1: Consider the function $f : A \to B$ with $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$



one-to-one

(all elements in *A* have a different image)



not one-to-one

(*a* and *b* have the same image)

Example 2: Is the function $f:\Re \to \Re$ defined by $f(x) = 5x - 3$ injective? Where, **R** is the set of real numbers .

Solution: Consider

$$f(x) = f(y)$$
$$\Rightarrow 5x - 3 = 5y - 3$$
$$\Rightarrow 5x = 5y$$
$$\Rightarrow x = y \;\Rightarrow \text{'}f\text{' is injective}$$

Example 3: Is the function $f:\Re \to \Re$ defined by $f(x) = x^2$ injective?

Solution: Consider

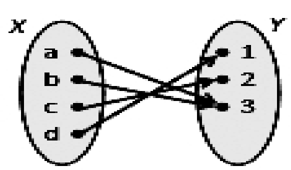
$$f(x) = f(y)$$
$$\Rightarrow 2x^2 = 2y^2$$
$$\Rightarrow x^2 = y^2$$
$$\Rightarrow x = \pm y \;\; Or \;\; y = \pm x \Rightarrow \text{'}f\text{' is not an injective function}$$

2.2.2 Onto Function (or surjective function)

A function $f: X \to Y$ is onto (or surjective), if for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$. In other words "Each element in the co-domain of f has pre- image".

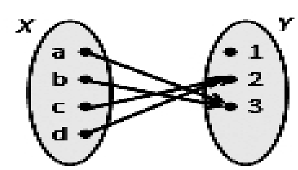
Equivalently; if $f: X \to Y$ is surjective then range = co-domain $\forall\; y \in Y \;\exists\; x \in X \;\; such\ that \;\; f(x) = y]$.

Example 1: Consider the function $f: A \to B$ with $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$



onto

(all elements in Y have a pre-image)



not onto

(1 has no pre-image)

Example 2: Is the function $f:\Re \to \Re$ defined by $f(x) = x^2$ onto?

Solution: Take an element $y = -1$, then for any

$x \in \Re,\;\; f(x) = x^2 \neq -1 = y$

Therefore, $f:\Re \to \Re$ is not onto.

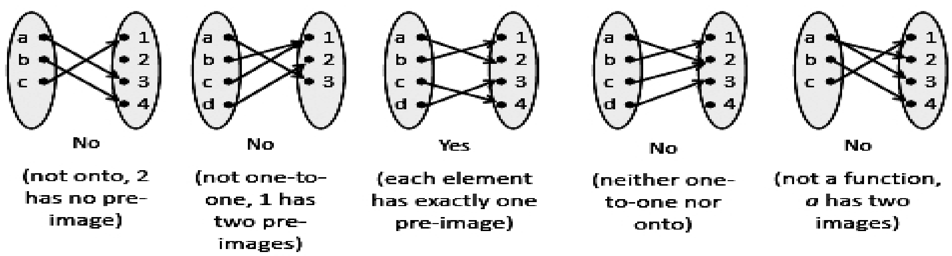
2.2.3 One-to-one Correspondence (Or Bijection)

A function f is one-to-one correspondence (or bijection), iff f is both one-one (or injective) and onto (or surjective).

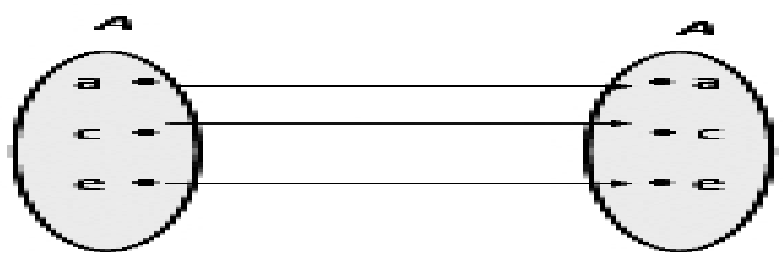
In other words;

" No element in the co-domain of f has two (or more) pre-images" and "Each element in the co-domain of f has pre-image".

Example 1: Consider the function $f: A \to B$ with $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$



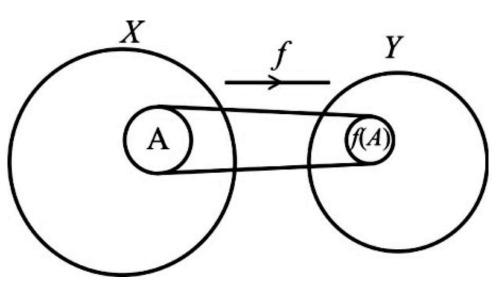
Example 2: Consider an identity function on set $A = \{a, c, e\}$ defined as $I: A \to A,\ I(x) = x \quad \forall\ x \in A$



Note: Identity map is bijective

2.3 Direct and Inverse image of sets

Definition: Let f: $X \to Y$ be a map and let $A \subseteq X$, $B \subseteq Y$, then the direct image of A under f denoted by $f(A)$ and is given by $f(A) = \{y \in Y \mid \exists\ x \in A$ with $f(x) = y\}$,



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that is $f(A)$ is the set of images of all the elements of A. the above diagram illustrates it. Thus $x \in A \Rightarrow f(x) \in f(A)$ the reserve implication viz

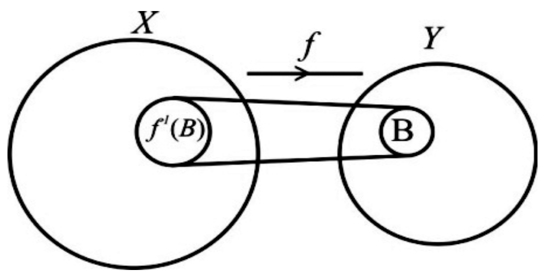
$f(x) \in f(A) \Rightarrow x \in A$ is only true when f is injective. If $x \in X$, then $f(\{x\}) = \{f(x)\}$ and $f(X)$ = range f and f $(\phi) = \phi$.

Note : We shall write $f^1(B)$ for $f^{-1}(B)$ etc

The inverse image of B under f denoted by $f^1(B)$ is given by $f^1(B) = \{x \in X: f(x) \in B\}$ thus $x \in f^1(B) \Rightarrow \exists\, x \in X$ *such that* $f(x) \in B$.

The reverse implication viz $f(x) \in B \Rightarrow x \in f^1(B)$ is also true.

Note : We shall write f^1 (B) for f^{-1} (B) etc.



In case there is no element $x \in X$ such that $f(x) \in B$

(which may happen when f is not surjective), then $f^1(B) = \phi$.

Example1: Let f: R→R be given by $f(x) = x^2$, $x \in$ R.

Let $A = \{x \in \text{R}: 1 \le x \le 2\} = [1, 2] \subset$ R.

Then $f(\text{A}) = \{y \in R: 1 \le y \le 4\} = [1, 4]$. [Since $1 \le x \le 2 \Rightarrow 1 \le x^2 \le 4$]

Let $B = \{y \in R: 4 \le y \le 9\} = [4, 9]$. Then $f^1(B) = [-3, -2] \cup [2, 3]$.

If $C = [-4, -1]$, then $f^1(C) = \phi$, since x∈R such that $f(x) = x^2$ is positive.

Example2: (a) Let $A = \{n\pi: n$ is an integer$\}$ and R be the set of real numbers.

Let $f: A \to R$ be defined by f $(\alpha) = \cos \alpha\ \forall \alpha \in A$. Find f (A) and $f^1(\{0\})$.

Now $f(n\pi) = \cos n\pi = +1$ or -1, Hence $f(A) = \{-1, 1\}$.

If f $(\alpha) = 0$ or $\cos \alpha = 0$ or $\alpha = (2n + 1)\dfrac{\pi}{2}$.

Hence $f^1(\{0\}) = \{(2n+1)\dfrac{\pi}{2} \mid n \in \text{Z}\}$

Now $(2n+1)\dfrac{\pi}{2} \notin \{n\pi\}$, So, $f^1(0) = \phi$.

Example3: Let $f: X \to Y$ be a map and let A and B be subsets of X, then

(i) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$

(ii) $f(A \cup B) = f(A) \cup f(B)$

(iii) $f(A \cap B) \subseteq f(A) \cap f(B)$. Equality holds when f is injective.

Proof: ***(i)*** *If* $A \subseteq B$, then $x \in A \Rightarrow x \in B$. Now $y \in f(A)$

$\Rightarrow \exists\, x \in A$ s.t. $f(x) = y$.

$\Rightarrow \exists\, x \in B$ s.t. $y = f(x)$. $\Rightarrow y = f(x) \in f(B)$ since $x \in B$, $\Rightarrow f(x) \in f(B)$

Therefore $y \in f(A) \Rightarrow y \in f(B)$ hence $f(A) \subseteq f(B)$.

(ii) $y \in f(A \cup B) \Rightarrow \exists\, x \in (A \cup B)$ s.t. $y = f(x)$

$\Rightarrow \exists x \in A$ or $x \in B$ s.t. $y = f(x)$

$\Rightarrow y = f(x) \in f(A)$ or $y = f(x) \in f(B)$. (since $x \in A \Rightarrow f(x) \in f(A)$ and $x \in B \Rightarrow f(x) \in f(B)$).

Hence, $y \in f(A \cup B) \Rightarrow y \in f(A) \cup f(B)$.

Therefore $f(A \cup B) \subseteq f(A) \cup f(B)$.

Again $A \subseteq A \cup B$, $B \subseteq A \cup B$ therefore by (i) $f(A) \subseteq f(A \cup B)$, $f(B) \subseteq f(A \cup B)$ therefore, $f(A) \cup f(B) \subseteq f(A \cup B)$.

From the above we get $f(A \cup B) = f(A) \cup f(B)$.

(iii) $A \cap B \subseteq A$, $A \cap B \subseteq B$, therefore by (i) $f(A \cap B) \subseteq f(A)$, $f(A \cap B) \subseteq f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Note: $f(A) \cap f(B) \subseteq f(A \cap B)$ is not true. Since $y \in f(A) \cap f(B)$

$\Rightarrow y \in f(A)$ and $y \in f(B) \Rightarrow \exists x_1 \in A \mid f(x_1) = y$ and

$\exists\, x_2 \in B$ *such that* $f(x_2) = y \nRightarrow \exists\, x \in A \cap B$ such that $f(x) = y$.

Since $x_1 \in A$ but x_1 may not be an element of B, similarly $x_2 \in$B but x_2 may not be an element of A, so there may not exist a common element x of A and B such that $f(x)=y$.

But if f is injective, then $f(A) \cap f(B) \subseteq f(A \cap B)$ will be true and hence in that case $f(A \cap B) = f(A) \cap f(B)$.

Example4: When $f(A) \cap f(B) \nsubseteq f(A \cap B)$.

Consider map f: $R \to R$ given by $f(x) = x^2$, It is clear f is not injective.

Let A= {-1, - 2, -3, 4} and B= {1, 2, -3} be subsets of Dom f.

Then $A \cap B = \{-3\}$. So, $f(A \cap B) = \{(-3)^2\}$.

Now $f(A) = \{(-1)^2, (-2)^2, (-3)^2, (4)^2\}$, $f(B) = \{1^2, 2^2, (-3)^2\}$, ($x^2 = (-x)^2$)

So, $f(A) \cap f(B) = \{1^2, 2^2, (-3)^2\} \nsubseteq \{(-3)^2\}$.

So, $f(A) \cap f(B) \not\subseteq f(A \cap B)$. (Since y = f($x_1$) = f($x_2$) => $x_1 = x_2$ because f is injective).

Example5: Let $f: X \to Y$ be a map and let A and B be subsets of Y.

Then **(i)** $A \subseteq B \Rightarrow f^{-1}(A) \Rightarrow f^{-1}(B)$

(ii) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

(iii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

***Proof*: (i)** $x \in f^{-1}(A) \Rightarrow f(x) \in A \Rightarrow f(x) \in B$ (since $A \subseteq B$)

So, $x \in f^{-1}(B)$. Therefore $f^{-1}(A) \subseteq f^{-1}(B)$.

(ii) $X \in f^{-1}(A \cup B) \Leftrightarrow f(x) \in A \cup B$

$\Leftrightarrow f(x) \in A$ or $f(x) \in B \Leftrightarrow x \in f^{-1}(A)$

or $x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cup f^{-1}(B)$.

Therefore $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(iii) $x \in f^{-1}(A \cap B) \Leftrightarrow f(x) \in A \cap B$

$\Leftrightarrow f(x) \in A$ and $f(x) \in B \Leftrightarrow x \in f^{-1}(A)$ and

$x \in f^{-1}(B) \Leftrightarrow x \in f^{-1}(A) \cap f^{-1}(B)$.

Therefore $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Thus (ii) and (iii) show that union and intersection are preserved under inverse image.

Check your progress

(2.1) Prove that $f: X \to Y$ is injective iff $f^{-1}(\{y\}) = \{x\}$ $\forall y \in f(X)$, and some $x \in X$

(2.2) Prove that $f: X \to Y$ is sujrective iff $f^{-1}(B) \neq \phi$ $\forall$ $B \subseteq Y$ and $B \neq \phi$.

(2.3) Prove that $f: X \to Y$ is bijective iff $\forall y \in Y$, $f^{-1}(\{y\}) = \{x\}$, $x \in X$.

(2.4) f $f: X \to Y$ and $A \subseteq X$, $B \subseteq Y$, prove that

(a). $f(f^{-1}(B)) \subseteq B$.

(b). $f^{-1}(f(A)) \supseteq A$.

(c). $f^{-1}(Y) = X$.

(d) let $f: X \to Y$ and let $A \subseteq Y$, then prove $f^{-1}(Y - A) = X - f^{-1}(A)$.

(2.5) Give examples when

(i) $f(f^{-1}(B))$ is a proper subset of B

(ii) A is a proper subset of $f^{-1}(f(A))$.

(2.6) Consider $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $f(x) = 4x + 1$. Is f injective, Is f surjective. Also find $f^{-1}(1/2)$.

(2.7) Consider $f: \boldsymbol{R} \to \boldsymbol{R}$ such that $f(x) = 2x^2 + 7$. Is f injective, Is f surjective.

Answer/solution

2.5 (i) Consider map $f: R \to R$ given by $f(x) = x^2$. So f is not surjective

Let $B = \{-1, -2, 3, 4\} \subseteq$ co-dom f. Then $f^{-1}(B) = \{\pm\sqrt{3}, \pm 2\}$,

hence $f(f^{-1}(B)) = \{(\pm\sqrt{3})^2, (\pm 2)^2\} = \{3, 4\}$.

Thus $f(f^{-1}(B))$ is a proper subset of B.

(ii) Consider the above map. Let $A = \{-1, -2, 3, 4\} \subset dom(f)$.

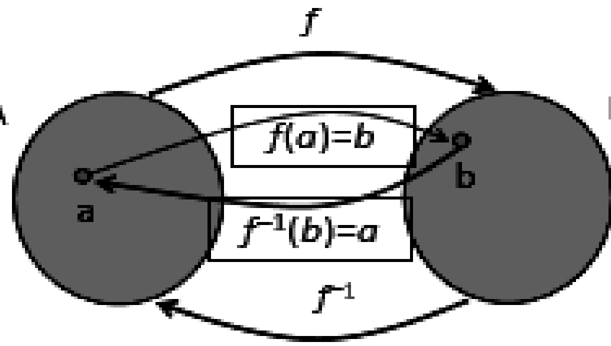
Then $f(A) = \{1^2, 2^2, 3^2, 4^2\}$.

Hence $f^{-1}\{f(A)\} = \{\pm 1, \pm 2, \pm 3, \pm 4\}$ (Prove).

Thus A is a proper subset of $f^{-1}((f(A))$.

2.5 Inverse Functions

Let $f : A \rightarrow B$ be a one-to-one correspondence (i.e., one-one and onto Or bijection). Then the inverse function of f, $f^{-1} : B \rightarrow A$ is defined as $f^{-1}(b)$ such that $f(a) = b$ for an unique element $a \in A$. Here f is also known as invertible.



Note:

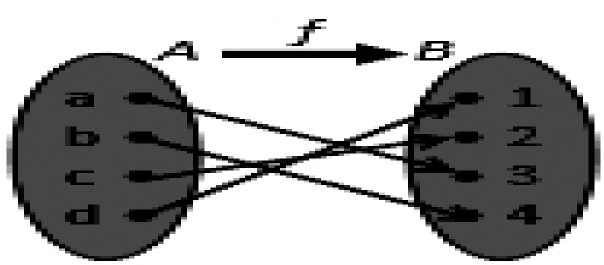
1) Functions that are one-to-one are invertible functions.

2) The inverse of one to one function f is obtained from f by interchanging the coordinates in each ordered pair of f.

3) if f is to be map, then every $b \in B$ must be the f image of some $a \in A$, that is f must be surjective. Further two different elements x_1 and x_2 of A must not have the same f-image $y \in B$, for in that case f (y) $= x_1$ also x_2, so f cannot be a map. Hence f must be injective. Thus when f is bijective we can define the above map f which is called inverse of f and will be denoted by f^{1}. Thus the inverse of a bijective map f is defined as: f^{1}: $B \rightarrow A$ given by $\forall y \in B$, $f^{1}(y) = x \in A$ such that $f(a) = b$.

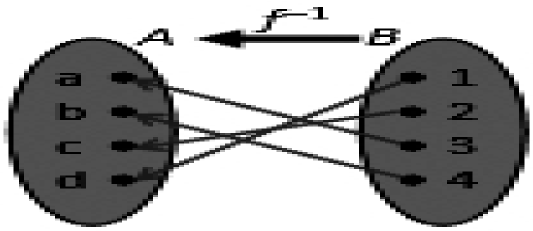
Remarks (2.1): Inverse map of f should not be confused with the inverse image of a subset under f, denoted by the same symbol viz f^{1}.

Example 1: Find the inverse function of $f : A \rightarrow B$ with $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$ Defined by



Solution: The function $f : A \rightarrow B$ is bijective. Therefore

$f^{-1} : B \rightarrow A$ exists.



Then,

$f^{-1}(b) = a \Leftrightarrow b = f(a)\ \forall\ b \in B$ *and* $\forall\ a \in A$. It is given by

Example 2: Find the inverse of $f : \Re \to \Re$ defined by $f(x) = 4x - 1$.

Solution: Let $y \in \Re$. Then, $y = f(x)$ gives

$$y = f(x)$$
$$y = 4x - 1 \quad Or \quad 4x = y + 1 \quad Or \quad x = \frac{y+1}{4}$$

$f^{-1}(y) = \frac{y+1}{4}$ is the inverse of f.

Example 3: Find the inverse of $f : \Re \to \Re$ defined by $f(x) = 4x^2$

Solution: $y = f(x)$ gives two values for $x = \pm\frac{\sqrt{y}}{2}$. Therefore its inverse doesn't exists.

Note: $y = f(x)$ can be made to have inverse with restricted domain to $[0, \infty)$.

Example 4: Find the inverse of $f : \Re \to \Re$ defined by $f(x) = (x-3)^2$ *for* $x \geq 3$.

Solution: f is a one-to-one function with domain $[3, \infty)$ and the range $[0, \infty)$. Therefore the domain of the inverse function is $[0, \infty)$ and its range is $[3, \infty)$. Now consider,

$$y = f(x)$$
$$y = (x-3)^2 \quad Or \quad x - 3 = \pm\sqrt{y} \quad Or \quad x = 3 \pm \sqrt{y}$$

We can have only $x = 3 + \sqrt{y}$ with domain of f^{-1} to be $[0, \infty)$

Inverse of the map f: $X \to Y$ only exists when f is bijective that is the inverse map f^{1}: $X \to Y$ only exists when f is bijective and the inverse map f^{-1}: $Y \to X$ is such that $f^{-1}((y) = x \Leftrightarrow f(x) = y$.

Example 5: Let X = [- π/2, π/2], Y = [- 1, 1]. Let f: $X \rightarrow Y$ be given by $f(x)$ = sin x, $x \in X$. It can be easily proved that f is a bijection. So f^{-1}: $Y \rightarrow X$ given by

$f^{-1}(y)$ = $\sin^{-1} y = x \in X$. such that sin x=y. Thus $\sin^{-1} y = x$. $x = y \Leftrightarrow$ sin

Example 6: If f: $X \rightarrow Y$ is a bijection, then the inverse map f^{-1}: $Y \rightarrow X$ is also a bijection. For let $f^{-1}(y_1) = x_1$, where $y_1 \in Y$ and $x_1 \in X$. Then $f(x_1) = y_1$, and $f^{-1}(y_2) = x_2$, $y_2 \in Y$ and $x_2 \in X$. Then $f(x_2) = y_2$. Now $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \Rightarrow y_1 = y_2$.

[since f is map Therefore f^{-1} is injective.

Again since f is bijective, every element $y \in Y$ is the f-image of a unique element $x \in X$. Hence every $x \in X$ is the f^{-1} image of an element $y \in Y$. Therefore f^{-1} is surjective.

Example7: Check whether the map f: $\boldsymbol{R} \rightarrow \boldsymbol{R}$ such that $f(x)$ = Sinx is bijective? If we take f: [- π/2, π/2], → [- 1, 1] as $f(x)$ = Sinx. Then f is bijective check it.

2.6 Graphs of Functions and Inverse Functions

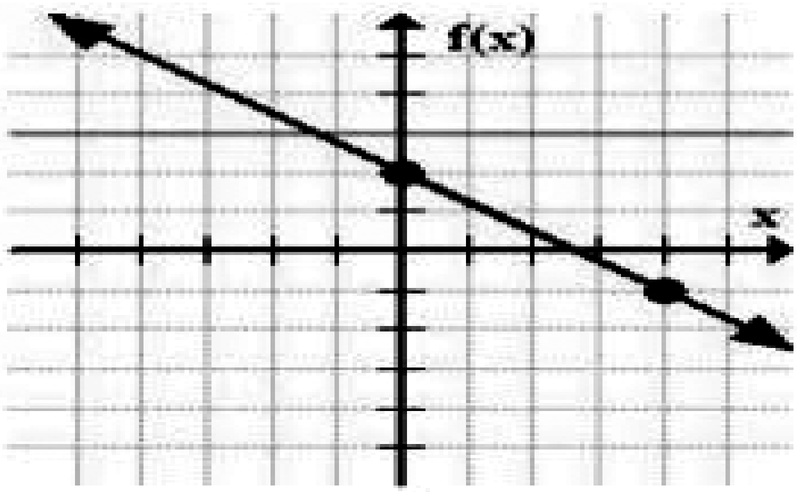
Horizontal Line Test: It says that a function is a one-to-one function if there is no horizontal line that intersects the graph of the function at more than one point.

Note: By applying the Horizontal Line Test, we can not only determine if a function is a one-to-one function, but more importantly we can determine if a function has an inverse or not.

Example 1: Determine if the function $f(x) = -\frac{3}{4}x + 2$ is one-to-one function.

Solution: To determine if $f(x)$ is an one-to-one function, we need to look at the graph of $f(x)$.

x	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
$f(x)$	**5**	**8.5**	**3.5**	**2.75**	**2**	**1.25**	**0.5**	**-0.25**	**-1**

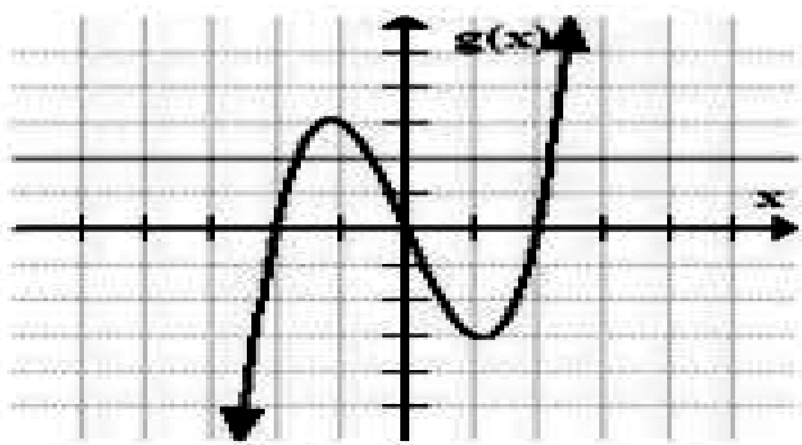


It can be seen in the graph that, any horizontal line drawn on the graph will intersect the graph of $f(x)$ only once. Therefore, $f(x)$ is an one-to-one function and it has an inverse.

Example 2: Determine if the function $g(x) = x^3 - 4x$ is one-to-one function.

Solution: To determine if $g(x)$ is an one-to-one function, we need to look at the graph of $g(x)$.

x	**-2**	**-1**	**0**	**1**	**2**	**3**
$g(x)$	**0**	**3**	**0**	**-3**	**0**	**15**



It can be seen in the graph that, any horizontal line drawn on the graph intersects the graph of $g(x)$ more than once. Therefore, $g(x)$ is not an one-to-one function and it does not have an inverse.

Example 3: Determine inverse of $f(x)$ if the function $f(x) = x^2$ and draw the graphs of $f(x)$ and $f^{-1}(x)$.

Solution: $y = f(x)$ then $y = x^2$ *Or* $x = \pm\sqrt{y}$.

Let's consider $f(x) = x^2$ *for* $x \geq 0$ and $f^{-1}(x) = \sqrt{x}$,s one-to-one

To look at the graphs of $f(x)$ and $f^{-1}(x)$.

Example 4: Determine inverse of the function $f(x) = \sqrt{x-1}$ and graph the $f(x)$ and $f^{-1}(x)$ on the same pairs of axes.

Solution: $y = f(x)$ then $y = \sqrt{x-1}$ *Or* $x = y^2 + 1$.

Let's consider $f(x) = \sqrt{x-1}$ and $f^{-1}(x) = x^2 + 1$ *for* $x \geq 0$. Since range of $f(x)$ is the set of nonnegative real numbers $[0, \infty)$. Therefore we must restrict the domain of $f^{-1}(x)$ to be $[0, \infty)$.

2.7 Operations on function

1) **Addition of two real functions:**

Let $f : A \to R$ and $g : A \to R$ be any two real functions, where $A \subseteq \Re$. Then, $f + g : A \to R$ defined by $(f + g)(x) = f(x) + g(x)\ \forall\ x \in A$.

Example: Let $f(x) = x^4 + 2x^2 + 1$ *and* $g(x) = 2 - x^2$ then,

$$(f + g)(x) = f(x) + g(x)$$
$$= (x^4 + 2x^2 + 1) + (2 - x^2) = x^4 + x^2 + 3$$

2) **Subtraction of real function from another real function:**

Let $f : A \to R$ and $g : A \to R$ be any two real functions, where $A \subseteq \Re$. Then, $f - g : A \to R$ defined by
$(f - g)(x) = f(x) - g(x)\ \forall\ x \in A$.

Example: Let $f(x) = x^4 + 2x^2 + 1$ *and* $g(x) = 2 - x^2$ then,

$$(f - g)(x) = f(x) - g(x)$$
$$= (x^4 + 2x^2 + 1) - (2 - x^2) = x^4 + 3x^2 - 1$$

3) **Multiplication by a scalar:**

Let $f : A \to R$ be any real function and α be any scalar belonging to $\Re$, Then, $\alpha f : A \to R$ defined by

$(\alpha f)(x) = \alpha\ f(x)\ \forall\ x \in A$.

Example: Let $f(x) = x^4 + 2x^2 + 1$ *and* $\alpha = 5$ then,

$$(\alpha f)(x) = (5)(x^4 + 2x^2 + 1) = 5x^4 + 10x^2 + 5$$

4) Multiplication of two real functions:

Let $f: A \to R$ and $g: A \to R$ be any two real functions, where $A \subseteq \Re$. Then, product of these two functions is $f\,g: A \to R$ defined by $(f\,g)(x) = f(x)\,g(x)\ \forall\ x \in A$.

Example: Let $f(x) = x^4 + 2x^2 + 1$ *and* $g(x) = 2 - x^2$ then,

$$(fg)(x) = f(x)g(x)$$
$$= (x^4 + 2x^2 + 1)(2 - x^2)$$
$$= 2x^4 - x^6 + 4x^2 - 2x^4 + 2 - x^2 = -x^6 + 3x^2 + 2$$

5) Quotient of two real functions:

Let $f: A \to R$ and $g: A \to R$ be any two real functions, where $A \subseteq \Re$. Then, the quotient of these two functions is $\frac{f}{g}: A \to R$

defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, *provoded* $g(x) \neq 0\ \forall\ x \in A$.

Note:

1) Domain of the **sum function** $f + g$, **difference function** $f - g$ and the product function fg is $\{x : x \in D_f \cap D_g\}$, where D_f is the domain of the function f and D_g is te domain of the function g.

2) Domain of the quotient function $\frac{f}{g}$ is $\{x : x \in D_f \cap D_g\}$ *and* $g(x) \neq 0$

Example 1: If $f(x) = x^2 - x - 2$ *and* $g(x) = x + 1$. Evaluate $f(x)$ *and* $g(x)$ *at* $x = -3$, hence find $(f + g)(-3)$.

Solution: We have $f(x) = x^2 - x - 2$ *and* $g(x) = x + 1$.

$$\therefore\ f(-3) = (-3)^2 - (-3) - 2 \quad and \quad g(-3) = (-3) + 1$$
$$= 9 + 3 - 2 \quad and \quad = -3 + 1$$
$$= 10 \quad and \quad = -2$$

Now, Consider

$$(f + g)(-3) = f(-3) + g(-3)$$
$$= 10 + (-2)$$
$$= 8$$

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Example 2: If $f(x) = 2x - 1$ *and* $g(x) = x + 4$. Find $(f + g)(x^2)$.

Solution: We have $f(x) = 2x - 1$ *and* $g(x) = x + 4$.

Consider,

$$(f + g)(x^2) = f(x^2) + g(x^2)$$
$$= [2(x^2) - 1] + [(x^2) + 4]$$
$$= 2x^2 - 1 + x^2 + 4$$
$$= 3x^2 + 3$$

Example 3: If $f(x) = 2x - 4$ *and* $g(x) = x^2 - x + 5$. Find $(f - g)(x)$.

Solution: We have $f(x) = 2x - 4$ *and* $g(x) = x^2 - x + 5$.

Consider,

$$(f - g)(x) = f(x) - g(x)$$
$$= (2x - 4) - (x^2 - x + 5)$$
$$= -x^2 + 3x - 9$$

Example 4: If $f(x) = 2x - 4$ *and* $g(x) = -3x + 1$. Evaluate $f(x)$ *and* $g(x)$ *at* $x = 5$, hence find $(f.g)(5)$.

Solution: We have $f(x) = 2x - 4$ *and* $g(x) = -3x + 1$.

$$\therefore\ f(5) = 2(5) - 4 \quad and \quad g(5) = -3(5) + 1$$
$$= 10 - 4 \quad and \quad = -15 + 1$$
$$= 6 \quad and \quad = -14$$

Now, Consider

$$(f.g)(5) = f(5).g(5)$$
$$= 6.(-14)$$
$$= -84$$

Example 5: If $f(x) = 2x - 1$ *and* $g(x) = x + 4$. Find $(f.g)(3x)$.

Solution: We have $f(x) = 2x - 1$ *and* $g(x) = x + 4$.

Consider,

$$(f.g)(3x) = f(3x).g(3x)$$
$$= [2(3x) - 1].[(3x) + 4]$$
$$= (6x - 1).(3x + 4)$$
$$= 18x^2 + 21x - 4$$

Example 6: If $f(x)=x^2-4x-5$ *and* $g(x)=x-5$. Find $\left(\frac{f}{g}\right)(x)$.

Solution: We have $f(x)=x^2-4x-5$ *and* $g(x)=x-5$.

Consider,

$$\left(\frac{f}{g}\right)(x)=\frac{f(x)}{g(x)}$$

$$=\frac{x^2-4x-5}{x-5} \quad x\neq 5$$

$$=\frac{(x-5).(x+1)}{(x-5)}$$

$$=(x-1)$$

Exercise Problems:

1. Let $f(x)=x^2-1$ and $g(x)=4x-3$. Find:

(1) $(f+g)(3)$ (2) $(f+g)(-2)$ (3) $(f-g)(0)$ (4) $(f-g)(-3)$

(5) $(f\cdot f)(3)$ (6) $(f\cdot g)(-2)$ (7) $(f\div g)(\frac{3}{4})$ (8) $(f\div g)(1)$

2.8 Composite Function (Or Composition of Functions)

It is nothing but the function of function. If $f(x)$ *and* $g(x)$ are two functions, then composition of these two functions is defined as $(fog)(x)=f[g(x)]$. The domain of $(fog)(x)$ is the set of all numbers x in the domain of g such that $g(x)$ is in the domain of f

To calculate the composition of function, we evaluate the inner function and substitute the answer into the outer function.

Example 1: If $f(x)=x^2-2x+1$ *and* $g(x)=x-5$. Evaluate $(fog)(3)$

Solution: Consider,

$$(fog)(x)=f\left[g(x)\right]$$

$$\Rightarrow (fog)(3)=f\left[g(3)\right]$$

$$Where\ g(3)=(3)-5=-2$$

$$\therefore (fog)(3)=f\left[-2\right]$$

$$=(-2)^2-2(-2)+1$$

$$=4+4+1$$

$$=9$$

Example 2: If $f(x) = x^2 - x \quad and \quad g(x) = x + 3$. Show that $(fog)(x) \neq (gof)(x)$ **Solution:** Consider,

$$(fog)(x) = f[g(x)]$$
$$Where \ \ g(x) = x + 3$$
$$\therefore (fog)(x) = f[(x+3)]$$
$$= (x+3)^2 - (x+3)$$
$$= x^2 + 6x + 9 - x - 3$$
$$= x^2 + 5x + 6$$

And consider,

$$(gof)(x) = g[f(x)]$$
$$Where \ \ f(x) = x^2 - x$$
$$\therefore (gof)(x) = g[(x^2 - x)]$$
$$= (x^2 - x) + 3$$
$$= x^2 - x + 3$$

Thus, $(fog)(x) \neq (gof)(x)$

Example 3: If $f(x) = 2x - 1 \quad and \quad g(x) = \frac{4}{x-1}$. Then, find $(fog)(2)$ and $(gof)(-3)$

Solution: Consider,

$$(fog)(x) = f[g(x)]$$
$$\Rightarrow (fog)(2) = f[g(2)]$$
$$Where \ \ g(2) = \frac{4}{2-1} = 4$$
$$\therefore (fog)(2) = f[4]$$
$$= 2(4) - 1 = 8 - 1 = 7$$

And consider,

$$(gof)(x) = g[f(x)]$$
$$\Rightarrow (gof)(-3) = g[f(-3)]$$
$$Where \ \ f(-3) = 2(-3) - 1 = -7$$
$$\therefore (gof)(-3) = g[-7]$$
$$= \frac{4}{-7-1} = \frac{4}{-8} = \frac{-1}{2}$$

Thus, In general $(fog)(x) \neq (gof)(x)$.

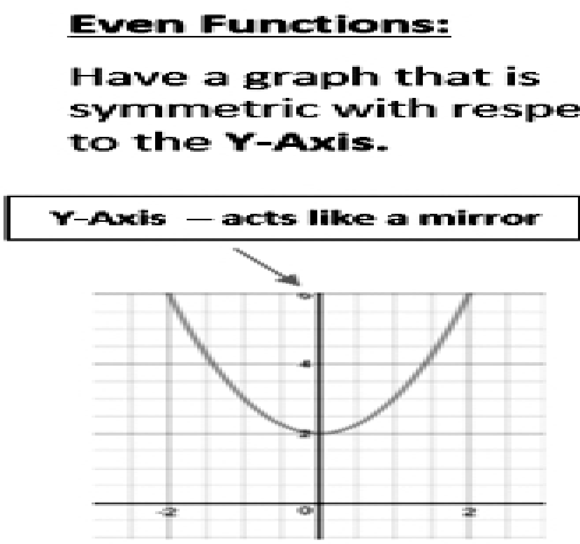
Problem: Let f: $\boldsymbol{R} \to \boldsymbol{R}$ and $\boldsymbol{g} : \boldsymbol{R} \to \boldsymbol{R}$ be defined as $f(x)$ = 2Sinx+ 1 and $g(x) = e^x$. Find (gof)(x) and (fog)(x).

2.9 Even and Odd Functions

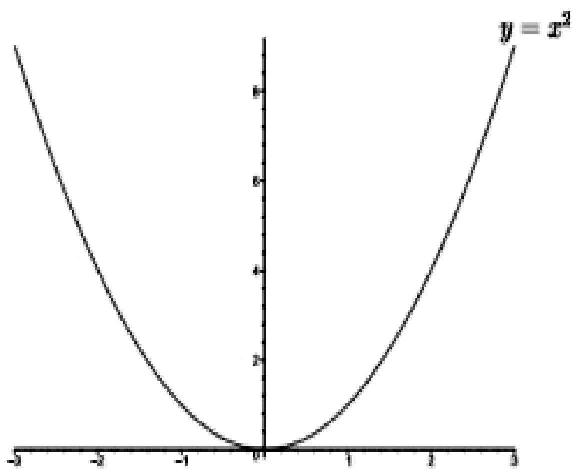
1) **Even Function:** The function $y = f(x)$ is said to be even (or symmetric), if for each x in the domain of $f(x)$, $f(-x) = f(x)$.

Geometrically, the graph of an even function is symmetrical about Y-axis.

For Example: (i) $f(x) = x^2$ (ii) $f(x) = x^4 + 1$ (iii) $f(x) = \cos x$



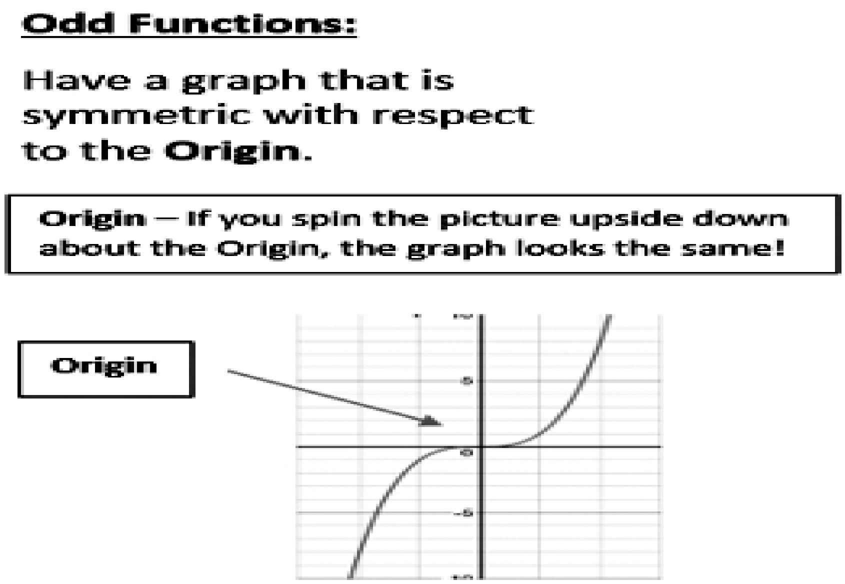
For Example: The graph of the function $f(x) = x^2$ is given by



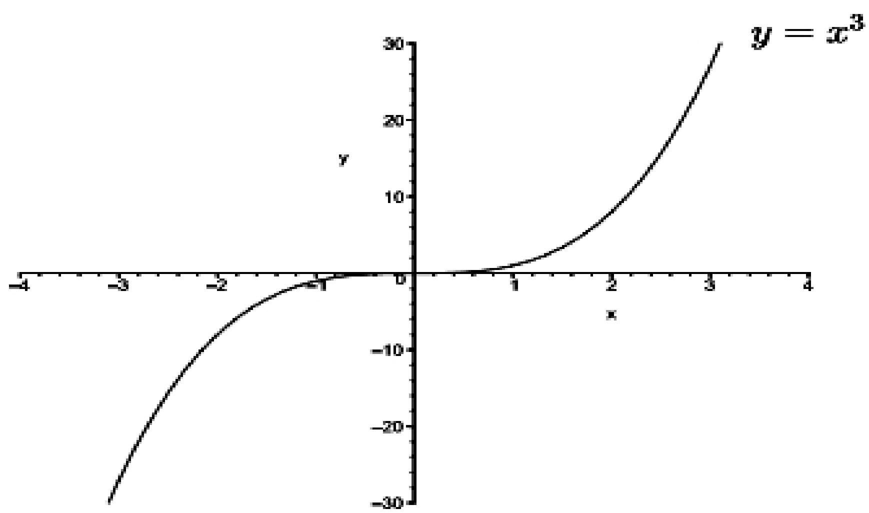
2) **Odd Function:** The function $y = f(x)$ is said to be odd (or antisymmetric), if for each x in the domain of $f(x)$, $f(-x) = -f(x)$.

Geometrically, the graph of an odd function is symmetrical about origin or in opposite quadrants (i.e., the graph has 180 degrees rotational symmetry about origin).

For Example: (i) $f(x) = x^3$ (ii) $f(x) = x^5 + 3$ (iii) $f(x) = \sin x$



For Example: The graph of the function $f(x) = x^3$ is given by



Some useful properties of even and odd functions, which are easy to verify from their definitions:

i) The product of two even functions is even $\Rightarrow (even) \times (even) = even$.

ii) The product of two odd functions is even $\Rightarrow (odd) \times (odd) = even$.

iii) The product of an odd function with an even functions is odd $\Rightarrow (odd) \times (even) = odd$.

iv) The sum of two even functions is even $\Rightarrow (even) + (even) = even$.

v) The sum of two odd functions is odd $\Rightarrow (odd) + (odd) = odd$.

vi) The sum of an odd function with an even functions is neither odd nor even $\Rightarrow (odd) + (even) = neither\ odd\ nor\ even$.

Example 1: Determine algebraically the function $f(x) = x^2 + 8$ is even or odd or neither.

Solution: Consider,

$$f(x) = x^2 + 8$$
$$\Rightarrow f(-x) = (-x)^2 + 8$$
$$= x^2 + 8 = f(x)$$

Thus, $f(x) = x^2 + 8$ is an even function.

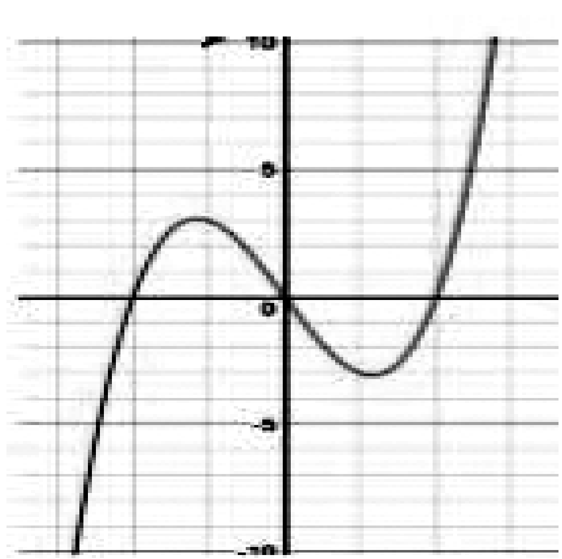
Example 2: Determine algebraically the function $f(x) = x^2 - 3x + 1$ is even or odd or neither.

Solution: Consider,

$$f(x) = x^2 - 3x + 1$$
$$\Rightarrow f(-x) = (-x)^2 - 3(-x) + 1$$
$$= x^2 + 3x + 1$$

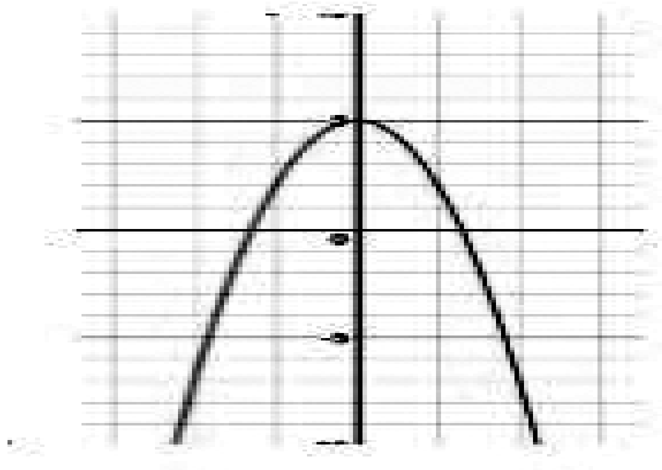
Thus, $f(x) = x^2 - 3x + 1$ is neither even nor odd function.

Example 4: Determine graphically the following function is even or odd or neither.



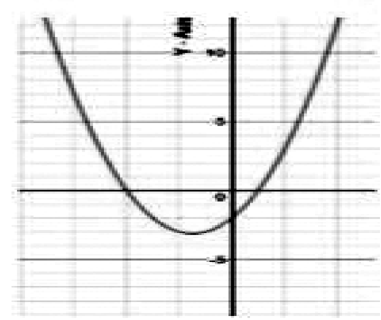
Solution: As symmetry is in opposite quadrants. Therefore, the function is odd.

Example 5: Determine graphically the following function is even or odd or neither.



Solution: As symmetry is about Y-axis. Therefore, the function is even.

Example 6: Determine graphically the following function is even or odd or neither.



Solution: As symmetry is not about Y-axis nor in opposite quadrants. Therefore, the function is neither even nor odd.

Practice Problems:

Determine algebraically the function $f(x) = x^2 - 3x + 1$ is even or odd or neither. Is $f(x)$ = Sinx+ cosx is an even fuction?.

1. $f(x) = x^3 - x^2 + 4x + 2$
2. $f(x) = -x^2 + 10$
3. $f(x) = x^3 + 4x$
4. $f(x) = -x^3 + 5x - 2$
5. $f(x) = \sqrt{x^4 - x^2} + 4$
6. $f(x) = |x + 4|$
7. $f(x) = |x| + 4$
8. $f(x) = x^4 - 2x^2 + 4$
9. $f(x) = \sqrt[3]{x}$
10. $f(x) = x\sqrt{x^2 - 1}$

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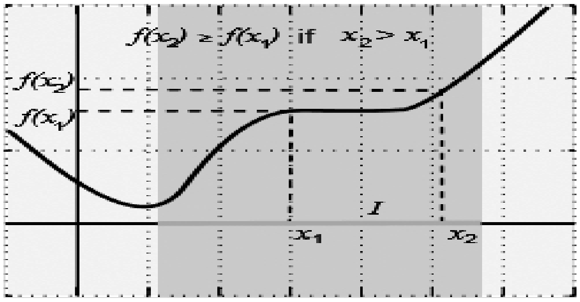
2.10 Monotone (Or Monotonic) Functions

- A function is said to be increasing, if $x_1 < x_2$ implies $f(x_1) \le f(x_2)$.

OR

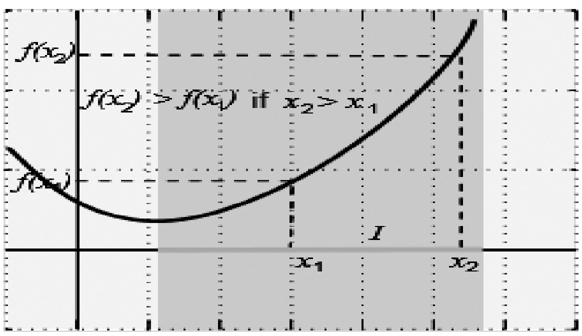
if $x_2 > x_1$ implies $f(x_2) \ge f(x_1)$.



- A function is said to be strictly increasing, if $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

OR

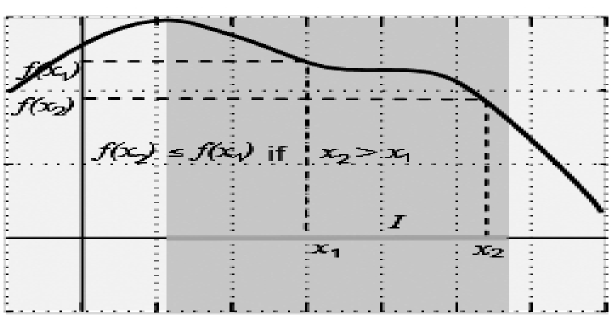
if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



- A function is said to be decreasing, if $x_1 < x_2$ implies $f(x_1) \ge f(x_2)$.

OR

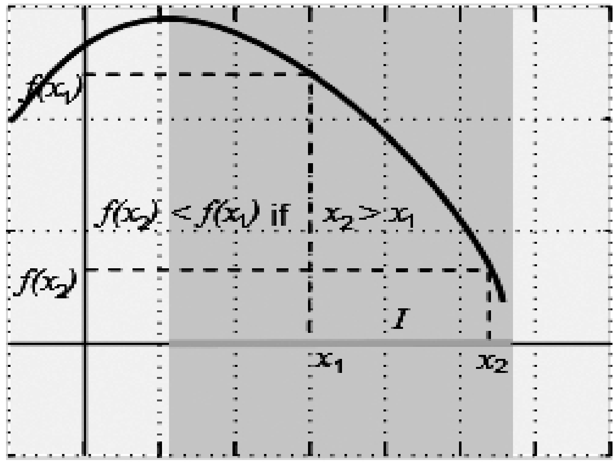
if $x_2 > x_1$ implies $f(x_2) \le f(x_1)$.



➢ A function is said to be strictly decreasing, if $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

OR

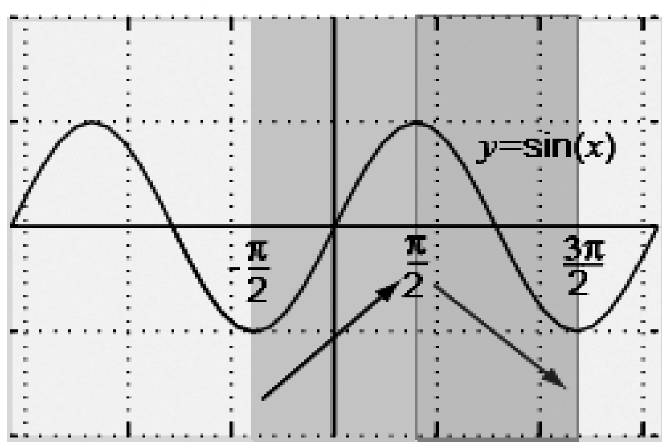
if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.



➢ **A function is said to be monotonic if it is either increasing or decreasing.**

➢ **A function is said to be strictly monotonic if it is either strictly increasing or strictly decreasing.**

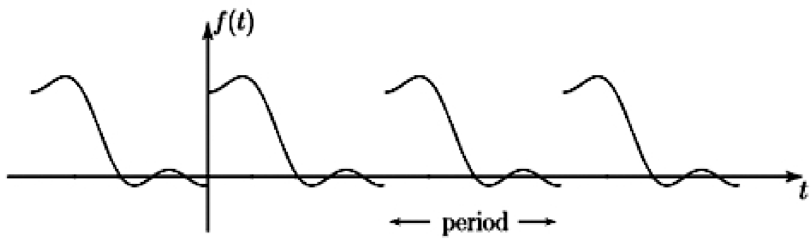
Example 1: The function $f(x) = \sin x$ is strictly monotonic in each interval of $\left[-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right]$; $k = 0, \pm 1, \pm 2, \pm 3, ...$ as it is strictly increasing in the interval $\left[-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right]$; $k = 0, \pm 2, \pm 4, \pm 6, ...$ And it is strictly decreasing in the interval $\left[-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right]$; $k = \pm 1, \pm 3, \pm 5, ...$



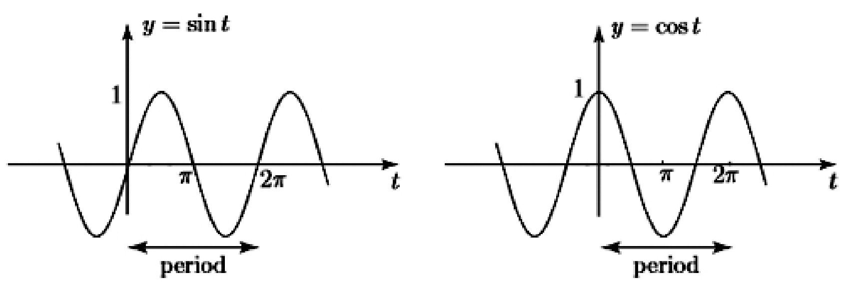
2.11 Periodic Functions

A function $f(t)$ is periodic if the function values repeat at regular intervals of the independent variable t. The regular interval is referred to as the period.

If P denotes the period, then $f(t+P) = f(t)$ *for any value of* t .



The most obvious examples of periodic functions are the trigonometric functions $\sin t$ and $\cos t$, both of which have period 2π. This follows since $\sin(t+2\pi) = \sin t$ and $\cos(t+2\pi) = \cos t$.

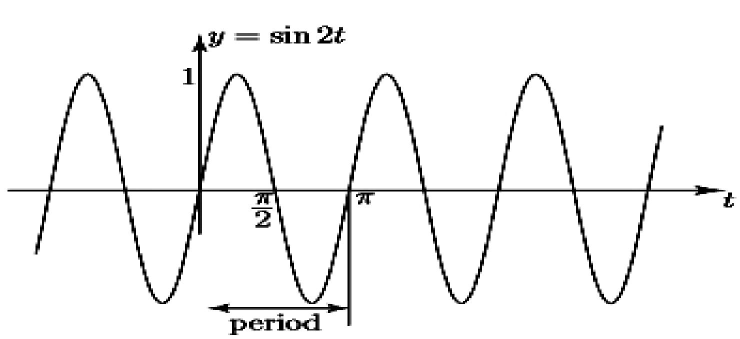


The amplitude of these sinusoidal (or trigonometric) functions is the maximum displacement from $y = 0$.

Consider $y = A\sin nt$, which has maximum amplitude A and period $\frac{2\pi}{n}$.

For example:

(i) $y = \sin 2t$ is a sinusoid of amplitude 1 and period $\frac{2\pi}{2} = \pi$ since $\sin 2(t+\pi) = \sin(2t+2\pi) = \sin 2t$ for any value of t.



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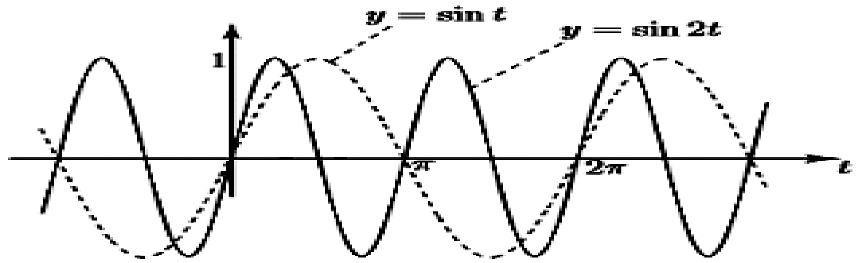
We can observe that $y = \sin 2t$ has half the period of $\sin t$, π as opposed to 2π in the following figure.



<u>Some standard results on periodic functions</u>

Functions	**Periods**
1. $\text{Sin}^n x$, $\text{Cos}^n x$, $\text{Sex}^n x$, $\text{Cosec}^n x$	π ; if n is even 2π ; if n is odd or fraction
2. $\tan^n x$, $\text{Cot}^n x$	π ; if n is even or odd
3. $\lvert sinx \rvert, \lvert cosx \rvert$ $\lvert tanx \rvert, \lvert cotx \rvert \lvert secx \rvert, \lvert cosecx \rvert$	π
4. Algebraic function e.g. $\sqrt{x}, x^2, x^3 + 5$ etc	Period does not exists.

Example1 : Find period for

(i) $\text{Cos}^4 x$, has period π as n is even.

(ii) $\text{Sin}^3 x$, has period 2 π as n is odd.

(iii) $\sqrt{cosx}$ has the period 2 π as n is in fraction.

(iv) $\cos\sqrt{x}$, is not periodic.

<u>Properties of periodic function:-</u>

(i) If f(x) is periodic with period T, then:

(ii) c. f(x) is periodic with period T.

(iii) f(x+c) is periodic with period T.

(iv) f(x)+c is periodic with period T.

Where c is any constant i.e. If constant is added, subtracted, multiplied or divided in periodic function, period remains same.

Example 2: We know that sinx has period 2π, then f(x) = 3sinx+2 is also periodic with period 2π .

If f(x) is periodic with period T, then

f(cx) is periodic with period $\frac{T}{|c|}$.

i.e. period is only affected by coefficient of x.

f(x) = sinx period 2π then period of sin2x = $\frac{2\pi}{2} = \pi$

sin3x = $\frac{2\pi}{3}$, sin4x = $\frac{2\pi}{4}$, tan4x = $\frac{\pi}{4}$

Example 3: State the amplitude and period of $y = 2\cos 5t$

$y = 2\cos 5t$ is a sinusoid of amplitude 2 and period $\frac{2\pi}{5}$ since

$$\cos 5\left(t + \frac{2\pi}{5}\right) = \cos(5t + 2\pi) = \cos 5t \text{ for any value of } t$$

State the amplitude and period of the following:

(a) $y = 5\cos 4t$ (b) $y = 6\sin\frac{2t}{3}$.

<u>**Non- Sinusoidal Periodic Functions:**</u>

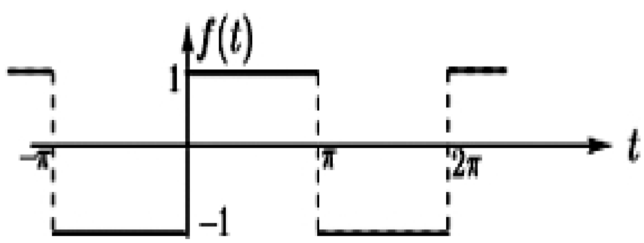
These non-sinusoidal periodic functions are often called "waves"

1) Square Wave:

This fuction is defined as;

$$f(t) = \begin{cases} -1 & ; -\pi < t < 0 \\ +1 & ; 0 < t < \pi \end{cases}$$

$$f(t + 2\pi) = f(t)$$



2) Saw-tooth wave:

This fuction is defined as;

$$f(t) = 2t \quad ; \; 0 < t < 2$$

$$f(t + 2) = f(t)$$

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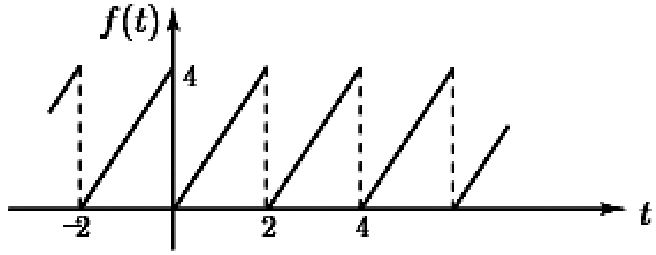


3) Triangular wave:

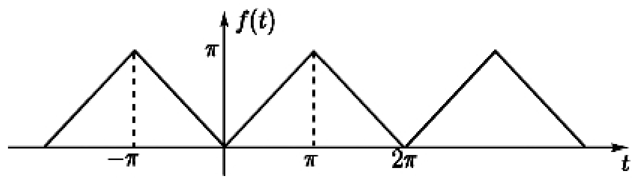
This fuction is defined as;

$$f(t) = \begin{cases} -t & ; -\pi < t < 0 \\ +t & ; 0 < t < \pi \end{cases}$$

Or more concisely,

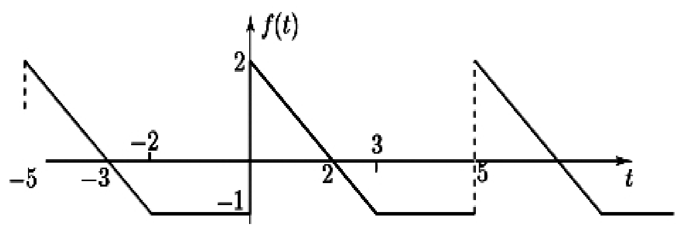
$$f(t) = |t|$$

$$f(t + 2\pi) = f(t)$$



Practice Problems:

1) Write down the function for the following graph;



2) Sketch the graph of the periodic function defined by;

$$f(t) = \begin{cases} \frac{t^2}{2} & ; 0 < t < 4 \\ 8 & ; 4 < t < 6 \\ 0 & ; 6 < t < 8 \end{cases}$$

$$f(t + 8) = f(t)$$

3) Sketch the graph of the periodic function defined by;

$$f(t) = 2t - t^2 \quad ; \ 0 < t < 2$$

$$f(t + 2) = f(t)$$

2.12 Axiomatic introduction of Real Numbers

Basic Properties of Real Numbers:

The basic properties of real numbers include:

1. Closure Property
2. Commutative Property
3. Associative Property
4. Distributive Property
5. Ordered Property
6. Completeness Property
7. Archimedean Property

These are known as algebraic properties of real numbers.

There are two binary operations defined on $\Re$, one called addition, denoted by $(a, b) \to a+b$, and other called multiplication, denoted by $(a, b) \to a \cdot b$.

1.	**Closure Property** $a+b \in \Re$ and $a, b \in \Re$ for every $a, b \in \Re$
2.	**Commutative Property:** For all $a, b \in \Re$ *then* $a+b=b+a$ *and* $a \cdot b = b \cdot a$ **(a) The commutative property of addition:** **Example :** a-b $\neq$ b-a i.e. difference is not commutative There exists an element in $\Re$, denoted by 0 read as zero is known as additive identity with the following properties; **(i)** $0+a=a+0=a$ *for all* $a \in \Re$ (Existence of additive identity) **(ii)** For every $a \in \Re$, there exists an unique element $-a \in \Re$ Such that $a+(-a)=-a+a=0$ *for all* $a \in \Re$ (*Existence of addi* **(iii)** Also there exist an element 1 in $\Re$ known as multiplicative identities with the following properties (i) 1.a=a.1,for ever $a \in \Re$ (ii) for ever $a \in \Re$ (a $\neq$ 0) there exist $\frac{1}{a} \in \Re$ s.t. $\frac{1}{a}.a = a.\frac{1}{a} = 1$ RIL-145

	(b) The commutative property of Multiplication: **Example:** (a) $\frac{1}{a}.b = b.\frac{1}{a}$ (a≠0) (b) $(x+2)\cdot 3 = 3\cdot(x+2) = 3(x+2)$ (c) $5 - y\cdot x = 5 - x\cdot y = 5 - xy$ There exists an element in $\Re$, denoted by 1 read as one or unity is known as multiplicative identity with the following properties; **(i)** $1\cdot a = a \quad for \;\; all \;\; a \in \Re$ (Existence of multiplicative identity) **(ii)** For every $a \neq 0 \in \Re$, there exists an unique element $a^{-1}\left(=\frac{1}{a}\right) \in \Re$ such that $a\cdot(a^{-1}) = 1 \;\; for \;\; all \;\; a \in \Re$ (Existence of multiplicative inverse)
3.	**Associative Property:** For all $a,b,c \in \Re \quad then \quad (a+b)+c = a+(b+c) \quad and \quad (a\cdot b)\cdot c = a\cdot(b\cdot c)$ **(a) The Associative property of addition:** **Example:** $(2+3)+6 = 2+(3+6)$ Let A={1,2,3,4,5} then addition holds in A. i.e. closed in A and addition is associative in A. **(b) Associative property of Multiplication:** **Example:** $(3\cdot x)\cdot x = 3\cdot(x\cdot x) = 3x^2$ $(x\cdot y)\cdot 5xy = 5\cdot(x\cdot x)(y\cdot y) = 5x^2y^2$
4.	**Distributive Property:** For all $a,b,c \in \Re \quad then \quad a\cdot(b+c) = ab+ac \;\; and \quad (b+c)\cdot a = ba+ca$ **Example:** $(\boldsymbol{a})\quad (\boldsymbol{a}+\boldsymbol{b}).(\boldsymbol{c}+\boldsymbol{d}) = (\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{c} + (\boldsymbol{a}+\boldsymbol{b}).\boldsymbol{d}$ ac+bc+ad+bd **(b)** difference is not associative since $(\boldsymbol{a}-\boldsymbol{b})-\boldsymbol{c} \neq \boldsymbol{a}-(\boldsymbol{b}-\boldsymbol{c})$ **Remarks:** On the basis of addition property, one can define the operation of subtraction by $a - b = a + (-b)$. On the basis of multiplication property, one can define the operation of division by $\frac{a}{b} = a\cdot\left(\frac{1}{b}\right) \quad (\; if \;\; b \neq 0)$.

5.	**Ordered Properties of Real Numbers:** There exists an order, denoted by <, between the elements of $\Re$ with the following properties; i) For $a, b \in \Re$, one and only one of the following relations hold; $a < b, \quad a = b, \quad b < a$ This is known as **Law of Trichotomy** ii) $a, b > 0 \Rightarrow a \cdot b > 0 \; and \; a + b > 0$ iii) $a > b \; and \; b > c \Rightarrow a > c$ Geometrically, set of all points on a line represent the set of all real numbers. There are some special subsets of $\Re$ which are important. These are the familiar number systems as shown in the above flowchart.
6.	**Completeness Property** Geometrically, rational numbers when represented by points on the line, do not cover every point of the line. Therefore, the property of the real numbers that distinguishes them from the rational numbers is called the completeness property.
7.	**Archimedean Property** For every $a \in \Re$, there exists $n \in N$ such that $n > a$

2.13 Absolute Value

The absolute value of a, denoted by $|a|$, read as "the absolute value of a", describes the distance 'a' on the number line from zero without considering which direction from zero the number lies. The absolute value the number is never negative.

Example 1: $|5| = 5$, as 5 is five units to the right of zero. But also $|-5| = 5$, because -5 is five units to the left of zero.

The absolute value does get a little more complicated when dealing with variables, since we don't know the sign of the variable.

For example: If $|a| = 5$, then we need to consider a is 5 or -5. Therefore solution to the equation is 5 or -5.

Definition: The absolute value of a real number 'a' is denoted by $|a|$ and is defined as

$$|a| = \begin{cases} a & if \quad a \geq 0 \\ -a & if \quad a < 0 \end{cases}$$

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For example: (i) $|7| = 7$ (ii) $\left|\frac{-3}{5}\right| = -\left(-\frac{3}{5}\right) = \frac{3}{5}$ (iii) $|0| = 0$

> **Note:**
>
> **1** The effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative.
>
> **2** $|a| = b \rightarrow a = \pm b \rightarrow |a| = |b| \rightarrow a = \pm b$

Example 1: Solve $|x-3| = 2$

Solution: Consider,

Positive side if x$>$ 3	**Negative side if x $<$ 3**
$x-3=2$	$x-3=-2$
$x-3+3=2+3$	$x-3+3=-2+3$
$x=5$	$x=1$

Verification: $|x-3| = 2$, $|5-3| = 2$, $|2| = 2$ Verification: $|x-3| = 2$, $|1-3| = 2$, $|-2| = 2$

Example 2: Solve $|2x-3| = 15$ if x $<$ 0

Solution: Consider,

Positive side	**Negative side**
$2x-3=15$	$2x-3=-15$
$2x-3+3=15+3$	$2x-3+3=-15+3$
$2x=18$	$2x=-12$
$x=9$	$x=-6$

Since solution of $|2x-3| = 15$ is x = -6

Example 3: Solve $|3x-1| = |x+7|$

Solution: Since $|a| = |b| \rightarrow a = \pm b$

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Consider,

Positive side	**Negative side**
$3x-1=x+7$	$3x-1=-(x+7)$
$3x-1+1=x+7+1$	$3x-1+1=-(x+7)+1$
$3x=x+8$	$3x=-x-6$
$2x=8$	$4x=-6$
$x=4$	$x=-\frac{3}{2}$

Check Your Progress

Solve the following

1) $|x-3|=5$ 2) $|2x-5|=10$ 3) $|5x+1|=5$

4) $|3x-7|=11$ 5) $|x+3|=1$

6) $|3x-2|=|5x+4|$ 7) $|x-2|=|3x+1|$ 8) $|6x+1|=|3x+4|$

9) $|3x-1|=|x+4|$

Properties of Absolute Value

If a and b are real numbers, then

1) $|a|\geq 0$ if and only if $a=0$, [that is $|a|=0$ if $a=0$ and conversely $a=0$ if $|a|=0$

2) $|-a|=|a|$ [A number and its negative have the same absolute value]

3) $|a|\leq b$ if and only if $-b\leq x\leq b$

4) $|a+b|\leq|a|+|b|$ [The Triangle Inequality]

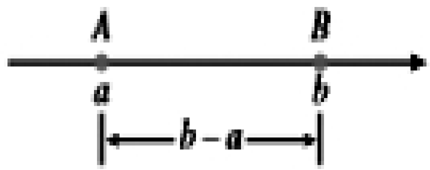
5) $||a|-|b||\leq|a-b|$

6) $|a\cdot b|=|a|\cdot|b|$ [The absolute value of a product is the product of the absolute values]

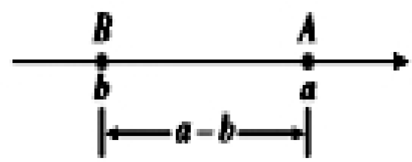
7) $\left|\frac{a}{b}\right|=\frac{|a|}{|b|}; b\neq 0$ [The absolute value of a ratio is the ratio of the absolute values]

Geometric Interpretation of Absolute Value:

The notion of absolute value arises naturally in distance problems. Suppose A and B are the points on a real number line that have coordinates a and b respectively. Depending on the relative positions of the points as shown in the figure, the distance between them will be



b-a

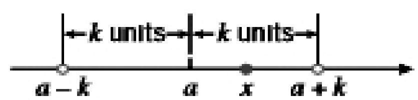


a-b

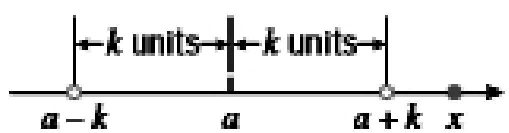
In either case, the distance can be written as $d = |b-a|$

<u>**Inequalities with Absolute values**</u>

i) For $k>0$, $|x-a|<k$, x is within k units of a;
$-k < x-a < k \;\; Or \;\; a-k < x < a+k$. It can be shown geometrically

(Figure) 

ii) For $k>0$, $|x-a|>k$, x is more than k units away from a;
$x-a<-k \;\; Or \;\; x-a>k \quad OR \;\; x<a-k \;\; Or \;\; x>a+k$. It can be



shown geometrically (Figure);

Example 1: Solve for x and express the solution in terms of intervals; $|x+6|<3$

Solution: We have $|x+6|<3$. Then,

$-3 < x+6 < 3$

Adding (–6) *throughout we obtain*

$-3-6 < x < 3-6$

$-9 < x < -3$

Which can be written in the interval notation as (-9, -3).

Example 2: Solve for x and express the solution in terms of intervals; $|7-x| \le 5$

Solution: We have $|7-x| \le 5$. Then,

$$(7-x) \le 5 \quad Or \quad (7-x) \ge -5 \quad Adding\ (-7)\ throughout$$

$$-x \le -2 \quad Or \quad -x \ge -12 \Rightarrow x \ge 2 \quad Or \quad x \le 12$$

Which can be written in the interval notation as (2, 12).

Example 3: Solve for x and express the solution in terms of intervals; $\frac{3}{|2x-1|} \ge 4$, x$\neq \frac{1}{2}$

Solution: We have $\frac{3}{|2x-1|} \ge 4$. Then,

$$\frac{|2x-1|}{3} \le 4 \quad Or \quad |2x-1| \le 12$$

$$-12 \le (2x-1) \le 12 \quad Adding\ 1\ throughout \quad -11 \le 2x \le 13$$

$$Diving\ thriughout\ by\ 2,\ \frac{-11}{2} \le x \le \frac{13}{2}$$

Which can be written in the interval notation as (-11/2, 13/2).

Check Your Progress
Solve for x **and express the solution in terms of intervals for the following**;   1) $|x-3| < 4$, 2) $|x+4| \ge 2$, 3) $|2x-3| \le 6$,   **2)** 4) , $\frac{1}{|2x-3|} > 5$, 5) $\frac{1}{|3x+1|} \ge 5$, 6). $\frac{2}{|x+3|} < 4$.

2.14 Intervals on the real line

A useful way of describing the set of real numbers is by using interval notation [In spite there exists many other ways viz, rosters, tables, number lines etc].

Interval notation is a frequent option to express a set of numbers between two values, a and b. Basically used two symbols are parentheses ( ) and brackets [ ].

( ) is used for less than <, or greater than >. This means that specified values for a or b are not included.

[ ] is used for less than or equal to $\le$, or greater than or equal to $\ge$. This means that specified values for a or b are included.

Definition:

i) Given any two extended real numbers $a < b$, the set; $(a,b) = \{x : x \in \Re,\ a < x < b\}$ is called an open interval.

Thus, Open Interval: $(a,b) = \{x : x \in \Re,\ a < x < b\}$

Half Open Intervals: $[a,b) = \{x : x \in \Re,\ a \le x < b\}$ **and** $(a,b] = \{x : x \in \Re,\ a < x \le b\}$

Infinite Open Intervals: $(-\infty,b) = \{x : x \in \Re, -\infty < x < b\}$ Or $(a,+\infty) = \{x : x \in \Re,\ a < x < \infty\}$.

ii) Given any two finite real numbers $a \le b$, the set; $[a,b] = \{x : x \in \Re,\ a \le x \le b\}$, $(-\infty,b] = \{x : x \in \Re,\ x \le b\}$ and $[a,+\infty) = \{x : x \in \Re,\ x \ge a\}$ are called closed intervals.

Thus, Closed Interval: $[a,b] = \{x : x \in \Re,\ a \le x \le b\}$

Infinite Closed Intervals: $(-\infty,b] = \{x : x \in \Re,\ x \le b\}$ Or $(-\infty,b] = \{x : x \in \Re, -\infty < x \le b\}$

And $[a,+\infty) = \{x : x \in \Re,\ x \ge a\}$ Or $[a,+\infty) = \{x : x \in \Re,\ a \le x < \infty\}$

iii) **Empty Open Interval:** $(a,a) = \phi$

Singleton Closed Intervals: $[a,a] = \{a\}$ Or $(a,+\infty) = \{x : x \in \Re,\ a < x < \infty\}$

Hence, We can summarize as follows

- A closed interval $[a,\ b]$ describes all real numbers x where $a \le x \le b$.
- An open interval $(a,\ b)$ describes all real numbers x where $a < x < b$.
- A half-open (or half-closed) interval describes one of the following
  - $[a,\ b)$ describes all real numbers x where $a \le x < b$.
  - $(a,\ b]$ describes all real numbers x where $a < x \le b$.

We use ∞ and $-\infty$ to signify that the values continue getting larger without end (unbounded to the right on the real number line) and smaller without end (unbounded to the left on the real number line) respectively.

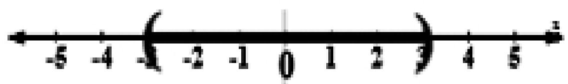
- $[a,\ \infty)$ describes all real numbers x where $x \ge a$.
- $(a,\ \infty)$ describes all real numbers x where $x > a$.
- $(-\infty,\ a]$ describes all real numbers x where $x \le a$.

- $(-\infty,\ a)$ describes all real numbers x where $x < a$.

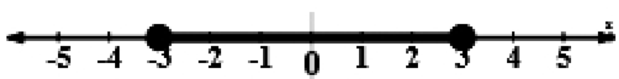
As stated above, shall assume that the set of real numbers can be identified with points on the straight line. If the point O represent the number 0, then the points on the left of O represent negative real numbers and points on the right of O represent positive real numbers. Intervals are part of the real number line.

This identification of the real numbers is useful in visualizing various properties of real numbers. An open interval of the type $(a-\delta,\ a+\delta)$ is called an δ- neighborhood of $a \in \Re$.

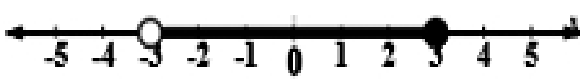
Example 1: The inequality $-3 < x < 3$ reflects all the real numbers between -3 and 3 without including both. Using interval notation, this inequality is written as $(-3,\ 3)$. The graph of the solution set is given by



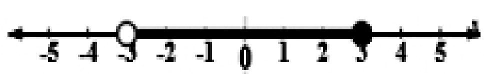
Example 2: The inequality $-3 \le x \le 3$ reflects all the real numbers between -3 and 3 including both. Using interval notation, this inequality is written as $[-3,\ 3]$. The graph of the solution set is given by



Example 3: The inequality $-3 \le x < 3$ reflects all the real numbers between -3 and 3 including -3 but not 3. Using interval notation, this inequality is written as $[-3,\ 3)$. The graph of the solution set is given by

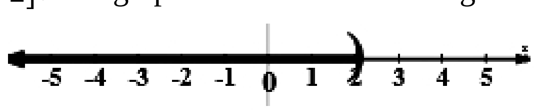


Example 4: The inequality $-3 < x \le 3$ reflects all the real numbers between -3 and 3 including 3 but not -3. Using interval notation, this inequality is written as $(-3,\ 3]$. The graph of the solution set is given by



Example 5: Write the inequality $-\infty \le x \le 2$ in interval notation and graph it.

Solution: The in equality $-\infty \le x \le 2$ reflects all the real numbers between -∞ and 2 including both. This inequality is written in the interval notation as $[-\infty,\ 2]$. The graph of the solution set is given by



Exercise Problems:

Write the following inequalities in interval notation and graph it.

1. $x \le 3$ 2) $-2 \le x \le 4$ 3) $-9 \le x \le 0$ 4) $x > -4$

5) $x < -3$ 6) $x \ge 6$

Express each of the following intervals in set-builder notation.

1) (2, 8) 2) [−5, 0) 3) (3, ∞) 4) (−∞, − 4]

5) [−7, 3]

2.15 Summary

Function or mapping as a relation with some conditions, types of mapping i.e. injective map , surjective map, bijective map is described in the unit. Direct image and inverse image of a subset i.e. f(A) and f ‘(B) and operation of union and intention on them is discussed . Inverse map and condition when it is defined, Graph of a function, Composite of two function f and g i.e. gof and fog properties are explained in the unit. Even and odd function, monotonic increasing and mono-tonic deceasing functions, periodic functions, axiomatic introduction for the set of real numbers as complete ordered field is explained. Basic properties of the set of real, absolute value and its properties are explained. Meaning of |x-a| ≤ b is studied in this unit.

2.16 Terminal Questions

1. Define a map f: R → R such that f(x) = -5x +3. Is f bijective? If yes find f ‘(-1/2).

2. Consider f: R → R such that f(x) = $4x^2$ – 3. is f injective?

3. Let f: A → B & g: B → C be maps. Show that

(i) If g 0 f is injective then f is injective

(ii) If g 0 f is subjective the g is surjective

4. Let A and B be two sets containing m and n elements respectively, then

(i) How many maps can be defined from A to B?

(ii) If m=n then how many bijective maps can be defined from A to B?

5. Let A = { nπ : n is an integer}

Define a map f : A → R such that f(x) = Sin x. then find f(A)

6. Define f : R → R such that f(x) = $x/(1+x^2)$. find the range of the function f.

Answers to Selected terminal Questions	
1.	Yes, f ‘(-1/2) = 7/10
2.	No
4.	(i) nm (ii) n!
5.	f(A) = {0}
6.	[-1/2, ½]

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UNIT-3

LIMITS

Structure

3.1. Introduction

3.2. Objectives

3.3. Definition of limit

3.4. Algebra of Limits

3.5. Infinite Limits (Limits as $x \to \pm\infty$)

3.6. One Sided Limits

3.7. Summary

3.8 Terminal Questions/Answers

3.1 INTRODUCTION

We begin the study of calculus, starting with the concept of 'limit'. As we read the later units, we will realize that the seeds of calculus were sown as early as the third century B.C. But it was only in the nineteenth century that a rigorous definition of a limit was given by **Weierstrass**. Before him, **Newton, D'Alembert and Cauchy** had a clear idea about limits, but none of them had given a formal and precise definition. They had depended, more or less, on intuition or geometry.

The introduction of limits revolutionized the study of calculus. The cumbersome proofs which were used by the Greek mathematicians have given way to neat, simpler ones. We already have an intuitive idea of limits. In Section.3 of this unit, we shall give a precise definition of this concept. This will lead to the study of continuous functions. We shall also give some examples of discontinuous functions in section 7 and section 8.

In the early development of mathematics the concept of limit was very vague. The calculation of a limit was so fundamental to understand certain aspects of calculus, that it required the precise definition. A more formal ε-δ (read epsilon-delta) definition of a limit was finally developed around the 1800's. This formalization resulted from the combined research into into limits developed by the mathematicians Weierstrass, Bolzano and Cauchy.

The concept of a limit or limiting process, essential to the understanding of the calculus, has been around for thousands of years. In fact, early mathematicians used a limiting process to obtain better and better approximations of areas of circles. Yet the formal definition of the limit- as we know and understand it today-did not appear until the 19^{th} century.

3.2 Objectives

After reading this unit students should be able to:

- understand the need and use of limits and its algebra
- calculate the limits of functions whenever they exits.
- Identify and evaluate one sided limits and infinite limits

3.3 Definition of Limit

In this section we will introduce you to the notion of 'limit'. We start with considering a situation which a lot of us are familiar with, such as train travel. Suppose we are travelling from Delhi to Agra by a train which will reach Agra at 10.00am. As the time gets closer and closer to 10.00 am., the distance of the train from Agra gets closer and closer to zero (assuming that the train is running on time!). Here, if we consider time as out independent variable, denoted by t and distance as a function of time, say f(t), then we see that f(t) approaches zero as t approaches 10. In this case we say that the limit of f(t) is zero as t tends to 10.

Now consider the function f: R $\rightarrow$R defined by $f(x) = x^2+1$. Let us consider Tables 1(a) and 1(b) in which we give the values of f(x) as x takes values nearer and nearer to 1. In Table 1(a) we see values of x which are greater than 1. We can also express this by saying that x approaches 1 from the right. Similarly, we can say that x approaches 1 from the left in Table 1(b).

Table 1(a)

X	1.2	1.1	1.01	1.001
f(x)	2.44	2.42	2.02	2.002

Table 1(b)

X	0.8	0.9	0.99	0.999
f(x)	1.64	1.81	1.9801	1.9989

We find that, as x gets closer and closer to 1, f(x) gets closer and closer to 2. Alternatively, we express this by saying that as x approaches 1 (or tends to 1), the limit of f(x) is 2. Let us now give a precise meaning of 'limit'.

To show that the limit of f(x) as x → p equals the number L, we need to show that the gap between f(x) and L can be made "as small as we choose" if x is kept "close enough" to p. Let us see what this would require if we specified the size of the gap between f(x) and L.

Let's consider the following example.

Example 1: Consider the function y = 2x – 1 near x = 4, Intuitively it appears that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x-1)=7$. However, how close to x = 4 does x have to be so that y = 2x – 1 differs from 7 by, say, less than 2 units?.

Solution: We are asked: For what values of x is $|y-7|<2$?.To find the answer we first express $|y-7|$ in terms of x:

$$|y-7| = |2x-1-7| = |2x-8|.$$

The question then becomes: what values of x satisfy the inequality $|2x-8|<2$?. To find out we solve the inequality:

$$|2x-8|<2$$
$$-2<2x-8<2$$
$$6<2x<10$$
$$3<x<5$$
$$-1<x-4<1$$

Keeping x with 1 unit of x = 4 will keep y within 2 units of y = 7 as shown in the following figure.

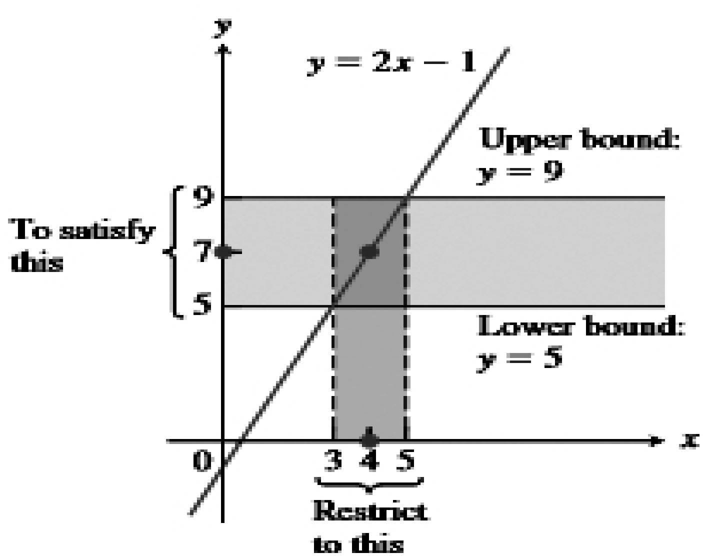


Figure: Keeping x within 1 unit of x = 4will keep y within 2 units of y = 7

In the above example we determined how close x must be to a particular value p to ensure that the outputs f(x) of some function lie within a prescribed interval about a limit value L. To show that the limit of f(x) as $x \to p$ actually equals L, we must be able to show that the gap between f(x) and L can be made less than any prescribed error, no matter how small, by holding x close enough to p.

Definition 1: Let f be a function defined at all points near p (except possibly at p). Let L be a real number. We say that f approaches the limit L as x approaches p if, for each real number $\varepsilon > 0$, we can find a real number $\delta > 0$ such that

$$0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

As you know from Unit 1, $|x - p| < \delta$ means that $x \in]p - \delta, p + \delta[$ and $0 < |x - p|$ means that $x \neq p$. That is, $0 < |x - p| < \delta$ means that x can take any value lying between $p - \delta$ and $p + \delta$ except p.

The limit L is denoted by $\lim_{x \to p} f(x)$. We also write $f(x) \to L$ as $x \to p$.

Note that, in the above definition, we take any real number $\varepsilon > 0$ and then choose some $\delta > 0$, so that $L - \varepsilon < f(x) < L + \varepsilon$, whenever $|x - p| < \delta$ that is, $p - \delta < x < p + \delta$.

In unit 1 we have also mentioned that $|x - p|$ can be thought of as the distance between x and p.

Equivalently,

Let f(x) be defined on an open interval about p, except possibly at p itself. We say the limit of f(x) as x approaches p is the number L, and write $\lim_{x \to p} f(x) = L$, if , for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $0 < |x - p| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

The relation of δ and ε in the definition of the limit is shown as in the following figure;

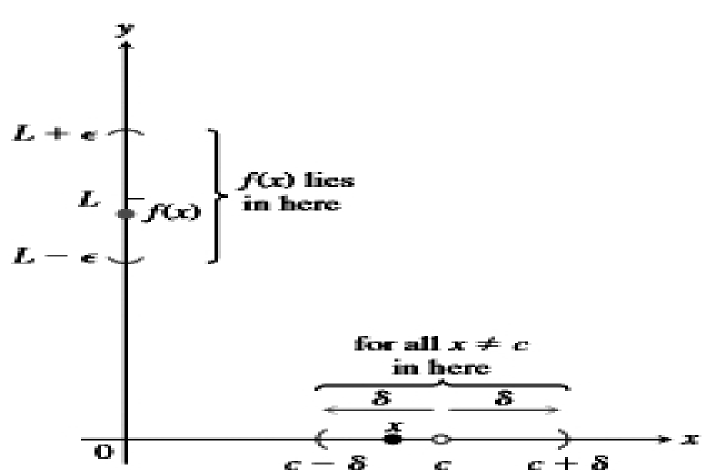


Figure : The relation of δ and ε in the definition of limit

Remark 1: The number ε is given first and the number δ is to be produced. An important point to note here is that while taking the limit of f(x) as x → p, we are concerned only with the values of f(x) as x takes values closer and closer to p, but not when x = p. For example, consider the function $f(x)=\frac{x^2-1}{x-1}$. This function is not defined for x = 1, but is defined for all other x ∈ R. However, we can still talk about its limit as x →1. This is because for taking the limit we will have to look at the values of f(x) as x tends to 1, but not when x = 1.

Example 2: Consider the function f : R →R defined by f(x) = x^3. How can we find $\lim_{x\to 0}$ f(x)? **Solution:**We will see that when x is small, x^3 is also small. As x comes closer and closer to 0, x^3 also comes closer and closer to zero. It is reasonable to except that lim f(x) = 0 as →0. Let us prove that this is what happens. Take any real number ε> 0.Then, | f(x) – 0 | <ε⇔ $|x^3|$ <ε⇔ |x| <$\varepsilon^{1/3}$. Therefore, if we choose δ = $\varepsilon^{1/3}$ we get | f(x) – 0 | <ε whenever 0 < |x – 0| < δ. This gives us $\lim_{x\to 0}$ f(x) = 0.

A useful general rule to prove $\lim_{x\to a}$ f(x) = L is to write down f(x) – L and then express it in terms of (x – a) as much as possible.

Let us now see how to use this rule to calculate the limit in the following examples.

Example 3: Show that $\lim_{x\to 1}(5x-3)=2$.

Solution: Set p=1, $f(x)=5x-3$ and L = 2 in the definition of the limit. For any given ε > 0, we have to find a suitable δ > 0 so that if x ≠ 1 and x is within distance δ of p = 1, that is, whenever $0<|x-1|<\delta$. It is true that f(x) is within distance ε of L = 2, so $|f(x)-2|<\varepsilon$. We find δ by working backward from the ε-inequality:

$$|5x-3-2|=|5x-5|<\varepsilon$$
$$=5|x-1|<\varepsilon$$
$$=|x-1|<\varepsilon/5.$$

Thus, we can take $\delta=\varepsilon/5$ (as shown in the following figure). If $0<|x-1|<\delta=\varepsilon/5$ then $|(5x-3)-2|=|5x-5|=5|x-1|<5(\varepsilon/5)=\varepsilon$, which proves that $\lim_{x\to 1}(5x-3)=2$. The value of $\delta=\varepsilon/5$ is not the only value that will make $0<|x-1|<\delta$ simply $|5x-5|<\varepsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work.

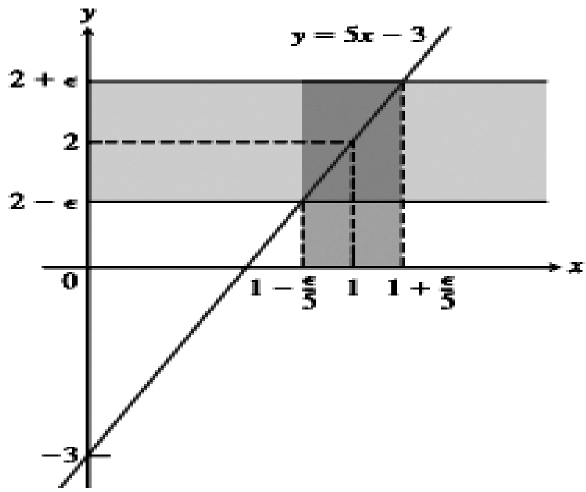


Figure: If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon / 5$ guarantees that $|f(x) - 2| < \varepsilon$.

Examples 4: Prove that $\lim_{x \to c} x = c$.

Solution: Let ε > 0 be given. We must find δ > 0 such that for all x,

$0 < |x - c| < \delta$ Implies $|x - c| < \varepsilon$

This implication will hold if δ equals ε or any smaller positive number (as shown in the following figure).

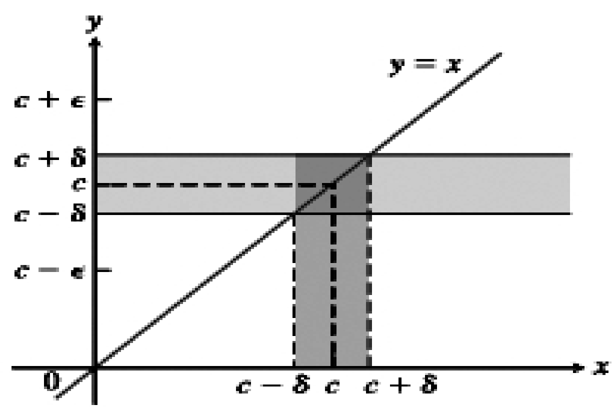


Figure: For the function f(x) = x, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever δ ≤ ε.

Remark 2: If f is a constant function on R, that is, if f(x) = k ∀ x ∈ R, where k is some fixed real number, then $\lim_{x \to p}$ f(x) = k.

Examples 5: Prove that $\lim_{x \to c} k = k$ .Where k is constant

Solution: Let ε > 0 be given. We must find δ > 0 such that for all x,

$0 < |x-c| < \delta$ Implies $|k-k| < \varepsilon$.

Since k – k = 0, we can use any positive number for δ and the implication will hold (as shown in the following figure). This proves that $\lim_{x \to c} k = k$

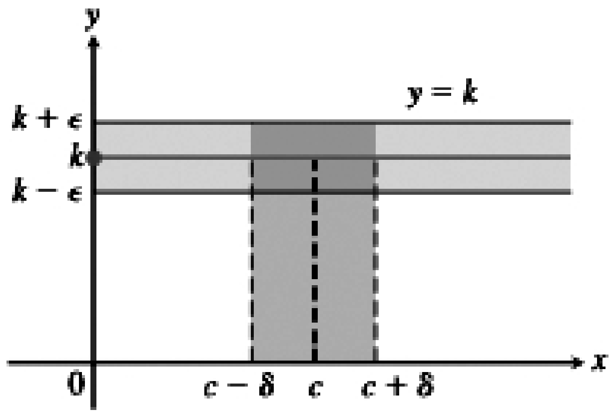


Figure: For the function f(x) = k, we find that $|f(x)-k| < \varepsilon$ for any positive δ.

Remark 3: In the above examples 2.3.2, 2.3.3 and 2.3.4, the interval of values about c (or p) for which $|f(x)-L|$ was less than ε and we could take δ to be half the length of the interval. When such symmetry is absent, as it usually is, we can take δ to be the distance from c(or p) to the interval's nearer endpoint.

Examples 6: For the limit, $\lim_{x \to 5} \sqrt{x-1} = 2$, find aδ> 0 that works for ε = 1. That is, find δ > 0 such that for all x, $0 < |x-5| < \delta \Rightarrow \left|\sqrt{x-1} - 2\right| < 1$.

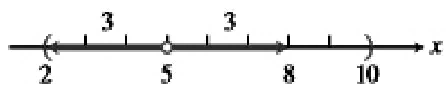
Solution: We organize the search into two steps.

i. Solve the inequality $\left|\sqrt{x-1} - 2\right| < 1$ to find an interval containing x = 5 on which the inequality holds for all x ≠ 5.

$$\left|\sqrt{x-1} - 2\right| < 1$$
$$-1 < \sqrt{x-1} - 2 < 1$$
$$1 < \sqrt{x-1} < 3$$
$$1 < x-1 < 9$$
$$2 < x < 10$$

ii. Find a value of δ > 0 to place the centered interval 5 – δ <x < 5 + δ (centered at x = 5) inside the interval (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (as shown in the following figure 1). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x-5| < \delta$ will automatically place x between 2 and 10 to make $\left|\sqrt{x-1}-2\right| < 1$ (as shown in the following figure 2):

$$0 < |x-5| < 3 \Rightarrow \left|\sqrt{x-1}-2\right| < 1 \quad .$$



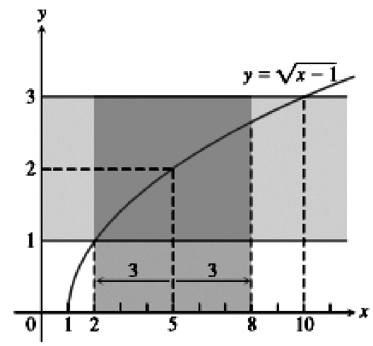


Figure 1: An open interval of radius 3 about x=5 will lie inside the open interval (2, 10)

Figure 2: The function and intervals in the example.

Examples 7: Let us calculate $\lim_{x\to 1}\frac{x^2-1}{x-1}$. We know that division by zero is not defined. Thus, the function $f(x) = \frac{x^2-1}{x-1}$ is not defined at x = 1. But, as we have mentioned earlier, when we calculate the limit as x approaches 1, we do not take the value of the function at x = 1. Now, to obtain $\lim_{x\to 1}\frac{x^2-1}{x-1}$, we first note that

$x^2 - 1 = (x - 1)(x + 1)$, so that $\frac{x^2-1}{x-1} = x + 1$ for $x \neq 1$. Therefore $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x + 1)$. As x approaches 1, we can intuitively see that this limit approaches 2. To prove that the limit is 2, we first write f(x) – L = x + 1 – 2 = x – 1, which is itself in the form x – a, since a = 1 in this case. Let us take any number $\varepsilon > 0$. Now,

$|(x + 1) - 2| < \varepsilon \Leftrightarrow |x - 1| < \in$

Thus, if we choose $\delta = \varepsilon \Rightarrow |f(x) - L| = |x - 1| < \varepsilon$. This shows that $\lim_{x\to 1}(x + 1) = 2$. Hence, $\lim_{x\to 1}\frac{x^2-1}{x-1} = 2$.

Example 8: Let us prove that ∀ε> 0 ∃δ> 0 such that $|x^2 + 4 - 13| < \varepsilon$ whenever $|x - 3| < \delta$. Here, $f(x) - L = (x^2 + 4) - 13 = x^2 - 9$, and $x - a = x - 3$.

Now we write $|x^2 - 9|$ in terms of $|x - 3|$: $|x^2 - 9| = |x + 3|\,|x - 3|$

Thus, apart from $|x - 3|$, we have a factor, namely $|x + 3|$ of $[x^2 - 9]$. To decide the limits of $|x + 3|$, let us put a restriction on δ.

Remark 4: we have to choose δ. So let us say we choose a $\delta \le 1$. $|x - 3| < \delta \Rightarrow |x - 3| < 1 \Rightarrow 3 - 1 < x < 3 + 1 \Rightarrow 2 < x < 4 \Rightarrow 5 < x + 3 < 7$.

Thus, we have $|x^2 - 9| < 7\,|x - 3| < \varepsilon$. Now when will this be true? It will be true when $\delta \le 1$. This means that given $\varepsilon > 0$, the δ choose should satisfy$\delta \le 1$ and also $\delta \le \varepsilon/7$. In other words, $\delta = \min\{1, \varepsilon/7\}$, should serve our purpose.

Examples 9: Prove that $\lim_{x \to 2} f(x) = 4$,if $f(x) = \begin{cases} x^2, & x \ne 2 \\ 1, & x = 2 \end{cases}$

Solution: our aim is to show that given ε > 0 there exists δ < 0 such that for all x,

$$0 < |x - 2| < \delta \quad \Rightarrow \quad |f(x) - 4| < \varepsilon$$

i. Solve the inequality $|f(x) - 4| < \varepsilon$ to find an open interval containing $x_0 = 2$ on which the inequality holds for all $x \ne x_0$.

For $x \ne x_0 = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$.

$$\begin{aligned} &|x^2 - 4| < \varepsilon \\ &-\varepsilon < x^2 - 4 < \varepsilon \\ &4 - \varepsilon < x^2 < 4 + \varepsilon \\ &\sqrt{4 - \varepsilon} < |x| < \sqrt{4 + \varepsilon} \\ &\sqrt{4 - \varepsilon} < x < \sqrt{4 + \varepsilon} \end{aligned}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \ne 2$ in the open interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$ (as shown in the following figure).

ii) Find a value of δ > 0 that places the centered interval (2 - δ, 2 + δ) inside the interval $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$.

Take δ to be the distance from $x_0 = 2$ to the nearer endpoint of $(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon})$. In other words, take $\delta = \min\{2 - (\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2)\}$, the minimum (the smaller) of the two numbers

$2-\sqrt{4-\varepsilon}$ *and* $\sqrt{4+\varepsilon}-2$. If δ has this or any smaller positive value, the inequality $0<|x-2|<\delta$ will automatically place x between $\sqrt{4-\varepsilon}$ *and* $\sqrt{4+\varepsilon}$ to make $|f(x)-4|<\varepsilon$. For all x, $0<|x-2|<\delta \Rightarrow |f(x)-4|<\varepsilon$.

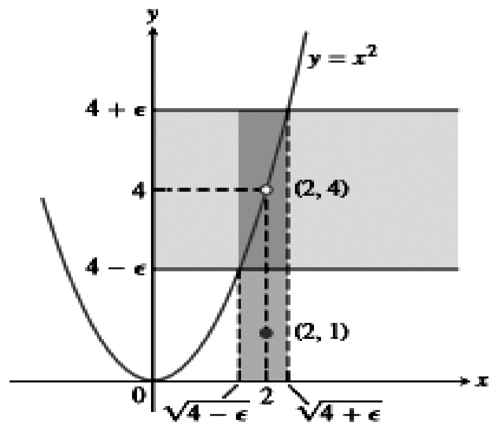


Figure: An interval containing x = 2 so that the function satisfies $|f(x)-4|<\varepsilon$.

3.4. Algbera of Limits

Let us state some basic properties of limits (their proofs are beyond the scope of this course).

Theorem 1: Let f and g be two functions such that $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = M$ exist. Where L, M, k, p are real numbers. Then,

i. $\lim_{x\to p}[f(x) + g(x)] = \lim_{x\to p} f(x) + \lim_{x\to p} g(x) = L + M$ (Sum Rule)

[i.e., Limit of the sum is sum of the limits]

ii. $\lim_{x\to p}[f(x) - g(x)] = \lim_{x\to p} f(x) - \lim_{x\to p} g(x) = L - M$ (Difference Rule)

[i.e., Limit of the difference is difference of the limits]

iii. $\lim_{x\to p} [k\, f(x)] = k\, [\lim_{x\to p} f(x)]\, [\lim_{x\to p} g(x) = kL$

(Constant Multiple Rule)

[i.e., Limit of the constant times the function is the constant time the limit of the function]

iv. $\lim_{x\to p} [f(x)\, g(x)] = [\lim_{x\to p} f(x)]\, [\lim_{x\to p} g(x)] = LM$ (Product Rule)

[i.e., Limit of the product is product of the limits]

v. $\dfrac{1}{g(x)} = \dfrac{1}{\lim\limits_{x \to p} g(x)} = \dfrac{1}{M}$, provided $\lim\limits_{x\to p}$ g (x) = M≠ 0

(Reciprocal Rule)

[i.e., Limit of the reciprocal relation exists provided limit of the denominator is non zero]

vi. $\lim\limits_{x \to p} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to p} f(x)}{\lim\limits_{x \to p} g(x)} = \dfrac{L}{M}$, provided $\lim\limits_{x\to p}$ g (x) = M≠ 0

(QuotientRule)

[i.e., Limit of the quotient is quotient of the limits provided limit of the denominator is non zero]

vii. $\lim\limits_{x \to p} [f(x)]^n = L^n$, n a positive integer (Power Rule)

viii. $\lim\limits_{x \to p} \left[\sqrt[n]{f(x)}\right]^n = \sqrt[n]{L} = L^{1/n}$, n is a positive integer (Root Rule)

[If n is even, we assume that $\lim\limits_{x\to p}$ f(x) = L > 0]

We have already proved two more results in the above Theorem 2.3.3 and Theorem 2.3.4 in addition to these. Those are :

ix. $\lim\limits_{x\to p}$ k = k

(Constant Function Rule)

x. $\lim\limits_{x\to p}$ x = p (Identity Function Rule)

We now state and prove a theorem whose usefulness will be clear to you in **Unit 4**.

Theorem 2: $\lim\limits_{x\to p}$ f(x)=L and $\lim\limits_{x\to p}$ f(x) = M, then L = M.

Proof. Suppose L ≠ M, then |L – M | > 0. Since $\lim\limits_{x\to p}$ f(x) = L. If we take ε= $\dfrac{|L - M|}{2}$ then $\exists \delta_1 > 0$ such that $|x - p| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon$

Similarly, since $\lim\limits_{x\to p}$ f(x) = M, $\exists \delta_2 > 0$ such that $|x - p| < \delta_2 \Rightarrow |f(x) - M| < \varepsilon$

If we choose δ = min $|\delta_1, \delta_2|$, then δ> 0 and |x – p | <δ we mean that $|x - p| < \delta_1$ and $|x - p| < \delta_2$.

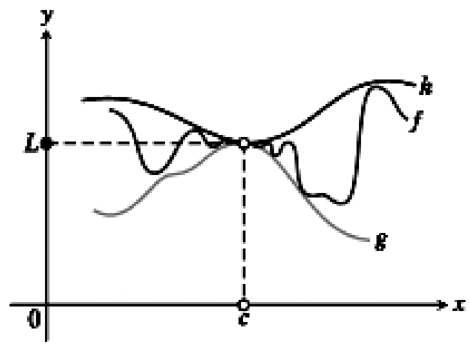
In this case we will have both |f(x) – L | <ε, as well as, |f(x) – M | <ε.

So that | L – M | = L – f(x) + f(x) + f(x) – M | ≤ | f(x) – L | + | f(x) – M | <ε + ε = 2ε = | L – M |. That is, we get | L – M | < | L – M |,

Which is a contradiction. Therefore, our supposition is wrong. Hence L = M.

The Sandwich Theorem

This theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions f and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (as shown in the following figure).



The graph of f is sandwiched between the graphs of g and h

Theorem 3: (The Sandwich Theorem):

Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at x = c itself. Suppose also that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. Then, $\lim_{x\to c} f(x) = L$.

Equivalently,

Theorem 4: Let f, g, and h be functions defined on an interval I containing a, except possibly at a, Suppose

(i) f(x) ≤ g (x) ≤ h(x) ∀x∈ I \ |a|

(ii) $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$

(iii) Then $\lim_{x\to a} g(x)$ exist and is equal to L.

Proof. By the definition of limit ε> 0, $\exists\delta_1 > 0$ and $\delta_2 > 0$ such that |f(x) – L | <ε for 0< | x – a|<$\delta_1$and |h(x) – L | <ε for 0 < | x – a| <δ_2.

Let δ = min {δ_1, δ_2}. Then 0 < | x – a| <δ⇒ | f(x) – L | <ε and | h(x) – L | <ε

⇒ L - ε≤ f(x) ≤ L + ε, and L - ε≤ h(x) ≤ L + ε

We also have f(x) ≤ g(x) ≤ h(x) ∀ x ∈ I/{a}. Thus, we get 0 < | x – a| <δ

$\Rightarrow$ L - $\varepsilon \le f(x) \le g(x) \le h(x) \le L + \varepsilon$. In other words, $0 < |x - a| < \delta$

$\Rightarrow |g(x) - L| < \varepsilon$. Therefore $\lim_{x \to a} g(x) = L$. Theorem 2 is also called the **sandwich theorem** (Or the S**queeze theorem**Or the **Pinching Theorem**), because g is being sandwiched between f and h.

Let us see how this theorem can be used.

Example 1: Given that $|f(x) - 1| \le 3(x+1)^2 \; \forall \; x \in R$, can we calculate $\lim_{x \to -1}$ f(x)?

We know that $-3(x+1)^2 \le f(x) - 1 \le 3(x+1)^2 \; \forall$ x. This means that

$-3(x+1)^2 + 1 \le f(x) \le 3(x+1)^2 + 1 \; \forall$ x. Using the sandwich theorem and the fact that

$\lim_{x \to -1} [-3(x+1)^2 + 1] = 1 = \lim_{x \to -1} [3(x+1)^2 + 1]$, we get $\lim_{x \to -1} f(x) = 1$.

$|f(x) - L| = |x - p| < \varepsilon$ whenever $|x - p| < \delta$, if we choose $\delta = \varepsilon$.

Example 2: Given that $1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$ for all $x \ne 0$. Find $\lim_{x \to 0} u(x)$.

Solution: Since $\lim_{x \to 0}\left[1 - \frac{x^2}{4}\right] = 1$ and $\lim_{x \to 0}\left[1 + \frac{x^2}{2}\right] = 1$. Then by Sandwich Theorem, it implies that $\lim_{x \to 0} u(x) = 1$. [We can observe the following figure).

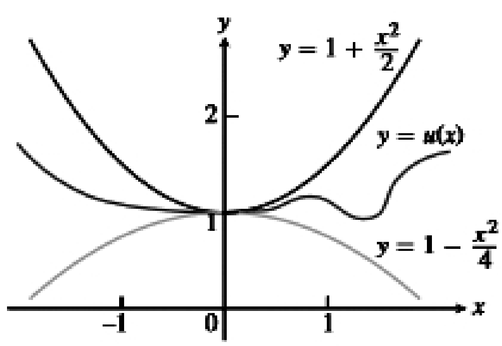


Figure: Any function u(x) whose graph lies in the region between $y = 1 + \frac{x^2}{2}$ and $y = 1 - \frac{x^2}{4}$ has limit 1 as $x \to 0$.

Example 3: Let us evaluate $\lim_{x \to 2} \frac{3x^2 + 4x}{2x + 1}$

Now $\lim_{x\to 2} 2x + 1 = \lim_{x\to 2} 2x + \lim_{x\to 2} 1$ by using (i) $= \lim_{x\to 2} 2 \lim_{x\to 2} x + \lim_{x\to 2} 1$ by using (ii)

$\therefore$ we can use (v) of theorem 2.4.1. Then the required limit is

$$\frac{\lim_{x\to 2}(2x^2+4x)}{\lim_{x\to 2}(2x+1)} = \frac{\lim_{x\to 2} 3x^2 + \lim_{x\to 2} 4x}{\lim_{x\to 2} 2x + \lim_{x\to 2} 1}$$ by using

(i) $$= \frac{\lim_{x\to 2} 3 \lim_{x\to 2} x \lim_{x\to 2} x + \lim_{x\to 2} 4 \lim_{x\to 2} x}{\lim_{x\to 2} 2 \lim_{x\to 2} x + \lim_{x\to 2} 1}$$ by using

(ii) $$= \frac{3\times 2\times 2 + 4\times 2}{2\times 2+1} = \frac{20}{5} = 4$$

Check your progress

(1) Show that

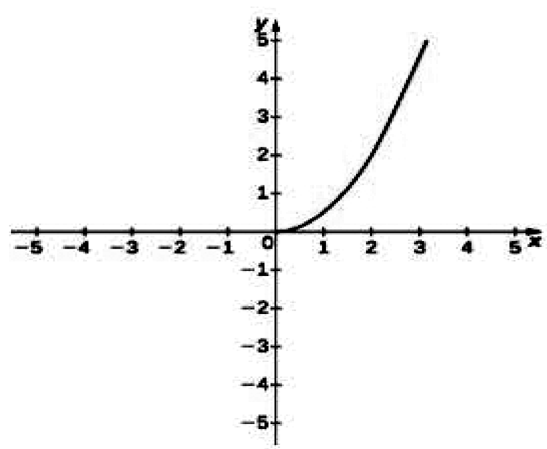
i. $\lim_{x\to 1} \frac{1}{x} = 1$

ii. $\lim_{x\to 1} \frac{x^3-1}{x-1} = 3.$

(2) Show that $\lim_{x\to 1} \frac{3}{x} = 3.$

(3) Calculate $\lim_{x\to 1} 2x + 5\left(\frac{x^2}{1+x^2}\right)$

(4) The following graph of the function fsatisfies $\lim_{x\to 2} f(x) = 2$. In the following exercises, determine a value of δ>0 that satisfies each statement.

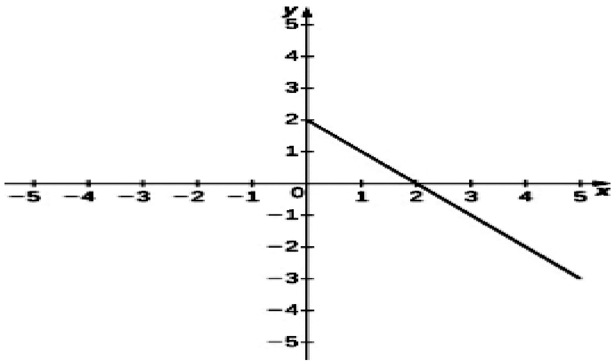


RIL-145

i. If $0 < |x-2| < \delta$ then $|f(x)-2| < 1$

ii. If $0 < |x-2| < \delta$ then $|f(x)-2| < 0.5$

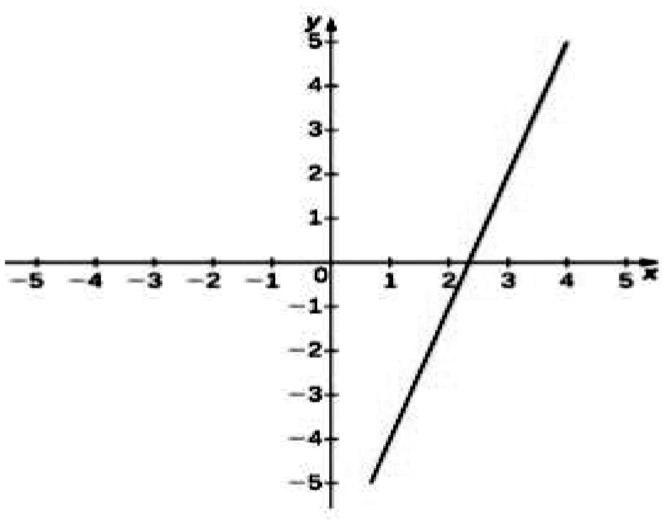
4) The following graph of the function fsatisfies $\lim_{x \to 3} f(x) = -1$. In the following exercises, determine a value of δ>0 that satisfies each statement.



i. If $0 < |x-3| < \delta$ then $|f(x)+1| < 1$

ii. If $0 < |x-3| < \delta$ then $|f(x)+1| < 2$

5) The following graph of the function fsatisfies $\lim_{x \to 3} f(x) = 2$. In the following exercises, for each value of ε, find a value of δ>0 such that the precise definition of limit holds true.



i. $\varepsilon = 1.5$

ii. $\varepsilon = 3$

6) In the following exercises, use the precise definition of limit to prove the given limits

i. $\lim_{x \to 2} (5x+8) = 18$

ii. $\lim_{x\to 3}\frac{x^2-9}{x-3}=6$

iii. $\lim_{x\to 2}\frac{2x^2-3x-2}{x-2}=5$

iv. $\lim_{x\to 2}(x^2+2x)=8$

3.5 Infinite Limits (Limits as $x \to \pm\infty$) Or Limits as x →∞ (or - ∞)

Take a look at the graph of the function f(x)=$\frac{1}{x}$, $x>0$ .This is a decreasing function of x. In fact, we see f(x) comes closer and closer to zero as x gets larger and larger. This situation is similar to the one where we have a function g(x) getting closer and closer to a value L as x comes nearer and nearer to some number p, that is when $\lim_{x\to p}$ g(x) = L. The only difference is that in the case f(x), x is not approaching any finite value , and is just becoming large and larger. We express this by saying that f(x) →0 as x →∞, or $\lim_{x\to\infty}$ f(x)=0.

Note that, ∞ is not a real number. We write x →∞ merely to indicate that x becomes larger and larger.

Let us look again at the function $f(x)=\frac{1}{x}$ as $x\to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (as shown in the following figure). Thus, f has no limit as $x\to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that f(x) approaches ∞ as $x\to 0^+$. We write $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{1}{x}=\infty$.

In writing this equation, we are not saying that the limit exists. Nor are we saying that there is a real number ∞, for there no such number. Rather, we are saying that $\lim_{x\to 0^+}\frac{1}{x}$ does not exists because $\frac{1}{x}$ becomes arbitrarily large and positive as $x\to 0^+$.

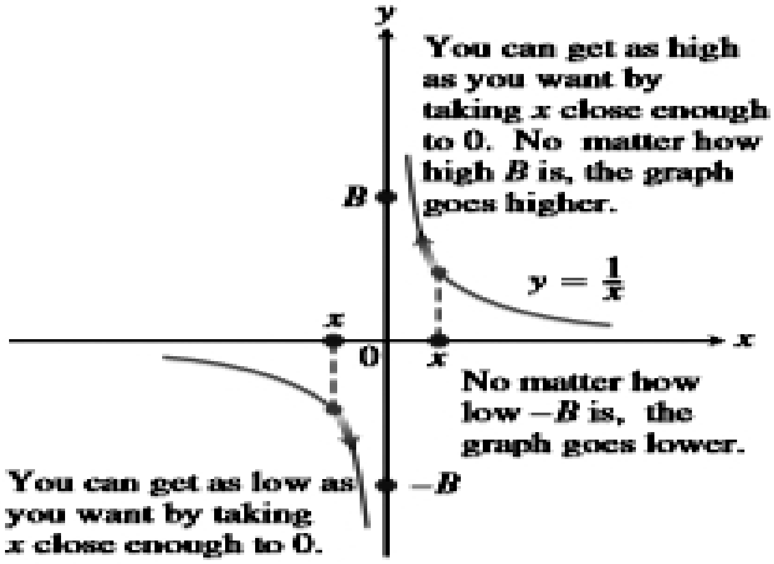


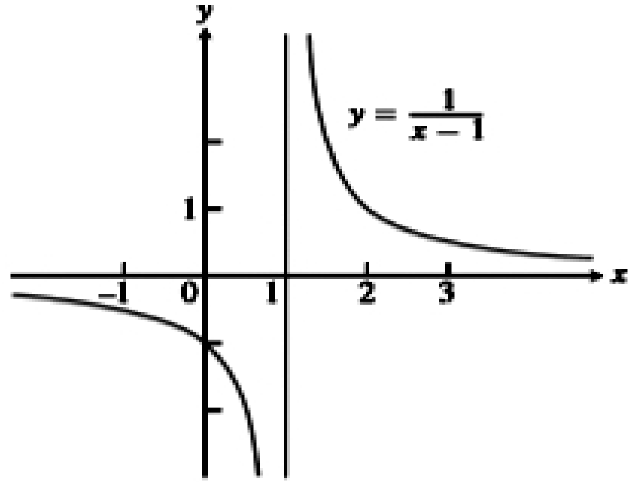
Figure: One sided infinite limits: $\lim_{x\to 0^+}\frac{1}{x}=\infty$ and $\lim_{x\to 0^-}\frac{1}{x}=-\infty$

As $x\to 0^-$, the values of $f(x)=\frac{1}{x}$ become arbitrarily large and negative. Given any negative real number –B, the values of f eventually lie below –B (as shown in the just above figure). We write $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{1}{x}=-\infty$.

Again, we are not saying that the limit exists and equals the number -∞. There is no real number -∞. We are describing the behavior of a function whose limit as $x\to 0^-$ does not exist because its values become arbitrarily large and negative.

Example 1: Find the $\lim_{x\to 1^+}\frac{1}{x-1}$ and $\lim_{x\to 1^-}\frac{1}{x-1}$

Solution: Geometrically, the graph of $y=\frac{1}{x-1}$ is the graph of $y=\frac{1}{x}$ shifted 1 unit to the right (as shown in the following figure). Therefore $y=\frac{1}{x-1}$ behaves near 1 exactly the way $y=\frac{1}{x}$ behaves near 0; $\lim_{x\to 1^+}\frac{1}{x-1}=\infty$ and $\lim_{x\to 1^-}\frac{1}{x-1}=-\infty$.



Analytically, we can think about the number x – 1 and its reciprocal. As $x \to 1^+$, we have $(x-1) \to 0^+$ and $\frac{1}{x-1} \to \infty$. As $x \to 1^-$, we have $(x-1) \to 0^-$ and $\frac{1}{x-1} \to -\infty$.

Example 2: Discuss the behavior of the function $f(x) = \frac{1}{x^2}$ as $x \to 0$.

Solution: As x approaches zero from either side, the value of $\frac{1}{x^2}$ are positive and become arbitrarily large (as shown in the just below figure). This means that $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty$

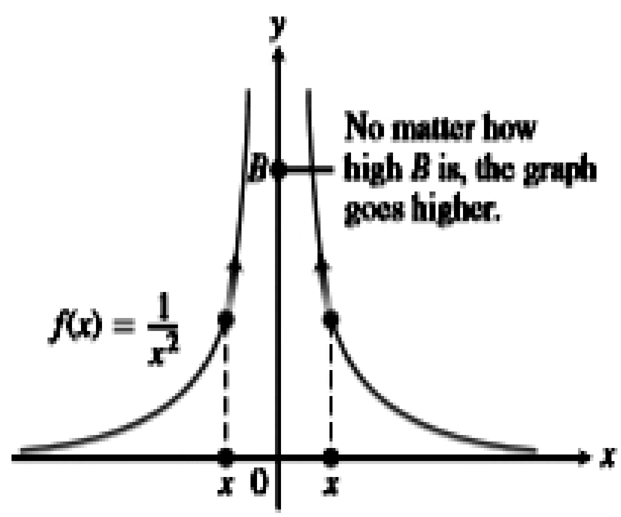


Figure: The graph of $f(x) = \frac{1}{x^2}$ approaches infinity as $x \to 0$

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The function $y = \frac{1}{x}$ shows no consistent behavior as $x \to 0$. We have $\frac{1}{x} \to \infty$ if $x \to 0^+$, but $\frac{1}{x} \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0}\frac{1}{x}$ is that it does not exist. The function $y = \frac{1}{x^2}$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0}\frac{1}{x^2} = \infty$.

Example 3: The following examples illustrate that rational functions can behave in various ways near zeros of the denominator.

i. $$\lim_{x\to 2}\frac{(x-2)^2}{x^2-4} = \lim_{x\to 2}\frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2}\frac{(x-2)}{(x+2)} = 0$$

ii. $$\lim_{x\to 2}\frac{(x-2)}{x^2-4} = \lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)} = \lim_{x\to 2}\frac{1}{(x+2)} = \frac{1}{4}$$

iii. $$\lim_{x\to 2^+}\frac{(x-3)}{x^2-4} = \lim_{x\to 2^+}\frac{(x-3)}{(x-2)(x+2)} = -\infty$$

iv. $$\lim_{x\to 2^-}\frac{(x-3)}{x^2-4} = \lim_{x\to 2^-}\frac{(x-3)}{(x-2)(x+2)} = \infty$$

v. $$\lim_{x\to 2}\frac{(x-3)}{x^2-4} = \lim_{x\to 2}\frac{(x-3)}{(x-2)(x+2)} \text{ does not exists}$$

vi. $$\lim_{x\to 2^+}\frac{2-x}{(x-2)^3} = \lim_{x\to 2^+}\frac{-(x-2)}{(x-2)^3} = \lim_{x\to 2^+}\frac{-1}{(x-2)^2} = -\infty$$

Precise Definitions of Infinite Limits:

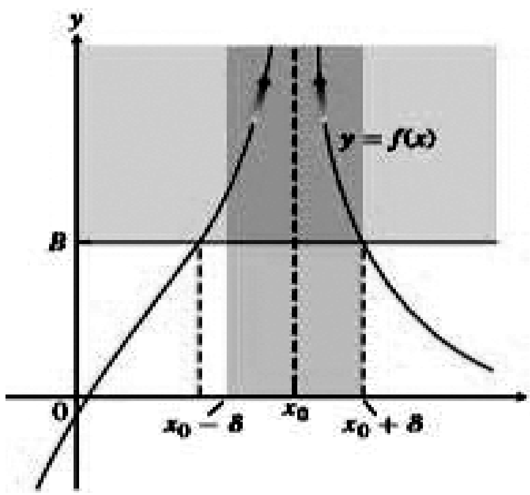
Instead of requiring f(x) to lie arbitrarily close to a finite number L for all x sufficiently close to x_0, the definition of the infinite limit require f(x) to lie arbitrarily far from zero. (See the figures below).

Definition 1: We say that f(x) approaches infinity as x approaches x_0, and write $\lim_{x\to x_0} f(x) = \infty$, if for every positive real number B there exists a corresponding $\delta > 0$ such that for all x,

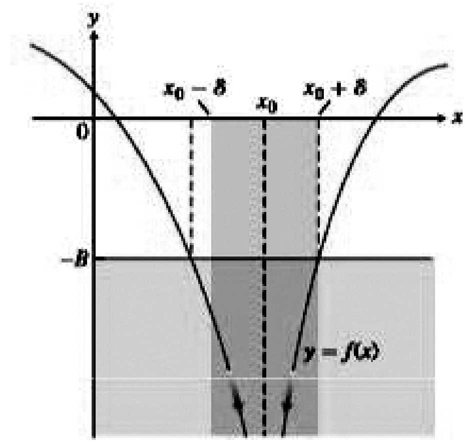
$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B.$$

Definition 2: We say that f(x) approaches minus infinity as x approaches x_0, and write $\lim_{x \to x_0} f(x) = -\infty$, if for every negative real number - B there exists a corresponding $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B.$$



For $x_0 - \delta < x < x_0 + \delta$, the graph of f(x) lies above the line y = B



For$x_0 - \delta < x < x_0 + \delta$, the graph of f(x) lies below the line y =- B

Equivalently,

Definition 3: A function f is said to tend to a limit L as x tends to ∞ if, for each $\varepsilon > 0$ it is possible to choose K > 0 such that $|f(x) - L| < \varepsilon$ whenever x > K.

In this case, as x gets larger and larger, f(x) gets nearer and nearer to L. We now give another example of this situation.

Example 4: Let f be defined by setting $f(x) = 1/x^2$ for all $x \in R - \{0\}$. Here f is defined for all real values of x other than zero. Let us substitute larger and larger value of x in $f(x) - 1/x^2$ and what happens (see table 2)

Table 2

X	100	1000	100,000
$f(x) = 1/x^2$	0001	000001	0000000001

We see that as x becomes larger and larger, f(x) comes closer and closer to zero. Now, let us choose any $\varepsilon > 0$. If $x > 1/\sqrt{\varepsilon}$, we find that $x > K \Rightarrow |f(x)| < \varepsilon$. Thus, $\lim_{x \to \infty} f(x) = 0$.

Sometimes we also need to study the behavior of a function f(x), as x takes smaller and smaller negative values. This can be examined by the following definition.

Definition 4: A function f is said to tend to a limit L as $x \to -\infty$ if, for each $\varepsilon > 0$, it is possible to choose K > 0, such that $|f(x) - L| < \varepsilon$ whenever $x < -K$.

Example 5: Consider the function $f : R \to R$ defined by $f(x) = \frac{1}{1+x^2}$

What happens to f(x) as x takes smaller and smaller negative values? Let us make a table (Table 3) to get some idea.

Table 3

x	-10	-100	-1000
$f(x) = \frac{1}{1+x^2}$	1/101	1/10001	1/1000001

We see that x takes smaller and smaller negative values, f(x) comes closer and closer to zero. In fact $1/1(1 + x^2) < \varepsilon$ whenever $1 + x^2 > 1/\varepsilon$, that is, whenever $x^2 > (1/\varepsilon) - 1$, that is, whenever either $x < -\left|\frac{1}{\varepsilon} - 1\right|^{1/2}$ *or* $x > \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$. Thus we find that if we take $K = \left|\frac{1}{\varepsilon} - 1\right|^{1/2}$, then $x < -k \Rightarrow |f(x)| < \varepsilon$. Consequently, $\lim_{x \to -\infty} f(x) = 0$. In the above example we also find that $\lim_{x \to \infty} f(x) = 0$.

In the above example, we have the function $f(x) = 1/(1+x^2)$, and as $x \to \infty$, or $x \to -\infty$, $f(x) \to 0$. From Fig. 5 you can see that , as $x \to \infty$ or $x \to -\infty$, or $x \to -\infty$, nearer and nearer the straight line y = 0, which is the x-axis.

Similarly, if we say that $\lim_{x \to \infty} g(x) = L$, then it means that, as $x \to \infty$ the curve y = g(x) comes closer and closer to the straight line y = L.

Example 6: Let us show that $\lim_{x \to \infty} \frac{x^2}{(1+x^2)} = 1$

Now, $|x^2/(1 + x^2) - 1| = 1/(1 + x^2)$. In the previous example we have shown that $|1/(1 + x^2)\} < \varepsilon$ for x > K, when $K = |1/\varepsilon - 1|^{1/2}$. Thus, given $\varepsilon > o$ we choose $K = |1/\varepsilon - 1|^{1/2}$, so that x > K then $\left|\frac{x^2}{1+x^2} - 1\right| < \varepsilon$. This means that $\lim_{x \to \infty} \frac{x^2}{1+x^2} = 1$

Example 7: Suppose, we want to find $\lim_{x\to\infty} \frac{3x+1}{2x+5}$

Note: We cannot apply Theorem 3 directly since the limits of the numerator and the denominator as x $\to\infty$ cannot be found.

Instead, we rewrite the quotient by multiplying the numerator and denominator by 1/x for x $\neq$0. Then, $\frac{3x+1}{2x+5} = \frac{3+(1/x)}{2+(5/x)}$. For x$\neq$ 0. Now we use theorem 3 and the fact that $\lim_{x\to\infty} 1/x = 0$, to get $\lim_{x\to\infty} \frac{3x+1}{2x+5} = \lim_{x\to\infty}$

$$\frac{3+(1/x)}{2+(5/x)} = \frac{\lim_{x\to\infty}(3+1/x)}{\lim_{x\to\infty}(2+5/x)} = \frac{3+0}{2+0} = \frac{3}{2}$$

Remark 1: In case we have to show that a function f does not tend to a limit L as x approaches p, we shall have to negate the definition of limit. Let us see what this means. Suppose we want to prove that $\lim_{x\to\infty}$ f(x) $\neq$L. Then, we should find some ε> 0 such that for every δ> 0, there is some x ϵ] p - δ, p + δ for which |f(x) – L | >δ. Through our next example we shall illustrate the negation of the definition of the limit of f(x) as x $\to\infty$.

Example 8: To show that $\lim_{x\to\infty}\frac{1}{x} \neq 1$, we have to find some ε> 0 such that for any K (however large) we can always find an x > K such that $\left|\frac{1}{x} - 1\right| > \varepsilon$.

This clearly shows that $\lim_{x\to\infty}\frac{1}{x} \neq 1$.

Example 9: Prove that $\lim_{x\to 0}\frac{1}{x^2} = \infty$.

Solution: Given B > 0, we want to find δ > 0 such that $0 < |x-0| < \delta \Rightarrow \frac{1}{x^2} > B$.

Now, $\frac{1}{x^2} > B$ if and only if $x^2 < \frac{1}{B}$

Or equivalently, $|x| < \frac{1}{\sqrt{B}}$

Thus choosing $\delta = \frac{1}{\sqrt{B}}$ (or any smaller positive number), we see that

$|x| < \delta \Rightarrow \frac{1}{x^2} > \frac{1}{\delta^2} \geq B$.

Therefore, by definition, $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Check your progress

(1) In the following exercises, use the precise definition of the limit to prove the given infinite limits.

i. $\lim_{x \to 0} \frac{1}{x^2} = \infty$

ii. $\lim_{x \to -1} \frac{3}{(x+1)^2} = \infty$

iii. $\lim_{x \to 2} \frac{1}{(x-2)^2} = \infty$

3.6. One-Sided Limits

In the last section, we consider limits at infinity (or negative infinity) by letting x approach the imaginary point ∞ (or - ∞). In this section we consider the limits at the point p on the real line by letting x approach p. Because x can approach p from the left-side or from the right side, we have left-side and right-side limits. They are called one-sided limits.

In this section we extend the limit concept to one sided limits, which are limits as x approaches the number p from the left-hand side (where x < p) or the right- hand side (x > p) only.

To have a limit L as x approaches p, a function f must be defined on both sides of p and its values f(x) must approach L as x approaches pfrom either side. Because of this, ordinary limits are called two-sided.

If f fails to have a two-sided limit at p, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit. From the left, it is a left-hand limit.

The function $f(x) = \frac{x}{|x|}$ (see the following figure) has limit 1 as x approaches 0 from the right, and limit – 1 as x approaches 0 from the left. Since these one sided limit values are not the same, there is no single number that f(x) approaches as x approaches 0. So f(x) does not have a (two-sided) limit at 0.

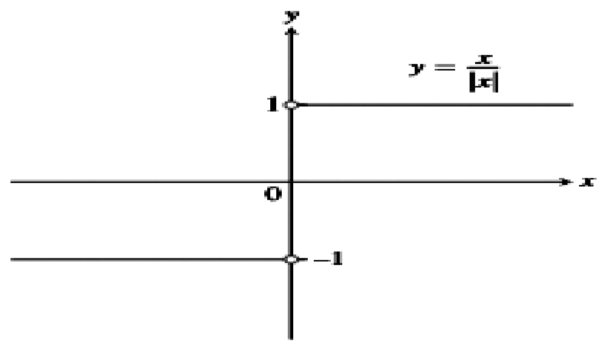


Figure: Different right-hand and left-hand limits at the origin

Intuitively, if f(x) is defined on an interval (c, b), where c < b, and approaches arbitrarily close to L as x approaches c from within that interval, then f has right-hand limit L at c. We write $\lim_{x \to c^+} f(x) = L$. The symbol $x \to c^+$ means that we consider only values x greater than c.

This informal definition of right-hand (one-sided) limit is illustrated in the below figure.

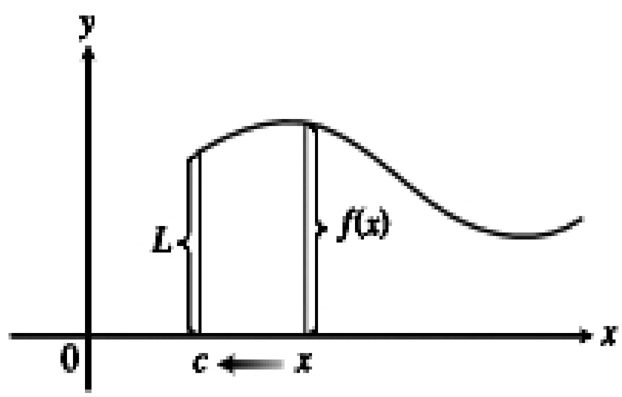


Figure : $\lim_{x \to c^+} f(x) = L$

Similarly, if f(x) is defined on an interval (a, c), where a < c, and approaches arbitrarily close to M as x approaches c from within that interval, then f has left-hand limit M at c. We write $\lim_{x \to c^-} f(x) = M$. The symbol $x \to c^-$ means that we consider only values x less than c.

This informal definition of left-hand (one-sided) limit is illustrated in the below figure.

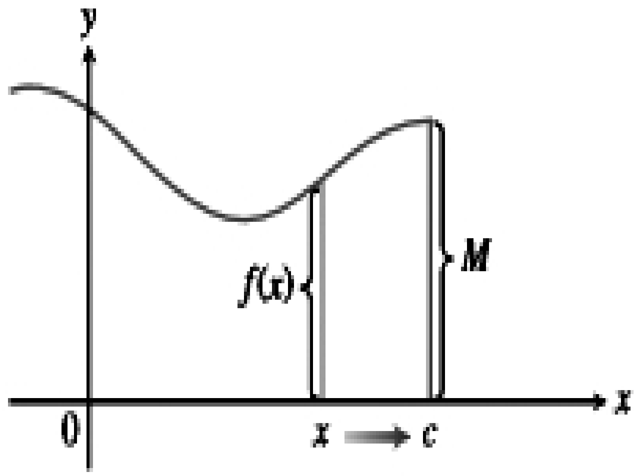


Figure : $\lim_{x \to c^-} f(x) = M$

The function $f(x) = \frac{x}{|x|}$ has limit 1 as x approaches 0 from the right, and limit – 1 as x approaches 0 from the left. Since these one sided limit values are not the same, there is no single number that f(x) approaches as x approaches 0. So f(x) does not have a (two-sided) limit at 0. Therefore, wehave $\lim_{x \to 0^+} f(x) = 1$ and $\lim_{x \to 0^-} f(x) = -1$. (Illustrated in the following figure).

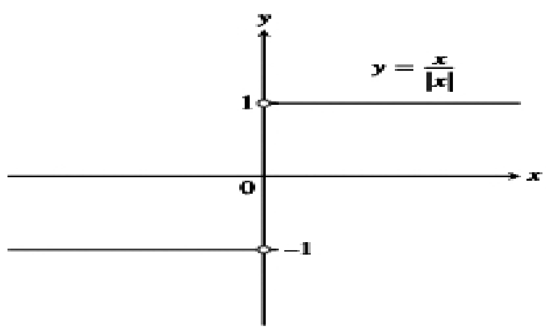


Figure: Different right-hand and left-hand limits at the origin

Consider the following example

Example 3.6.1: The domain of $f(x) = \sqrt{4 - x^2}$ is [-2, 2]; its graph is the semicircle in the following figure, we have

$\lim_{x \to 2^+} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$.

The function does not have a left-hand limit at x = -2 or a right-hand limit at x = 2. It does not have ordinary two- sided limits at either -2 or 2.

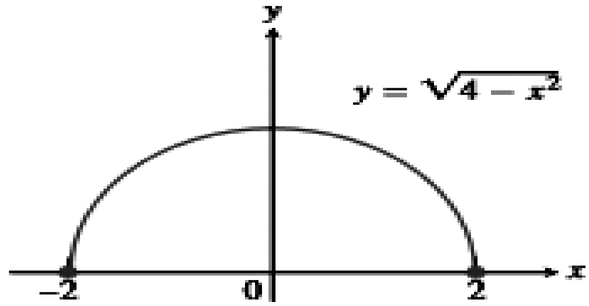


Figure: $\lim_{x \to 2^+} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2^-} \sqrt{4 - x^2} = 0$

Remark 3.6.1: One sided limits have all the properties listed in Theorem 2.4.1 in section 2.4. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as do the Sandwich Theorem.

Theorem 3.6.1: A function f(x) has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = L \quad and \quad \lim_{x \to c^+} f(x) = L \text{ .}$$

Equivalently,

Theorem 3.6.1: The following statements are equivalent:

(i) $\lim_{x \to p}$ f(x) exists

(ii) $\lim_{x \to p^+}$ f(x) and $\lim_{x \to p}$ exist and are equal

Proof. To show that (i) and (ii) are equaling, we have to show that (i) ⇒ (ii) and (ii) ⇒ (i). We first prove that (i) = (ii). For this we assume that $\lim_{x \to p}$ f(x) = L. Then given ⇒ε> 0.∃δ> 0 such that |f (x) – L | <ε for 0 < | x – p | <δ.

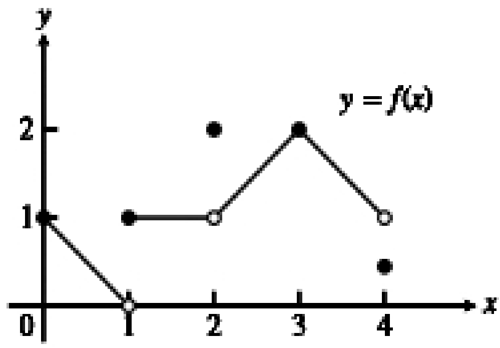
Now, 0 < | x – p | <δ⇒ p < x < p + δ and p - δ< x < p. Thus, we have |f(x)–L|<ε for p < x < p + δ and p - δ< x < p. This means that $\lim_{x \to p}$ f(x) = L $\lim_{x \to p^+}$ f(x). we now prove the converse, that is, (ii) ⇒ (i). For this, we assume that $\lim_{x \to p}$ f(x) = $\lim_{x \to p^+}$ f(x) = L. Then given ε> 0, ∃δ_1> 0. Such that

|f(x) – L | <ε for p – δ_1< x < p ⇒|f(x) – L | <ε for p < x < p + δ_2

Let δ = min {δ_1, δ_2}. Then for both p - δ<x < p and p < x < p + δ, we have |f(x) – L| <ε. Hence $\lim_{x \to p}$ f(x) = L.

Thus, we have shown that (i) ⇒ and (ii) ⇒ (i), proving that they are equivalent. Also exist and further. $\lim_{x\to p} f(x) = \lim_{x\to p^+} f(x) = \lim_{x\to p^-} f(x)$.

Example 3.6.2: Consider the graph of the function y = f(x)



At x = 0: $\lim_{x\to 0^+} f(x) = 1,$

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exists. The function is not defined to the left of x = 0.

At x = 1: $\lim_{x\to 1^-} f(x) = 0$, even though $f(1) = 1$

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exists. The right and left-hand limits are not equal.

At x = 2: $\lim_{x\to 2^-} f(x) = 1,$

$\lim_{x\to 2^+} f(x) = 1$, $\lim_{x\to 2} f(x) = 1,$

Even though $f(2) = 2$.

At x = 3: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At x = 4: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exists. The function is not defined to the right of x = 4.

At every other point c in [0, 4], f(x) has limit f(c).

Precise Definitions of One Sided Limits

Definition 3.6.1: We say that f(x) has right-hand limit L at x_0, and write

$$\lim_{x \to x_0^+} f(x) = L \text{ (see the figure below)}$$

If for every number ε > 0 there exists a corresponding number δ > 0 such that for all x

$$x_0 < x < x_0 + \delta \;\Rightarrow\; |f(x) - L| < \varepsilon .$$

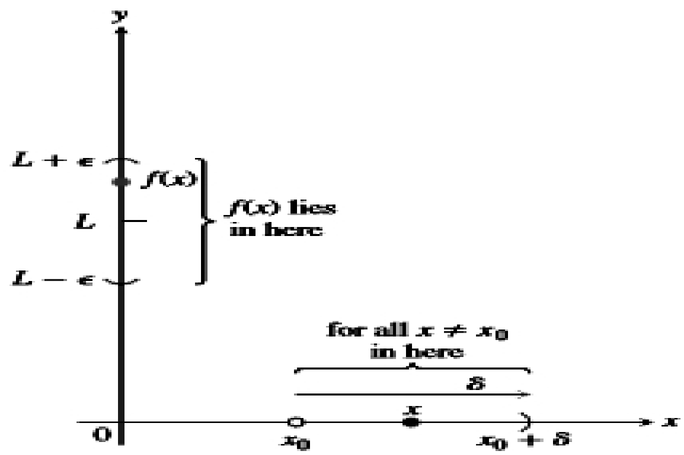


Figure: Intervals associated with the definition of right hand limit

Definition 3.6.2: We say that f(x) has left-hand limit L at x_0, and write

$$\lim_{x \to x_0^-} f(x) = L \text{ (see the figure below)}$$

If for every number ε > 0 there exists a corresponding number δ > 0 such that for all x

$$x_0 - \delta < x < x_0 \;\Rightarrow\; |f(x) - L| < \varepsilon .$$

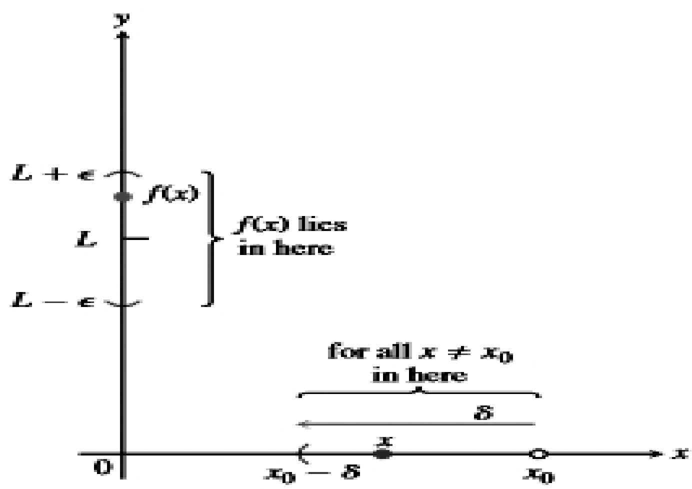


Figure: Intervals associated with the definition of left hand limit

Equivalently,

Definition 3.6.1: Let f be a function defined for all x in the interval]p, q[, f is said to approach a limit L as x approaches p from right if, given any $\varepsilon > 0$, there exist a number $\delta > 0$ such that $p < x < p + \delta \Rightarrow |f(x) - L| < \varepsilon$.

In symbols we denote this limit by $\lim_{x \to p^+} f(x) = L$.

Similarly, the function f:]a, p[$\to$ R is said to approach a limit L as x approaches p from the left if, given any $\varepsilon > 0$, $\exists \delta > 0$ such that $p - \delta < x < p \Rightarrow |f(x) - L| < \varepsilon$.

This limit is denoted by $\lim_{x \to p^-} f(x)$.

If we consider the graph of function f(x) = [x], (as shown in the figure) we see that f(x) does not see to approached any fixed value as x approaches 2. But from the graph we cansay that if x approaches 2 from the left then f(x) seems to tend to 1. At the same time, if x approaches 2 from the right, then f(x) seems to tend to 2. This means that the limit of f(x) exists if x approaches 2 from only one side (left or right) at a time. This example suggests that we introduce the idea of a one-sided limit.

Note that in computing these limits the values of f(x) for x lying on only one side of p. Let us apply this definition to the function f(x) = [x], we know that for x $\in$x [1, 2], [x] = 1. That is, [x] is a constant function [1, 2[, Hence $\lim_{x \to 2^-} [x] = 1$. Arguing similarly, we find that since [x] = 2 for all x $\in$ [2, 3], x [x] is, again, a constant function on [2, 3[, and $\lim_{x \to 2^+} [x] = 2$.

Remark 3.6.1: if you apply theorem 3.6.1 to the function f(x) = x – [x] { see example 3.6.2), you will see that $\lim_{x \to 1} \{x - [x]\}$ does not exist as $\lim_{x \to 1^+} \{x - [x]\} \neq \lim_{x \to 1} \{x - [x]\}$.

Let us improve our understanding of the definition of one-sided limits by looking at some more examples.

Example 3.6.1: Let be defined on R by setting f(x) = $\frac{|x|}{x}$, when $x \neq 0$.

f(0) = 0

we still show that $\lim_{x \to 0^-} f(x)$ equals –1 . When x < 0, |x| = - x , and therefore,

f(x)=(-x)/x= -1. In order to show that $\lim_{x \to 0^-} f(x)$ exists and equal – 1. We have to start with any $\varepsilon > 0$ and then find a $\delta > 0$ such that, if $-\delta < x < 0$, then $|f(x) - (-1)| < \varepsilon$.

Since f(x) = - 1 for all x < 0, |f(x) – (- 1) | = 0 and, hence any number δ> 0 will work. Therefore, whatever δ> 0 we may choose, if - δ< x < 0, then |f (x) – (- 1) |=0 <ε. Hence $\lim_{x\to 0^-}$ f(x) = - 1.

Example 3.6.2: f is a function defined on R by setting f(x) = x – [x], for all x ∈ R.

Let us examine whether $\lim_{x\to 1^-}$ f(x) exists. This function is given by f(x) = x, if 0 ≤ x < 1.f(x) = x – 1 if 1 ≤ x < 2, and, in general f(x) = x – n if n ≤ x < n + 1

Since f(x) = x for values of x less than 1 but close to 1, but it is reasonable to except that $\lim_{x\to 1^-}$ f(x) = 1. Let us prove this by taking any ε> 0 and choosing δ = min | 1, ε |. We find 1 - δ< x < 1 ⇒ f(x) = x and |f(x) – 1 | = x – 1| <δ≤ε.

Therefore $\lim_{x\to 1^-}$ f(x) = 1. Processing exactly as above, the noting that f(x) = x – 1 if 1 ≤ x < 2, we can similarly prove that f(x) = 0.

Example 3.6.3: Prove that $\lim_{x\to 0^+}\sqrt{x} = 0$

Solution: Let ε > 0 be given. Here x_0 = 0 and L = 0, so we want to find a δ > 0 such that for all x

$0 < x < \delta \quad \Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$, or $0 < x < \delta \quad \Rightarrow \sqrt{x} < \varepsilon$

Squaring both sides of this last inequality gives $x < \varepsilon^2$ if $0 < x < \delta$.

If we choose $\delta = \varepsilon^2$ we have

$0 < x < \delta = \varepsilon^2 \quad \Rightarrow \sqrt{x} < \varepsilon$ or $0 < x < \varepsilon^2 \quad \Rightarrow \left|\sqrt{x} - 0\right| < \varepsilon$

According to the definition, this shows that $\lim_{x\to 0^+}\sqrt{x} = 0$ (see the figure below)

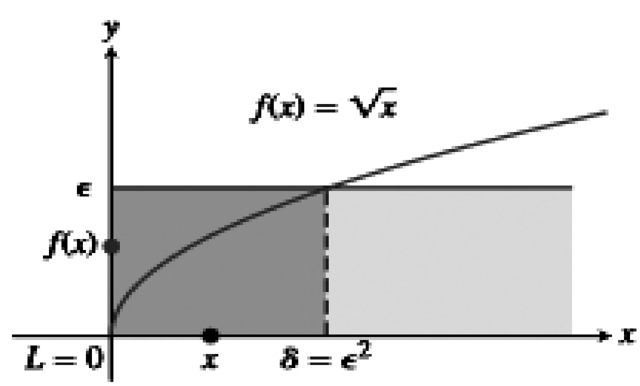


Figure: $\lim_{x\to 0^+}\sqrt{x} = 0$

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case. Let's consider the following example to see this fact.

Example 3.6.4: Show that $y = \sin\left(\frac{1}{x}\right)$ has no limit as x approaches zero from either side.

Solution: As x approaches zero, its reciprocal, 1/x , grows without bound and the values of $\sin\left(\frac{1}{x}\right)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand nor a left-hand limit at x = 0.

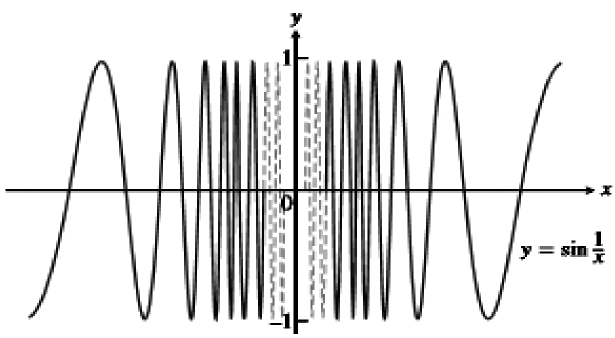


Figure: The function $y = \sin\left(\frac{1}{x}\right)$ has neither a right-hand nor a left-hand limit as x approaches zero.

<u>Check your progress</u>

In the following exercises, use the precise definition of limit to prove the given one-sided limits.

i. $\lim_{x \to 4^+} \sqrt{x-4} = 0$.

ii. $\lim_{x \to 5^-} \sqrt{5-x} = 0$.

iii. $\lim_{x \to 0^+} f(x) = -2$ where $f(x) = \begin{cases} 8x-3, & if \;\; x < 0 \\ 4x-2, & if \;\; x \geq 0 \end{cases}$

iv. $\lim_{x \to 1^-} f(x) = 3$ where $f(x) = \begin{cases} 5x-2, & if \;\; x < 0 \\ 7x-1, & if \;\; x \geq 0 \end{cases}$

3.7 Summary

Definition of limit of a function at a given point and to find $\delta > 0$ for given $\varepsilon > 0$, uniqueness of limit of a function is discussed. Algebra of limits i.e. the limit of sum of two functions i.e. f(x)+g(x), limits of production of two function i.e. f(x).g(x), limit of $\frac{f(x)}{g(x)}$ when g(x) $\neq 0$ etc. is also described in the unit. Infinite limits i.e. limit of f(x) when $x \to \pm\infty$, one sided limit i.e. left hand limit and right hand limit at a given point is studied.

3.8 Terminal Questions

1. By $\varepsilon - \delta$ method prove the following:

(i) $\lim_{x\to 0} x \, Sin \, \frac{1}{x} = 0$

(ii) $\lim_{x\to 0} x^2 \, Sin \, \frac{1}{x} = 0$

(iii) $\lim_{x\to 2} |3\,x - 1| = 5$

2. If $\lim_{x\to a} f(x) = l$ then show that $\lim_{x\to a} | f(x)\} = |l|$

(Hint: Use $|f(x) - l| \geq | \, |f(x)| - |l| \, |$)

3. By $\varepsilon - \delta$ method show that

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = 2$$

4. Give an example to show that if $\lim_{x\to a} |f(x)|$ exists then $\lim_{x\to a} f(x)$ may not exist.

Selected Answers To Terminal Questions		
1. (i) $\delta = \varepsilon$	(ii) $\delta = \sqrt{\varepsilon}$	(iii) $\delta = \varepsilon/3$
3. $\delta = \varepsilon$		

UNIT-4

CONTINUITY

Structure

4.1 INTRODUCTION

A continuous process is one that goes on smoothly without any abrupt change. Continuity of a function can also be interpreted in similar way.

Continuous functions play a very important role in calculus. As you proceed, you will be able see that many theorems which we have stated in this course are true only for continuous functions. You will also see that continuity is a necessary condition for the derivability of a function. But let us give a precise meaning to "a continuous function" now.

4.2. Objectives

After reading this unit students should be able to:

- Define and evaluate the continuity of a function
- understand the need and use of algebra and properties of continuous functions
- Understand the Local boundedness and local maintenance of sign

- Understand the Boundedness and intermediate value properties of continuous functions over closed intervals
- Understand the Image of a closed interval under continuous maps

4.3 Continuity (Definitions and Examples)

In this section we shall give you the definition and some examples of a continuous function. We shall also give you a short list of conditions which a function must satisfy in order to be continuous at a point.

Definition 4.3.1: let f be a function defined on a domain D, and let r be a positive real number such that the interval]p – r, p + r[$\subset$ D. f is said to be continuous at x = p if $\lim_{x \to p} f(x) = f(p)$. By the definition of limit this means that f is continuous at p is given $\varepsilon > 0, \exists\ \delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta$.

Example 4.3.1: Let us check the continuity of the function f: R $\to$R such that f(x) = x at the point x = 0. Now, f(0) = 0. Thus we want to know if $\lim_{x \to 0} f(x) = 0$. This is true because given $\varepsilon > 0$, we can choose $\delta = \varepsilon$ and verify that $|x| < \delta \Rightarrow |f(x)| < \varepsilon$ Thus f is continuous at x = 0.

Remark 4.3.1: f is continuous at x = p provided the following two criteria are met :

(i) $\lim_{x \to p} f(x)$ exists

(ii) $\lim_{x \to p} f(x) = f(p)$

Criterion (i) is not met by f, whereas f fails to meet criterion (ii). If you read Remark 4.3.1 again, you will find that f(x) = x – [x] is not continuous as x = 1. $[x]$ is the largest integer $\leq$ x. But we have seen that we can calculate one-side limits of f(x) = x – [x] at x = 1. This leads us to the following definition.

Definition 4.3.2: A function f :]p, q[$\to$ R is said to be continuous from the right at x=p

if $\lim_{x \to p^+} f(x) = f(p)$. We say that f is continuous from the left at q if $\lim_{x \to q^-} f(x) = f(q)$.

Thus, f(x) = x – [x] is continuous from the right but not from the left at x = 1

since $\lim_{x \to ^+} f(x) = f(1) = 0$.

Definition 4.3.3: Let f be a real valued function defined on an interval I. Then,

(i) The function f is continuous at the point $p \in I$ if $\lim_{x \to p} f(x) = f(p)$

(ii) The function f is said to be continuous on I if f is continuous at every $p \in I$

Using this definition of the limit, it follows that

- F is continuous at $p \in I$ if and only if for every $\in > 0$, there exists δ>0 such that $|f(x) - f(p)| < \in$ whenever $x \in I,\ |x - p| < \delta$.

Remark 4.3.2: Suppose the domain of the function f is not specified explicitly. Even then we can say that f is continuous at a point $p \in \Re$ to mean that f is defined on an interval containing p and f is continuous at p.

Theorem 4.3.1: A function $f : I \to \Re$ is continuous at $p \in I$ if and only if for every sequence $< x_n >$ in I with $x_n \to p$ we have $f(x_n) \to f(p)$.

Remark 4.3.3: The greatest integer function f : R →R : f(x) = [x] is discontinuous at x = 2. To prove this $\lim_{x \to 2^-}$ f(x) = 1 and $\lim_{x \to 2^+}$ f(x) = 2. Thus, since these two limits are not equal $\lim_{x \to 2}$ f(x) =1. Therefore, f is not continuous at x = 2. In fact $f(x) = [x]$ is discontinues at each integral point.

Example 4.3.2: Let f(x) = |x| for all x ∈ R. This is continuous at x = 0. Here f(x) = x, if x ≥0, and f(x) = -x if x < 0. You can show that $\lim_{x \to 0^+}$ f(x) = $\lim_{x \to o^-}$ (- x) = 0 = f(0). Thus $\lim_{x \to 0}$ f(x) exists and equal f(0). Hence f is continuous at x = 0.

Note: f is also continuous at every other point of R.

4.4. Algebra of Continuous Functions

Suppose f and g are defined on an interval I and both f and g are continuous at $p \in I$. Then we have the following facts.

i. $f \pm g$ is continuous at p

ii. $k\ f$ is continuous at p

iii. fg is continuous at p

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iv. $\frac{f}{g}$ *for* $g(p) \neq 0$ is continuous at p

Example 4.4.1: If f(x) is a polynomial say $f(x) = a_0 + a_1 x + ... + a_k x^k$, then f is continuous on $\Re$.

Example 4.4.2: Let f(x) = x^n for all x∈R and an n ∈ Z^+. Show that f(x) is continuous at x = p for all p∈ R. We know that $\lim_{x\to p}$ x = p for any p ∈ R. Then, by the product rule in 4.4.3, we get $\lim_{x\to p} x^n$ =($\lim_{x\to p}$ x) ($\lim_{x\to p}$ x) ……. ($\lim_{x\to p}$ x) (n times)

= p.p ……… p (n times) = p^n. Therefore, $\lim_{x\to p}$ f(x) exists and equals f(p). Hence f is continuous at x = p. Since p was any arbitrary number in R, we can say that f is continuous.

Example 4.4.3: For given $x_0 \in \Re$, let $f(x) = |x - x_0|$, $x \in \Re$. Then f is continuous on $\Re$. To see this, note that, for $p \in \Re$, $|f(x) - f(p)| = ||x - x_0| - |p - x_0|| \leq |(x - x_0) - (p - x_0)| = |x - p|$.

Hence, for every ε>0, we have $|x - p| < \varepsilon \Rightarrow |f(x) - f(p)| < \varepsilon$

Example 4.4.4: Suppose we want to find whether f(x) = $\frac{x^2 - 1}{x - 1}$ is continuous at x = 0. It is the line y = x + 1 except for the point (1, 2).

Example 4.4.5: Let $f(x) = \frac{x^2 - 4}{x - 2}$ for $x \in \Re \setminus \{2\}$ and $f(2) = 4$. Then, f is continuous on $\Re$.

Example 4.4.6: The function $f : \Re \to \Re$ given by $f(x) = \frac{x + 3x^3 + 5x^5}{1 + x^2 + x^4}$ is continuous on $\Re$ since it is a rational function whose denominator never vanishes.

Theorem 4.4.1: Suppose f and g are defined on an interval I and both f and g are continuous at $p \in I$. Then, if $g(p) \neq 0$, then there exists $\delta_0 > 0$ such that $g(x) \neq 0$ for every x in the interval $I_0 = I \bigcap (p - \delta, p + \delta)$, and the function $\frac{f}{g}$ defined on I_0 is continuous at p.

Theorem 4.4.2: Suppose $f : A \to R$ and $g : B \to R$ where $f(A) \subset B$. If f is continuous at $p \in A$ and g is continuous at $f(p) \in B$ then $g \, o \, f : A \to R$ is continuous at p.

Theorem 4.4.3: Suppose $f : A \to R$ and $g : B \to R$ where $f(A) \subset B$. If f is continuous on A and g is continuous at $f(A)$ then $g \, o \, f$ is continuous on A.

Theorem 4.4.4: Every polynomial function is continuous on $\Re$ and every rational function is continuous on its domain.

Proof: The constant function f(x)= 1 and the identity function g(x) = x are continuous on $\Re$. Repeated application of the article **4.4** for scalar multiples, sums and products implies that every polynomial is continuous on $\Re$. It also follows that a rational function $\frac{f}{g}$ is continuous at every point where $g \neq 0$.

Example 4.4.4: The function, $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$

Is continuous on $\Re \setminus \{0\}$,

Example 4.4.5: The function, $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$

is continuous on $\Re \setminus \{0\}$, since it is the product of the functions that are continuous on $\Re \setminus \{0\}$ and $y \mapsto \sin y$, which is continuous on $\Re$. f is also continuous at 0 as $|f(x) - f(0)| = \left| x\sin\left(\frac{1}{x}\right) \right| \leq |x|$ for $x \neq 0$. So f is continuous on $\Re$.

Example 4.4.5: Using ε-δ definition to show that the function $f : (0, \infty) \to \Re$ defined by $f(x) = \sqrt{x}$ is continuous on (0, ∞).

Solution: Let a > 0 so $a \in (0, \infty)$. Given ε > 0 we want $\left| f(x) - \sqrt{a} \right| < \varepsilon$.
This is true

$$\Leftrightarrow \left| \sqrt{x} - \sqrt{a} \right| < \varepsilon$$

$$\Leftrightarrow \frac{|x - a|}{\sqrt{x} + \sqrt{a}} < \varepsilon$$

But $\dfrac{|x-a|}{\sqrt{x}+\sqrt{a}} < \dfrac{|x-a|}{\sqrt{a}}$

So let $\delta_\varepsilon = \varepsilon\sqrt{a}$. With this choice, if $|x-a| < \delta_\varepsilon$ we have $|f(x) - f(a)| < \varepsilon$. To show this we need a chain of implications.

Because ,

$$|x-a| < \delta_\varepsilon$$
$$\Rightarrow |x-a| < \varepsilon\sqrt{a}$$
$$\Rightarrow \frac{|x-a|}{\sqrt{x}+\sqrt{a}} < \frac{|x-a|}{\sqrt{a}} < \varepsilon$$
$$\Rightarrow |f(x) - f(a)| < \varepsilon$$

Therefore f is continuous at x = a. Since this is true for every a in the open interval (0, ∞), f is continuous on (0, ∞).

Check your progress

i. Show that $\lim_{x\to\infty}(1/x + 3/x^2+5)=5$

ii. If for some ε> 0, and for every K,∃ x > K s.t.|f(x) – L | >ε, what will you infer?

iii. If $\lim_{x\to\infty}$ f(x) ≠ L, how can you express it in the ε - δ form?.

iv. Prove that (a) $\lim_{x\to 3^-} x\,[x] = 1$ (b) $\lim_{x\to 0^+}\dfrac{|x|}{x}=1$ (c) $\lim_{x\to 0^-}\dfrac{(x^2+2)|x|}{x}=-2$

v. Give ε - δ definition of continuity at a point from the right as well as from the left.

vi. Show that function f :]p – r, p + r [→ R is continuous at x = p if and only if f is continuous from the right as well as from the left at x = p (use theorem 4).

i. Let $f:\Re\to\Re$ be defined by, $f(x) = x^4$. Show that f is continuous on $\Re$.

ii. Let $f:\Re\to\Re$ be defined by, $f(x) = 0$. If x is rational and f(x)=1 if x is irrational. Show that f is not continuous at any point of $\Re$.(see the example 4.8.5)

4.5. Properties of Continuous Functions

Theorem 4.5.1: (Intermediate value theorem). The function which is continuous on an interval and which is positive at some point and negative at another must be zero somewhere on the interval. (Proved in Theorem 4.7.3)

Theorem 4.5.2: (Boundedness theorem). A continuous function on a closed bounded interval is bounded and attains its bounds. (Proved in Theorem 4.7.2)

Equivalently, Suppose f is a real valued function defined on a closed and bounded interval $[a, b]$. Then f is a bounded function.

4.6. Local Boundedness supremum and infimum of a function

Definition 4.6.1: Let $f : I \to \Re$ be a function defined on some set $I \subset \Re$. The supremumof f on I, denoted by $\sup_{x\in I} f(x)$, is defined as $\sup_{x\in I} f(x) =$ the smallest M such that $f(x) \leq M$ for all x Є I.

If the function is bounded above, its supremum is finite. If f is not bounded above on I , then we set $\sup_{x\in I} f(x) = \infty$.

Definition 4.6.2: Let $f : I \to \Re$ be a function defined on some set $I \subset \Re$. The infimumof f on I, denoted by $\inf_{x\in I} f(x)$, is defined as $\inf_{x\in I} f(x) =$ the largest m such that $f(x) \geq m$ for all x Є I.

If the function is bounded below, its infimum is finite. If f is not bounded below on I, then we set $\inf_{x\in I} f(x) = -\infty$.

Remark 4.6.1:

i) We say that f attains its upper bound if there is some c Є I such that $f(c) = \sup_{x\in I} f(x)$. In this case we also write $\max_{x\in I} = \sup_{x\in I} f(x)$, f(c) is called maximum of the function f on I.

ii) We say that f attains its lower bound if there is some d Є I such that $f(d) = Inf_{x\in I} f(x)$. In this case we also write $\min_{x\in I} = \inf_{x\in I} f(x)$, f(d) is called minimum of the function f on I.

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However, in many situations it is important to talk about local maximum or a local minimum. Hence,

Definition 4.6.3: We can say that functionf has a local maximum at c if there is some interval containing c such that f attains its maximum in this interval at the point c, note that the interval may be very small so no information about the behavior of f outside this interval is included in the definition.

Thus,

We define f(c) to be local maximum of the function f if there is δ > 0 such that $|x-c| < \delta$ implies $f(x) \le f(c)$.

Definition 4.6.4: We can say that functionf has a local minimum at d if there is some interval containing d such that f attains its minimum in this interval at the point d, note that the interval may be very small so no information about the behavior of f outside this interval is included in the definition.

Thus,

We define f(d) to be local minimum of the function f if there is δ > 0 such that $|x-d| < \delta$ implies $f(x) \ge f(d)$.

Equivalently,

Definition 4.6.5: If f defined on some deleted neighborhood D of a point p or on some open interval with endpoint p (i.e., some set of the form (p-δ, p) or (p, p+δ), but f is not necessarily defined at p, then f is locally unbounded at p if and only if f is unbounded on every deleted neighborhood of p.

Symbolically, f is "locally unbounded at p" if and only if for every M ≥ 0 and δ > 0 there exists x such that $0 < |x-p| < \delta$ and $|f(x)| > M$.

Definition 4.6.6: If f defined on some deleted neighborhood D of a point p or on some open interval with p as endpoint, then f is locally bounded at p if and only if f is bounded on some deleted neighborhood of p.

Symbolically, f is "locally bounded at p" if and only if $\exists M \ge 0$, δ > 0 and for every x, $0 < |x-p| < \delta$ and $|f(x)| \le M$.

Remark 4.6.2: The basic property of continuous functions is **"Continuous functions are locally bounded".**

Example 4.6.1: consider the function defined by $f(x) = \dfrac{1}{x}$ can be locally bounded at every point of (0, 1), since it is continuous at every point of (0, 1).

Example 4.6.2: Consider the function $f:(0, 1] \to \Re$ defined by $f(x) = \dfrac{1}{x}$ for $x \in (0, 1]$ is continuous, but does not attain its supremum.

Example 4.6.3: Consider the function $f:[0, 1] \to \Re$ defined by $f(x) = \begin{cases} \dfrac{1}{x}, & x \in (0, 1] \\ 1, & x = 0 \end{cases}$ is neither continuous nor the domain is closed and bounded.

Example 4.6.4: Consider the function $f:[0, 1) \to \Re$ defined by $f(x) = \begin{cases} 0, & x \in [0, 1/2) \\ 1, & x \in [1/2, 1) \end{cases}$ is neither continuous nor the domain is closed, but f attains both its maximum and minimum.

4.7. Boundedness and Intermediate Value Properties of Continuous Functions over Closed Intervals

Continuous functions on closed intervals not only have to be bounded, but much more is true: they also have to attain their bounds.

Definition 4.7.1: A real valued function f defined on a set S of $\Re$ is said to be bounded if and only if there exists some real number $M \geq 0$ such that for every $x \in S$, $|f(x)| \leq M$.

Remark 4.7.1:

i) A real valued function f defined on a finite set S of $\Re$, thenevery function defined on S is bounded on S.

ii) If S is bounded subset of $\Re$, then S has infimum and supremum

iii) Suppose S is bounded, and let $\inf S = \alpha$ and $\sup S = \beta$, then there exists the sequences $< s_n >$ and $< t_n >$ in S such that $s_n \to \alpha$ and $t_n \to \beta$.

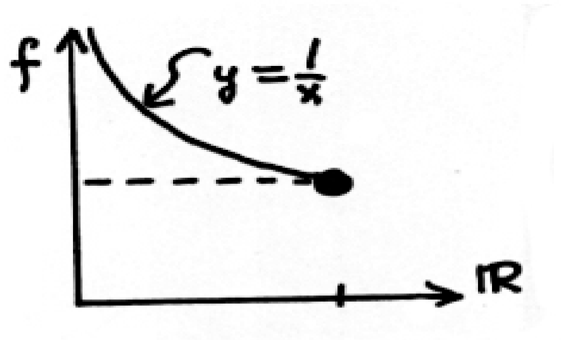
iv) A subset S of $\Re$ is unbounded is and only if there exists a sequence $< s_n >$ in S such that $|s_n| \to \infty$ as $n \to \infty$.

Definition 4.7.2: A real valued function f defined on a set S is unbounded if and only if it is not bounded on S, i.e., iff no matter how large we take M to be, there is always some point x in S with $|f(x)| > M$.

Equivalently,

Definition 4.7.3: A real valuedfunction defined on a set $S \subseteq \Re$ is said to be a bounded function if the set $\{f(x) : x \in S\}$ is bounded.

A function is said to be an unbounded if it is not bounded.



An unbounded continuous function on $(0, 1]$.

Remark 4.7.2:

i) A function $f : S \to \Re$ is bounded if and only if there exists M>0 such that $|f(x)| \leq M \;\; \forall \; x \in S$.

ii) A function $f : S \to \Re$ is unbounded if and only if there exists a sequence $(x_n) \in S$ such that $|f(x_n)| \to \infty \;\; as \;\; n \to \infty$.

Theorem 4.7.1: Let f is a real valued function defined on a closed and bounded interval [a, b]. Then, f is a bounded function.

The following result explains why closed bounded interval have nicer properties than other ones.

Theorem 4.7.2: (Boundedness Theorem): A continuous function on a closed bounded interval is bounded and attains its bounds.

Proof: Suppose f is defined and continuous at every point of the interval [a, b].

Let us assume f is not bounded above. Then, we could find a point x_1 with $f(x_1)>1$, a point x_2 with $f(x_2)>2$,………,.. . Now look at the sequence (x_n). By the Bolzano Weierstrass theorem it has a subsequence (x_i) which converges to a point $\alpha \in [a, b]$. By our construction the sequence $(f(x_i))$ is unbounded, by the continuity of f, this sequence should converge to $f(\alpha)$ and we have a contradiction.

Similarly it can be proved that f is bounded below.

Now, to show that f attains its bounds. Let M be the least upper bound of the set $S = \{f(x) / x \notin [a, b]\}$. We need to find a point $\beta \in [a, b]$ with $f(\beta) = M$. To do this, we can construct the sequence in the following manner. For each n Є N, let x_n be a point for which $\left|M - f(x_n)\right| < \frac{1}{n}$. Such a point must exists otherwise $M - \frac{1}{n}$ would be an upper bound of X. Some subsequence of (x_1, x_2, …….) converges to β (say) and $(f(x_1), f(x_2), ...) \to M$ and by continuity $f(\beta) = M$ as required.

Similarly, it can be proved that f attains its lower bound.

Let us consider the following examples to show why we must have a close bounded interval for the above result to work.

Example 4.7.1: Consider the cases where interval is not closed

i) The function $f : (0, 1] \to \Re$ defined by $f(x) = \frac{1}{x}$ is continuous but not bounded as it does not attain its supremum.

ii) The function $f : [0, 1) \to \Re$ defined by $f(x) = x$ is continuous and bounded but does not attain its least upper bound of 1

Example 4.7.2: Consider the cases where interval is not bounded

i) The function $f : [0, \infty) \to \Re$ defined by $f(x) = x$ is continuous but not bounded as it does not attain its infimum.

ii) The function $f : [0, \infty) \to \Re$ defined by $f(x) = \frac{1}{1-x}$ is continuous and bounded but it does not attain its least upper bound of 1.

Theorem 4.7.3: (Intermediate Value Theorem): Let f be continuous real valued function defined on a set $S \subseteq \Re$ in the closed interval [a, b]. Suppose c is a real number lying between f(a) and f(b). (That is, f(a) < c < f(b) or f(a) > c > f(b)). Then there exists some $x_0 \in]a, b[$, such that $f(x_0)=c$.

Proof: Without loss of generality, assume that $a < b$. Let $S = \{x \in [a, b] : f(x) < c\}$. Then,

S is non-empty (since $a \in S$) and bounded above (since $x \leq x_2$ for all $x \in S$).

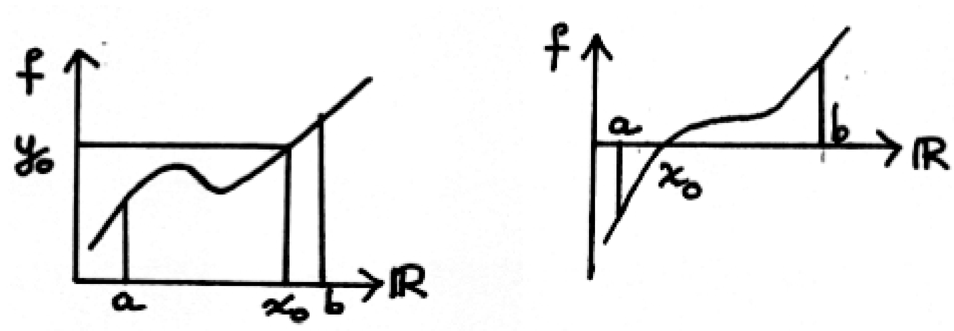
Let sup S = α. Then there exists a sequence (x_n) in S such that $x_n \to \alpha$. Note that $\alpha \in [a, b]$.

Hence, by continuity of f , $f(x_n) \to f(\alpha)$. Since $f(x_n) < c$ for all $n \in N$, we have $f(\alpha) \geq c$. Note that $\alpha \neq b$ since $f(\alpha) \leq c < f(b)$.

Now, let (y_n) be a sequence in (α, b) such that $y_n \to \alpha$. Then, again by the continuity of f, $f(y_n) \to f(\alpha)$. Since $y_n > \alpha$, $y_n \notin S$ and hence $f(y_n) \geq c$. Therefore, $f(\alpha) \geq c$.

Thus, we have proved that there exists $x_0 = \alpha$ such that $f(x_0) \leq c < f(x_0)$ so that $f(x_0)=c$.

> **Note:** How can we interpret this geometrically? We have already seen that the graph of a continuous function is smooth. It does not have any breaks or jumps. This theorem says that, if the points {a, f(a)) and (b, f(b)) lie on two opposite sides of a line y = c then the graph of f must cross the line y = c.



The following corollaries are immediate consequences of the above theorem.

Corollary 4.7.1: Let f be a continuous function defined on an interval. Then range of f is an interval.

Corollary 4.7.2: Suppose f is a continuous real valued function defined on an interval I. If a, b Є I satisfy a < b and f(a)f(b) < 0, then there exists $x_0 \in I$ such that $a \leq x_0 \leq b$ and $f(x_0) = 0$.

Example 4.7.3: Consider the continuous function $f : [0, 1] \to [0, 1]$. Then there is some point $c \in [0, 1]$ such that f(c) = c.

Solution: First we observe that clearly f(c) = c means f(c) – c = 0. This motivates one to introduce function g(x) = f(x) – x. We immediately see that g is continuous (on [0, 1]) as the difference of two continuous functions. We also have g(0) = f(0) ≥ 0 and g(1) = f(1) -1 ≤ 0 because of the assumption that 0 ≤ f(x) ≤ 1 for all x Є [0, 1]. But now the intermediate value theorem assures us that there is some c Є [0, 1] such that g(c)=0, i.e., we have f(c) = c as required.

Theorem 4.7.4 Let a function f be defined and is continuous on a closed interval $[a, b]$ then f attains its supremum.

Proof Let M be the supremum of f on the interval $[a,b]$ then we show that there exist a point $x_0 \in [a,b]$ such that $f(x_0) = M$ if possible let $f(x) < M$, $for\ all\ x \in I$

then $M - f(x) > 0$ for all $x \in [a,b]$.

Let g be the function defined on $[a,b]$ as

$$g(x) = \frac{1}{M - f(x)} \text{ for all } x \in [a,b]$$

since f(x) is continuous on $[a,b]$ and therefore g(x) is also continuous on $[a,b]$. Since every continuous function on a closed interval is bounded therefore g(x) is bounded that is there exist a real number k such that g(x)≤k for all $x \in [a,b]$

therefore $\frac{1}{M - f(x)} \leq$ k for all $x \in [a,b]$

or $f(x) \leq M - \frac{1}{k}$ $for\ all\ x \in [a,b]$

and so $M - \frac{1}{k}$ is an upper bound of f and this contradicts the fact that M is the supremum of f and therefore there exist some $x_0 \in [a,b]$ such that f(x_0) = M.

Theorem 4.7.5. Let a function f be defined and continuous on a closed interval $[a,b]$ then f attains its infimum.

Proof Let m be the infimum of f then we show that there exist a point $x_0 \in [a,b]$

such that f(x_0) = m .

Now we apply the theorem 4.7.5 to the function – f

Note: If the interval is not closed then there 4.7.4 and 4.7.5 will not old. For example

Example4.7.4: Let f be defined on open interval (0, 1) such that f(x) =x, for all x∈ (0,1) then f is continues in (0,1). It is bounded above in (0,1). The supf = 1but there is no point in (0,1) at which f(x) = 1.

Example 4.7.5: Let f be defined as

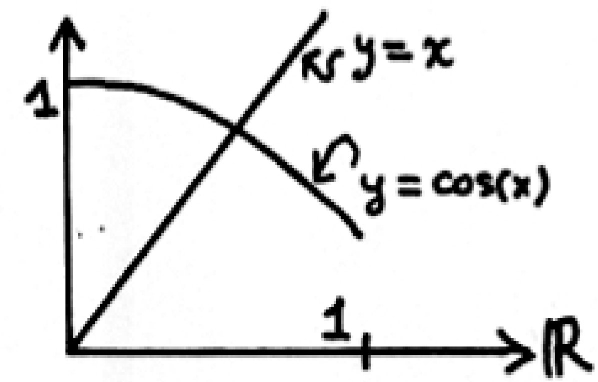
$$f(x) = \frac{1}{1+x^2} \text{ for all } x \in \mathbf{R}.$$

then f is continous on **R**. It is bounded both below and above. Supf = 1 and inf f = 0 supf is attend at x = 0 but inf f i.e. not attained is there is no point in **R** at which

$f(x) = Inf\ f = 0$

Example 4.7.6: Show that there exists an $x_0 \in (0, 1)$ such that $\cos(x_0) = x_0$.

Solution: Let $f : [0, 1] \to \Re$ be defined by $f(x) = \cos x - x$. Then, f(x) is continuous function on [0, 1] and f(1) < 0 < f(0). Therefore by the intermediate value theorem there exist a point $x_0 \in (0, 1)$ such that $f(x_0) = 0$. i.e., $\cos(x_0) = x_0$.



Using Intermediate Value Theorem to show the root exists

Remark4.7.3: Continuous functions on closed interval are bounded.

Example 4.7.7: The function f(x) = x^2 is bounded on the interval I = (-1, 1). Indeed, we have

0 ≤ f(x) < 1 for all x Є I.

Example 4.7.8: The function $f(x) = \frac{1}{x}$ on I = (0, 1) is bounded below but not above. Indeed the image of I under f is f (I) = (1, ∞), so f on I is bounded below but not above.

Example 4.7.9: The function $f(x) = \tan x$ on $I = (-\pi/2, \pi/2)$ is bounded neither above nor below on I. Indeed we can easily see that f (I) = (-∞, ∞).

Note: Thus, we may have all the possibilities of functions being bounded or unbounded on open intervals. However, this changes completely if the interval I is closed.

4.8 Type of discontinuity

Definition 4.8.1: Let f be a function defined on an interval $[a,b]$. of f is discontinuous at a point p$\in[a,b]$ then we have the following types of discontinuity

1. **Removable discontinuity:-** f has removable discontinuity at the point p if $\lim_{x\to p} f(x)$ exist but is not equal to the value f(p).
2. **Discontinuity of the first kind:-** f has discontinuity of the first kind at the point p if the left hand limit (LHL) and right hand limit(RHL) limit exist at p but they are not equal.
3. **Discontinuity of the second kind:-** f has discontinuity of the second kind at the point p if neither left hand limit nor right hand limit exist at the point p.
4. **Infinite discontinuity:-** A function is said to have infinite discontinuity at a point $x = p$ if f(x) is infinite at x = p

Example 4.8.1: Let f be defined on **R** as

$$f(x) = \frac{\sin x}{x}, \; x \neq 0$$

$$then \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1$$

$\therefore \lim_{x\to 0} f(x)$ exist but is not equal to f(0).

$\therefore$ f has a removable discontinuity.

Example 4.8.2: Let f be defined on **R** as

$$f(x) = \begin{cases} 0, & when\; x = 0 \\ \dfrac{e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & when\; x \neq 0 \end{cases}$$

$$then\, at\; x = 0\; LHL = \lim_{h\to 0} f(o-h) = \lim_{h\to 0} \frac{e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}} = 0 \qquad (\therefore \lim_{h\to 0} e^{-\frac{1}{h}} = 0)$$

$$then\, at\; x = 0\; RHL = \lim_{h\to 0} f(o+h) = \lim_{h\to 0} \frac{e^{\frac{1}{h}}}{1+e^{\frac{1}{h}}}$$

$$= \lim_{h\to 0} \frac{1}{e^{-\frac{1}{h}}+1}$$

$$= 1$$

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$\therefore$ LHL and RHL at x = 0 both exist but they are not equal. Therefore, f has discontinuity of first kind.

Example 4.8.3: Let f be defined on **R** such that

$$f(x)=\begin{cases}0, & when\ x=0\\ \sin\frac{1}{x} & when\ x\neq 0\end{cases}$$

Here none of the limits $\lim_{h\to 0} f(o-h)$ and $\lim_{h\to 0} f(o+h)$ exists. Therefore, f has discontinuity of second kind.

SOME TYPICAL EXAMPLES

Example 4.8.4: (Dirichlet function) Let f defined on **R** as

$$f(x)=\begin{cases}1, & when\ x\, is\, rational\\ -1, & when\ x\, is\, irrational\end{cases}$$

then f is discontinuous at every point in **R**

Case 1 Let p be a rational number, for each positive integer n let p_n be an irrational number such that $|p_n-p|<\frac{1}{n}$ then the sequence $\{p_n\}$ Converse to p it is given that $f(p_n)=-1$, for all n ($\therefore$ p_n is irrational)

$\lim_{n\to\infty} f(p_n)=-1\, and\ f(p)=1$ since p is rational

Thus $\lim_{n\to\infty} f(p_n)\neq f(p)$ so f is discontinuous at p.

Case 2 Let p be irrational. Now for each positive integer n let p_n be a rational number such that

$|p_n - p| < \frac{1}{n}$ Then $\{p_n\}$ converges to p

Now $\lim_{n\to\infty} f(p_n)=1$ (science p_n is a rational number) and f(p) = - 1 science p is irrational Therefore $\lim_{n\to\infty} f(p_n)\neq f(p)$ .Thus f is discontinuous at p.

Example 4.8.5: Let f be defined on the interval $[-1,1]$ such that

$$f(x)=\begin{cases}0, & if\ \ x\, is\, rational\\ x, & if\ \ x\, is\, irrational\end{cases}$$

Thus f is continuous only at x = 0.

Let p be any point of $[-1,1]$ for each positive integer n choose a rational number a_n and irrational number b_n belonging to $[-1,1]$ such that

$|a_n - p| < \frac{1}{n}$ and $|b_n - p| < \frac{1}{n}$

Then $\lim_{n\to\infty} a_n = p = \lim_{n\to\infty} b_n$ ....(1)

f is continuous at the point p then we must have

$\lim_{n\to\infty} f(a_n) = f(p) = \lim_{n\to\infty} f(b_n)$ for all n

But $f(a_n) = 0$ $(\therefore a_n$ is rational)

And $f(\mathrm{b}_n) = \mathrm{b}_n$ $(\therefore \mathrm{b}_n$ is irrational)

Therefore we have

$$0 = f(p) = \lim_{n\to\infty} f(b_n)$$
$$or\ 0 = f(p) = p \qquad form(1)$$

Therefore 0 is the only positive point of discontinuity we show that f is discontinuous at the point 0

For $\ni > 0$ let δ = ϵ/2 then

$|x-0| < \delta \Rightarrow |f(x) - f(0)| = 0$ if x is rational

And $|x-0| < \delta \Rightarrow |f(x) - f(0)| = |x| < \delta < \ni$ if x is irrational

Therefore $|x-0| < \delta \Rightarrow |f(x) - f(0)| < \ni$ and so f is continuous at x = 0.

<u>Bracket function:-</u>

Definition 4.8.21: Let x be a real number then $[x]$ (bracket x) is defined as largest integer ≤ x. i.e. if x is an integer then $[x]$ = x

And if x≠ an integer but x lies between the integer n and n+1 then $[x]$ = x

Example 4.8.6:

$$[0] = 0,\ [1.9] = 1,\ [-1.9] = -2$$
$$0 \le x < 1 \Rightarrow [x] = 0,$$
$$2 \le x < 3 \Rightarrow [x] = 2,$$
$$-2 \le x < -1 \Rightarrow [x] = -2, \text{etc}.$$

Example 4.8.7: Let $f(x) = [x]$ where $[x]$ is the largest integer ≤ x then f(x) is continuous at each point in **R** except at the integral points.

Case1 First we show that if x = a is an integer then f (x) is discontinuous at x = a

$LHL\,(\text{at x = a}) = \lim_{h\to 0} f(\text{a}-h)$

$= \lim_{h\to 0}[\text{a}-h]$ ($\therefore$where h$\to 0$ then a – h lies between the

$= a-1$

integer a – 1 and a)

$RHL\,(\text{at x = a}) = \lim_{h\to 0} f(\text{a}+h)$

$= \lim_{h\to 0}[\text{a}+h]$ ($\therefore$where h$\to 0$ then a + h lies between the

$= a$

integer a and a +1)

Since LHL$\neq$ RHL at x = a

Therefore f is not continues at x = a.

Case 2 Let $x_0 \neq$ an integer. We show that f is continues at x_0

Let x_0 lies between the integer x and x+1

$LHL\,(\text{at x} = x_0) = \lim_{h\to 0} f(x_0-h)$

$= \lim_{h\to 0}[x_0-h]$ ($\therefore$where h$\to 0$ then x_0 – h lies between

$= x$

the integer x and x+1)

$RHL\,(\text{at x} = x_0) = \lim_{h\to 0} f(x_0+h)$

$= \lim_{h\to 0}[x_0+h]$ ($\therefore$where h$\to 0$ then x_0 + h lies between the

$= x$

integer x and x +1)

Value at $x = x_0 = f(x_0)$

$[x_0] = x$ Since x_0 lies between the integer x and x +1

Example 4.8.8: Let f defined on **R** as

$$f(x) = \begin{cases} -x^2 & where\ x \le 0 \\ 5x-4 & where\ 0 < x \le 1 \\ 4x^2-3x & where\ 1 < x \le 2 \\ 5x+4 & where\ x \ge 2 \end{cases}$$

Discuss for continuity

(Hint test for continuity at the points x = 0, 1, 2 the function will be continuous at all other points).

Example 4.8.9: Let f defined on **R** as

$$f(x)=\begin{cases}0, & when\ x=0\\ \dfrac{e^{\frac{1}{x}}\sin 1/\mathrm{x}}{1+e^{\frac{1}{x}}} & when\ x\neq 0\end{cases}$$

Test for continuity at the point x = 0.

Example 4.8.10: Let f defined on **R** as

$$f(0) = 0$$

$$f(x0) = \frac{1}{2} - x\ where\ 0 < x < \frac{1}{2}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$f(x0) = \frac{3}{2} - x\ where\ \frac{1}{2} < x < 1$$

$$f(1) = 1$$

Show that f is discontinues at x = 0, $\frac{1}{2}$, 1

4.9. Image of a Closed Interval under Continuous Maps

Theorem 4.9.1: The image of an interval under a continuous map is also an interval.

Proof: Let $f : I \to \Re$ be a function defined on some interval $I \subset \Re$. If $f(a),\ f(b) \in f(I)$ and c lies between them, then by the intermediate value theorem there is an x between a and b with

f(x) = c.

Theorem 4.9.2: The image of a closed bounded interval under a continuous map is closed and bounded.

Proof: By theTheorem 4.7.1, the image of an interval I = [a, b] is bounded and is a subset of [m, M], where m and M are the least upper bound and greatest lower bound of the image. Since the function attains its bounds, $m,\ M \in f(I)$ and so the image is [m, M].

4.10 Summary

Definition of continuity by ε-δ method, to find $\delta > 0$ for a given function for a given $\in > 0$ is studied. Algebra of continuous function i.e. continuity of a $f(x) \pm g(x)$, $f(x).g(x)$, $\frac{f(y)}{g(x)}$,$g(x) \neq 0$ c.f(x) is discussed. Continuity of composite of two functions, properties of a continuous function on a closed interval with reference to bounded function is described. Supermem and infemum of a function, intermediate value theorem, image of a closed interval under continuity is described. Examples of continuous function with the help of left hand limit and right hand limit, types of discontinuity of a function at a given point is discussed. Examples to be done with the help of LHL & RHL are also provided for better understanding.

4.11 Terminal Questions

1. Show that $\lim_{x\to 0} \frac{e^{\frac{1}{x}} - e^{\frac{-1}{x}}}{e^{\frac{1}{x}} + e^{\frac{-1}{x}}}$ does not exist.

2. Let f be defined on R as

$$f(x) = \begin{cases} \dfrac{1}{1 - e^{\frac{1}{x}}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Is f continuous at x = 0? (Answer: No)

3. Let f be defined on R as

$$f(x) = \begin{cases} 1 & if\ x > 0 \\ -1, & if\ x < 0 \\ 0, & if\ x = 0 \end{cases}$$

Show that f has a discontinuity of the first kind at x =0.

Uttar Pradesh Rajarshi Tandon
Open University

UGMM-101
Differential Calculus

BLOCK

2

Differential Calculus

RIL-145

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UNIT-5

DIFFERENTIABILITY AND DERIVATIVES

Structure

INTRODUCTION

The history of mathematics presents the development of calculus as being accredited to Sir Isaac Newton (1642-1727) an English physicist, mathematician and Gottfried Wilhelm Leibnitz (1646-1716) a German physicist, mathematician. The introduction of calculus created an explosion in the development of physical sciences and other areas of science as calculus provided a way of describing natural and physical laws in a mathematical format which is easily understood.

It is another property of a function**s**. The co-founders of calculus are generally recognized to be Gottfried Wilhelm von Leibnitz (1646-1716) and Sir Isaac Newton (1642-1727). Newton approached calculus by solving a physics problem involving falling objects, while Liebnitz approached calculus by solving a geometry problem. Surprisingly, solution of these two problems led to the same mathematical concept called the derivative.

In the previous units we have studied the concepts of limits and continuity. In this unit we can see the interpretations of the derivative of a function at a point. We then extend the concept of derivative from a single point to the derivative of a function; we develop the rules of finding derivatives of functions easily, without having to calculate any limits directly. These rules are used to find derivatives of most of the common functions as well as various combinations of them.

The differentiability (Or derivative) is one of the key ideas in calculus, and is used to study a wide range of problems in mathematics, science, economics, medicine, engineering etc.

5.2. Objectives

After reading this unit you should be able to:

- define the differentiability of a function and its geometrical interpretation
- evaluate the derivative of the given function
- identify the rule of differentiation to find the derivative of a given function
- Differentiate the trigonometric, inverse trigonometric, hyperbolic and inverse hyperbolic functions.
- Understand the need of logarithmic differentiation, implicit functions differentiation and chain rule.

5.3. Differentiability of a function at a point

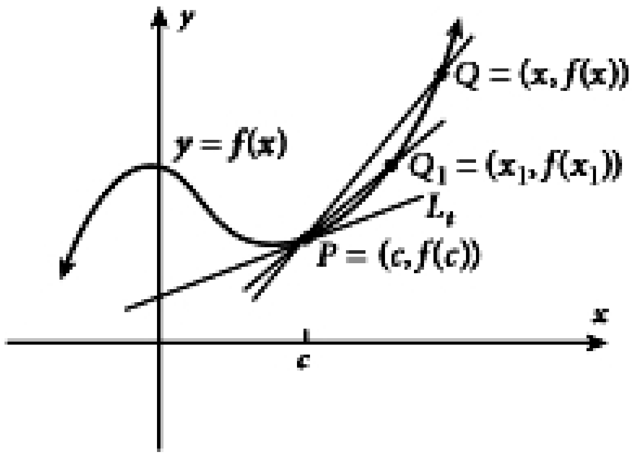
Definition 5.3.1: The derivative of a function f at a point a, denoted by $f^{/}(a)$ is defined as

$$f^{/}(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ Provided this limit must exist

OR

Geometrically,

The important generalization of the ‘the tangent line to the graph of a function $y = f(x)$ at a point $P = (c, f(c))$ on its graph is defined as the line containing the point P whose slope is $m = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}$ provided this limit exists. If the slope m exists, an equation of the tangent is given by $y - f(c) = m(x - c)$.



Now, let's define the derivative of the function $y = f(x)$ at the point c .

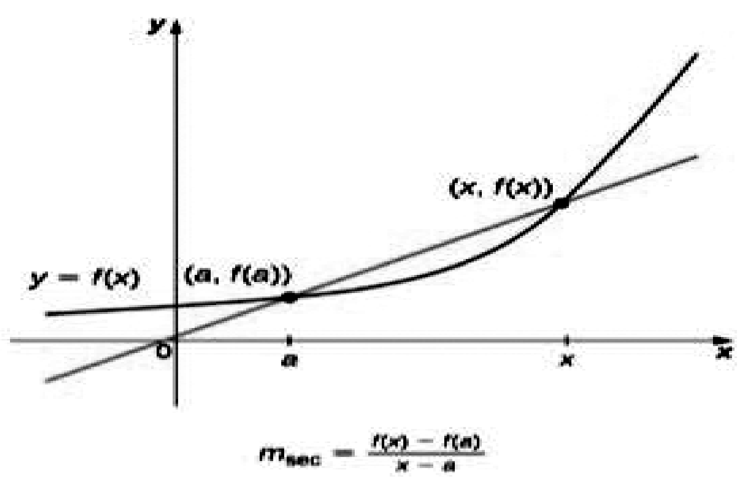
5.4. Definition of derivative of a function and its geometrical interpretation

Definition 5.4.1: Let f be a function defined on an interval I containing a.

If $x \neq a$ is in I, then the expression $m_{\sec} = \dfrac{f(x) - f(a)}{x - a}$ is called

difference quotient.

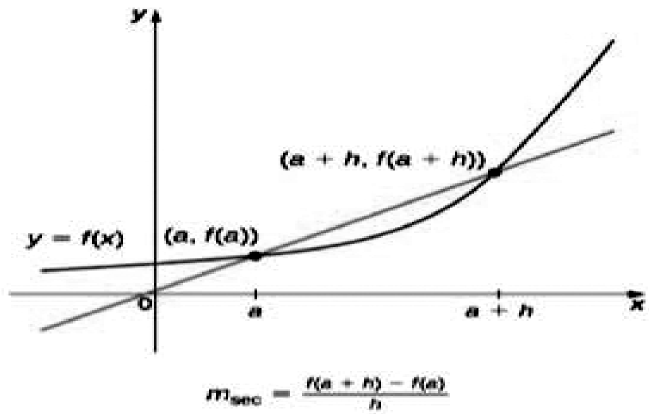
Geometrically, This gives the slope of the secant line by choosing a value of x near a and drawing the line through the points (a, f(a)) and (x, f(x)) as shown in the following figure



Also, if $h \neq 0$ is chosen so that $a + h$ is in I, then the **difference quotient** is given by $m_{\sec} = \dfrac{f(a+h) - f(a)}{h}$.

Geometrically,

We can also obtain the slope of the secant line by replacing the value of x with a + h , where h ia a value close to a and drawing a line through the points (a, f(a)) and (a+h, f(a+h)) with the increment h as shown in the following figure.

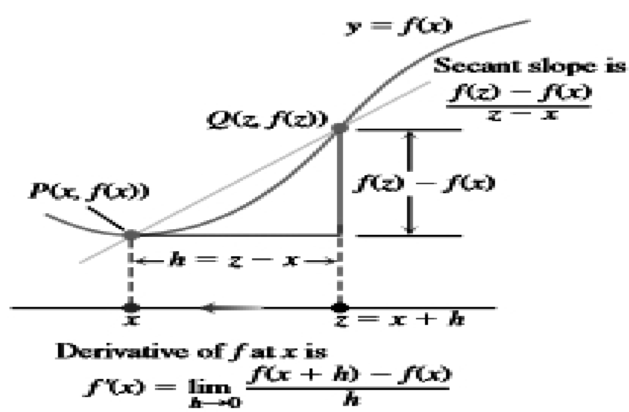


Definition 5.4.2: The derivative of the function f(x) with respect to the variable x is the function $f^{/}$ whose value at x is

$$f^{/}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$ Provided this limit must exist.

We use the notation f(x) in the definition to emphasize the independent variable x with respect to which the derivative function $f^{/}(x)$ is being defined. The domain of $f^{/}$ is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If $f^{/}$ exists at a particular x, we say that f is differentiable (has a derivative) at x. If $f^{/}$ exists at every point in the domain of f, we call f is differentiable.

If we write $z = x + h$ then $h = z - x$ and h approaches to zero if and only if z approaches to x. Therefore, an equivalent definition of the derivative is as follows (see the figure). This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.



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Thus, the alternate formula for the derivative is given by

$$f^{/}(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

Remark 5.4.1: The process of calculating a derivative is called differentiation. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation $\frac{d}{dx} f(x)$ as another way to denote the derivative $f^{/}(x)$. Thus, for the function $y = f(x)$, each of the following notations represents the derivative of f(x):

$f^{/}(x)$, $\frac{dy}{dx}$, $y^{/}$, $\frac{d}{dx}\left(f(x)\right)$.

Equivalently,

- The notation $\frac{dy}{dx}$, often referred to as the Leibnitz's notation.
- It can also be written as $\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{d}{dx}\left[f(x)\right]$. Where $\frac{d}{dx}\left[f(x)\right]$ is an instruction to compute the derivative of the function f with respect to its independent variable x.
- A change in the symbol used for the independent variable does not affect the meaning. If $s = f(t)$ is a function of t, then $\frac{ds}{dt}$ is an instruction to differentiate f with respect to t.

5.5. Differentiability on an interval; One-Sided Derivatives

A function $y = f(x)$ is differentiable on an interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval [a, b] if it is differentiable on the interior (a, b) and if the limits

$\lim_{h \to 0^{+}} \frac{f(a+h) - f(a)}{h}$ **Right-hand derivative at a** and, $h > 0$

$\lim_{h \to 0^{-}} \frac{f(b+h) - f(b)}{h}$ **Left-hand derivative at b, h** < 0

exist at the endpoints (as shown in the following figure).

Right-hand and Left-hand derivatives may be defined at any point of a function's domain. Because, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

Note:

If h > 0 then left hand derivative at 'a' is $\lim_{h \to 0^-} \frac{f(a-h) - f(a)}{-h} = Lf'(a)$

And right hand derivative at 'a' is $\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = Rf'(a)$

$$f \text{ will be differenciable at } x = a$$
$$\text{if } Lf'(a) = Rf'(a)$$

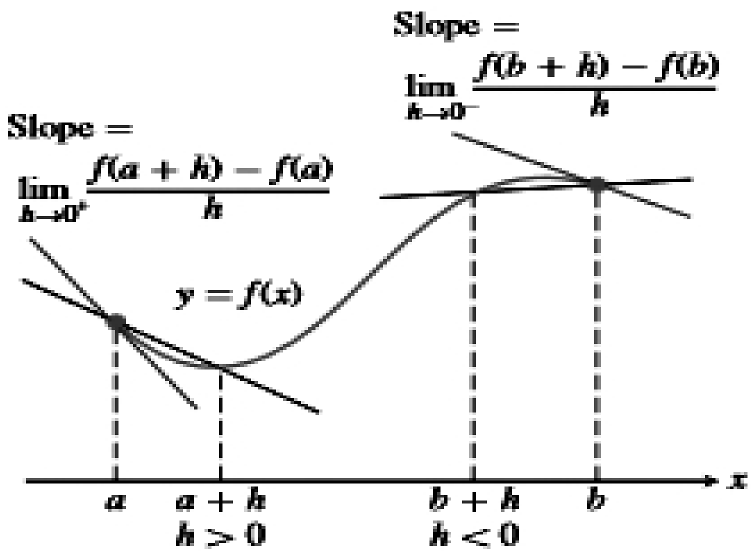


Figure: Derivatives at the end points are one sided limits Consider the following example,

Example 5.5.1: Show that the function $y = |x|$ is differentiable on (-∞, 0) and (0, ∞) but has no derivative at x = 0.

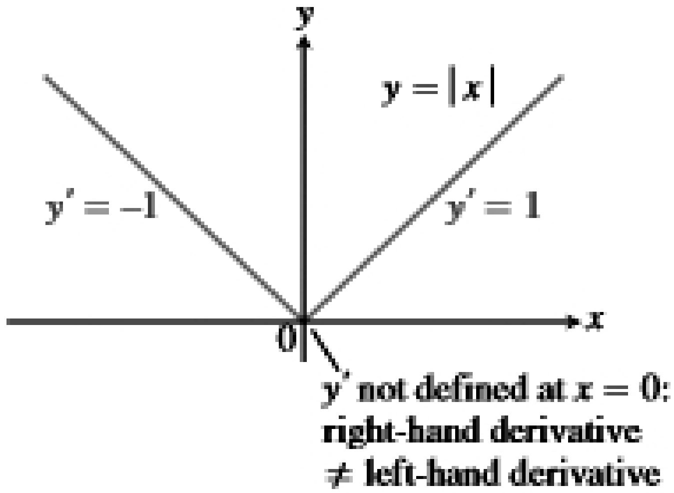
Solution: From the section 3.3, the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1..x) = 1.$$

And, to the left of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1..x) = -1.$$

There is no derivative at the origin (See the following figure), because the one-sided derivatives differ there.



Right-hand derivative of $|x|$ at zero = $\lim\limits_{h \to 0^+} \dfrac{|0+h|-|0|}{h} = \lim\limits_{h \to 0^+} \dfrac{|h|}{h}$

$$= \lim_{h \to 0^+} \frac{h}{h} \qquad |h| = h \ \ when\ h > 0$$
$$= \lim_{h \to 0^+} 1 = 1$$

Left-hand derivative of $|x|$ at zero = $\lim\limits_{h \to 0^+} \dfrac{|0-h|-|0|}{-h} = \lim\limits_{h \to 0^+} \dfrac{|h|}{-h} = -1$

Example 5.5.2: Show that $f(x) = \begin{cases} 0, & when\ \ x = 0 \\ x \sin \dfrac{1}{x} & when\ \ x \neq 0 \end{cases}$ is continuous but not differentiable at x = 0.

First we check limit at x = 0

$$LHL \qquad \lim_{x \to 0^-} f(x) = \lim_{h \to 0} f(0-h) = \lim_{h \to 0} -h \sin(-\frac{1}{h}) = \lim_{h \to 0} h \sin \frac{1}{h}$$
$$= 0 \times (\ finite\, number\, between -1 to +1)$$
$$= 0$$

$$RHL \qquad \lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(0+h) = \lim_{h \to 0} h \sin \frac{1}{h}$$
$$= 0 \times (\ finite\, number\, between -1 to +1)$$
$$= 0$$

LHL = RHL = Value of the function at x = 0 is 0 i.e. f(0) = 0.

This shows that function is continuous at x = 0 .

Now we check differentiability at x = 0.

$$LHD \quad \lim_{h\to 0}\frac{f(o-h)-f(0)}{-h}=\lim_{h\to 0}\frac{-h\sin(-\frac{1}{h})-0}{-h}$$

$$=\lim_{h\to 0}-\sin(\frac{1}{h})$$

$$=\text{a number which is } oscillates\ btween -1 to 1$$

$$= dose\ not\ existy$$

$$RHD \quad \lim_{h\to 0}\frac{f(o+h)-f(0)}{h}=\lim_{h\to 0}\frac{h\sin(\frac{1}{h})-0}{h}$$

$$=\lim_{h\to 0}\sin(\frac{1}{h})$$

$$=\text{a number which is } oscillates\ btween -1 to 1$$

$$= dose\ not\ existy$$

RHD and LHD does not exists, thus f(x) is not differentiable at x = 0 but it is continuous.

Example 5.5.3:

Show that $f(x)=\begin{cases} x^2\sin\frac{1}{x} & when\ x\neq 0 \\ 0, & when\ x=0 \end{cases}$ is continuous and differentiable at x = 0.

First we check limit at x = 0

$$LHL \qquad \lim_{x\to 0^-} f(x)=\lim_{h\to 0} f(0-h)=\lim_{h\to 0}(0-h)^2\sin(\frac{1}{0-h})=\lim_{h\to 0}h^2\sin-\frac{1}{h}$$

$$=0\times(\ finite\ number\ between -1 to +1)$$

$$=0$$

$$RHL \qquad \lim_{x\to 0^+} f(x)=\lim_{h\to 0} f(0+h)=\lim_{h\to 0}(0+h)^2\sin\frac{1}{0+h}=\lim_{h\to 0}h^2\sin\frac{1}{h}$$

$$=0\times(\ finite\ number\ between -1 to +1)$$

$$=0$$

LHL = RHL = Value of the function at x = 0 = f(0) = 0.

This shows that function is continuous at x = 0.

Now to check differentiability at x = 0.

$$LHD \quad \lim_{h\to 0}\frac{f(o-h)-f(0)}{-h}=\lim_{h\to 0}\frac{(-h)^2\sin(-\frac{1}{h})-0}{-h}$$

$$=\lim_{h\to 0} \text{h}\sin(\frac{1}{h})$$

$$=0\times(\text{a number which is } oscillates\ btween-1to1)$$

$$=0$$

$$RHD \quad \lim_{h\to 0}\frac{f(o+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin(\frac{1}{h})-0}{h}$$

$$=\lim_{h\to 0}\text{h}\sin(\frac{1}{h})$$

$$=0\times(\text{a number which is } oscillates\ btween-1to1)$$

$$=0$$

RHD and LHD are exists, thus f(x) is differentiable at x = 0 and it is continuous

$$F\left(x\right)=x^2\sin\frac{1}{x}$$

$$f\,'(x)=x^2.\cos\frac{1}{x}\left(-\frac{1}{x^2}\right)+\sin\frac{1}{x}.2x \qquad .$$

$$f\,'(x)=\begin{cases}2x\sin\frac{1}{x}-\cos\frac{1}{x} & for\ x\neq 0\\ 0 & for\ x=0\end{cases}$$

To check continuity of f'(x) at x = 0.

$$LHL \qquad \lim_{x\to 0^-} f(x)=\lim_{h\to 0} f\,'(0-h)=\lim_{h\to 0}2(0-h)\sin(\frac{1}{0-h})-\cos(\frac{1}{0-h})$$

$$=\lim_{h\to 0}2h\sin\frac{1}{h}-\cos\frac{1}{h}$$

$$=0-\lim_{h\to 0}\cos\frac{1}{h}$$

$$=0-does\ not\ exist$$

$$=does\ not\ exist$$

$$RHL \qquad \lim_{x\to 0^+} f\,'(x)=\lim_{h\to 0} f\,'(0+h)=\lim_{h\to 0}2(0+h)\sin\frac{1}{h}-\cos\frac{1}{h}$$

$$==0-does\ not\ exist$$

$$=does\ not\ exist$$

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Hence f'(x) are not continues at x = 0.

Check Your Progress

Find the derivatives of the following functions at their given points

1) $f(x) = x^2 + 9x$ at the point x = 2

2) $f(x) = \sqrt{x-2}$ at the point x = 6

3) $f(x) = \dfrac{1}{x-3}$ at the point x = -1

4) $f(x) = \dfrac{1}{\sqrt{x}}$ at the point x = 4

5) $f(x) = \dfrac{1}{x^3}$ at the point x = 1

6) Discuss the continuity and differentiability of f(x) at x = 0

$$f(x) = \begin{cases} \dfrac{1}{x}\sin x^2 & when\ x \neq 0 \\ 0, & when\ x = 0 \end{cases}$$

7) To discuss the nature of the function f(x) at x = 1

$$f(x) = \begin{cases} (x-1)\sin\dfrac{1}{x-1} & when\ x \neq 1 \\ 0, & when\ x = 1 \end{cases}$$

5.6 Derivatives of some simple functions

Example 5.6.1: Differentiate the function $\dfrac{x}{x-1}$.

Solution: Consider, $f(x) = \dfrac{x}{x-1}$.

By the definition 5.4.2; $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h}$

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} = \lim_{h\to 0}\frac{(x+h)(x-1) - x(x+h-1)}{h.(x-1)(x+h-1)}$$

$$= \lim_{h\to 0}\frac{x^2 - x + hx - h - x^2 - xh + x}{h(x-1)(x+h-1)} = \lim_{h\to 0}\frac{-h}{h(x-1)(x+h-1)} = -\frac{1}{(x-1)^2}$$

Example 5.6.2: Find the derivative of the function $f(x) = \dfrac{1-x}{2x}$ at the point x = -1.

Solution: By the definition 5.3.1; $f'(a) = \lim\limits_{h\to 0}\dfrac{f(a+h) - f(a)}{h}$

$$\therefore\ f'(-1) = \lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$= \lim_{h\to 0}\frac{\dfrac{[1-(-1+h)]}{2(-1+h)}+1}{h}$$

$$= \lim_{h\to 0}\frac{(2-h)+(2h-2)}{h(2h-2)} = \lim_{h\to 0}\frac{h}{h(2h-2)} = \lim_{h\to 0}\frac{1}{2h-2} = \frac{-1}{2}$$

Example 5.6.3: Differentiate the function $\sin x$.

Solution: Consider, $f(x) = \sin x$.

By the definition 5.4.2; $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cosh + \cos x \sinh - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x + \cos x \sinh - \sin x}{h} = \cos x \lim_{h\to 0}\frac{\sinh}{h} = \cos x.\ \ \sin ce\ \ \lim_{x\to 0}\frac{\sin x}{x} = 1$$

Example 5.6.4: Find the derivative of the function $f(x) = \sqrt{x}$ at the point x = 4.

Solution: By the definition 5.3.1; $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

$$\therefore\ f'(4) = \lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{4+h}-\sqrt{4}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\times\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}$$

$$= \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0}\frac{h}{h\left(\sqrt{4+h}+2\right)} = \frac{1}{4}$$

Example 5.6.5: Differentiate the function $\sqrt{x-3}$.

Solution: Consider, f(x) = $\sqrt{x-3}$.

By the definition 5.4.2; $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h-3}-\sqrt{x-3}}{h}\times\frac{\sqrt{x+h-3}+\sqrt{x-3}}{\sqrt{x+h-3}+\sqrt{x-3}}$$

$$= \lim_{h\to 0}\frac{(x+h-3)-(x-3)}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \lim_{h\to 0}\frac{h}{h(\sqrt{x+h-3}+\sqrt{x-3})} = \frac{1}{2\sqrt{x-3}}$$

Example 5.6.6: Differentiate the function $\frac{1}{\sqrt{x}}$.

Solution: Consider, $f(x) = \frac{1}{\sqrt{x}}$.

By the definition 5.4.2; $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore \; f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{(\sqrt{x}) - (\sqrt{x+h})}{h\sqrt{x}(\sqrt{x+h})} \times \frac{(\sqrt{x}) + (\sqrt{x+h})}{(\sqrt{x}) + (\sqrt{x+h})}$$

$$= \lim_{h \to 0} \frac{x - (x+h)}{h\sqrt{x}(\sqrt{x+h})\left((\sqrt{x}) + (\sqrt{x+h})\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{x}(\sqrt{x+h})\left((\sqrt{x}) + (\sqrt{x+h})\right)} = -\frac{1}{\sqrt{x}(\sqrt{x})\left(\sqrt{x} + \sqrt{x}\right)} = \frac{-1}{2x\sqrt{x}}$$

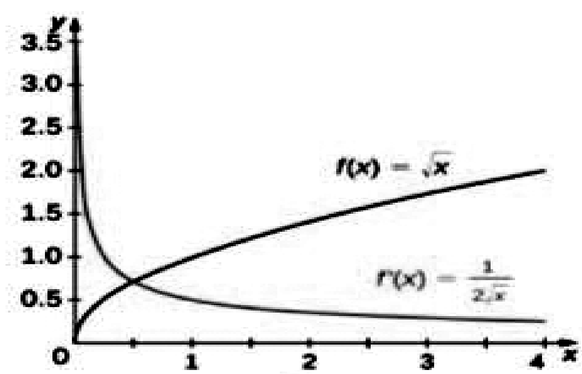
Example 5.6.7: Differentiate the function $f(x) = \sqrt{x}$. Graph both $f(x)$ and $f'(x)$

Solution: By the definition 5.4.2; $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore \; f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x + h - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \frac{1}{2\sqrt{x}}$$

It we graph the functions $f(x)$ and $f'(x)$ as shown in the following figure. First we notice that $f(x)$ is increasing over its entire domain, which means that the slopes of its tangent lines at all points are positive. Consequently, we expect $f'(x) > 0$ for all values of x in its domain. Furthermore, as x increases, the slopes of the tangent lines to $f(x)$ are decreasing and we expect to see a corresponding decrease in $f'(x)$. We also observe that $f(0)$ is undefined and that $\lim_{x \to 0^+} f(x) = +\infty$, corresponding to a vertical tangent to $f(x)$ at 0.



The derivative is positive everywhere because function f(x) is increasing.

Example 5.6.8: Differentiate the function $f(x) = x^2 - 2x$. Graph both $f(x)$ and $f'(x)$.

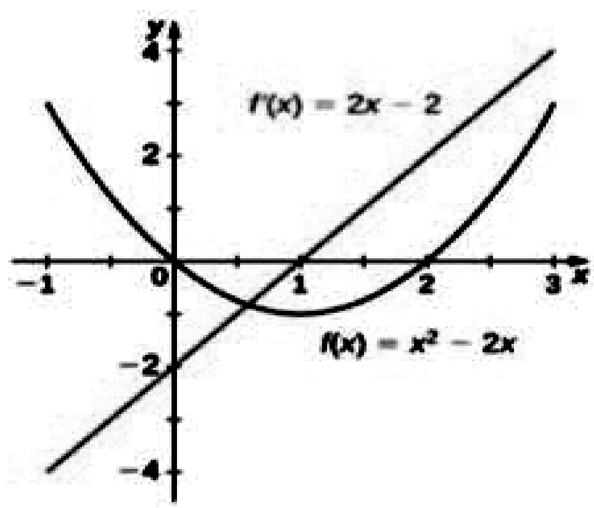
Solution: By the definition 5.4.2; $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$\therefore \ f'(x) = \lim_{h \to 0} \frac{[(x+h)^2 - 2(x+h)] - (x^2 - 2x)}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0} \frac{h(2x + h - 2)}{h} = 2x - 2.$$

The graphs of $f(x)$ and $f'(x)$ as shown in the following figure. Observe that $f(x)$ is decreasing fir x < 1. For these same values of x, $f'(x) < 0$. For values of x > 1, $f(x)$ is increasing and $f'(x) > 0$. Also $f(x)$ has horizontal tangent at x = 1 and $f'(1) = 0$.



The derivative $f'(x) < 0$ where the function $f(x)$ is decreasing and $f'(x) > 0$ where $f(x)$ is increasing. The derivative is zero where the function has a horizontal tangent.

Example 5.6.9: To graph the derivative of the function $y = f(x)$ as shown in the figure (a).

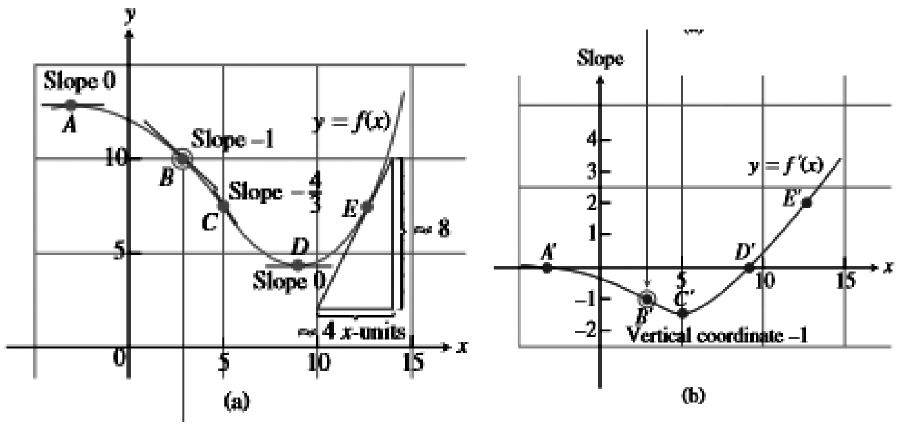
Solution: We sketch the tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as shown in the figure (b).

The following points can be noted from the graph of $y = f^{/}(x)$.

i. where the rate of change of f is positive, negative or zero;

ii. the rough size of the growth rate at any x and its size in relation to the size of of f(x);

iii. where the rate of change itself is increasing or decreasing.



In the above figures, we made the graph of $y = f^{/}(x)$ in (b) by plotting slopes from the graph $y = f(x)$ in (a). The vertical coordinate of $B^{/}$ is the slope at B and so on. The slope at E is approximately 8/4 = 2. In (b) we see that the rate of change of f is negative for x between $A^{/}$ and $D^{/}$; the rate of change is positive for x to the right of $D^{/}$.

Differentiate the following functions and graph it

1) $f(x) = \dfrac{x}{2x+1}$

2) $f(x) = \dfrac{1}{x^2}$

3) $f(x) = \sqrt{x}$

4) $f(x) = \cos x$

5.7. Algebra of derivatives

5.7.1. Derivative of a Constant Function

If f has the constant value $f(x) = c$ then, $\dfrac{df}{dx} = \dfrac{d}{dx}(c) = 0$.

Proof: By the definition 5.4.2; $f^{/}(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\therefore \; f^{/}(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

5.7.2. Derivative of a Positive Integer Power

If ‘n’ is a positive integer, then $\frac{d}{dx}(x^n) = nx^{n-1}$

Proof: By the definition 5.4.2; $f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f^{/}(x) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{x^n + nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + ...+ {}^nC_n h^n . - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + ...+ {}^nC_n h^n .}{h}$$

$$= \lim_{h\to 0}\frac{h(nx^{n-1} + {}^nC_2 x^{n-2}h + ...+ {}^nC_n h^{n-1}).}{h} = nx^{n-1}.$$

Example 5.7.2 (i): Find the derivative of the function $f(x) = x^{\sqrt{2}}$

Solution: $f^{/}(x) = \frac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$.

Example 5.7.2 (ii): Find the derivative of the function $f(x) = x^{-4/3}$

Solution: $f^{/}(x) = \frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-4/3-1} = -\frac{4}{3}x^{-7/3}$.

5.7.3. Derivative of a Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then $\frac{d}{dx}(cu) = c\frac{du}{dx}$.

Proof: By the definition 5.4.2; $f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f^{/}(x) = \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} = c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} = c\frac{du}{dx}$$

Example 5.7.3: Find the derivative of the function $f(x) = 5x^3$

Solution: $f^{/}(x) = \frac{d}{dx}(5x^3) = 5\frac{d}{dx}x^3 = 5(3x^2) = 15x^2$.

5.7.4. Derivative Sum/Difference Rule

If u and v are differentiable functions of x, then their sum $u \pm v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}.$$

Proof: By the definition 5.4.2; $f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f^{/}(x) = \lim_{h\to 0}\frac{[u(x+h)\pm v(x+h)]-[u(x)\pm v(x)]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \pm \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} = \frac{du}{dx} \pm \frac{dv}{dx}$$

Example 5.7.4(i): Find the derivative of the function $f(x) = (\sin^2 x + \tan^2 x)$

Solution: $f^{/}(x) = \frac{d}{dx}(\sin^2 x + \tan^2 x) = \frac{d}{dx}(\sin^2 x) + \frac{d}{dx}(\cos^2 x)$.

$$= 2\sin x.\mathrm{cosx} - 2\,\mathrm{cosx}.\mathrm{sinx}$$

5.7.5. Derivative Product Rule

If u and v are differentiable at x, then so is their product uv and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Proof: By the definition 5.4.2; $f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f^{/}(x) = \lim_{h\to 0}\frac{[u(x+h)v(x+h)]-[u(x)v(x)]}{h}$$

Add and Subtract $u(x+h)v(x)$ *in the numerator, we obtain,*

$$f^{/}(x) = \lim_{h\to 0}\frac{[u(x+h)v(x+h)] + u(x+h)v(x) - u(x+h)v(x) - [u(x)v(x)]}{h}$$

$$= \lim_{h\to 0}\frac{u(x+h)[v(x+h)-v(x)] + v(x)[u(x+h)-u(x)]}{h}$$

$$= \lim_{h\to 0}u(x+h)\lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + \lim_{h\to 0}v(x)\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}$$

$$= u(x)\frac{dv(x)}{dx} + v(x)\frac{du(x)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 5.7.5: Find the derivative of the function $f(x) = (x^5 - 3)(5x^4 + 3x^2)$

Solution: Consider,

$$f^{/}(x)=\frac{d}{dx}\left[(x^5-3)(5x^4+3x^2)\right]$$

$$=(x^5-3)\frac{d}{dx}(5x^4+3x^2)+(5x^4+3x^2)\frac{d}{dx}(x^5-3)$$

$$=(x^5-3)(20x^3+6x)+(5x^4+3x^2)(5x^4)$$

5.7.6. Derivative Quotient Rule

If u and v are differentiable at x and if $v(x)\neq 0$, then the quotient $\frac{u}{v}$ is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Proof: By the definition 5.4.2; $f^{/}(x)=\lim\limits_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\therefore\ f^{/}(x)=\lim_{h\to 0}\frac{\frac{u(x+h)}{v(x+h)}-\frac{u(x)}{v(x)}}{h}=\lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x+h)}{hv(x)v(x+h)}$$

Add and Subtract $u(x)v(x)$ in the numerator, we obtain,

$$f^{/}(x)=\lim_{h\to 0}\frac{u(x+h)v(x)-u(x)v(x)+u(x)v(x)-u(x)v(x+h)}{hv(x)v(x+h)}$$

$$=\lim_{h\to 0}\frac{v(x)[u(x+h)-u(x)]-u(x)[v(x+h)-v(x)]}{hv(x)v(x+h)}$$

$$=\lim_{h\to 0}\frac{v(x)\frac{u(x+h)-u(x)}{h}-u(x)\frac{v(x+h)-v(x)}{h}}{v(x)v(x+h)}$$

$$\therefore\ f^{/}(x)=\frac{v(x)\frac{du(x)}{dx}-u(x)\frac{dv(x)}{dx}}{v(x)v(x)}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

5.8. Continuity versus Differentiability

Differentiable functions are continuous. In other words, a function is continuous at every point where it has a derivative.

Definition 5.8.1: A function $f(x)$ is called differentiable at the point x = c if $f^{/}(x)$ exists and $f(x)$ is called differentiable on an interval if the derivative exists for each point in the interval.

Theorem 5.8.1: (Differentiability implies continuity). If f has a derivative at x = c, then f is continuous at x = c.

Proof: Given that $f'(c)$ exists, we must show that $\lim_{x \to c} f(x) = f(c)$. If $h \neq 0$, then

$$f(c+h) = f(c) + \left(f(c+h) - f(c)\right)$$
$$= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h$$

Now, take limits as $h \to 0$, then

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} f(c) + \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h \to 0} h$$
$$= f(c) + f'(c) \cdot 0$$
$$= f(c) + 0$$
$$= f(c).$$

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at x = c then f is continuous from that side at x = c.

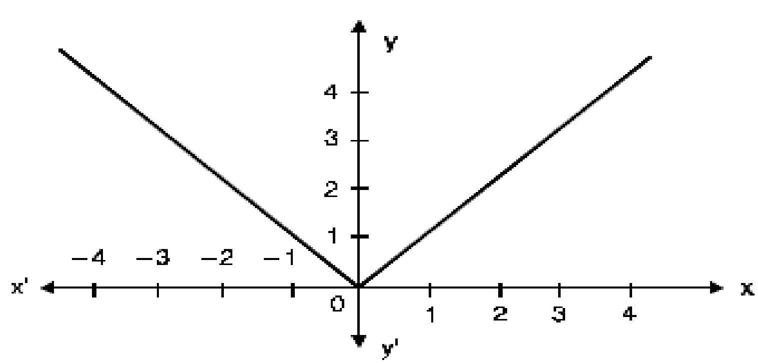
If a function has a discontinuity at a point, then it cannot be differentiable there.

The converse of this theorem is not true. “A function need not have a derivative at a point where it is continuous”.

To show this, consider the following example. To discuss the continuity of $f(x) = |x|$ at $x = 0$.

The function $f(x) = |x|$ at $x = 0$ is defined as follows,

$f(x) = |x| = \begin{cases} -x & ; \ x < 0 \\ x & ; \ x \geq 0 \end{cases}$ The graph of the function is given by,



The left hand limit of $f(x)$ as x approaches 0 is given by,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |-x|$$

$$= \lim_{h\to 0} |-(0-h)| = 0$$

The right hand limit of $f(x)$ as x approaches 0 is given by,

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} |x|$$

$$= \lim_{h\to 0} |(0-h)| = 0$$

The value of the function $f(x)$ at $x=0$ is given by, $f(0)=|0|=0$.

Thus, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$. Therefore, the function $f(x)=|x|$ is continuous.

Now, We know that, the Derivative of the function $y = f(x)$ at x is given by $f^{/}(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

Therefore, the derivative of the function $y = f(x)$ at $x=0$ is given by

$f^{/}(0) = \lim_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$, to evaluate this limit ,

Consider, the left hand limit of it; $f^{/}(0) = \lim_{h\to 0^-} \dfrac{f(0-h)-f(0)}{h}$ and

$$= \lim_{h\to 0^-} \frac{|0-h|-|0|}{-h}, \qquad h>0$$

$$= \lim_{h\to 0^-} \frac{|-h|}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

The right hand limit of it; $f^{/}(0) = \lim_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h}$.

$$= \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \frac{h}{h} = 1$$

Since the left and right hand limits are not equal.

$\therefore\ f^{/}(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ does not exists.

Thus, the function defined by $f(x)=|x|$ is continuous but not differentiable at a point $x=0$.

5.9. Chain Rule

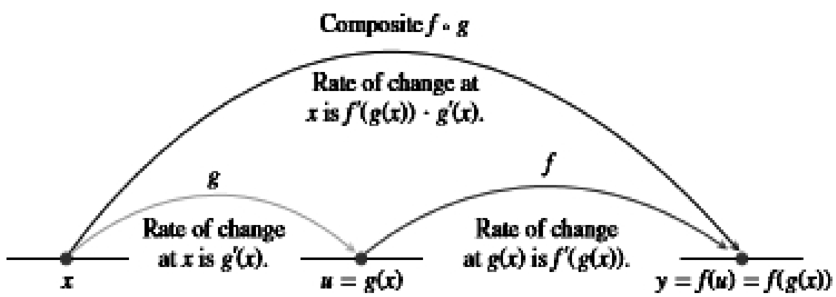
If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(fog)(x) = f(g(x))$ is differentiable at x, and

$$(fog)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibnitz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \text{ where } \frac{dy}{du} \text{ is evaluated at } u = g(x).$$

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (See the following figure).



Note: The chain rule provides us to decompose the composite function into simple functions to find its derivative.

<u>"Outside-Inside" Rule</u>

A difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives in the chain rule are supposed to be evaluated. So it sometimes helps to think about the Chain Rule using functional notation. If $y = f(g(x))$, then $\frac{dy}{dx} = f'(g(x)) \cdot g'(x)$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function".

Example 5.9.1: Find $\frac{d}{dx}(\sin t)\, if\ t = x^2$

$$\frac{d}{dx}(\sin t) = \frac{d}{dt}(\sin t).\frac{dt}{dx} = \cos t.2x = \cos x^2.2x$$

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Example 5.9.2: Find $\frac{d}{du}(\tan x)$

$$\frac{d}{du}(\tan x) = \frac{d}{dx}(\tan x)\frac{dx}{du}$$

Example 5.9.3: Find $\frac{d}{dx}(5t-9)^5$

$$= 5(5t-9)^4.5 = 25(5t-9)^4$$

5.10. Sign of derivatives and monotonicity of functions

5.10.1 <u>Monotone Function:</u>

We consider two types of functions: (i) Increasing and (ii) Decreasing

Any function which conforms to any one of these types is called a monotone function. Does the profit of a company increase with production? Does the volume of gas decrease with increase in pressure? Problems like these require the use of increasing or decreasing functions. Now let us see what we mean by an increasing function. Consider the function g defined by $g(x) = \begin{cases} -x, \ if \ \ x \le 0 \\ 1, \ if \ x \ > 0 \end{cases}$

Note that whenever $x_2 > x_1$, implies $g(x_2) > g(x_1)$.

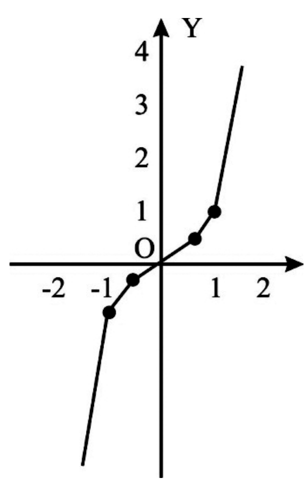


Fig. 15

In other words, as x increases, g(x) also increases. In this case we see that if $x_2 > x_1$, Equivalently, we can say that g(x) increase (or does not decreases) as x increases. Function like g is called increasing or non-decreasing function.

Thus, a function f defined on a domain D is said to be **increasing (or non-decreasing**) if, for every pair of elements $x_1, x_2 \in$ D, $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$. Further, we say that f is strictly increasing if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$ (strict inequality).

Clearly, the function g (x) = x^3 , is a strictly increasing function We shall now study another concept which is, in some sense, complementary to that of an increasing function. Consider the function f_1 defined on R by setting.

$$f_1(x) = \begin{cases} 1, if\ x \leq -1 \\ -x, if\ -1 < x < 1 \\ -1, if\ x \geq 1 \end{cases}$$ The graph of f_1 is as shown in

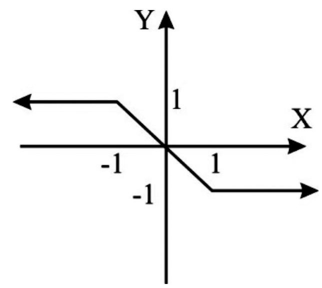
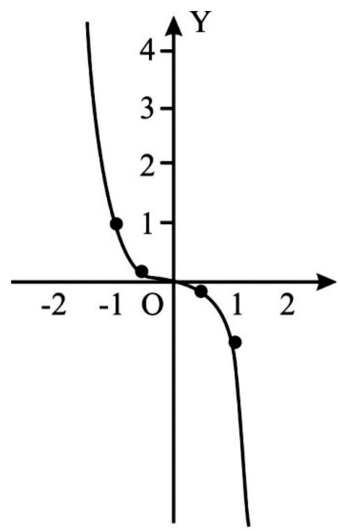


Fig. (16)

From the graph we can easily see that as x increases f_1 does not increase.

That is, $x_2 > x_1 \Rightarrow f_1(x_2) \leq f_1(x_1)$. Now consider the function $f_2(x) = -x^3$ $(x \in R)$

The graph of f_2 is shown in



Graph of f_2

Fig. (17).

Since $x_2 > x_1 \Rightarrow -x_2^3 < -x_1^3 \Rightarrow x_2^3 \Rightarrow< f_2(x_2) < f_2(x_1)$, we find that as x increases, $f_2(x)$ decreases. Functions like f_1 and f_2 are called decreasing or non-increasing functions. A function f defined on a domain D is said to be **decreasing (or non-increasing)** if for every pair of elements x_1, x_2, $x_2 > x_1 \Rightarrow f(x_2) \le f(x_1)$. Further, f is said to be strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$.

We have seen that, f_2 is strictly decreasing, while f_1 is not strictly decreasing.

A function f defined on a domain D is said to be a **monotone function** if it is either increasing or decreasing on D. The functions (g, f_1, f_2) discussed above are monotone functions. The word 'monotonically increasing' and 'monotonically decreasing' are used for 'increasing' and 'decreasing', respectively.

There are many other functions which are not monotonic. $f(x) = x^2$ ($x \in R$).

This function is neither increasing nor decreasing. If we find that a given function is not monotonic, we can still determine some subsets of the domain on which the function is increasing or decreasing. The function $f(x) = x^2$ is strictly decreasing on] - ∞, 0] and is strictly increasing in [0, ∞ [.

5.11. Derivatives of Exponential Functions

Exponential and logarithmic functions are pivotal mathematical concepts; they play central roles in advanced mathematics including calculus, differential equations and complex analysis.

If we apply the definition of the derivative to $f(x) = a^x$, we obtain

$$\frac{d}{dx}(a^x) = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h}$$

$$= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = \underbrace{\left(\lim_{h \to 0} \frac{a^h - 1}{h} \right)}_{a \ \ fixed \ \ number} \cdot a^x .$$

Thus, we see that the derivative of a^x is a constant multiple of a^x. The constant multiple is a limit unlike any we have encountered earlier section. Note, however, that it equals the derivative of $f(x) = a^x \ \ at \ \ x = 0$.

$$f'(0) = \lim_{h \to 0} \frac{a^h - a^0}{h} = \lim_{h \to 0} \frac{a^h - 1}{h} = \text{Constant.}$$

This limit is therefore the slope of the graph of $f(x) = a^x$ where it crosses the Y-axis. In earlier section, we have seen the logarithmic and exponential functions, we prove that the limit L exists and has the value $\ln a$ $(i.e., \log_e a)$. For now we investigate the values the limit by graphing the function $y = \frac{a^h - 1}{h}$ and studying its behavior as h approaches 0.

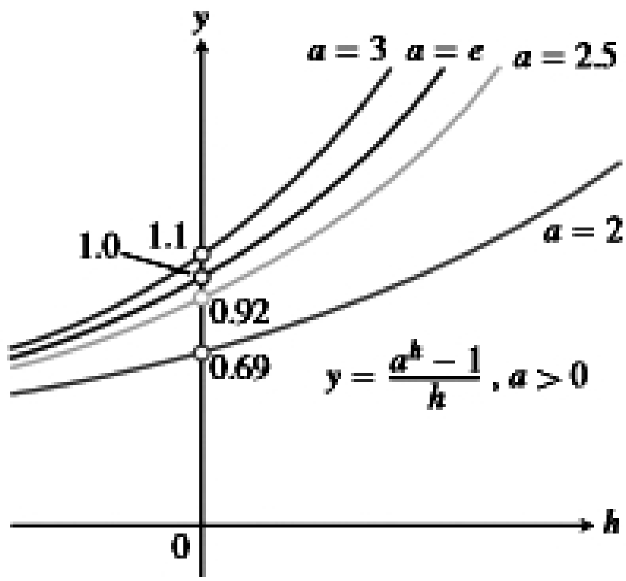


Figure: The position of the curve $y = \frac{a^h - 1}{h}$, a > 0, varies continuously with 'a'

The figure shows the graphs of $y = \frac{a^h - 1}{h}$ for four different values of a. The valu of the limit is approximately 0.69 if a = 2, about 0.92 if a = 2.5, and about 1.1 if a = 3. It appears that the value of L is 1 at some number a chosen between 2.5 and 3. That number is given by a =e = 2.718281828. With this choice of base we obtain the natural exponential function $f(x) = e^x$ and it satisfies the property,

$$f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$

This limit is 1 implies an important relationship between the natural exponential function e^x and its derivative.

$$\frac{d}{dx}(e^x) = \lim_{h \to 0}\left(\frac{e^h - 1}{h}\right) \cdot e^x = 1 \cdot e^x = e^x$$

Therefore the natural exponential function is its own derivative.

Consider, $y = f(x) = e^x$

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Derivative of the function $y = f(x)$ at x is given by

$$f^{/}(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

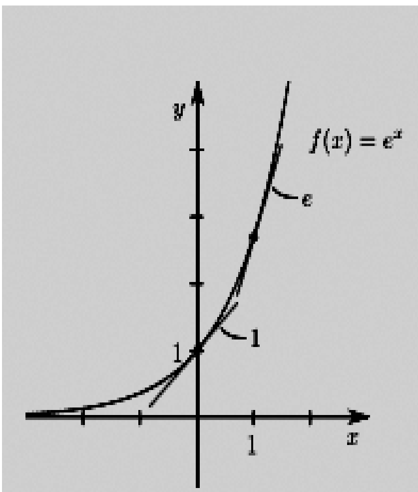
$$f^{/}(x) = \frac{d}{dx}(e^{x}) = \lim_{h\to 0}\left[\frac{e^{x+h}-e^{x}}{h}\right]$$

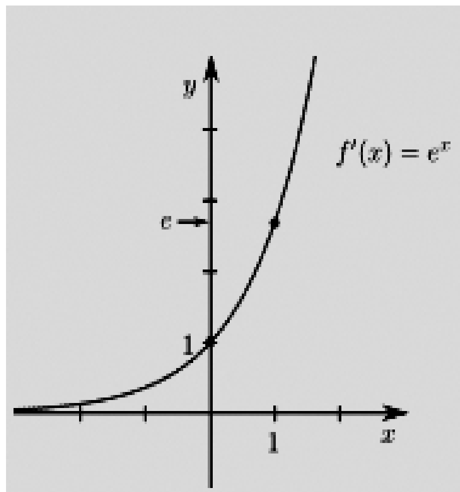
$$= \lim_{h\to 0}\left[\frac{e^{x}e^{h}-e^{x}}{h}\right] = \lim_{h\to 0}\left[\frac{e^{x}(e^{h}-1)}{h}\right]$$

$$= e^{x}\lim_{h\to 0}\left[\frac{(e^{h}-1)}{h}\right] = e^{x}\log_{e} e = e^{x}$$

$$Since\ \lim_{x\to 0}\left[\frac{a^{x}-1}{x}\right] = \log_{e} a\ ;\ a > o$$

Geometrically,





Example 5.11.1 Find the derivative of $a^{\tan^{-1}x}$.

$$\frac{d}{dx}(a^{\tan^{-1}x}) = a^{\tan^{-1}x}.\log a.\frac{d}{dx}(\tan^{-1}x)$$

$$= a^{\tan^{-1}x}.\log a.\frac{1}{1+x^{2}}$$

Example 5.11.2 Find the derivative of $e^{x^4-2x^2+8}$

$$\frac{d}{dx}(e^{x^4-2x^2+8}) = \frac{d}{dt}(e^{t})\frac{d}{dx}(x^4-2x^2+8)\quad where\ t = x^4-2x^2+8$$

$$= e^{t}.(4x^3-4x)$$

$$= e^{(x^4-2x^2+8)}.(4x^3-4x)$$

Example 5.11.3 Find the derivative of

a). 2^{2x}, b). $7^{\cos x}$, c). $e^{\frac{x+1}{x}}$.

5.12. Derivatives of Logarithmic Functions

Derivative of the natural logarithmic function $\log_e x$:

We know that the function e^x and $\log_e x$ are inverse to each other. Therefore, Finding derivative of e^x is relatively easier than it's inverse function, $\log_e x$.

Consider,

$$y = \log_e x \quad then \quad x = e^y, \ Taking\ derivatives\ on\ both\ sides$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(e^y) \ Or\ 1 = e^y \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

$$Thus, \ \frac{d}{dx}[\log_e x] = \frac{1}{x}$$

Note: $\frac{d}{dx}[\log_e u] = \frac{1}{u}\frac{du}{dx}$

Derivative of the general logarithmic function $\log_a x$:

To find the derivative of $\log_a x$ for an arbitrary base ($a > 0$, $a \neq 1$), we use the change of base formula for logarithms to express $\log_a x$ in terms of a natural logarithms, as $\log_a x = \frac{\log x}{\log a}$. Then,

$$\frac{d}{dx}(\log_a x) = \frac{d}{dx}\left[\frac{\log x}{\log a}\right]$$

$$= \frac{1}{\log a}\frac{d}{dx}(\log x) = \frac{1}{x \log a}$$

Note: If u is a differentiable function of x and $u > 0$, the formula is as follows.

For $a > 0$ and $a \neq 1$, then $\frac{d}{dx}(\log_a u) = \frac{1}{u \log a} \cdot \frac{du}{dx}$.

Example 5.12.1. Find the differential coefficient of $(\cos x)^{\log x}$.

Let $y = (\cos x)^{\log x}$.

Taking logs, $\log y = \log x . \mathrm{logcosx}$

Differentiating on both sides, we get

$$\frac{1}{y}.\frac{dy}{dx} = \log x.\frac{1}{\cos x}.(-\sin x) + \frac{1}{x}\log\cos x$$

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Therefore $\frac{dy}{dx} = (\cos x)^{\log x}\left[\log x.(-\tan x) + (\log\cos x)/x\right]$

5.13 Summary

Differentiability of a function at a point, Left hand dentine L f '(a) and right hand derivative R f '(a) at a given point x=a, Geometrical meaning of differentiability is discussed in the unit. Examples of differentiability with the help of left hand derivative & right hand derivatives, algebra of derivatives are studied in the unit. Differentiability implies continuity but not the converse. Chain rule, sign of derivatives and their implications for monotonic increasing and decreasing functions are also described. Derivative of exponential and logarithmic functions, derivative of function of the type $[f(x)]^{g(x)}$ is also described.

5.14 Terminal Questions

1. Find $\frac{dy}{dx}$ $when\ y = x^x + x^{\frac{1}{x}}$

[**Hint :** Let u = xx & v = x1/x the y = u +v

Therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} - - - - - (1)$

⇨ log u = x log x

Therefore differentiating $\frac{1}{u}\frac{du}{dx} = x\frac{1}{x} + \log x = 1 + \log x$

Therefore $\frac{du}{dx} = u(1 + x)$

similarly find $\frac{dv}{dx}$]

2. Find $\frac{dy}{dx}$ $when\ y = (1 + \frac{1}{x})^x + x^{(1+\frac{1}{x})}$

[**Hint**: Do as above]

3. If y = (cot x$)^{1/x}$ + (tan x)$^{\cos x}$ then find $\frac{dy}{dx}$

[**Hint**: Do as above]

4. Differentiate tan $^{-1}$ x with respect to log x^2.

[**Hint**:$\frac{d(tan^{-1}x)}{d(logx^2\ \ )} = \frac{\frac{d}{dx}tan\ \ ^{-1}x}{\frac{d}{dx}logx^2} = \frac{\frac{1}{1+x^2}}{\frac{1}{x^2}.2x}\ etc$]

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5. Differentiate x $^{\sin x}$ with respect to (tanx $)^x$.

[**Hint**: Do as above]

6. If $x^y = e^{x-y}$ then prove that $\frac{dy}{dx} = \frac{\log x}{(1+logx)^2}$

[**Hint**: Taking lograthimic on both sides;

y log x = (x-y) (since log e =1)

therefore, $y = \frac{x}{1+\log x}$ $, Now\ find\ \frac{dy}{dx}$]

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UNIT-6

DERIVATIVE OF HYPERBOLIC FUNCTIONS AND SOME SPECIAL FUNCTIONS

Structure

INTRODUCTION

In many of the applications of advance mathematical sciences, we very often come across with the functions involving the combinations of e^x and e^{-x} .These are known as hyperbolic functions. Therefore, hyperbolic functions are nothing more than simple combinations of the exponential functions e^x and e^{-x}. The basic hyperbolic functions are hyperbolic sine denoted by $\sinh x$ and hyperbolic cosine denoted by $\cosh x$. These name suggests that hyperbolic functions are analogs of trigonometric functions and they will have similar properties to trigonometric functions. The first systematic consideration of hyperbolic functions was done by Swiss mathematician Johana Heinrich Lambert in 17^{th} century. Now let's define these functions. Why these functions are called "hyperbolic"?.

For the trigonometric functions, if we substitute $x = \cos\theta$ and $y = \sin\theta$. Then we get $x^2 + y^2 = 1$, this is an equation of an unit circle.

On the same lines, if we substitute $x = \cosh\theta$ and $y = \sinh\theta$. Then we get $x^2 - y^2 = 1$, this is an equation of an hyperbola.

For any real number x, the hyperbolic sine and the hyperbolic cosine functions are defined as the following combination of exponential functions.

The derivatives of the hyperbolic functions follow immediately from their basic definitions as simple combinations of exponential functions. For any real number $'x'$, we define the their derivative results.

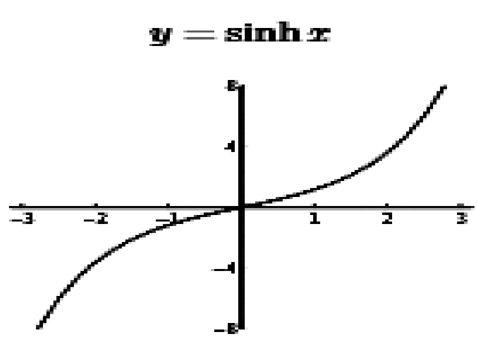
Objectives

After studying this unit you should be able to;

- find the derivative of exponential and logarithmic functions.
- define hyperbolic functions and discuss the existence of their inverse.
- differentiate hyperbolic functions and Inverse hyperbolic functions.
- use the method of logarithmic differentiation for solving some problems.
- differentiate implicit function and also those functions which are defined with the help of a parameter.

6.2 Definition of Hyperbolic functions

Definition 6.2.1: The hyperbolic sine function, written as $\sinh x$, is defined by the relation $\sinh x = \dfrac{e^x - e^{-x}}{2}$. It graph is shown as below



Theorem 6.2.1: Prove that, $\frac{d}{dx}(\sinh x) = \cosh x$

Proof: By the definition, we have $\sinh x = \frac{e^x - e^{-x}}{2}$

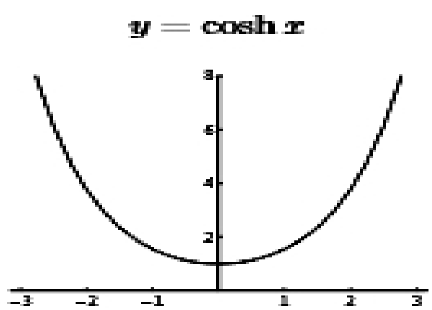
$$\therefore \frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\frac{d}{dx}(e^x - e^{-x})$$

$$= \frac{1}{2}\left[\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right]$$

$$= \frac{1}{2}\left[(e^x) - (-e^{-x})\right] = \frac{e^x + e^{-x}}{2} = \cosh x$$

Definitions 6.2.2: The hyperbolic cosine function, written as $\cosh x$, is defined by the relation $\cosh x = \frac{e^x + e^{-x}}{2}$. Its graph is as shown below,



Theorem 6.2.2: Prove that, $\frac{d}{dx}(\cosh x) = \sinh x$

Proof: By the definition, we have $\cosh x = \frac{e^x + e^{-x}}{2}$

$$\therefore \frac{d}{dx}(\cosh x) = \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)$$

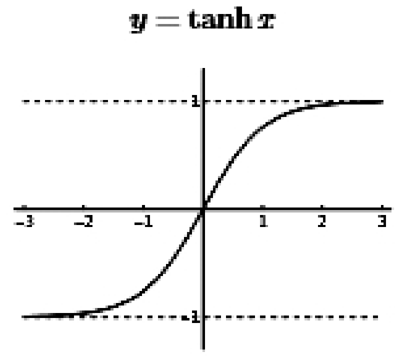
$$= \frac{1}{2}\frac{d}{dx}(e^x + e^{-x})$$

$$= \frac{1}{2}\left[\frac{d}{dx}(e^x) + \frac{d}{dx}(e^{-x})\right]$$

$$= \frac{1}{2}\left[(e^x) + (-e^{-x})\right] = \frac{e^x - e^{-x}}{2} = \sinh x$$

We can define four additional hyperbolic functions from hyperbolic sine and hyperbolic cosine as follows.

Definitions 6.2.3: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$. Its graph is as shown below,



Theorem 6.2.3: Prove that, $\dfrac{d}{dx}(\tanh x) = \sec h^2 x$

Proof: By the definition, we have $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\therefore \frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$$

$$\therefore \frac{d}{dx}(\tanh x) = \frac{(e^x + e^{-x})\frac{d}{dx}(e^x - e^{-x}) - (e^x - e^{-x})\frac{d}{dx}(e^x + e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

$$= \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{(e^x + e^{-x})^2}$$

$$= \frac{4}{(e^x + e^{-x})^2} = \left(\frac{2}{(e^x + e^{-x})}\right)^2 = \sec h^2 x$$

Alternatively,

Consider, $\tanh x = \dfrac{\sinh x}{\cosh x}$

$$\therefore \frac{d}{dx}(\tanh x) = \frac{(\cosh x)\frac{d}{dx}(\sinh x) - (\sinh x)\frac{d}{dx}(\cosh x)}{(\cosh x)^2}$$

$$= \frac{(\cosh x)(\cosh x) - (\sinh x)(\sinh x)}{(\cosh x)^2}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \sec h^2 x$$

Similarly, we can also define $\cos echx$, $\sec hx$ and $\coth x$ as follows

1) $\cos echx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

2) $\cos echx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

3) $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Remark 6.2.1: The derivatives of remaining hyperbolic functions can be proved. Thus, they are given as follows

1) $\frac{d}{dx}(\cos echx) = -\cos echx \coth x$

2) $\frac{d}{dx}(\sec hx) = -\sec hx \tanh x$

3) $\frac{d}{dx}(\coth x) = -\cos ech^2 x$

Remark 6.2.2: Like trigonometric identities, we can have hyperbolic identities also. For this purpose there exists a rule called Osborn's Rule, this rule is to find the formula for hyperbolic functions from the corresponding identity for trigonometric function. "Replace the trigonometric function by the corresponding hyperbolic function, and change the sign of every product of sine terms".

Example 6.2.1: consider the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

To get the hyperbolic identity by using the Osborn's rule as follows.

Step 1: Wtrite down the given trigonometric identity in terms of hyperbolic functions as; $\sinh^2 x + \cosh^2 x = 1$

Step 2: Change the sign of every product of sine terms; $-\sinh^2 x + \cosh^2 x = 1$

Therefore, the required hyperbolic identity is $\cosh^2 x - \sinh^2 x = 1$.

Alternate Method:

Consider,

$$LHS = \cosh^2 x - \sinh^2 x$$

$$= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = \frac{4}{4} = 1 = RHS$$

Example 6.2.2: consider the trigonometric identity $\cos(x+y)=\cos x\cos y-\sin x\sin y$.

To get the hyperbolic identity by using the Osborn's rule as follows.

Step 1: Write down the given trigonometric identity in terms of hyperbolic functions as; $\cosh(x+y)=\cosh x\cosh y-\sinh x\sinh y$

Step 2: Change the sign of every product of sine terms; $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$

Therefore, the required hyperbolic identity is $\cosh(x+y)=\cosh x\cosh y+\sinh x\sinh y$.

Alternate Method:

Consider,

$$LHS=\cosh(x+y)$$
$$=\left(\frac{e^{x+y}+e^{-(x+y)}}{2}\right)$$

$$RHS=\cosh x\cosh y+\sinh x\sinh y$$
$$=\left(\frac{e^{x}+e^{-x}}{2}\right)\left(\frac{e^{y}+e^{-y}}{2}\right)+\left(\frac{e^{x}-e^{-x}}{2}\right)\left(\frac{e^{y}-e^{-y}}{2}\right)$$
$$=\frac{e^{x+y}+e^{x-y}+e^{y-x}+e^{-(x+y)}}{4}+\frac{e^{x+y}-e^{x-y}-e^{y-x}+e^{-(x+y)}}{4}$$
$$=\frac{e^{x+y}+e^{x-y}+e^{y-x}+e^{-(x+y)}+e^{x+y}-e^{x-y}-e^{y-x}+e^{-(x+y)}}{4}$$
$$=\frac{2e^{x+y}+2e^{-(x+y)}}{4}=\frac{e^{x+y}+e^{-(x+y)}}{2}$$

Thus, LHS = RHS

Example 6.2.3: Consider the trigonometric identity $\sin(x-y)=\sin x\cos y-\cos x\sin y$.

To get the hyperbolic identity by using the Osborn's rule as follows.

Step 1: Wtrite down the given trigonometric identity in terms of hyperbolic functions as; $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$

Step 2: Change the sign of every product of sine terms; as there are no such terms to change the sign, therefore identity remains as it is $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$

Therefore, the required hyperbolic identity is $\sinh(x-y)=\sinh x\cosh y-\cosh x\sinh y$.

Alternate Method:

Consider,
$$LHS = \sinh(x-y) = \left(\frac{e^{x-y} - e^{-(x-y)}}{2}\right)$$

$$RHS = \sinh x \cosh y - \cosh x \sinh y$$
$$= \left(\frac{e^{x} - e^{-x}}{2}\right)\left(\frac{e^{y} + e^{-y}}{2}\right) - \left(\frac{e^{x} + e^{-x}}{2}\right)\left(\frac{e^{y} - e^{-y}}{2}\right)$$
$$= \frac{e^{x+y} + e^{x-y} - e^{y-x} - e^{-(x+y)}}{4} - \frac{e^{x+y} - e^{x-y} + e^{y-x} - e^{-(x+y)}}{4}$$

$$\therefore RHS = \frac{e^{x+y} + e^{x-y} - e^{y-x} - e^{-(x+y)} - e^{x+y} + e^{x-y} - e^{y-x} + e^{-(x+y)}}{4}$$
$$= \frac{2e^{x-y} - 2e^{y-x}}{4} = \frac{e^{x-y} - e^{-(x-y)}}{2}$$

Thus, LHS = RHS.

Further, we can notice the following results,

i. $\sinh(-x) = -\sinh x$.

Proof: Since, $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

$$\Rightarrow \sinh(-x) = \frac{e^{-x} - e^{-(-x)}}{2}$$
$$= \frac{e^{-x} - e^{x}}{2} = \frac{-\left(e^{x} - e^{-x}\right)}{2} = -\sinh x$$

ii. $\cosh(-x) = \cosh x$.

Proof: Since, $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

$$\Rightarrow \cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2}$$
$$= \frac{e^{-x} + e^{x}}{2} = \frac{e^{x} + e^{-x}}{2} = \sinh x$$

Similarly, we can prove, $\tanh(-x) = -\tanh x$, $\sec h(-x) = \sec hx$, $\cos ech(-x) = -\cos echx$ and $\coth(-x) = -\coth x$.

6.3. Derivative of Inverse hyperbolic functions

For a function to have an inverse, it must be one-to-one. Just like in case of inverse trigonometric functions, by restricting the domains on which they are one-to-one. The notations we will use for the hyperbolic inverses are $\sinh^{-1} x$, $\cosh^{-1} x$, $\tanh^{-1} x$, $\cos ech^{-1} x$, $\sec h^{-1} x$ and $\coth^{-1} x$. Since the hyperbolic functions are defined as the combinations of exponential functions. It would seem reasonable to expect that their inverses could be expressed in terms of logarithmic functions.

Definition 6.3.1: The Inverse Hyperbolic Sine Function ($\sinh^{-1} x$):

It is defined as $y = \sinh^{-1} x$ iff $\sinh y = x$ with $y \in (-\infty, \infty)$ and $x \in (-\infty, \infty)$.

To obtain an expression for $\sinh^{-1} x$;

Consider,

$$\sinh x = \frac{e^x - e^{-x}}{2} \Rightarrow \sin y = \frac{e^y - e^{-y}}{2}$$

$$\therefore\ x = \frac{e^y - e^{-y}}{2}$$

$$\therefore\ 2x = e^y - e^{-y} \ Or\ 2x = e^y - \frac{1}{e^y}$$

$$2xe^y = e^{2y} - 1$$

$$Let\ \ e^y = z\ then\ 2xz = z^2 - 1\ Or\ z^2 - 2xz - 1 = 0$$

This represents a quadratic equation in z.

$$\therefore\ z = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}$$

Since $z = e^y > 0 \Rightarrow z = x \pm \sqrt{x^2 + 1}$

$$\therefore\ e^y = x + \sqrt{x^2 + 1} \quad Or\ \ y = \log\left[x + \sqrt{x^2 + 1}\right]$$

$$Thus,\ \ y = \sinh^{-1} x = \log\left[x + \sqrt{x^2 + 1}\right]$$

Derivative of Inverse Hyperbolic Sine Function:

Theorem 6.3.1: Prove that $\frac{d}{dx}\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}}; \quad -\infty < x < \infty$.

Proof: Consider $y=\sinh^{-1} x$ then $\sinh y = x$

$$\therefore \frac{d}{dx}\left(\sinh y\right)=\frac{d}{dx}(x)$$

$$\cosh y \cdot \frac{dy}{dx}=1 \;\; Or \;\; \frac{dy}{dx}=\frac{1}{\cosh y}$$

$By\ u\sin g\ \cosh^2 y = 1+\sinh^2 y$

$$\frac{dy}{dx}=\frac{1}{\sqrt{1+\sinh^2 y}}=\frac{1}{\sqrt{1+x^2}}$$

Thus, $\frac{d}{dx}\left(\sinh^{-1} x\right)=\frac{1}{\sqrt{x^2+1}}; \; -\infty < x < \infty$

Alternatively;

We have, $y = sinh^{-1} x = log\left[x^2+\sqrt{x^2+1}\right]$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(sinh^{-1} x\right)=\frac{d}{dx}\left\{log\left[x+\sqrt{x^2+1}\right]\right\}$$

$$=\frac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\frac{d}{dx}\left(x+\sqrt{x^2+1}\right)$$

$$=\frac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\frac{1}{2\sqrt{x^2+1}}\cdot\frac{d}{dx}(x^2+1)\right)$$

$$=\frac{1}{\left(x+\sqrt{x^2+1}\right)}\cdot\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}(\sinh^{-1} x)=\frac{x+\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}}$$

Definition 6.3.2: The Inverse Hyperbolic Cosine Function ($\cosh^{-1} x$ **):**

It is defined as $y=\cosh^{-1} x$ iff $\cosh y = x$ with $y\in[0,\infty)$ and $x\in[1,\infty)$.

To obtain an expression for $\cosh^{-1} x$ **;**

Consider,

$$\cosh x = \frac{e^x + e^{-x}}{2} \Rightarrow \cosh y = \frac{e^y + e^{-y}}{2}$$

$$\therefore\ x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$\therefore\ 2x = e^y + \ e^{-y} Or\ 2x\ = e^y + \frac{1}{e^y}$$

$$2xe^y = e^{2y} + 1$$

$$Let\ \ e^y = z\ then\ 2xz = z^2 + 1\ Or\ z^2 - 2xz + 1 = 0$$

This represents a quadratic equation in z.

$$\therefore\ z = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$$

Since $z = e^y > 0 \Rightarrow z = x \pm \sqrt{x^2 - 1}$

$$\therefore\ e^y = x + \sqrt{x^2 - 1} \quad Or \quad y = \log\left[x + \sqrt{x^2 - 1}\ \right]$$

$$Thus,\ \ y = \cosh^{-1} x = \log\left[x + \sqrt{x^2 - 1}\ \right]$$

<u>Derivative of Inverse Hyperbolic Cosine Function:</u>

Theorem 6.3.2: Prove that $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}};\ \ x > 1$**.**

Proof: Consider $y = \cosh^{-1} x$ then $\cosh y = x$

$$\therefore\ \frac{d}{dx}\left(\cosh y\right) = \frac{d}{dx}(x)$$

$$\sinh y \cdot \frac{dy}{dx} = 1\ \ Or\ \ \frac{dy}{dx} = \frac{1}{\sinh y}$$

$$By\ u\sin g\ \ \cosh^2 y - 1 = \sinh^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

Thus, $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2 - 1}};\ \ x > 1$

Alternatively;

We have, $y = \cosh^{-1} x = \log\left[x + \sqrt{x^2 - 1}\,\right]$

$$\therefore\ \frac{dy}{dx} = \frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{d}{dx}\left\{\log\left[x + \sqrt{x^2 - 1}\,\right]\right\}$$

$$= \frac{1}{\left(x + \sqrt{x^2 - 1}\right)} \cdot \frac{d}{dx}\left(x + \sqrt{x^2 - 1}\,\right)$$

$$= \frac{1}{\left(x + \sqrt{x^2 - 1}\right)} \cdot \left(1 + \frac{1}{2\sqrt{x^2 - 1}} \cdot \frac{d}{dx}(x^2 - 1)\right)$$

$$= \frac{1}{\left(x + \sqrt{x^2 - 1}\right)} \cdot \left(1 + \frac{2x}{2\sqrt{x^2 - 1}}\right)$$

$$\therefore\ \frac{dy}{dx} = \frac{x + \sqrt{x^2 - 1}}{\left(x + \sqrt{x^2 - 1}\right)\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}\ ; \quad x > 1$$

Definition 6.3.3: The Inverse Hyperbolic Tangent Function ($\tanh^{-1} x$ **):**

It is defined as $y = \tanh^{-1} x$ iff $\tanh y = x$ with $y \in [-1, 1]$ and $x \in [-\infty, \infty]$.

To obtain an expression for $\tanh^{-1} x$**;**

Consider,

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

$$\therefore\ x = \tanh y = \frac{e^y - \frac{1}{e^y}}{e^y + \frac{1}{e^y}} = \frac{e^{2y} - 1}{e^{2y} + 1}$$

Let's solve for y , then we obtain

$$(e^{2y} + 1)x = e^{2y} - 1$$

$$\therefore\ (x - 1)e^{2y} = -(x + 1)\ \ Or\ \ e^{2y} = \frac{1 + x}{1 - x}$$

$$\therefore\ 2y = \log\left(\frac{1 + x}{1 - x}\right)\ \ Or\ \ y = \frac{1}{2}\log\left(\frac{1 + x}{1 - x}\right)$$

$$Thus,\ \tanh^{-1} x = \frac{1}{2}\log\left(\frac{1 + x}{1 - x}\right)$$

Derivative of Inverse Hyperbolic Tangent Function:

Theorem 6.3.3: Prove that $\frac{d}{dx}\left(\tanh^{-1} x\right)=\frac{1}{1-x^2}; \quad -1<x<1$**.**

Proof: Consider $y=\tanh^{-1} x$ then $\tanh y = x$

$$\therefore \frac{d}{dx}\left(\tanh y\right)=\frac{d}{dx}(x)$$

$$\sec h^2 y\cdot\frac{dy}{dx}=1 \ Or \ \frac{dy}{dx}=\frac{1}{\sec h^2 y}$$

$$By\ u\sin g,\ 1-\tanh^2 y=\sec h^2 y$$

$$\frac{dy}{dx}=\frac{1}{1-\tanh^2 y}=\frac{1}{1-x^2}$$

Thus, $\frac{d}{dx}\left(\tanh^{-1} x\right)=\frac{1}{1-x^2}; \quad -1<x<1$

Alternatively;

We have, $y=\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\tanh^{-1} x\right)=\frac{d}{dx}\left\{\frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right\}$$

$$=\frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

$$=\frac{1}{2}\cdot\frac{1-x}{1+x}\left(\frac{(1-x)\frac{d}{dx}(1+x)-(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right)$$

$$=\frac{1}{2}\cdot\frac{1-x}{1+x}\left(\frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}\right)$$

$$\therefore \frac{dy}{dx}=\frac{2(1-x)}{2(1+x)(1-x)^2}=\frac{1}{1-x^2}; \quad -1<x<1$$

Definition 6.3.4: The Inverse Hyperbolic Secant Function ($\sec h^{-1} x$**):**

It is defined as $y=\sec h^{-1}x$ iff $\sec hy = x$ with $y\in[1,\infty)$ and $x\in[0,\infty)$

To obtain an expression for $\sec h^{-1}x$**;**

Consider,

$$\sec hx = \frac{2}{e^x + e^{-x}} \Rightarrow \sec hy = \frac{2}{e^y + e^{-y}}$$

$$\therefore\ x = \sec hy = \frac{2}{e^y + e^{-y}}$$

$$\therefore\ x = \frac{2}{e^y + \frac{1}{e^y}}\ Or\ x = \frac{2e^y}{e^{2y} + 1}$$

$$Or\ \ xe^{2y} - 2e^y + x = 0$$

$$Let\ \ z = e^y\ \ then\ xz^2 - 2z + x = 0$$

$$\therefore\ z = \frac{2 \pm \sqrt{4 - 4x^2}}{2x} = \frac{1 \pm \sqrt{1 - x^2}}{x}$$

$$Since\ z = e^y > 0 \Rightarrow\ z = \frac{1 + \sqrt{1 - x^2}}{x}$$

$$\therefore\ e^y = \frac{1 + \sqrt{1 - x^2}}{x}\quad Or\ \ y = \log\left[\frac{1 + \sqrt{1 - x^2}}{x}\right]$$

$$Thus,\ \ y = \sec h^{-1}x = \log\left[\frac{1 + \sqrt{1 - x^2}}{x}\right] = \log\left(1 + \sqrt{1 - x^2}\right) - \log x$$

Derivative of Inverse Hyperbolic Secant Function:

Theorem 6.3.4: Prove that $\frac{d}{dx}\left(\sec h^{-1}x\right) = \frac{-1}{x\sqrt{1 - x^2}};\ 0 < x < 1$**.**

Proof: Consider $y = \sec h^{-1}x$ then $\sec hy = x$

$$\therefore\ \frac{d}{dx}\left(\sec hy\right) = \frac{d}{dx}(x)$$

$$-\sec hy \tanh y \cdot \frac{dy}{dx} = 1\ \ Or\ \ \frac{dy}{dx} = \frac{-1}{\sec hy \tanh y}$$

$$By\ u\sin g\ 1 - \tanh^2 y = \sec h^2 y$$

$$\therefore\ \frac{dy}{dx} = \frac{-1}{\sec hy\sqrt{1 - \sec h^2 y}} = \frac{-1}{x\sqrt{1 - x^2}}$$

Thus, $\frac{d}{dx}\left(\sec h^{-1}x\right) = \frac{-1}{x\sqrt{1 - x^2}};\ 0 < x < 1$

Alternatively;

We have, $y = \sec h^{-1}x = \log\left[\frac{1 + \sqrt{1 - x^2}}{x}\right] = \log\left(1 + \sqrt{1 - x^2}\right) - \log x$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\sec h^{-1}x\right)=\frac{d}{dx}\left\{\log\left[1+\sqrt{1-x^2}\right]-\log x\right\}$$

$$=\frac{1}{\left(1+\sqrt{1-x^2}\right)}\cdot\frac{d}{dx}\left(1+\sqrt{1-x^2}\right)-\frac{1}{x}$$

$$=\frac{1}{\left(1+\sqrt{1-x^2}\right)}\cdot\left(\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2)\right)-\frac{1}{x}$$

$$=\frac{1}{\left(1+\sqrt{1-x^2}\right)}\cdot\left(\frac{-2x}{2\sqrt{1-x^2}}\right)-\frac{1}{x}$$

$$\therefore \frac{dy}{dx}=\frac{-x}{\left(1+\sqrt{1-x^2}\right)\sqrt{1-x^2}}-\frac{1}{x}$$

$$=\frac{-x^2-\left[\sqrt{1-x^2}+(1-x^2)\right]}{\left[\sqrt{1-x^2}+(1-x^2)\right]x}=\frac{-1}{x\sqrt{1-x^2}}\ ;\quad x>1$$

Similarly, we can prove the following two more results.

- $y=\cos ech^{-1}x=\log\left[\frac{1+\sqrt{1+x^2}}{x}\right]=\log\left(1+\sqrt{1+x^2}\right)-\log x$
- $\frac{d}{dx}\left(\cos ech^{-1}x\right)=\frac{-1}{x\sqrt{1+x^2}};\ x\neq 0$
- $\coth^{-1}x=\frac{1}{2}\log\left(\frac{x+1}{x-1}\right)$
- $\frac{d}{dx}\left(\coth^{-1}x\right)=\frac{-1}{x^2-1}=\frac{1}{1-x^2}\ ;\quad -1<x<1$

6.4. Methods of differentiation (Derivative of x^r)

Already we know that, for any rational number ‘ r ’, $\frac{d}{dx}\left(x^r\right)=r\cdot x^{r-1}$.

Now we will see this result, when ‘ r ’ any real number.

Consider $y=x^r$; $x>0$ and for any $r\in\Re$

We can write it as $y=e^{\log(x^r)}=e^{r\log x}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(e^{r\log x}\right)$$

$$=e^{r\log x}\frac{d}{dx}(r\log x)$$

$$=e^{r\log x}\cdot r\cdot\frac{d}{dx}(\log x)=x^r\cdot r\cdot\frac{1}{x}=r\cdot x^{r-1}$$

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Thus, $\frac{d}{dx}\left(x^r\right) = r \cdot x^{r-1}$; for $x > 0$ and $r \in \Re$.

Example6.4.1. Differentiate $x^{\sqrt{2}}$

Do yourself

6.5. Logarithmic Differentiation

This rule provides the method to differentiate the algebraically complicated functions or functions for which the ordinary rules of differentiation do not apply. For example, the problems those involve expressions where the variable is raised to a variable power.

Let's see some illustrations of this method.

Example6.5.1. Find the differential coefficient of $(\sin x)^{\log x}$ **.**

Let $\mathrm{y} = (\sin x)^{\log x}$.

Taking logs, $\log y = \log x . \log \sin x$

Differentiating on both sides, we get

$$\frac{1}{y}.\frac{dy}{dx} = \log x . \frac{1}{\sin x}.\cos x + \frac{1}{x}\log \sin x$$

Therefore $$\frac{dy}{dx} = (\sin x)^{\log x}\left[\log x . \cot x + (\log \sin x)/x\right]$$

Example6.5.2. Find the differential coefficient of $(\sin x)^{\log x}$ **.**

Do your self

<u>Methods of finding differencial coefficient of logarithmic function of $\left[f_1(\mathrm{x})\right]^{f_2(x)}$</u>

Let $y = \left[f_1(\mathrm{x})\right]^{f_2(x)}$

Taking log, $\log \mathrm{y} = f_2(x)\log f_1(x)$

Differentiating w.r.t.x we have

$$\frac{1}{y}\frac{dy}{dx} = f_2(x).\frac{1}{f_1(x)}.f_1^{'}(x) + f_2^{'}(x).\log \mathrm{f}_1(x)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[f_1(\mathrm{x})\right]^{f_2(x)} = \left[f_1(\mathrm{x})\right]^{f_2(x)}\left[f_2(x).\frac{1}{f_1(x)}.f_1^{'}(x) + f_2^{'}(x).\log \mathrm{f}_1(x)\right]$$

$$\frac{d}{dx}\left[f_1(\mathrm{x})\right]^{f_2(x)} = \left[f_1(\mathrm{x})\right]^{f_2(x)-1}.f_2(x)\,f_1^{'}(x) + \left[f_1(\mathrm{x})\right]^{f_2(x)}.\left[f_2^{'}(x).\log \mathrm{f}_1(x)\right]$$

i.e. to differentiate $\left[f_1(\mathrm{x})\right]^{f_2(x)}$,differentiate first as if $f_2(x)$ were constant,then differentiate as if $f_1(x)$ were costant, and add the two results.

Example6.5.3. Find the differential coefficient of 1. $x^{\sin x}$ **,**

2. $\tan x^{\cos x}$ **,**

3. $x^x + (\sin x)^{\log x}$.

6.6. Derivatives of functions defined in terms of a parameter

Some relationships between two quantities or variables are so complicated that sometimes, it is essential to introduce a third quantity or variable in order to make things simple to handle. In mathematics this third quantity or variable is called parameter.i.e., Instead of a function y being defined explicitly in terms of the independent variable x, it is sometimes useful to define both x and y in terms of a third variable t (say), known as a parameter. In other words, Instead of single equation relating two variables x and y , we have two equations, one relating x with the parameter and another relating y with parameter.

The process of differentiating such functions is known as parametric differentiation.

Working Method

$$If \quad x = f(t)\, and\ y = \phi(t) \text{ Then find } \frac{dy}{dx}\, and\ \frac{d^2y}{dx^2}$$

$$Here,\ x = f(t)\, and\ y = \phi(t)$$

$$\text{so} \frac{dx}{dt} = f'(\text{t}), \frac{dy}{dt} = \phi'(\text{t})$$

$$Now\, consider \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{\phi'(\text{t})}{f'(\text{t})} = \frac{\phi'}{f'}$$

$$again\, differentintion \frac{d^2y}{dx^2} = \left[\frac{f'\phi'' - \phi' f''}{(f')^2}\right] . \frac{1}{f'}$$

$$= \frac{f'\phi'' - \phi' f''}{(f')^3}$$

Example6.6.1. Find $\frac{dy}{dx}$ $if\ x = at^2, y = 2at\ where\, t\ is\ parameter.$

We differentiate the given equations w.r.t.,t, and get

$$\frac{dy}{dt} = 2a, \qquad \frac{dx}{dt} = 2at$$

$$Now\, consider \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2a}{2at} = \frac{1}{t}$$

Example6.6.2. If $x = a\cos\theta, y = b\sin\theta$ where θ is parameter.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

We differentiate the given equations w.r.t.,t, and get

$$\frac{dy}{d\theta} = b\cos\theta, \qquad \frac{dx}{d\theta} = -a\sin\theta$$

$$Now\,consider\, \frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$Again\ find\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{b}{a}\cot\theta\cos ec\theta.\frac{1}{-a\sin\theta}$$

Example6.6.3. If $x = \log\phi, y = \phi^2 - 1.\ Find\ \frac{dy}{dx}\, and\ \frac{d^2y}{dx^2}$

$$\frac{dy}{dx} = \frac{dy}{d\phi}\bigg/\frac{dx}{d\phi} = \frac{2\varphi}{1/\phi} = 2\phi^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(2\phi^2) = \frac{d}{d\phi}(2\phi^2).\frac{d\phi}{dx} = 4\phi^2$$

Example6.6.4. If $x = t^8 + 1, y = t^{10} + 1.\ Find\ \frac{d^2y}{dx^2}$

$$Now\,consider\, \frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{10t^9}{8t^7} = \frac{5}{4}t^2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\frac{5}{4}t^2)\frac{dt}{dx} = \frac{5}{16}t^6$$

Example 6.6.5 $If\ x = 2\cos t - \cos 2t, y = 2\sin t - \sin 2t.\ Find\ \frac{dy}{dx}\, and\ \frac{d^2y}{dx^2}$

6.7 Derivatives of Implicit Functions

Instead of expressing y as a direct function of x written as $y = f(x)$, is known as explicit function, if the relation between x and y is expressed implicitly, written as $f(x, y) = 0$ or $f(x, y) = c$. Then also it is possible to find the derivative.

A function which we cannot represent $y = f(x)$ is called implicit function. Suppose $f(x, y) = c$ is a implicit function i.e. $f(x, y) = c$

Then by differentiation of total differential coefficient of $f(x, y) = c$

$$\frac{\partial f}{\partial x}\frac{dx}{dx}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

$$\frac{\partial f}{\partial y}\frac{dy}{dx}=-\frac{\partial f}{\partial x}$$

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

$$=\frac{\textit{differenction of } f \textit{ w.r.t. } x\text{, treated y as cnstant}}{\textit{differenction of } f \textit{ w.r.t. } y\text{, treated } x \text{ as cnstant}}$$

Example 6.7.1.*If* $x^y + y^x = c$. *Find* $\frac{dy}{dx}$

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

$$=\frac{\textit{differenction of } f \textit{ w.r.t. } x\text{, treated y as cnstant}}{\textit{differenction of } f \textit{ w.r.t. } y\text{, treated } x \text{ as cnstant}}$$

$$=-\frac{yx^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$$

Example 6.7.2 Find $\frac{dy}{dx}$ if x and y are related by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Here $f(x,\ y)=ax^2+2hxy+by^2+2gx+2fy+c$

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$$

$$=\frac{\textit{differenction of } f \textit{ w.r.t. } x\text{, treated y as cnstant}}{\textit{differenction of } f \textit{ w.r.t. } y\text{, treated } x \text{ as cnstant}}$$

$$=-\frac{ax+hy+g}{hx+by+f}$$

Example 6.7.3 Find $\frac{dy}{dx}$ if x and y are related by

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

Example 6.7.4 Find $\frac{dy}{dx}$ if x and y are related as

$x^3y^3+x^2y^2+xy+1=0$ $\quad Ans-\frac{3x^2y^3-2xy+y}{3x^3y^2+2xy+x}$

$x^2+y^2=1$ $\quad Ans-\frac{x}{y}$

6.8 Derivatives of Trigonometric Functions

Trigonometry is the branch of Mathematics that has made itself indispensable for other branches of higher mathematics. Otherwise just cannot be processed without encountering trigonometric functions. Further within the specific limit, trigonometric functions give us the inverses as well. The purpose is to explore the rules of finding the derivatives studied by us so far in developing the formulae for derivatives of trigonometric functions and their inverses.

Let us note the important limits of trigonometric functions.

- $\lim_{x\to 0}\sin x = 0$
- $\lim_{x\to 0}\frac{\sin x}{x} = 1$
- $\lim_{x\to 0}\cos x = 1$
- $\lim_{x\to 0}\frac{\tan x}{x} = 1$

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

6.8.1 Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$ for x measured in radians, we combine the limits with the angle sum identity for the sine function:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

If $f(x) = \sin x$, then

$$f^{/}(x) = \lim_{h\to 0}\frac{fx+h) - f(x)}{h} \quad \text{by the definition of derivative}$$

$$= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h\to 0}\left[\sin x \cdot \frac{(\cos h - 1)}{h}\right] + \lim_{h\to 0}\left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \lim_{h\to 0}\left[\frac{\cos h - 1}{h}\right] + \cos x \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right) = 0 + \cos x \cdot 1 = \cos x$$

Thus, $\frac{d}{dx}(\sin x) = \cos x$.

6.8.2 Derivative of the Cosine Function

Consider,

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h}$$ by the definition of derivative

Using $\cos(x+h) = \cos x \cos h - \sin x \sin h$

$$\therefore \frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h}$$

$$= \lim_{h\to 0}\frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h\to 0}\cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0}\sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1 = -\sin x$$

Thus, $\frac{d}{dx}(\cos x) = -\sin x$

6.8.3 The derivatives of the other trigonometric functions

i. $\frac{d}{dx}(\tan x) = \sec^2 x$

ii. $\frac{d}{dx}(\cot x) = -\cos ec^2 x$

iii. $\frac{d}{dx}(\sec x) = \sec x \tan x$

iv. $\frac{d}{dx}(\cos ecx) = -\cos ecx \cot x$

Let's now find the derivative of tanx and remaining are left as an exercise.

6.8.4 Derivative of the Tangent Function

Consider,

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right]$$

$$= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\therefore \frac{d}{dx}(\tan x) = \sec^2 x.$$

6.9. Derivatives of inverse functions

Theorem 6.9.1: (Immediate consequence of inverse function and chain rule of derivatives)

Let f be a function defined in the interval (a, b). If f has an inverse function $f^{-1}(x) = g(x)$ (say), then $g(x)$ is differentiable for all $x \in (a, b)$. Moreover, $g(x) = \frac{1}{f^{/}[g(x)]}$ Or $\left(f^{-1}\right)^{/}(x) = \frac{1}{f^{/}\left[f^{-1}(x)\right]}$.

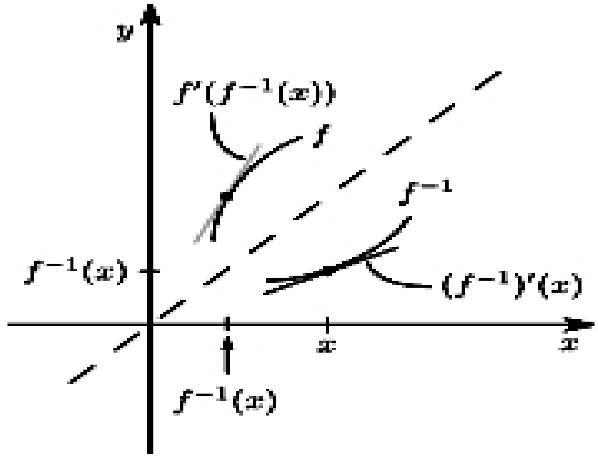
Proof: Since $f(x)$ and $f^{-1}(x) = g(x)$ are inverse functions of each other and x is in the domain of $f^{-1}(x) = g(x)$.

Then, $f[g(x)] = f\left[f^{-1}(x)\right] = x$,

Taking the derivative w.r.t. x on both sides

$$f^{/}[g(x)] \cdot g^{/}(x) = f^{/}\left[f^{-1}(x)\right] \cdot \left(f^{-1}(x)\right)^{/} = 1$$

$$\therefore g^{/}(x) = \frac{1}{f^{/}[g(x)]} \quad Or \quad \left(f^{-1}(x)\right)^{/} = \frac{1}{f^{/}\left[f^{-1}(x)\right]}$$



Theorem 6.9.2: (Inverse Function Theorem)

If f is differentiable and strictly monotonic on an interval, then f^{-1} is differentiable at the corresponding point $y = f(x)$ and $\left(f^{-1}\right)^{/}(y) = \dfrac{1}{f^{/}(x)}$.

This can also be written as $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$.

Equivalently,

Let $f:[a, b] \to \Re$ be a continuous function. If f is differentiable on (a, b) and $f^{/}(x) > 0 \ \forall \ x \in (a, b)$ (Or $f^{/}(x) < 0 \ \forall \ x \in (a, b)$) then f has an inverse function f^{-1} which is differentiable. If $y = f(x)$ then, $\left(f^{-1}\right)^{/}(y) = \dfrac{1}{f^{/}(x)}$ Or equivalently, $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$.

Proof: We begin by noting that if $f^{/}$ is always positive then f is strictly increasing on (a, b). Therefore it is one to one function also. Hence an inverse function f^{-1} exists. To find the derivative of f^{-1} at the point $y = f(x)$. We are required to look at,

$$\lim_{k\to 0}\frac{f^{-1}(y+k) - f^{-1}(y)}{k} = \lim_{k\to 0}\frac{f^{-1}(y+k) - x}{k} \quad \text{since} \quad x = f^{-1}(y).$$

If we take $h = f^{-1}(y+k) - x$ then $x + h = f^{-1}(y+k)$

so that $f(x+h) = (y+k)$ Or $f(x+h) - y = k = f(x+h) - f(x)$.

For $k \neq 0$, we get $h \neq 0$.

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$$\therefore \frac{f^{-1}(y+k)-f^{-1}(y)}{k} = \frac{h}{f(x+h)-f(x)} = \frac{1}{\frac{f(x+h)-f(x)}{h}}$$

This gives rise to; $\lim_{k\to 0}\frac{f^{-1}(y+k)-f^{-1}(y)}{k} = \lim_{h\to 0}\frac{1}{\frac{f(x+h)-f(x)}{h}} = \frac{1}{f^{/}(x)}$

This shows that $f^{-1}(y)$ is differentiable.

$$\therefore \left(f^{-1}\right)^{/}(y) = \frac{1}{f^{/}(x)} \quad \text{Or} \quad \frac{d}{dy}\left(f^{-1}(y)\right) = \frac{1}{\frac{d}{dx}f(x)} \quad \text{Or} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

6.10. Derivatives of inverse trigonometric functions

Already we are familiar of the inverse function theorem. Now we shall see the how this theorem is useful in finding the derivatives of inverse trigonometric functions. Inverse of a function exists if the function is one-to-one and onto (i.e., bijective). Since the trigonometric functions are many one over their domains. We restrict their domains and codomains in order to make them one-to-one and onto and then find their inverses.

Theorem 6.10.1: Derivative of Inverse Sine function ($\sin^{-1}x$):

In order to define the inverse sine function, we will restrict its domain to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. The function $f(x) = \sin x$ is an increasing function in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore $f(x) = \sin x$ is one-to-one and consequently it has an inverse written as $f^{-1}(x) = \sin^{-1} x$. This function is called as inverse sine function with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Proof: By inverse function theorem, we have

$$\left(f^{-1}\right)^{/}(y) = \frac{1}{f^{/}(x)}$$

$$i.e., \frac{d}{dy}\left[f^{-1}(y)\right] = \frac{1}{\frac{d}{dx}\left(f(x)\right)} \Rightarrow \frac{d}{dy}\left(\sin^{-1} y\right) = \frac{1}{\frac{d}{dx}\left(\sin x\right)}$$

$$Or\ \frac{d}{dy}\left(\sin^{-1} y\right) = \frac{1}{\cos x}$$

Using the identity, $\sin^2 x + \cos^2 x = 1 \; Or \; \cos x = \sqrt{1-\sin^2 x}$

$$\therefore \frac{d}{dy}\left(\sin^{-1} y\right) = \frac{1}{\sqrt{1-\sin^2 x}} = \frac{1}{\sqrt{1-y^2}}$$

Thus, $\frac{d}{dx}\left(\sin^{-1} x\right) == \frac{1}{\sqrt{1-x^2}}$

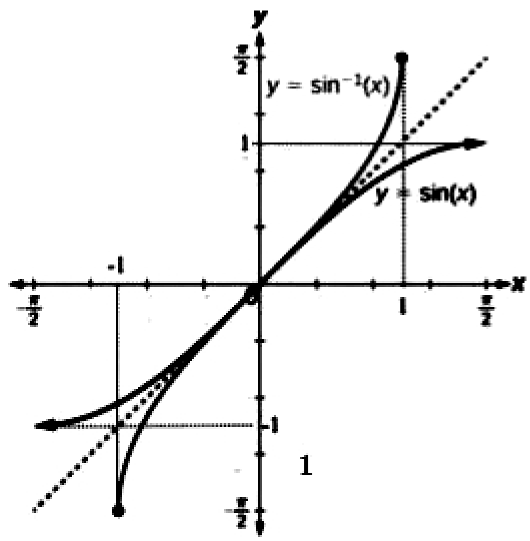
Remark 6.10.1: **1)** $\sin^{-1} x \neq (\sin x)^{-1}$. Where as , $(\sin x)^{-1} = \frac{1}{\sin x}$ and

$$\sin x^{-1} = \sin \frac{1}{x}$$

2) $\sin(\sin^{-1} x) = x \;\; \forall \;\; x \in [-1, 1]$

3) $\sin^{-1}(\sin x) = x \;\; \forall \;\; x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

4) $y = \sin^{-1} x$ if and only if $x = \sin y$



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Theorem 6.10.2: Derivative of Inverse Cosine function ($\cos^{-1}$x)

In order to define the inverse cosine function, we will restrict the function $f(x) = \cos x$ over the interval $[0, \pi]$ as it is always decreasing in this interval. Therefore $f(x)$ is ono-to –one function. Hence its inverse is written as $f^{-1}(x) = \cos^{-1} x$. This is called as inverse cosine function.

Proof: By inverse function theorem, we have

$$\left(f^{-1}\right)^{/}(y) = \frac{1}{f^{/}(x)}$$

$$i.e., \frac{d}{dy}\left[f^{-1}(y)\right] = \frac{1}{\frac{d}{dx}\left(f(x)\right)} \Rightarrow \frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{\frac{d}{dx}\left(\cos x\right)}$$

$$Or \; \frac{d}{dy}\left(\cos^{-1} y\right) = \frac{1}{-\sin x}$$

Using the identity, $\sin^2 x + \cos^2 x = 1 \; Or \; \sin x = \sqrt{1-\cos^2 x}$

$$\therefore \frac{d}{dy}\left(\cos^{-1} y\right)=\frac{-1}{\sqrt{1-\cos^2 x}}=\frac{-1}{\sqrt{1-y^2}}$$

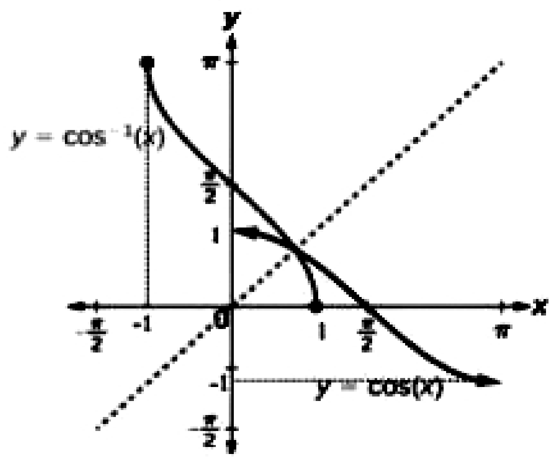
Thus, $\frac{d}{dx}\left(\cos^{-1} x\right)==\frac{-1}{\sqrt{1-x^2}}$

Remark 6.10.2: 1) $\cos^{-1} x \neq (\cos x)^{-1}$. Where as , $(\cos x)^{-1}=\frac{1}{\cos x}$ and $\cos x^{-1}=\cos\frac{1}{x}$

2) $\cos(\cos^{-1} x)=x \;\; \forall \; x\in[-1, 1]$

3) $\cos^{-1}(\cos x)=x \;\; \forall \; x\in[-0, \pi]$

4) $y=\cos^{-1} x$ if and only if $x=\cos y$



Theorem 6.10.3: Derivative of Inverse Tangent function ($\tan^{-1}x$)

In order to define the inverse tangent function, we will restrict the function $f(x)=\tan x$ over the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ as it is increasing in this interval. Therefore $f(x)$ is one-to –one function. Hence its inverse is written as $f^{-1}(x)=\tan^{-1} x$. This is called as inverse tangent function.

Proof: By inverse function theorem, we have

$$\left(f^{-1}\right)^{/}(y)=\frac{1}{f^{/}(x)}$$

$$i.e., \frac{d}{dy}\left[f^{-1}(y)\right]=\frac{1}{\frac{d}{dx}\left(f(x)\right)} \Rightarrow \frac{d}{dy}\left(\tan^{-1} y\right)=\frac{1}{\frac{d}{dx}\left(\tan x\right)}$$

$$Or \;\; \frac{d}{dy}\left(\tan^{-1} y\right)=\frac{1}{\sec^2 x}$$

Using the identity, $1+\tan^2 x=\sec^2 x$

$$\therefore \frac{d}{dy}\left(\tan^{-1} y\right) = \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

Thus, $\frac{d}{dx}\left(\tan^{-1} x\right) == \frac{1}{1+x^2}$

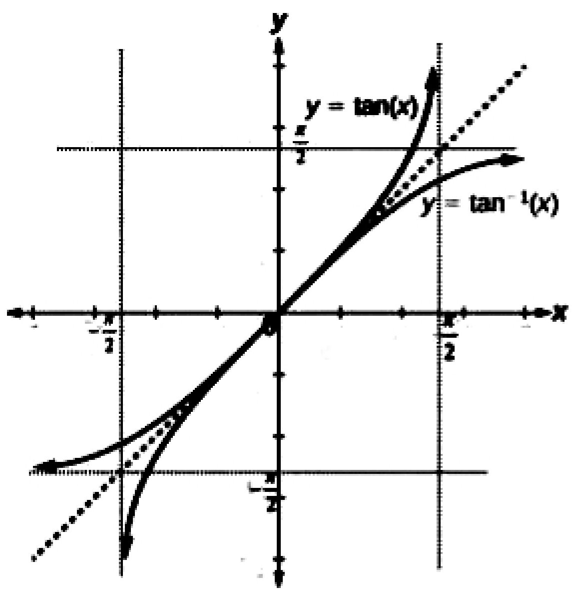
Remark 6.10.3: **1)** $\tan^{-1} x \neq (\tan x)^{-1}$. Where as , $(\tan x)^{-1} = \frac{1}{\tan x}$ and

$$\tan x^{-1} = \tan\frac{1}{x}$$

2) $\tan(\tan^{-1} x) = x \quad \forall\ x$

3) $\tan^{-1}(\tan x) = x \quad \forall\ x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

4) $y = \tan^{-1} x$ if and only if $x = \tan y$



Theorem 6.10.4: Derivative of Inverse Secant function ($\sec^{-1}$x):

In order to define the inverse cosine function, we will restrict the function $f(x) = \sec x$ over the domain $\Re - (-1, 1)$ and the range $[0, \pi] - \frac{\pi}{2}$. Therefore $f(x)$ is ono-to –one function. Hence its inverse is written as $f^{-1}(x) = \sec^{-1} x$. This is called as inverse secant function. Consider,

$$y = \sec^{-1} x \quad Or \quad \sec y = x$$

$$\therefore \frac{1}{\cos y} = x \quad Or \quad \cos y = \frac{1}{x}$$

$$\therefore\ y = \cos^{-1}\left(\frac{1}{x}\right)$$

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Using the result $\dfrac{d}{dx}\left(\cos^{-1} x\right) == \dfrac{-1}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}\left(\frac{1}{x}\right)\right] == \frac{-1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{-x}{\sqrt{x^2-1}} \cdot \left(\frac{-1}{x^2}\right) = \frac{1}{x\sqrt{x^2-1}}$$

$$\therefore \frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \quad for \quad x > 1$$

Theorem 6.10.5: Derivative of Inverse Cosecant function (cosec^{-1}x):

In order to define the inverse cosine function, we will restrict the function $f(x) = \cos ecx$ over the domain $R-(-1, 1)$ and the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$-{0} .Therefore $f(x)$ is ono-to-one function. Hence its inverse is written as $f^{-1}(x) = \cos ec^{-1} x$. This is called as inverse cosecant function. Consider,

$$y = \cos ec^{-1} x \quad Or \quad \cos ecy = x$$

$$\therefore \frac{1}{\sin y} = x \quad Or \quad \sin y = \frac{1}{x}$$

$$\therefore \; y = \sin^{-1}\left(\frac{1}{x}\right)$$

Using the result $\dfrac{d}{dx}\left(\sin^{-1} x\right) == \dfrac{1}{\sqrt{1-x^2}}$

$$\therefore \frac{dy}{dx} = \frac{d}{dx}\left[\sin^{-1}\left(\frac{1}{x}\right)\right] == \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{x}{\sqrt{x^2-1}} \cdot \left(\frac{-1}{x^2}\right) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\therefore \frac{d}{dx}\left(\cos ec^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}} \quad for \quad x > 1$$

Theorem 6.10.6: Derivative of Inverse Cotangent function (cot^{-1}x):

In order to define the inverse cotangent function, we will restrict the function $f(x) = \cot x$ over the interval $[0, \pi]$. Therefore $f(x)$ is one-to –

one function. Hence its inverse is written as $f^{-1}(x) = \cot^{-1} x$. This is called as inverse cotangent function.

Proof: By inverse function theorem, we have

$$\left(f^{-1}\right)^{/}(y) = \frac{1}{f^{/}(x)}$$

$$i.e., \frac{d}{dy}\left[f^{-1}(y)\right] = \frac{1}{\frac{d}{dx}\left(f(x)\right)} \Rightarrow \frac{d}{dy}\left(\cot^{-1} y\right) = \frac{1}{\frac{d}{dx}\left(\cot x\right)}$$

$$Or\ \frac{d}{dy}\left(\cot^{-1} y\right) = \frac{1}{-\operatorname{cosec}^2 x}$$

Using the identity, $1+\cot^2 x = \operatorname{cosec}^2 x$

$$\therefore \frac{d}{dy}\left(\cot^{-1} y\right) = \frac{1}{-\left(1+\cot^2 x\right)} = \frac{-1}{1+y^2}$$

Thus, $\frac{d}{dx}\left(\cot^{-1} x\right) == \frac{-1}{1+x^2}$

List of Derivatives of Inverse Trigonometric Functions:

- $\frac{d}{dx}\left(\sin^{-1} x\right) == \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\cos^{-1} x\right) == \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\left(\tan^{-1} x\right) == \frac{1}{1+x^2}$
- $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \quad for \quad x > 1$
- $\frac{d}{dx}\left(\operatorname{cosec}^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}} \quad for \quad x > 1$
- $\frac{d}{dx}\left(\cot^{-1} x\right) == \frac{-1}{1+x^2}$

6.11. Use of transformations

We can observe by considering some illustrations here, how the making use of suitable transformations reduces the process of finding the derivatives to simplest form.

1. x^2+a^2 , $\dfrac{1}{x^2+a^2}$

   $\sqrt{x^2+a^2}$, $\dfrac{1}{\sqrt{x^2+a^2}}$

   Putting $x = a\tan\theta$ *or* $x = a\cot\theta$

2. x^2-a^2 , $\dfrac{1}{x^2-a^2}$

   $\sqrt{x^2-a^2}$, $\dfrac{1}{\sqrt{x^2-a^2}}$

   Putting $x = a\sec\theta$ *or* $x = a\,\text{cosec}\,\theta$

3. a^2-x^2 $\dfrac{1}{a^2-x^2}$

   $\sqrt{a^2-x^2}$ $\dfrac{1}{\sqrt{a^2-x^2}}$

   Putting $x = a\sin\theta$ *or* $x = a\cos\theta$

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Example6.11.1 *If* $y=\tan^{-1}\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$, find $\dfrac{dy}{dx}$

putting $x^2=\cos 2\theta$

$$y=\tan^{-1}\frac{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}$$

$$=\tan^{-1}\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}$$

$$=\tan^{-1}\frac{\sqrt{2}\cos\theta(1+\tan\theta)}{\sqrt{2}\cos\theta(1-\tan\theta)}$$

$$=\tan^{-1}\tan(\pi/4+\theta)$$

$$y=\pi/4+\theta$$

$\sin ce$ $x^2=\cos 2\theta$ $so\ 2\theta=\cos^{-1}x^2 and$ $\theta=\dfrac{1}{2}\cos^{-1}x^2$

Therefore $y=\pi/4+\dfrac{1}{2}\cos^{-1}x^2$

Now differentiating , we get $\dfrac{dy}{dx}=\dfrac{1}{2}\left[-\dfrac{1}{\sqrt{1-x^4}}.2x\right]$

$$\frac{dy}{dx}=-\frac{x}{\sqrt{1-x^4}}$$

Example6.11.2 *If* $y=\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$, find $\frac{dy}{dx}$

$$Ans\ \frac{dy}{dx}=\frac{x}{\sqrt{1-x^4}}$$

Example6.11.3 *If* $y=\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, find $\frac{dy}{dx}$

$$y=\tan^{-1}\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$$

putting $x=\cos 2\theta$ *then*

$$y=\tan^{-1}\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}$$

$$=\tan^{-1}\frac{1-\tan\theta}{1+\tan\theta}$$

$$=\tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$$

$$y=\frac{\pi}{4}-\theta$$

$$y=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x \quad \text{since} \quad x=\cos 2\theta$$

$$\frac{dy}{dx}=\frac{1}{2\sqrt{1-x^2}}$$

6.12 Summary

Definition of hyperbolic function and the differentiation, logarithmic differentiation, differentiation of a function defined in parametric form x=f(t) ,y=g(t), differentiation of an implicit function, differentiation of trignometrical function, derivative of inverse function f^{-1}, derivative of inverse trignometrical function, differentiation by using transformation in polar coordinates i.e. taking x=rcosθ & y=r sinθ is discussed and studied in this unit.

6.13 Terminal Questions

Find the derivatives of the fallowing functions using suitable transformations

1) $\sin^{-1}(3x-4x^3)$
2) $\cos^{-1}(4x^3-3x)$
3) $\cos^{-1}(1-2x^2)$
4) $\sin^{-1}\dfrac{2x}{1+x^2}$
5) $\cos^{-1}\dfrac{1-x^2}{1+x^2}$
6) $\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$
7) If x = a(t – sin t), y = a(1- cos t) then find $\frac{dy}{dx}$
8) Find $\frac{dy}{dx}$ if $x = a\cos^3 t$, $y = a\sin^3 t$

Answers to Selected Terminal Questions
[Hint for Q1): Put x = sinθ, then $y = \sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1}x$] rest step do your self
[Hint for Q1): Put x = cosθ, then $y = \cos^{-1}(\cos 3\theta) = 3\theta = 3\cos^{-1}x$] rest step your self
[Hint for Q4): Put x = tan θ
[Hint for Q5): Put x = tan θ
[Hint for Q6): Put x = tan θ

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UNIT-7

SUCCESSIVE DIFFERENTIATION

Structure

7.1 Introduction

Derivative is one of most important idea of differential calculus which measures rate of change of variable. Derivative is very useful in engineering, science, medicine, economic and computer science. The higher order derivatives of a given function used in Taylor's Theorem by which we can express any differentiable function in power of series form.

In this unit we will introduce second, third and higher order derivatives. Then we will discuss Leibnitz's Theorem. We will also introduce Taylor's Series and Maclaurin's Series.

Objectives:

After reading this unit you should be able to;

- Calculate higher order derivatives of a given function.
- Use maxima and minima of functions.
- Use increasing and decreasing functions.

- Use the Leibnitz's formula to find the n[th] order derivatives of products of functions.
- Use in curve tracing for concavity and convexity of curves.
- Calculate velocity, acceleration, rate of change of temperature, curvature of curves etc.

7.2 Second and third order Derivatives

Let us consider $y = f(x)$ be a function of x, then $\frac{dy}{dx} = \frac{d}{dx} f(x)$ is called the first derivative of y with respect to x. If this derivative is again differentiable, then its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called the second derivative of y with respect to x and is denoted by $\frac{d^2y}{dx^2}$. Similarly, if $\frac{d^2y}{dx^2}$ is differentiable then its derivative $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ is called the third derivative of y and is denoted by $\frac{d^3y}{dx^3}$ and so on.

The different notations are used for the successive derivatives of y with respect to x. They are as follows

$$Let \qquad y = f(x) \quad \text{then}$$

$$\frac{dy}{dx} = \frac{d}{dx} f(x) = \mathrm{Dy} = \mathrm{f'(x)} = \mathrm{y'} = \mathrm{y_1} \quad where \quad D = \frac{d}{dx}$$

$$so \ \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y = y'' = f''(\mathrm{x}) = \mathrm{y_2}$$

$$Similarly \quad \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3y = y''' = f'''(\mathrm{x}) = \mathrm{y_3}$$

this shows that the process of differentiating given again and again in succession is called successive differentiation

Example7.2.1: If $y = a\cos(\log x) + b\sin(\log x)$, show

$$\text{That } x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

We have $y = a\cos(\log x) + b\sin(\log x)$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = [-a\sin(\log x)].\frac{1}{x} + [b\cos(\log x)].\frac{1}{x}$$

$$x\frac{dy}{dx} = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t. x, we get

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = [-a\cos(\log x)].\frac{1}{x} + [-b\sin(\log x).\frac{1}{x}$$

$$= -\frac{1}{x}[\cos(\log x) + b\sin(\log x)] = -\frac{y}{x}$$

Hence $\quad x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

Example7.2.2: If $y = 6x^3 + 11x + 2 \quad Find\ \frac{dy}{dx}$

Here $y = 6x^3 + 11x + 2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 18x^2 + 11$$

Again differentiating w.r.t. x, we have

$$\frac{d^2y}{dx^2} = 36\,\mathrm{x}$$

$$\frac{d^3y}{dx^3} = 36$$

Example 7.2.3: If $\mathrm{x}\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that

$$\frac{d^2y}{dx^2} = -\frac{a}{(1-x^2)^{3/2}}$$

We have $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ ………. (1)

Let $x = \cos\alpha$ and $y = \cos\beta$, then (1) becomes

$$\cos\alpha\sqrt{1-\cos^2\beta} + \cos\beta\sqrt{1-\cos^2\alpha} = a$$

or, $\cos\alpha \sin\beta + \cos\beta\sin\alpha = a$ or, $\sin(\alpha + \beta) = a$

or, $\alpha + \beta = \sin^{-1}a$ or $\cos^{-1}x + \cos^{-1}y = \sin^{-1}a$

Differentiating w.r.t., x, $\frac{-1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \, dy\!\!\diagup\!\! dx = 0$ or.

$$\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \qquad (2)$$

Differentiating again w.r.t.x, we get

$$\frac{d^2y}{dx^2} = \frac{\sqrt{1-x^2}.\frac{1}{2\sqrt{1-y^2}}(-2y)\frac{dy}{dx} - \sqrt{1-y^2}.\frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$= -\frac{-\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}.y\left(-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right) + -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.x}{1-x^2}$$

$$= -\frac{y\sqrt{1-x^2} + x\sqrt{1-y^2}}{(1-x^2)^{3/2}} \qquad \text{[by (2)]}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}$$

7.3 n^th^ Order Derivatives

Definition and Notation: If y be a function of x, its differential coefficient dy/dx will be in general, a function of x which can be differentiated. The differential coefficient of dy/dx is called *the second differential coefficient* of y. Similarly, the differential coefficient of the second differential coefficient is called *the third differential coefficient*, and so on. The successive differential coefficients of y are denoted by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$.........,

then n^{th} differential coefficient *of* y being $\frac{d^ny}{dx^n}$

Alternative methods of writing the nth differential coefficient are $\left(\frac{d}{dx}\right)^n y, D^n y, y_n, \frac{d^ny}{dx^n}, y^{(n)}$

In the last case, the first, second, third etc, differential coefficients would be written as y_1, y_2, y_3, etc. The value of a differential coefficient at $x = a$ is usually indicated by adding a suffix; thus $(y_n)_{x=a}$ or $(y_n)_a$. If $y = f(x)$, the same thing can also be indicated by $f^{(n)}$ (a).

7.3.1 Some Standard Results of the n^{th} derivative

(1) If $y = e^{ax}$,

then $y_1 = ae^{ax}$,

$y_2 = a^2e^{ax}$,

$y_3 = a^3e^{ax}$, etc.

In general, $\boldsymbol{D^n e^{ax} = a^n e^{ax}}$. Or $y_n = \boldsymbol{a^n e^{ax}}$

(2) If $y = (ax + b)^m$,

then $y_1 = m.\, a\, (ax + b)^{m-1}$.

$Y_2 = a^2 m(m-1).\, (ax + b)^{m-2}$

$y_3 = a^3 m(m-1)(m-2).\, (ax + b)^{m-3}$

Hence

$$D^n (ax+b)^m = m(m-1)(m-2)......(m-n+1)a^m (ax+b)^{m-n}$$

In particular if m = n

$$i.e.\quad y = (ax+b)^n$$

$$D^n (ax+b)^n = n!a^n$$

$$if\ \ a = 1, b = 0\ we\ get\ y = x^n$$

$$y = n!$$

If m is a positive integer, the $(m+1)$th and all the successive differential coefficients of $(ax + b)^m$ would be zero.

(3) If $y = a^x$ to find the n^{th} differential co-efficient.

$$y = a^x$$

$$then\ \ y_1 = a^x \log_e a,$$

$$y_2 = a^x (\log_e a)^2$$

$$y_n = a^x (\log_e a)^n$$

(4) If $y = \log(ax+b), then \quad y_1 = a(ax+b)^{-1}$

$$y_2 = (-1)a^2(ax+b)^{-2},$$

$$y_3 = (-1)^2 2!a^3 (ax+b)^{-3},$$

$$y_n = (-1)(-2)(-3)........-(n-1)(ax+b)^{-n} a^n$$

In general, $y_n = \mathbf{D^n \log(ax+b) = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}}$

Cor. $if\ a=1,\ b=0,\ \ y=\log x$

$then\ y_{n=}\dfrac{\mathbf{(-1)^{n-1}(n-1)!}}{x^n}$

(5) If *y* = sin (*ax* + *b*) then

$$y_1 = a\cos(ax+b) = a\sin\left(ax+b+\frac{1}{2}\pi\right),$$

$$y_2 = a^2\cos\left(ax+b+\frac{1}{2}\pi\right) = a^2\sin(ax+b+\pi),$$

$$y_3 = a^3\sin\left(ax+b+\frac{3}{2}\pi\right);\text{etc.}$$

In general, $\mathbf{D^n\sin(ax+b)} = y_n = \mathbf{a^n\sin\left(ax+b+\frac{1}{2}n\pi\right)}$

(6) To find the n^th^ differential co-efficient of cos(ax+b)

If y = cos(ax+b) then

$$y_1 = -a\sin(ax+b) = a\cos\left(ax+b+\frac{1}{2}\pi\right),$$

$$y_2 = -a^2\sin\left(ax+b+\frac{1}{2}\pi\right) = a^2\cos(ax+b+\pi),$$

$$y_3 = a^3\cos\left(ax+b+\frac{3}{2}\pi\right);\text{etc.}$$

In general ;

$$\mathbf{D}^n\cos(ax+b) = y_n = a^n\cos\left(ax+b+\frac{1}{2}n\pi\right)$$

> **Note:** Putting *a* = 1 and *b* = 0, we have $D^n\sin x = \sin\left(x+\frac{1}{2}n\pi\right)$,
>
> And $D^n\cos x = \cos\left(x+\frac{1}{2}n\pi\right)$

(7) To find the n^th^ differential co-efficient of $y\ = 1/(ax+b) = (ax+b)^{-1}$ where $x \neq -\dfrac{b}{a}$

Then $y_1 = (-1)(ax+b)^{-2}a$

$$y_2 = (-1)(-2)(ax+b)^{-3}a^2 = (-1)^2 2!(ax+b)^{-3}a^2$$
$$y_3 = (-1)(-2)(-3)(ax+b)^{-4}a^3 = (-1)^3 3!(ax+b)^{-4}a^3$$
$$similarly\ \ y_n = (-1)(-2)(-3)......(-n)(ax+b)^{-(n+1)}a^n = \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}$$

(8) If $y = e^{ax}$ sin (bx + c), to find y_n

then $y_1 = e^{ax}b\cos(bx + c) + ae^{ax}\sin(bx + c)$.

$y_1 = e^{ax(}b\cos(bx + c) + a\sin(bx + c))$.

$$Putting\ \ a = rcos\alpha\ \ and\ \ b = rsin\alpha,$$
$$\mathrm{r}^2 = \mathrm{a}^2 + \mathrm{b}^2 \to r = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}$$
$$\tan\alpha = \frac{b}{a} \to \alpha = \tan^{-1}\frac{b}{a}$$
$$y_1 = e^{ax}\left[r\cos\alpha.sin\ (bx\ +\ c) + \mathrm{r}\sin\alpha\cos(\mathrm{bx}+\mathrm{c})\right]$$
$$= re^{ax}\left[\cos\alpha.sin\ (bx\ +\ c) + \sin\alpha\cos(\mathrm{bx}+\mathrm{c})\right]$$
$$= \sqrt{\mathrm{a}^2+\mathrm{b}^2}\ e^{ax}\left[sin\ (bx\ +\ c+\alpha)\right]$$
$$we\ have\quad y_1 = re^{ax}sin\ (bx\ +\ c+\alpha).$$
$$Similarly\quad y_2 = r^2e^{ax}sin\ (bx\ +\ c+2\alpha);\ etc.\ In\ general,$$

Putting $a = r\cos\phi$ and $b = r\sin\phi$, we have $y_1 = re^{ax}\sin(bx + c + \phi)$.

Similarly $y_2 = r^2e^{ax}\sin(bx + c + 2\phi)$; etc. In general

$$D^n\{e^{ax}\sin(bx+c)\} = r^n e^{ax}\sin(bx+c+n\alpha),\ \text{where}$$

$$r = (a^2+b^2)^{1/2},\ \text{and}\ \alpha = \tan^{-1}(b/a)$$

(9) To find the n[th] differential co-efficient of

$$y_n(e^{ax}\cos(bx+c)) = D^n\{e^{ax}\cos(bx+c)\}$$

$$If\ \ y = e^{ax}\cos(bx+c))$$
$$\text{then } \mathrm{y}_1 = ae^{ax}\cos(bx+c) - \mathrm{b}\,e^{ax}\sin(bx+c)$$
$$= e^{ax}\left[\mathrm{a}\cos(bx+c) - \mathrm{bsin}(\mathrm{bx}+\mathrm{c})\right]$$

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Putting $a = r\cos\alpha$ *and* $b = r\sin\alpha$,

$$r^2 = a^2 + b^2 \rightarrow r = \sqrt{a^2+b^2}$$

$$\tan\alpha = \frac{b}{a} \rightarrow \alpha = \tan^{-1}\frac{b}{a}$$

$$y_1 = e^{ax}\left[r\cos\alpha.\cos\ (bx\ +\ c) - r\sin\alpha\sin(bx+c)\right]$$

$$= re^{ax}\left[\cos\alpha.\cos\ (bx\ +\ c) + \sin\alpha\sin(bx+c)\right]$$

$$= \sqrt{a^2+b^2}\ e^{ax}\left[\cos(bx\ +\ c+\alpha)\right]$$

Thus y_1 is obtained from y on multiplying it by the constant r and increasing the angle by the constant α repeating the same rule to y_1, we have

$$y_2 = r^2 e^{ax} \cos\ (bx\ +\ c + 2\alpha\ );$$

$$y_3 = r^3 e^{ax} \sin\ (bx\ +\ c + 2\alpha\ );$$

Similarly

$$y_n(e^{ax}\cos(bx+c)) = D^n\{e^{ax}\cos(bx+c)\} = r^n e^{ax}\cos(bx+c+n\alpha).$$

Example 7.3.1: Find the n^{th} differential coefficient of $\tan^{-1}(x/a)$.

If $y = \tan^{-1}(x/a)$, then $y_1 = a/(a^2+x^2)$

Now, $$\frac{1}{a^2+x^2} = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ai}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

Therefore, $$y_n = \frac{a(-1)^{n-1}(n-1)!}{2ia}\left\{\frac{1}{(x-ia)^n} - \frac{1}{(x+ia)^n}\right\}.$$

Put $x = r\cos\phi$, $a = r\sin\phi$; then

$$y_n = \frac{1}{2}(-1)^n(n-1)!ir^{-n}\{(\cos\phi - i\sin\phi)^{-n} - (\cos\phi + i\sin\phi)^{-n}\}$$

$$= \frac{1}{2}(-1)^n(n-1)!ir^{-n}\{(\cos n\phi - i\sin n\phi) - (\cos n\phi + i\sin n\phi)\}$$

$$= \frac{1}{2}(-1)^{n+1}(n-1)!ir^{-n}\sin n\phi$$

But $\quad r^{-n} = a^{-n}\sin^n\phi$ [since $a = r\sin\phi$]

Hence $D^n\tan^{-1}(x/a) = (-1)^{n-1}(n-1)!a^{-n}\sin^n\phi\sin n\phi$,

where $\phi = \tan^{-1}(a/x)$.

Example 7.3.2: Find the *n*the differential coefficient of $y = \cos x \cos 2x \cos 3x$

Hence, $y = \cos x \cos 2x \cos 3x$

$$= \tfrac{1}{2}(2\cos 3x.\cos x).\cos 2x = \tfrac{1}{2}(\cos 4x + \cos 2x).\cos 2x$$

$$= \tfrac{1}{2}(2\cos 4xos 2x + 2\cos^2 2x) = \tfrac{1}{4}(2\cos 4x \cos 2x + 2\cos^2 2x)$$

$$= \tfrac{1}{4}(\cos 6x \cos 2x + 1\cos 4x) \text{ i. e. } y = \tfrac{1}{4}(\cos 6x + \cos 4x + \cos 2x + 1)$$

Hence, $$y_n = \frac{1}{4}\left[6^n \cos\left(6x + n\frac{\pi}{2}\right) + 4^n \cos\left(4x + n\frac{\pi}{2}\right) + 2^n \cos\left(2x + n\frac{\pi}{2}\right)\right]$$

Check your progress

1. Find the second differential coefficients of

$$x^4 e^{5x}, e^{\sin x^2}, \sin(\cos x), x^3 \tan^{-1} x^2, \tan e^{5x}$$

2. If $y = A\sin mx + B\cos mx$, prove that, $y_2 + m^2 y = 0$.

3. If $y = e^{ax} \sin bx$, prove that $y_2 - 2ay_1 + (a^2 + b^2)\, y = 0$.

4. $y = e^{ex+b}$

5. $y = \sin^3 x$

6. $y = \cos^4 x$.

7. $y = e^{ax} \cos^2 bx$.

8. If $y = \tan^{-1} x$,

7.4 Leibnitz's Theorem:

This theorem is useful for finding the n^{th} differential coefficient of product of two functions. This theorem states that if u and v be two functions of x, then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0 u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \ldots\ldots\ldots\ldots {}^nC_r u_{n-r} v_r + \ldots\ldots\ldots + {}^nC_n u v_n.$$

or

$$D^n(uv) = (D^n u).\, v + {}^nC_1 D^{n-1} u.\, Dv + {}^nC_2 D^{n-2} u.\, D^2 v + \ldots.. + {}^nC_r D^{n-r} u.\, D^r v + \ldots. + u.\, D^n v.$$

Remark Theorem is true for all positive integral values of n.

Proof: This theorem will be proved by induction method, by actual differentiation we know that

$$\frac{d(\text{uv})}{dx} = uv_1 + u_1 v$$

$$\frac{d^2}{dx^2}(\text{uv}) = \frac{d}{dx}(uv_1 + u_1 v)$$

$$= uv_2 + 2u_1 v_1 + u_2 v$$

$$= uv_2 + {}^2C_1 u_1 v_1 + u_2 v$$

Thus, the theorem is true for n = 1, 2.

Now Let us assume that the theorem is true for n = m.

$$\frac{d^m}{dx^m}(uv) = {}^mC_0 u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \ldots\ldots\ldots\ldots {}^mC_r u_{m-r} v_r + \ldots\ldots\ldots + {}^mC_m u v_m$$

Differentiating both sides, we get

$$\frac{d^{m+1}}{dx^{m+1}}(uv) = {}^mC_0(u_{m+1}v + u_m v_1) + {}^mC_1(u_m v_1 + u_{m-1}v_2) + {}^mC_2(u_{m-1}v_2 + u_{m-2}v_3) + \ldots\ldots\ldots\ldots +$$

$$+ {}^mC_r(u_{m-r+1}v_r + u_{m-r}v_{r+1}) + \ldots\ldots\ldots + {}^mC_m(u_1 v_m + uv_{m+1})$$

$$= {}^mC_0 u_{m+1} v + ({}^mC_0 + {}^mC_1)u_m v_1 + ({}^mC_1 + {}^mC_2)u_{m-1}v_2 + ({}^mC_2 + {}^mC_3)u_{m-2}v_3 + \ldots\ldots\ldots$$

$$+ ({}^mC_{m-1} + {}^mC_m)u_1 v_m + {}^mC_m uv_{m+1}.$$

$$= {}^mC_0 u_{m+1} v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + {}^{m+1}C_3 u_{m-2} v_3 + \ldots\ldots\ldots + {}^{m+1}C_m u_1 v_m + {}^{m+1}C_{m+1} uv_{m+1}.$$

Therefore, Theorem is true for n = m +1

Example.7.4.1. If $y^{1/m} + y^{-1/m} = 2x$,

prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$

We have $y^{1/m} + y^{-1/m} = 2x$

i.e. $y^{2/m} - 2xy^{1/m} + 1 = 0$

Or, $y^{1/m} = \dfrac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$

i.e. $y = (x \pm \sqrt{x^2-1})^m$

Differentiating w.r.t. x, we get

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$$y_1 = m(x \pm \sqrt{x^2-1})^{m-1}\left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$= \frac{m(x \pm \sqrt{x^2-1})^m}{\sqrt{x^2-1}}$$

$$= \frac{my}{\sqrt{x^2-1}}$$

Or $(\sqrt{x^2-1})y_1 = my$

Squaring both sides, we get $(x^2 - 1)y_1^2 = m^2y^2$

Differentiating again w.r.t. x, we get $2(x^2 -1)\, y_1y_2 + 2xy_1^2 = 2\, m^2yy_1$

$\Rightarrow (x^2 -1)\, y_2 + xy_1 - m^2\, yy_1 = 0.$

Applying Leibnitz's Theorem, we get (differentiating n times)

$$(x^2-1)y_{n+2} + {}^nC_1 y_{n+1}.2x + {}^nC_2 y_n.2 + xy_{n+1}\,{}^nC_1 y_n - m^2 y_n = 0$$

$\Rightarrow (x^2 -1)y_{n+2} + (2n +1)\, xy_{n+1}\, (n^2 - m^2)\, y_n = 0$

Example.7.4.2. If $y = e^{a\sin^{-1}x}$,

Prove that $(1 - x^2)y_{n+2} - (2n + 1)xy_{n+1} - (n^2 + a^2)y = 0$

Since $y = e^{a\sin^{-1}x}$,

we have $y_1 = e^{a\sin^{-1}x}\dfrac{a}{\sqrt{1-x^2}} = \dfrac{ay}{\sqrt{1-x^2}}$

Squaring both sides, we get $(1 - x^2)y_1^2 = a^2\, y^2$

Differentiating w.r.t. x, we get $(1 - x^2)2y_1 y_2 - 2xy_1^2 = 2a^2y\, y_1$

or $(1 - x^2)y_2 - xy_1 = a^2y$.

Applying Leibnitz's theorem, we get (differentiating n times)

$$(1-x^2)y_{n+2} - 2nxy_{n+1}n(n-1)y_n - xy_{n+1} - ny_n = a^2 y_n$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Example7.4.3. Differentiate n times the equation

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$$

Here $D^{n-1}\{(1-x^2)y_2\} =$

$$(1-x^2)y_{n+2} + n.(-2x)y_{n+1} + \{n(n-1)/2!\}(-2)y_n$$

$$D^n(-xy_1) = -xy_{n+1} - ny_n{}'\ D^n(a^2y) = a^2y_n.$$

Adding $0=(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-a^2)y_n$

i. e. $(1-x^2)\dfrac{d^{n+2}y}{d\ x^{n+2}}-(2n+1)x\dfrac{d^{n+1}y}{dx^{n+1}}-(n^2-a^2)\dfrac{d^n y}{dx^n}=0$

Example7.4.4. If $y=\log\{x+\sqrt{1+x^2}\}$,

Prove that $(1+x^2)y_{n+1}+(2x+1)xy_{n+1}+n^2y_n=0$

Let $y=\log\{x+\sqrt{1+x^2}\}$,

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{x+\sqrt{1+x^2}}\{1+\frac{1}{2}(1+x^2)^{-1/2}.2x\}$$

$$=\frac{1}{x+\sqrt{1+x^2}}.\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}$$

$$=\frac{1}{\sqrt{1+x^2}}$$

Squaring both sides, we get $(1 + x^2)y_1^2 = 1$,

By differentiation, we get $(1 + x^2)2y_2y_2 + 2xy_1^2 = 0$.

Or, $(1 + x^2)\ y_2 + xy_1 = 0$. [Dividing by y_1 throughout]

Differentiating this *n* times, using Leibnitz's theorem, we get

$$(1+x^2)y_{n+2}+n.2x.y_{n+1}+\{n(n-1)/2!\}.2.y_n+xy_{n+1}+n.y_n=0$$

Or, $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

Value of the n^th^ derivative of a function for x = 0

Working rule to find the value of n^th^ derivative of a function for x = 0

Step 1 Equate the given function to y.

Step 2 Find y_1

Step3 Again find y_2

Step 4 Differentiate both sides n times by Leibnitz theorem

Step 5 Put x = 0 two cases arises

(1) When n = 0 odd integer

(2) When n = 0 even integer

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Example7.4.5 $y=\sin(m\sin^{-1}x)$

Then prove that $(1-x^2)y_{n-2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$.

and also find the value of y_n when x = 0.

We have $y=\sin(m\sin^{-1}x)$ **…….(1)**

Then $y_1=\cos(m\sin^{-1}x).\frac{m}{\sqrt{1-x^2}}$

$$\sqrt{1-x^2}.y_1=m\cos(m\sin^{-1}x)$$

$$(1-x^2)y_1^2=m^2(1-y^2).......(2)$$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1.y_2-2xy_1^2=-m^2.2y.y_1$$
$$(1-x^2).y_2-xy_1+m^2y=0...........(3)$$

Differentiating n times using Leibnitz rule, we get

$$(1-x^2)y_{n+2}-2nxy_{n+1}-n(n-1)y_n-xy_{n+1}-ny_n+m^2y_n=0$$
$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0........(4)$$

So we have $y=\sin(m\sin^{-1}x)........(1)$

$$(1-x^2)y_1^2=m^2(1-y^2).......(2)$$

$$(1-x^2).y_2-xy_1+m^2y=0...........(3)$$

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0........(4)$$

Putting x = 0 in equation (1), (2), (3) and (4), we get

$$y(0)=\sin(m\sin^{-1}0)=0\Rightarrow y(0)=0$$

$$(1-0)y_1^2(0)=m^2(1-y^2(0))$$
$$y_1^2(0)=m^2\Rightarrow y_1(0)=m$$
$$y_2(0)=0$$
$$y_{n+2}=(n^2-m^2)y_n$$

For n = 1, 2, 3, 4, ….. we have

For n = 1

$$\text{For } n=1 \quad y_3(0)=(1^2-m^2)y_1=(1^2-m^2)y_1(0)=m(1^2-m^2)$$
$$\text{For } n=2, \quad y_{2+2}(0)=y_4(0)=(2^2-m^2)y_2(0)=(1^2-m^2).0=0$$
$$\text{For } n=3, \quad y_{3+2}(0)=y_5(0)=(3^2-m^2)y_3(0)=m(1^2-m^2)(3^2-m^2)$$
$$\text{For } n=4, \quad y_{4+2}(0)=y_6(0)=(4^2-m^2)y_4(0)=(4^2-m^2).0$$
$$\text{For } n=5, \quad y_{5+2}(0)=y_7(0)=(5^2-m^2)y_5(0)=(5^2-m^2).m(1^2-m^2)(3^2-m^2)$$

So, we have $y(0)=y_2(0)=y_4(0)=y_6(0)=y_8(0)=y_{10}(0)............=0$

and

$$y_1(0)=m$$

$$y_3(0)=m(1^2-m^2)$$
$$y_5(0)=(3^2-m^2)(1^2-m^2)\,\mathrm{m}$$
$$y_7(0)=(5^2-m^2)(3^2-m^2)(1^2-m^2)\,\mathrm{m}$$
$$\text{So } y_n(0) \text{ if n is odd} =\left[(\mathrm{n}-2)^2-m^2\right]\left[(\mathrm{n}-4)^2-m^2\right].......(5^2-m^2)(5^2-m^2)(3^2-m^2)(1^2-m^2)\,\mathrm{m}$$
$$y_n(0)=\begin{cases}0; \text{when n is even}\\ \left[(\mathrm{n}-2)^2-m^2\right]\left[(\mathrm{n}-4)^2-m^2\right].......(5^2-m^2)(5^2-m^2)(3^2-m^2)(1^2-m^2)\,\mathrm{m}\end{cases}$$

Example7.4.6 If $\log y=\tan^{-1}x$ show that

$$(1+\mathrm{x}^2)y_{n+2}+(2(n+1)x-1)y_{n+1}+n(n+1)\,\mathrm{y}_n=0$$

and hence find $y_3, y_4, and\ y_5\ at\ x=0$

Solution We have $\log y=\tan^{-1}x$

If $y=e^{\tan^{-1}x}$........(1)

Differentiating w.r.t. x, we get

$$y_1=e^{\tan^{-1}x}\frac{1}{1+x^2}$$
$$(1+x^2)y_1=y................(2)$$

Again differentiating w.r.t. x, we have

$$(1+x^2)y_2+2xy_1=y_1..........(3)$$

Differentiating n times applying Leibnitz's theorem, we get

$$(1+x^2)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+2\,\mathrm{x}y_{n+1}+2ny_n=y_{n+1}$$

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$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$$

$$(1+x^2)y_{n+2}+(2(n+1)x-1)y_{n+1}+n(n+1)y_n=0$$

So we have

$$y=e^{\tan^{-1}x}\text{........(1)}$$

$$(1+x^2)y_1=y\text{................(2)}$$

$$(1+x^2)y_2+2xy_1=y_1\text{..........(3)}$$

$$(1+x^2)y_{n+2}+(2(n+1)x-1)y_{n+1}+n(n+1)y_n=0\text{......(4)}$$

Putting x = 0 in equation (1), (2), (3) and (4), we get

$$y=e^{\tan^{-1}0}=e^0=1\Rightarrow y(0)=1$$

By equation (2), we get

$$(1+0)y_1(0)=y(0)=1\Rightarrow y_1(0)=1$$

By equation (3), we get

$$y_2(0)=1$$

and by equation (4), we get by putting x = 0

$$y_{n+2}-y_{n+1}+n(n+1)y_n=0$$

For n = 1, 2, 3, 4, ….. we have

For $n=1\quad y_{1+2}-y_{1+1}+1(1+1)y_1=0,\ \ y_3(0)-y_2(0)+2y_1=0$

$\Rightarrow y_3-1+2=0\Rightarrow y_3=-1$

For $n=2,\ \ y_{2+2}(0)-y_3(0)+6y_2(0)=0\Rightarrow y_4(0)=-7$

For $n=3,\ \ y_{3+2}(0)=y_5(0)-y_4(0)+12y_3(0)\Rightarrow y_5(0)=5$

Example.7.4.7. If $y=e^{a\sin^{-1}x}$,

Then prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$

Deduce that $\lim_{x\to 0}\dfrac{y_{n+2}}{y_n}=n^2+a^2$,Hence find $y_n(0)$.

Since $\quad y=e^{a\sin^{-1}x}$...........(1)

$$y_1=e^{a\sin^{-1}x}\frac{a}{\sqrt{1-x^2}}=\frac{ay}{\sqrt{1-x^2}}$$

Squaring both sides, we get $(1 - x^2)y_1^2 = a^2 y^2$……..(2)

Differentiating w.r.t. x, we get $(1 - x^2)2y_1 y_2 - 2xy_1^2 = 2a^2 y\, y_1$

or$(1 - x^2)y_2 - xy_1 - a^2 y$. ………..(3)

Applying Leibnitz's theorem, we get

$$(1-x^2)y_{n+2} - 2nxy_{n+1} n(n-1)y_n - xy_{n+1} - ny_n = a^2 y_n$$

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0.........(4)$$

Putting x = 0 in equation (1), (2), (3) and (4), we get

$$y(0) = e^{a\sin^{-1} 0} = 1,$$

$$y_1(0) = ay(0) \Rightarrow y_1(0) = a$$
$$y_2(0) = a^2$$

$$y_{n+2} = (n^2 + a^2)y_n$$

For n = 1, 2, 3, 4, ….. we have

$$\text{For } n=1 \quad y_{1+2}(0) = y_3(0) = (1^2+a^2)y_1 = (1^2+a^2)a$$
$$\text{For } n=2, \quad y_{2+2}(0) = y_4(0) = (2^2+a^2)y_2(0) = (2^2+a^2)\,a^2$$
$$\text{For } n=3, \quad y_{3+2}(0) = y_5(0) = (3^2+a^2)y_3(0) = (3^2+a^2)(1^2+a^2)\,a$$
$$\text{For } n=4, \quad y_{4+2}(0) = y_6(0) = (4^2+m^2)y_4(0) = (4^2+a^2).a^2$$

$$\text{So } y_n(0) = \begin{cases} a^2(2^2+a^2)(4^2+a^2).......((n-2)^2-a^2) & \text{when } n \text{ is even} \\ a(1^2+a^2)(3^2+a^2)(5^2+a^2).......((n-2)^2+a^2) & \text{when n is odd} \end{cases}$$

Divides equation (4) by y_n and then taking limit $x \to 0$,we get

$$(1-x^2)\frac{y_{n+2}}{y_n} - (2n+1)\frac{xy_{n+1}}{y_n} - (n^2+a^2) = 0$$

$$\lim_{x\to 0}\left[(1-x^2)\frac{y_{n+2}}{y_n} - (2n+1)\frac{xy_{n+1}}{y_n} - (n^2+a^2) = 0\right]$$

$$\lim_{x\to 0}\frac{y_{n+2}}{y_n} = (n^2+a^2)$$

Check your progress

1. If $y = a\cos(\log x) + b\sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

2. If $y = Ae^{-kt}\cos(pt + c)$ show that $\frac{d^2y}{dt^2} + 2k\frac{dy}{dx} + n^2 y = 0$

where $n^2 = p^2 + k^2$

3. If $y = x^2 e^x$, show that

$$\frac{d^n y}{dx^n} = \frac{1}{2}n(n-1)\frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2)y.$$

7.5 Expansion of Functions

7.5.1 Infinite series

We have seen that the ordinary processes of addition, subtraction, multiplication, division, rearrangement of terms, raising to a given power, taking limits, differentiation, etc., though applicable to the sum of a finite number of terms, may break down for an infinite series. The expansions in the form of infinite series obtained by the methods given below therefore to be regarded merely as formal expansion, which may not be true in exceptional cases

7.5.2 Maclaurin's Theorem

Let $f(x)$ be a function of x which can be expanded in ascending powers of x and let the expansion be differentiable term by term any number of times. Then $f(x)$ can be expanded as

$$f(x) = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots\ldots. A_nx^n + \ldots\ldots\ldots$$

where $A_0, A_1, A_2, A_3, \ldots\ldots. A_n \ldots\ldots\ldots$ are constants.

Then by successive differentiation we have

$$f'(x) = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 \ldots\ldots.,$$

$$f''(x) = 2.1\,A_2 + 3.2A_3x + 4.3A_4x^2 + \ldots\ldots.,$$

$$f'''(x) = 3.2.1\,A_3 + 4.3.2A_4x + \ldots\ldots., \text{ etc.}$$

Putting $x = 0$ in each of these, we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2!\,A_2,\ f'''(0) = 3!\,A_3, \ldots \text{ etc.}$$

Substituting all these values in $f(x)$, we have

Hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0)$

$$+\frac{x^3}{3!} f'''(0) + ... + \frac{x^n}{n!} f^n(0) + ...$$

This result is generally known as **Maclaurin's Theorem.**

Working Rule for Maclaurin's Theorem

Step 1 Put the given function equal to step to *f*(*x*).

Step 2 Differentiate *f*(*x*), a number of times and find *f'*(*x*), *f''*(*x*), *f'''*(*x*), *f''''*(*x*) ….. and show on.

Step 3 Put x = 0 in the results obtained in step 2 and find *f*(*0*). *f'*(0), *f''*(*0*), *f'''*(*0*), *f''''*(*0*) ….. and so on.

Step 4 Now substituting the values of *f*(*0*), *f'*(0), *f''*(*0*), *f'''*(*0*), *f''''*(*0*) ….. and so on.

In $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... + \frac{x^n}{n!} f^n(0) + ...$

Example:7.5.1. Expand $\tan^{-1} x$ by Maclaurin's theorem.

Let $f(x) = \tan^{-1} x$ then $f(0) = \tan^{-1} 0 = 0$,

$$f'(x) = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 ....., \quad f'(0) = 1$$

$$f''(x) = -2x + 4x^3 - 6x^5 ......, \quad f''(0) = 0$$

$$f'''(x) = -2 + 12x^2 - 30x^4 + ..., \quad f'''(0) = -2$$

$$f''''(x) = 24x - 120x^3 + ..., \quad f''''(0) = 0$$

$$f'''''(x) = 24 - 360x^2 + ..., \quad f'''''(0) = 24$$

Putting these values in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + ... + \frac{x^n}{n!} f^n(0) + ...$$

$$= 0 + x.1 + 0 - 2\frac{x^3}{3!} + 24\frac{x^5}{5!} ....$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} ....$$

Example:7.5.2. Expand sin*x* by Maclaurin's theorem.

Let $f(x) = \sin x$ then $f(0) = 0$,

$f'(x) = \cos x$, $f'(0) = 1$,

$f''(x) = -\sin x,$ $f''(0) = 0,$

$f'''(x) = -\cos x,$ $f'''(0) = -1,$ etc.

$f^{(n)}(x) = \sin(x + \frac{1}{2}n\pi)$, $f^{(n)}(0) = \sin + \frac{1}{2}n\pi$ = 0 if $n = 2m$,

and = $(-1)^m$ if $n = 2m + 1$.

Putting these values in

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n}{n!}f^n(0) + ...$$

$$\sin x = 0 + x.1 + 0 + \frac{x^3}{3!}(-1) + 0 + ... + 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + ...$$

$$= x - \frac{x^3}{3!} + \frac{x^3}{5!} - .... + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + ...$$

Example7.5.3. Expand tan x ascending powers of x by Maclaurin's theorem.

Let $y = \tan x$ then $y(0) = 1$

$y_1 = \sec^2 x$ $y_1(0) = 1$

$y_2 = 2\sec^2 x \tan x = 2yy_1$, $y_2(0) = 0$

$y_3 = 2y_1y_1 + 2yy_2 = 2y_1^2 + 2yy2$ $y_3(0) = 2$

$y_4 = 4y_1y_2 - 2y_1y_2 + 2yy_3 = 6y_1y_2 + 2yy_3$, $y_4(0) = 0$

$y_5 = 6y_2y_2 + 6y_1y_3 + 2y_1y_3 - y_1y_4 = 6y_2^2 + 8y_1y_3 + 2yy_4$, $y_5(0) = 16$

Using Maclaurin's theorem, we get

$$\tan x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x3}{3!}.2 + \frac{x^4}{4!}.0 + \frac{x^5}{5!}.16 + ...$$

$$\text{Or, } \tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + ...$$

Example7.5.4. Apply Maclaurin's theorem to obtain the expansion of log $(1 + \sin x)$.

Let $y = \log(1 + \sin x)$, then $y(0) = 0$

$$y_1 = \frac{\cos x}{1 + \sin x}, \ y_1 = 0 \ \text{---------------(1)}$$

$$y_2 = \frac{(-\sin x)(1 + \sin x) - \cos^2 x}{(1 + \sin x)^2} = \frac{-(\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} = \frac{(1 + \sin x)}{(1 + \sin x)^2}$$

Or, $y_2 = -\dfrac{1}{1+\sin x}$, $y_2(0) = -1$ -----------(2)

$$y_3 = \frac{(1+\sin x).0+\cos x}{(1+\sin x)^2} = \frac{\cos x}{(1+\sin x)^2}$$ Or, $y_3 = -y_1y_2$, $y_3(0) = 1$ ------- (3)

Using (1), (2) and (3), we get $y_4 = -y_2y_2 - y_1y_3 = -y_2^2 - y_1y_3$

$y_4(0) = -2$ ------------(4)

Using (1), (2), (3) and (4), we get $y_5 = -2y_2y_3 - y_2y_3 = -y_1y_3 - y_1y_4$ $= -3y_2y_3 - y_1y_4$

$y_5(0) = 5$, Hence by Maclaurin's theorem,

$$\log(1+\sin x) = 0 + x.1 + \frac{x^2}{2!}(-1) + \frac{x3}{3!}.1 + \frac{x4}{4!}(-2) + \frac{x^5}{5!}.5 + ...$$

$$\text{Or, } \log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24} + ...$$

Example:7.5.5. Expansion of $\cos^{-1} x$

$$\begin{aligned}
&Let \quad y = \cos^{-1} x \\
&differenction\, w.r.t.x, \quad we\ get \\
&\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}} = -\left(\sqrt{1-x^2}\right)^{-1} \\
&\quad = -\left[1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{15x^6}{48} + ...\right]
\end{aligned}$$

By integration between limits 0 to x, we get

$$\begin{aligned}
\int dy &= \int(1 + \frac{x^2}{2} + \frac{3x^4}{8} + \frac{15x^6}{48} + ...)dx \\
&= -(\text{x} + \frac{x^3}{6} + \frac{3x^5}{40} + ...)
\end{aligned}$$

> **NOTE**: When the *n*th differential coefficient of the function cannot be found, the *n* th term of the expansion cannot be ascertained. It is possible, however, that the *n* th differential coefficient be known for $x = 0$.

Check your progress

Expand the fallowing function by Maclaurin's theorem.

1. $\cos x$,
2. $\sec x$
3. $\sin^{-1}x$
4. $\cot^{-1}x$
5. $\cos^{-1}x$
6. Secx
7. $(1+x)^m$
8. $\log(1+x)$
9. $e^x \log(1+x)$
10. $\log(1+\sin^2 x)$

Remark $\log x$, $\cot x$, $\operatorname{cosec} x$ and $x^{\frac{1}{2}}$ cannot be expanded by Maclaurin's theorem because function and its derivative does not exist at x = 0.

7.6 Taylor's Theorem

Let $f(a+h)$ be a function of h which can be expanded in powers of h, and let the expansion be differentiable any number of times with respect to h. Its expansion is given as

$$f(a+h) = A_0 + A_1h + A_2h^2 + A_3h^3 + \ldots..$$

By successive differentiation with respect to h, we have

$$f'(a+h) = A_1 + 2A_2h + 3A_3h^2 + 4A_4h^3 + \ldots.$$

since $\frac{d}{dh} f(a+h) = \frac{d}{dh} f(t).\frac{dt}{dh}$, where $t = a + h$,

$$f''(a+h) = 2.1A_2 + 3.2A_3h + 4.3A_4h^2 + \ldots.$$

$$f'''(a+h) = 3.2.1A_3 + 4.3.2A_4h + \ldots \qquad \text{etc.}$$

Putting $h = 0$ in $f(a+h)$ and its derivatives, we get

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$$f(a) = A_0,$$
$$f'(a) = A_1$$
$$\frac{f''(a)}{2!} = A_2$$
$$\frac{f'''(a)}{3!} = A_3, \ldots\ldots \quad \text{so on}$$

Putting all these values in equation (1), we get

Hence $$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots + \frac{h^n}{n!} f^n(a) + \ldots \quad (1)$$

This is known as Taylor's Theorem, If we put a = 0 and h = x we get the particular case know as Maclaurin's Theorem. A more useful form is obtained on replacing h by (x – a). Thus

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots + \frac{(x-a)^n}{n!} f^n(a) + \ldots$$

Which is an expansion in powers of (x – a).

Example:7.6.1. Expand log(x+h) in powers of h by Taylor's Theorem.

Here we have expand log(x+h) in powers of x. hence we shall ues the following form of Taylor's theorem.

$$f(x+h) = f(h) + xf'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \ldots + \frac{x^n}{n!} f^n(h) + \ldots$$

Let $\text{f}(x+h) = \log(x+h) \quad (1)$

$\therefore \text{f}(x) = \log x \quad \text{hence} \quad \text{f}(h) = \log h$

$\text{f}'(x) = \frac{1}{x} \quad \text{hence} \quad \text{f}'(h) = \frac{1}{h}$

$\text{f}''(x) = -\frac{1}{x^2} \quad \text{hence} \quad \text{f}''(h) = -\frac{1}{h^2}$

$\text{f}'''(x) = \frac{1}{x^3} \quad \text{hence} \quad \text{f}'''(h) = \frac{1}{h^3}$

Substituting these values of $\text{f}(h)\ \text{f}'(h)\ \text{f}''(h), \text{f}'''(h)$ etc. in (1), we get

$$f(x+h) = \log h + x\frac{1}{h} + \frac{x^2}{2!}\left[-\frac{1}{h^2}\right] + \frac{x^3}{3!}\left[\frac{2}{h^3}\right] + \ldots\ldots\ldots$$

$$\because \log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} \ldots\ldots$$

Example7.6.2. Expand log sin x in powers of $(x - 2)$

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Let $f(x) = \log \sin x$.………..(1).

This can be written as f $(2 + x - 2) = \log \sin (2 + x - 2) = f(a + h)$

where a=2 & h = x-2, we apply (1)

since (1) is to be expanded in powers of $(x - 2)$. Then

$f(x) = \log \sin x$ $\quad$ $f(2) = \log \sin 2$

$f'(x) = \cot x,$ $\quad$ $f'(2) = \cot 2,$

$f''(x) = -\operatorname{cosec}^2 x,$ $\quad$ $f''(2) = \operatorname{cosec}^2 2$

$f'''(x) = 2\operatorname{cosec}^2 x.\cot x\ ,$ $\quad$ $f'''(2) = 2\operatorname{cosec}^2 2 \cot 2$, etc.

Hence by Taylor's theorem, we have

$$f(x) = f(2 + x - 2) = f(2) + (x-2) f'(2)$$
$$+ \frac{(x-2)^2}{2!} f''(2) + \frac{(x-2)^3}{3!} f'''(2) + \ldots.$$

Or, $\log \sin x = \log \sin 2 + (x-2)\cot 2 + \dfrac{(x-2)^2}{2!}(-\operatorname{cosec}^2 2)$

$$+ \frac{(x-2)^3}{3!}(2\operatorname{cosec}^2 2 \cot 2) + \ldots.$$

Example:7.6.3. Expand $\sin x$ in powers of $\left(x - \dfrac{\pi}{2}\right)$.

Let $f(x) = \sin x$ $\quad$ then $f\left(\dfrac{\pi}{2}\right) = 1$

$f'(x) = \cos x$ $\quad$ $f'\left(\dfrac{\pi}{2}\right) = 0$

$f''(x) = -\sin x$ $\quad$ $f''\left(\dfrac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x$ $\quad$ $f'''\left(\dfrac{\pi}{2}\right) = 0$

$f^{iv}(x) = \sin x$ $\quad$ $f^{iv}\left(\dfrac{\pi}{2}\right) = 1$ etc.

By applying Taylor's theorem, we get

Sin x= $\sin(\pi/2 + x - \pi/2)$

Therefore,

$a = \pi/2$ and $h = x - \pi/2$

therefore from (1)

$$\sin x = f(\pi/2) + (x-\pi/2) f'(\pi/2) + \frac{1}{2!}(x-\pi/2)^2 f''(\pi/2)$$
$$+\frac{1}{3!}(x-\pi/2)^3 f'''(\pi/2) + \frac{1}{4!}(x-\pi/2)^4 f^{iv}(\pi/2) + ...$$

$$\text{Or, } \sin x = 1 - \frac{1}{2!}(x-\pi/2)^2 + \frac{1}{4!}(x-\pi/2)^4 - .....$$

Example:7.6.4. Expand $3x^3 + 7x^2 + x - 6$ in powers of $x - 2$.

$$\text{Let} \quad f(x) = f(2 + x - 2) = 2x^3 + 7x^2 + x - 6$$

$$\text{Then} \quad f(2) = 22^3 + 72^2 + 2 - 6 = 40$$

$$f'(x) = 6x^2 + 14x + 1, \; f'(2) = 53$$
$$f''(x) = 12x + 14 \qquad f''(2) = 38$$
$$f'''(x) = 12 \qquad f'''(2) = 12$$

and f^{iv}, f^{v}, …… are all zero. Hence by Taylor's theorem,

$$f(x) = f(2) + (x-2) f'(2) + \frac{1}{2!}(x-2)^2 f''(2) + \frac{1}{3!}(x-2)^3 f'''(2) + ...$$

$$= 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

7.7 Summary

Second and third order differentiations order differentiation, n^{th} order differentiation of some standard functions as: f(x)=e^{ax} ,f(x)=$(ax+b)^m$, f(x) = log(ax+b), f(x) = a^x, f(x) = sin(ax+b), f(x) = cos (ax+b), f(x) = e^{ax} sin(bx+c), f(x) = e^{ax}cos(bx+c), Leibritiz theorem for nth derivatives of product of two functions, n^{th} derivative at origin, expansion of functions like Maclaurin's theorem, Taylor's theorem for expansion of f(x+h) or f(a+h), expansion of function in some given power is discussed in this unit.

7.8 Terminal Questions

1. Expand log (x + a) in powers of x by Taylor's theorem.
2. tan (x +h) in powers of h up to h^4
3. $\tan^{-1}$ x in powers of $(x-\frac{1}{4}\pi)$
4. Prove that

$$f(mx)= f(x)+(m-1)x.f'(x)+\frac{1}{2!}(m-1)^2 x^2 .f''(x)+\frac{1}{3!}(m-1)^3 x^3 f'''(x)+...$$

5. Prove that $\frac{1}{x+h}=\frac{1}{x}-\frac{1}{x^2}+\frac{h}{x^3}-\frac{h^3}{x^4}+......$

UNIT-8

MEAN VALUE THEOREMS

Structure

8.1 Introduction

In the unit we shall study four important theorems in differential calculus which are Roll's theorem, Lagrange form of mean value theorem, Cauchy form of mean value theorem and General mean value theorem. For this we shall need the concept of continuity & differentiability of a function in an interval.

(i) A function is said to be continuous at a given point c € R If

$\lim_{h\to 0} f(c - h) = \lim_{h\to 0} f(c + h) = f(c)$

Left limit at $(x = c)$ = right limit at $(x = c)$ = value at $(x = c)$

A function is said to be continuous in an interval if it is continuous at each point of the interval.

(ii) A function is said to be differentiable at given point c € R if

L$f'(c) = \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ = R$f'(c)$

Left derivations at $(x = c) =$ Right derivations at $(x = c)$

A function is said to be differential in an interval if it is differentiable at each point in the interval.

Note: - Every differentiable function is continuous but a continuous function need not be differentiable.

Objectives: After reading this unit you should be able to;

- Familiarize with the concept of continuity & differentiability of a function in an interval
- Understand Roll's theorem and its proof.
- Understand Lagrange form of mean value theorem and its proof.
- Understand Cauchy form of mean value theorem and its proof.
- Understand General mean value theorem and its proof

8.2 Rolle's Theorem

Let f be defined on the closed in table [a , b] such that

(i) $f(x)$is continuous in the closed interval [a, b]

i.e. $a \leq x \leq b$

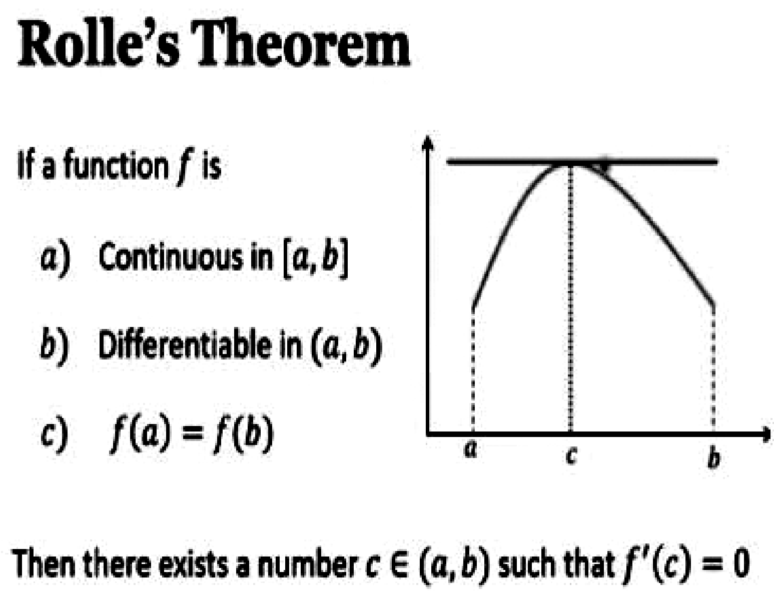
(ii) $f(x)$ is differentiable is an open interval (a , b)

i.e $a < x < b$

If $f(a) = f(b)$ then there exist a real number x_0 between a & b such that $f'(x_0) = 0$

i.e a<$x_0 < b$

(or tangent at $x = x_0$ is parallel to X axis)



Proof:- since f is continues on the closed interval [a,b] and therefore $f(x)$ is bounded ns [a,b].

Let M be the least upper bound (or supremum) & m be the greatest lower bound (or infenum) at [a, b]

Then we have the following cases:

Case 1- When M = m then $f(x)$ is a constant function and there for $f'(x) = 0, \quad \forall x \in [a, b]$

Case 2- M ≠ m

Now suppose that M ≠ $f(a)$ where $f(a) = f(b)$

Since f is continuous in the closed interval [a , b]

Therefore f attains its supremum & so there exist a real number x_0 in [a, b] such that $f(x_0) = M$

Since $f(a) \ddagger f(b)$ therefore x_0 is different from a & b therefore x_0 exists in the open interval (a, b)

Since $f(x_0) =$ M is the supremum of f in [a, b]

$\therefore f(x) \leq f(x_0) \quad , \quad \forall x €$ [a, b]

& therefore $f(x_0 - h) \leq f(x_0)$

$$\Rightarrow f(x_0 - h) - f(x_0) \leq 0$$

$$\Rightarrow \quad \frac{f(x_0-h)-f(x_0)}{-h} \geq 0$$

$$\Rightarrow \quad \lim_{h\to 0} \frac{f(x_0-h)-f(x_0)}{-h} \geq 0$$

$$\Rightarrow \quad \text{L}f'(\quad x_0) \geq 0$$

……….. (1)

Also $\quad f(x_0 + h) \leq f(x_0) \qquad (\because f(x_0) = M = supremum\ f)$

$$\Rightarrow \quad f(x_0 + h) - f(x_0) \leq 0$$

$$\Rightarrow \quad \frac{f(x_0+h)-f(x_0)}{h} \leq 0$$

$$\therefore \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} \leq 0$$

Or $\quad = \quad Rf'(x_0) \leq 0 \qquad$ …………… (2)

Since f is differentiable at x_0

Therefore L $f'(x_0)$ = R$f'(x_0) = f'(x_0)$

& so from (1) and (2) $f'(x_0) = 0$

Note 1:- In Rolle's theorem there exist at least one point or more that one point at which $f'(x) = 0$

Note 2: if $f(x)$ is polynomial and all the condition of the Roll's theorem are satisfied and $f(a) = f(b) = 0$

Then between any two roots of the equation $f(x) = 0$

There is at least one root of $f'(x) = 0$

Example 1: Examine the validity and conclusion of the roll's theorem for the function $f(x) = (x-1)^{2/3}$.

Solution ;- (i) $f(x)$ is continuous at each point in [0,2]

(ii) $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}} = \frac{2}{3(x-1)^{\frac{1}{3}}}$

$\therefore$ $f'(x)$ does not exist at $x = 1$ or $f(x)$ is not differentiable is [0 , 2]

(iii) $f(0) = f(2)$

From (ii) roll's theorem is not valid for the given function $f(x)$

Example 2: Is the roll's theorem valid for $f(x) = sinx$ in the interval $[0, \pi]$?

Solution:- $f(x) = sinx$ is continuous & differentiable at each point in the interval also

$f(0) = sin0 = 0$ & $f(\pi) = sin\pi = 0)$ therefore the conditions &of the Roll's

theorem are satisfied & so there exist $x_0$between 0 & π for which $f'(x_0) = 0$

Or $cosx_0 = 0$ or $x_0 = \frac{\pi}{2}$

Test your knowledge

(1) Examine the validity & conclusion & the roll's theorem in the following function

(a) $f(x) = |x|$ is [-1, 1]

(b) $f(x) = x^3 - 6x^2 + \frac{1}{x} - 6$ is [1 ,3]

(c) $f(x) = \cos x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$

8.3 Langrange's mean value theorem

Theorem:- let f be a function defined in [a, b] such that

(1) $f(x)$ is continuous is the closed interval [a, b]

(2) $f(x)$ is differentiable in the open interval (a, b)

Then there exist a real number $x_0 \epsilon$ (a, b)

Such that $\frac{f(b)-f(a)}{b-a} = f'(x_0)$

Or $f(b) - f(a) = (b-a)f'(x_0)$

Lagrange's Mean Value Theorem

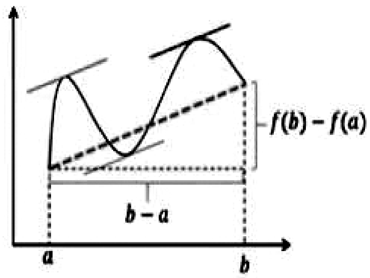
If a function f is

a) Continuous in $[a, b]$

b) Differentiable in (a, b)

Then there exists a number $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



In other words, there is at least one tangent line in the interval that is parallel to the line segment that goes through the endpoints of the curve in {a,b}.

Proof:- Let $\emptyset$ be a function defined on [a, b] such that $\emptyset(x) = f(x) + \lambda . x$ ………………...(1)

Where λ is a constant to be chosen suitably. From (1), since f is continuous on the closed interval [a, b] & $f_1(x) = \lambda\, x$ is also continuous on [a, b] therefore

From (1)

(i) $\emptyset(x)$ is continuous because sum & two continuous functions is continuous

(ii) since f is differentiable on (a, b) and $f_1(x) = \lambda\, x$ is also differentiable & so

from (1) $\emptyset(x)$ is differentiable is (a, b) as the sum of two differentiable function is differentiable.

We now choose the constant λ such that

$\emptyset(a) = \emptyset(b)$

$\therefore$ from (1)$f(a) + \lambda\, a = f(b) + \lambda\, b$

$\therefore \frac{f(b) - f(a)}{a - b} = \lambda$ …………………… (2)

Then for this value of λ the function $\emptyset(x)$ satisfied the conditions of Roll's theorem. & therefore there exist a point x_0 is (a, b)

Such that $\emptyset'(x_0) = 0$

$\therefore$from (1) $f'(x_0) + \lambda = 0$

Or $\lambda = -f'(x_0)$ …………………… (3)

From (2)& (3)……….

$$\frac{f(b)-f(a)}{b-a} = f'(x_0)$$

Or $f(b) - f(a) = (b-a)f'(x_0)$

Note:- In the above theorem of we takes b = a + h then b −a = h therefore

$$x_0 = a + \theta h \ , \ 0 < \theta < 1$$

Where θ is a real number than the roll's theorem can be written or

$$f(a+h) - f(a) = hf'(a+\theta h)$$

Example 1: Discuss the applicability of Langrange's mean value theorem for $f(x) = \sqrt{x^2-4}$ in the interval [2, 4]

Solution:-

(1) $f(x) = \sqrt{x^2-4}$ is continuous in [2, 4] and

(2) $f'(x) = \frac{x}{\sqrt{x^2-4}}$ is defined is (2, 4)

∴ $f(x)$ is differential is (2, 4)

Therefore the conditions of the mean value theorem are satisfied and so there exist a number $x_0 \, \epsilon \, (2, 4)$ such that

$$f(4) - f(2) = f'(x_0).(4-2)$$

Or $2\sqrt{3} - 0 = 2.\frac{x_0}{\sqrt{x_0^2-4}}$

Or $\sqrt{3} = \frac{x_0}{\sqrt{x_0^2-4}}$

Or $2x_0^2 = 12$ or $x_0 = \pm\sqrt{6}$

Hence the value $\sqrt{6}$ lies between 2 & 4 hence the Lagrange mean value theorem is applicable in the given interval .

Example 2: Find x_0 of the mean value theorem if $f(x) = x(x-1)$

$(x-2)$ is interval $[0, \frac{1}{2}]$

Since $f(x)$ is a polynomial in x and so f is continuous in [0, ½] & differentiable in (0 , ½) therefore all the conditions of mean value theorem are satisfied for f is [0 , ½]. Therefore by language's mean value theorem these exist a point $x_0 \epsilon \left(0, \frac{1}{2}\right)$ such that

$$f(b) - f(0) = f'(x_0)(b-a)$$

Or $f\left(\frac{1}{2}\right) - f(0) = f'(x_0)\left(\frac{1}{2} - 0\right)$............... 2

Now for (1), $f(0) = 0 \, , f\left(\frac{1}{2}\right) = \frac{3}{8}$

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Also $f'(x) = 3x^2 - 6x + 2$

$$\therefore f'(x_0) = 3{x_0}^2 - 6x_0 + 2$$

∴ From (2); $\frac{3}{8} - 0 = \frac{1}{2}\, f'(x_0) = \frac{1}{2}(3x_0^2 - 6x_0 + 2)$

Or $12x_0^2 - 24x_0 + 5 = 0$

Or $x_0 = \frac{6 \pm \sqrt{21}}{6}$ but $x_0 = \frac{6+\sqrt{21}}{6} \notin (0, \frac{1}{2})$

Therefore $x_0 = \frac{6-\sqrt{21}}{6}$ ϵ(0, ½)

Test your knowledge

Example 3: Verify the hypothesis' and conclusion of the Langrange's mean value theorem of the following function

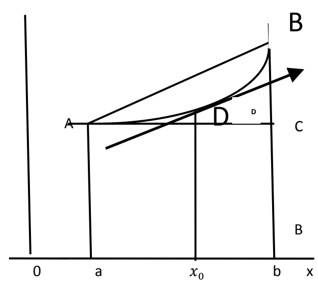
(1) $f(x) = x^2 - 2x + 3$ $in\ [1, \frac{1}{2}]$

(2) $f(x) = 1 + x^{\frac{2}{3}}$ $in\ [-8, 1]$

(3) $f(x) = |x|$ in [-1 , 2]

Geometrical interpretation of Langrange's mean Value theorem

In the adjacent diagram.



b – a = AC

$f(b) - f(a) = BC$ s

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{BC}{AC}$$

$= \tan(\angle BCA) = tan\theta$ where θ is the angle made by chord AB with X axis.

$\therefore \quad \frac{f(b)-f(a)}{b-a}$ in the slope of tangent at D = $f'(x_0)$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = f'(x_0)$$

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8.4 Cauchy's means value theorem

Theorem : Let f & g be functions defined on the interval [a, b] such that

(1) $f(x)$ and $g(x)$ are continuous are [a, b]

(2) $f(x)$ and $g(x)$ are differentiable in (a, b)

(3) $g'(x) \neq 0$, $\forall x \epsilon (a, b)$

Then there exist a real number $x_0 \epsilon\ (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f\prime(x_0)}{g'(x_0)}$$

Proof: Consider a function ∅ defined m [a, b] such that

$$\emptyset(x) = f(x) + \curlywedge . g(x) \quad \text{for } x\epsilon\ [a, b].$$

Where ⋋ is a constant …………….. (1)

Now **(1)** since $f(x) and\ g(x)$ are continuous on the interval [a, b], therefore From (1) $\emptyset(x)$ is continuous in [a, b]

(2) since $f(x)$ & $g(x)$ are differentiable in (a, b) therefore from (1) $\emptyset(x)$ is differentiable in (a, b)

(3) we now choose the number ⋋ such that $\emptyset(x) = \emptyset(b)$

∴ from $f(a) + \curlywedge\ g(a) = f(b) + \curlywedge\ g(b)$

Or $\frac{f(b)-f(a)}{g(a)-g(b)} = - \curlywedge$ …………….. (2)

Therefore for this value of ⋋ & from (1) & (2) the function $\emptyset(x)$ satisfies the conditions of the

Roll's theorem in [a, b] & so there exist a real number x_0 in (a, b) such that .

$\emptyset'(x_0) = 0$

∴ form (1) $f'(x_0 + \curlywedge\ g'(x_0) = 0$

$\therefore - \curlywedge = \frac{f'(x_0)}{g'(x_0)}$ ……………. (3)

∴ form (2) & (3)

$$\frac{f(b)-f(a)}{g(a)-g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

Note: (1) if $b = a + h$, & $x_0 = a + \theta h$, $0 < \theta < I$

Then above theorem can be stated as follows: if f & g are continues or $[a, a + h]$ and differentiable in (a, a+h) & $g'(x) \ddagger 0$ for all $x\epsilon(a, a + h)$ then there in a real number θ between 0 & 1 such that

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)} = \frac{f'(a+\theta h)}{g'(a+\theta h)}$$

Note : if we take $g(x) = x$ for all x in [a, b] then from the Cauchy's mean value theorem.

$$\frac{f(b)-f(a)}{b-a} = \frac{f'(x_0)}{1}$$

Or $\frac{f(b)-f(a)}{b-a} = f'(x_0),$ $\quad x_0 < (a,b)$

Which is the Langrange's mean value theorem

Example 1: Calculate the value of x_0 for which.

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

For the following function

1. $f(x) = sinx\ ,\ g(x) = cosx\ in\ [-\frac{\pi}{2}\ ,0]$

2. $f(x) = e^x,\ g(x) = e^{-x}$ in [0 , 1]

8.5 General mean value theorem

Theorem: Let the three function f ,g & h be defined on [a, b] such that

1. f, g & h are continuous in [a , b]

2. f, g & h are differentiable in (a, b)

Then ∃ a real number $x_0 \epsilon (a\ , b)$ such that-

$$\begin{vmatrix} f'(x_0) & g'(x_0) & h'(x_0) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Proof : We define a function $\emptyset(x)$ such that

$$\emptyset(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \quad for\ x\epsilon[a,h] \ \ldots\ldots\ldots\ldots\ldots.. (1)$$

Since the function $f(x)$, $g(x)$ & $h(x)$ are continuous is [a, b] and differentiable in (a, b)

Therefore from (1) ∅ is continuous in [a, b] and differentiable in (a, b)

Also,

$$\emptyset(a) = \begin{vmatrix} f(a) & g(a) & h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 \qquad \text{(two rows are same)}$$

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Similarly:- $\emptyset(b) = 0$ & so $\emptyset(a) = \emptyset(b)$

Therefore $\emptyset(x)$ satisfied the conditions of the Roll's theorem & so there exist $x_0 \epsilon (a,b)$ such that $\emptyset'(x_0) = 0$ …………………. (2)

From (1) $\emptyset'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$

$\therefore \ \emptyset'(x_0) = \begin{vmatrix} f'(x_0) & g'(x_0) & h'(x_0) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$ ……………. (3)

This proves the theorem.

Note 1: If we take $g(x) = x$ & $h(x) = 1$, $\forall x \epsilon [a,b]$ then form (3)

$\theta^1(x_o) \quad \begin{vmatrix} f'(x_0) & 1 & 0 \\ f(a) & a & 1 \\ f(b) & b & 1 \end{vmatrix} = 0$

Or $f'(x_0)\{a-b\} - 1\{f(a) - f(b)\} = 0$

Or $f'(x_0) = \frac{f(a)-f(b)}{a-b} = \frac{f(b)-f(a)}{b-a}$

which is Langrange's form of mean value theorem.

Note 2: If we take $h(x) = 1$, $\forall x \epsilon [a,b]$ then from (3)

$$\begin{vmatrix} f'(x_0) & g'(x_0) & 0 \\ f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \end{vmatrix} = 0$$

Or $f'(x_0)\{g(a) - g(b)\} - g'(x_0)\{f(a) - f(b)\} = 0$

Or $\frac{f(a)-f(b)}{g(a)-g(b)} = \frac{f'(x_0)}{g'(x_0)}$

Or $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$

Which in Cauchy's form of the mean value theorem.

Note 3: if Cauchy's form of mean value theorem we have –

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

Hence $g(a) \ddagger g(b)$. because If $g(a) = g(b)$ then function $g(x)$ will satisfy all the

Conditions of the Roll's theorem in the interval [a, b] and therefore for some x in (a,b)

we must have $g'(x) = 0$ which contradicts that $g'(x) \ddagger 0, \forall x \epsilon (a,b)$

Example 1: Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distend roots is [0,1]

Solution:- we know that if $f(x)$ is a polynomial and all the conditions of the roll's theorem are satisfied is [a, b] & $f(a) = f(b) = 0$ then between any two roots of the equation $f(x) = 0$ there is at least one root of $f'(x) = 0$

on the contrary suppose that there is a number k_1 such that the equation

$x^3 - 3x + k_1 = 0$ has two distinct root's x_1 & x_2 is [0, 1 where $x_1 < x_2$ $[x_1 \ddagger x_2]$

Since x_1, x_2 are roots of equation (1)

$\therefore x_1^3 - 3x_1 + k_1 = 0$ & $x_2^3 - 3x_2 + k_1 = 0$ ……….. (2)

Now let $f(x) = x^3 - 3x + k^1$ for $x \epsilon [x_1, x_2]$

Then $f(x)$ is continuous on $[x_1, x_2]$ and differentiable is (x_1, x_2) , also $f(x_1) = f(x_2) = 0$ from (2)

Therefore all the condition of the Roll's theorem are satisfied for the function $f(x)$ in (3)

And so ∃ $x_0 \epsilon (x_1, x_2)$ such that $f'(x_0) = 0$ ……………….(4)

Or $3x_0^2 - 3 = 0$ or $x_0^2 = 1$, or $x_0 = \pm 1$

& therefore $x_0 \notin 0, 1)$

$=> x_0 \notin (x_1, x_2)$

Which is a contradiction to (4)

Therefore there exist no real number k for which the given equation has two distinct roots in [0, 1].

Example 2: Find the value of c in the Cauchy's mean value theorem for the functions

$$f(x) = e^x \ \& \ g(x) = e^{-x} \ \ on\ [a, b]$$

Solution: We have $f'(x) = e^x$ & $g'(x) = -e^{-x} \ddagger 0$, $\forall x \epsilon (a, b)$

Theorem by Cauchy's mean value theorem there exist a number $c \epsilon (a, b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

Or $$\frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

Or $$\frac{(e^b - e^a)e^{a+b}}{e^b - e^a} = e^{2c}$$

Or $e^{a+b} = e^{2c}$

Or $a + b = 2c \quad or \quad c = \frac{a+b}{2} \; \epsilon (a, b)$ is the required value

Example 3: Verify Roll's theorem.

For $f(x) = e^{-x}(sinx - cosx)$ in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ …………….. (1)

Solution :- since $e^{-x}, sinx, cosx$ are all continuous function and the product & difference of two continuous function is continuous and also $e_9^{-x}\; sinx\;, cosx$ are differentiable therefore;

$f(x) = e^{-x}\,(sinx - cosx)$ is continuous in

$\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and differentiable in $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ also

$f\left(\frac{\pi}{4}\right) = e^{-\frac{\pi}{4}}\left(sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) = 0$

$f\left(\frac{5\pi}{4}\right) = e^{-\frac{5\pi}{4}}\left(sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right) = 0$

Therefore $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right) = 0$

Thus all the conditions of the Roll's theorem are satisfied therefore there is at leat one value of $x_0 \, \epsilon \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ such that

$f'(x_0) = 0$ …………….. (2)

But $f'(x) = e^{-x}(cosx + sinx) - e^{-x}(sinx - cosx)$

$= e^{-x}[cosx + sinx - sinx + cosx]$

$= e^{-x}[2cosx]$

$$\therefore f'(x_0) = e^{-x_0}\,[2cosx_0]$$

From (1) $e^{-x_0}[2cosx_0] = 0$

Or $e^{-x_0}cosx_0 = 0$

Or $cosx_0 = 0 \qquad (\therefore \quad e^{-x_0} \ddagger 0)$

Or $x_0 = \frac{\pi}{2}$ & $\frac{\pi}{2}\;\epsilon \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

8.6 Summary

Rolle's Theorem and its geometrical meaning and applications i.e. to find roots of f '(x)=0 if roots of f(x) =0 is given. Langrang's mean value theorem and its geometrical meaning, Cauchy mean value theorem, the general mean value theorem and deduction of Langrang's form of mean value theorem and Cauchy form of mean value theorem from the general mean value theorem is discussed. Applicability and conclusion of the above theorems for the given function f(x) or functions f(x) and g(x) in a given interval is also discussed.

8.7 Terminal Questions

1. Verify the truth of Rolle's theorem for the functions

(a) $f(x) = x^2 - 3x + 2$ on [1,2]

(b) $f(x) = (x-1)(x-2)(x-3)$ on [1,3]

(c) $f(x) = \sin x$ on $[0,\pi]$

2. The function $f(x) = 4x^3 + x^2 - 4x - 1$ has roots 1 and -1. Find the root of the derivative $f'(x)$mentioned in Rolle's theorem.

3. Verify Lagrange's formula for the function $f(x) = 2x - x^2$ on [0,1].

4. Apply Lagrange theorem and prove the inequalities

(i) $e^x \geq 1 + x$ **(ii)** $\ln(1+x) < x$ $(x > 0)$

(iii) $b^n - a^n < nb^{n-1}(b-a)$ for $(b > a)$

5. Using Cauchy's mean value theorem show that $\lim_{x\to 0} \frac{Sin\ x}{x} = 1$.

6. Write the Cauchy formula for the functions $f(x) = x^2$, $g(x) = x^3$ on [1, 2].

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Rough Work