Uttar Pradesh Rajarshi Tandon
Open University

Bachelor of Science

UGMM-102

Analytical Geometry

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BLOCK

1

CONIC SECTION

UNIT-1

Conic section

UNIT-2

Curve Tracing

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BLOCK INTRODUCTION

Unit-1 **Conic Section :** Homogeneous equation of second degree and conditions on it to represent different types of conics. Polar coordinates. Polar equation of a line, parabola, ellipse and hyperbola when focus is taken as pole. Polar equations of the chord joining two points.

Unit-2 **Curve Tracing :** Tangent, normal polar (chord of contact), pair of tangent lines, asymptotes, Tracing of a conic.

UNIT-I CONIC SECTIONS

Structure

1.1 INTRODUCTION

In this unit, our aim is to re-acquaint with some essential elements of two dimensional geometry.The French philosopher mathematician **Rene Descartes** (1596--1650) was the first to realize that geometrical ideas can be translated into algebraic relations. The combination of Algebra and Plane Geometry came to be known as **Coordinate Geometry** or **Analytical Geometry**. A basic necessity for the study of Coordinate Geometry is thus, the introduction of a coordinate system and to define coordinates in the concerned space. We will briefly touch upon the distance formula and various ways of representing a straight line algebraically. Then we shall look at the polar representation of a point in the plane. Next, we will talk about symmetry with respect to origin or a coordinate axis. Finally, we shall consider some ways in which a coordinate system can be transformed.This collection of topics may seem random to us .

We have read about lines, angles and rectilinear figures in geometry.

Recall that a line isthe join of two points ina plane continuing e ndlesslyin both directions. We have also seen that graphs of linear equations,

Which came out to be straight lines. Interestingly, the reverse problem Of the above is finding the equations of straight lines, under different conditions in a plane. The Analytical Geometry, more commonly called Coordinate geomatry, comes to our help in this regard.

In this unit we shall find equations of a straight line in different forms And try to solve t h e problem based on these.

1.2 OBJECTIVES

After studying this unit you should be able to find:

1. Equation of a pair of lines passing through the origin
2. Angle between pair of lines
3. Bisectors of the angles between two lines.
4. Pair of bisectors of angles between the pair of lines.
5. Equation of pair of lines passing through given point and parallel/perpendicular to the given pair of lines.
6. Condition for perpendicular and coincident lines
7. Area of the triangle formed by given pair of lines and a line.
8. Pair of lines of second degree general equation
9. Conditions for parallel lines distance between them.
10. Point of intersection of the pair of lines.
11. Homogeneous equation of second degree equation w.r.t a 1^{st}degree equation in x and y.
12. Relate the polar coordinates and cartesian coordinates of a point.
13. Equation of bisectors
14. Obtain the polar form of an equation and the equation of Directrices
15. Equation of the chord when the vectorial angles of the extremities are given

1.3 EQUATION OF A PAIR OF STRAIGHT LINES

Definition: Let $L_1 = 0$, $L_2 = 0$be the equations of two straight lines. If $P(x_1,y_1)$ is a point on L_1 then it satisfies the equation $L_1=0$. Similarly, if $P(x_1,y_1)$ is a point on $L_2 = 0$,then it satisfies the equation. If $P(x_1,y_1)$ lies on L_1or L_2, then $P(x_1,y_1)$ satisfies the equation $L_1L_2 = 0$.

$\therefore L_1 L_2 = 0$ represents the pair of straight lines $L_1 = 0$ and $L_2 = 0$ and the joint equation of

$L_1= 0$ and $L_2= 0$ is given by $L_1.L_2 = 0$.-----(1)

On expanding equation (1) we get and equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ which is a second degree (non-homogeneous) equation in x and y.

1.4 GENERAL EQUATION OF SECOND DEGREE

General equation of second degree is $ax^2+by^2+2hxy+2gx+2fy+c = 0$, if this equation represents a pair of straight lines, suppose that general equation of straight lines be, $l_1x+m_1y+n_1 =0$ and $l_2x+m_2y+n_2 = 0$ then product of these two lines

$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0 \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c$.

Compairing the coefficients, we get

$l_1l_2= a$, $m_1m_2= b$, $n_1n_2 = c$, $l_1m_2 + l_2m_1 = 2h$, $m_1n_2 + m_2n_1= 2f$, $n_1l_2 + n_2l_1 = 2g$.

or, $a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0$$

That is if $\mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$then

Case(1). $ab - h^2 = 0$, it represents a pair of straight lines.

Case(2.) $ab - h^2 \neq 0$, it represents a pair of intersecting straight lines.

Case(3) . $ab - h^2 < 0$, it represents a pair of real or imaginary straight lines.

Case(2.) $ab - h^2 > 0$, it represents a point.

Note: Here a, b, c, stand for coefficients of x^2, y^2 and constant term respectively and f, g, h stand for half of the coefficients of y, x and xy.

Again, If **abc + 2fgh - af²- bg²- ch² ≠ 0**then

Case (1). $h = 0, a = b$, then it represent a circle.

Case (2). $ab - h^2 = 0$, it represents a parabola.

Case (3). $ab - h^2 > 0$, it represents an ellipse.

Case (4). $ab - h^2 < 0$, it represents a hyperbola.

Case (5). $ab - h^2 < 0 \; and \; a + b = 0$, it represents a rectangular hyperbola.

Note (1). If **abc + 2fgh - af²- bg²- ch² ≠ 0**in the above equationthen it represents the **non- degenerate conic**.

(2). If **abc + 2fgh - af²- bg²- ch² = 0**in the above equationthen it represents the **degenerate conic**.

Example 1: What conic does $13x^2 - 18hxy + 37y^2 + 2x + 14y - 2 = 0$ represent?

Solution: Compare the given equation with

$ax^2 + 2hxy\, by^2 + 2gx + 2fy + c = 0$,

we get that $a = 13, h = -9, b = 37, g = 1, f = 7, c = -2$.

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 13 \times 37 \times -2 + 2 \times 7 \times 1 \times -9 - 13 \times 7 \times 7 - 37 \times 1 \times 1 + 2 \times -9 \times -9$$

$$= -962 - 126 - 637 - 37 + 162 = -1600 \neq 0$$

also, $h^2 = (-9)^2 = 81 \; and \; ab = 13 \times 37 = 481$.

Here, $ab - h^2 = 400 > 0$. So, it represents an ellipse.

Example 2: What conic is represented by the equation $\sqrt{ax} + \sqrt{by} = 1$?

Solution: The given conic is $\sqrt{ax} + \sqrt{by} = 1$. Squaring on both sides then $ax + by + 2\sqrt{axby} = 1$, or, $ax + by - 1 = -2\sqrt{axby}$

Now squaring on both sides, we get $(ax + by - 1)^2 = 4axby$

$$= a^2x^2 - 2abxy + b^2y^2 - 2ax - 2by + 1 = 0$$

Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2$

$= a^2b^2 - 2a^2b^2 - a^2b^2 - a^2b^2 - a^2b^2 = -4a^2b^2 \neq 0, and\; h^2 = a^2b^2$.

So, we have $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

and h^2 - $a^2b^2 = 0$. Hence, the given equation represents a parabola.

Example 3: If the equation $x^2 - y^2 - 2x + 2y + \lambda = 0$ represents a degenerate conic then find the value of λ.

Solution: For degenerate conic,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Comparing the given equation of conic with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we get,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 1 \times -1 \times \lambda + 1 \times (-1)^2 - 0 = 0 \Rightarrow -\lambda - 1 + 1 = 0$ So, $\lambda = 0$

Example 4: For what value of λ the equation of conic $2xy + 4x - 6y + \lambda = 0$ represents two intersecting straight lines? if $\lambda = 17$ then this equation represent ?

Solution: Comparing the given equation of conic with $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, since we know that for two intersecting lines, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and $ab - h^2 \neq 0$

Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2$

$= 0 + 2 \times -3 \times 2 \times 1 - 0 - 0 - \lambda(1)^2 = -12 - \lambda = 0.$

So, $\lambda = -12.$

For $\lambda = 17$, the given equation of conic $2xy + 4x - 6y + 17 = 0$

According to the above condition, here $c = 17$,

so $abc + 2fgh - af^2 - bg^2 - ch^2 = -29 \neq 0$

and $ab - h^2 = 0 - 1 = -1 < 0.$

Hence the given equation represents a Hyperbola.

1.5 HOMOGENOUS EQUATION OF PAIR OF STRAIGHT LINES

$ax^2 + by^2 + 2hxy = 0$ is a homogenous equation of second degree and it will represent a pair of straight lines or two straight lines passing through the origin.suppose these lines be $y = m_1x$and $y = m_2x$.

Therefore $(y - m_1x).(y - m_2x) = 0$ represent a pair of straight lines

i.e. $y^2 - (m_1 + m_2).xy + m_1m_2x^2 = 0$………………………(1)

Also, $ax^2 + by^2 + 2hxy = 0$ or, $y^2 + \frac{a}{b} x^2 + \frac{2h}{b} xy = 0$ …..(2)

We compare the coefficients of equation (1) and (2) we get

$m_1 + m_2 = \frac{-2h}{b}$, and $m_1m_2 = \frac{a}{b}$ ……………(3)

1.6 ANGLE BETWEEN THE LINES $ax^2 + by^2 + 2hxy = 0$

Suppose that the two lines represented by $ax^2 + by^2 + 2hxy = 0$ be

$y= m_1x$ and $y= m_2x$.

Suppose that angle between the lines $y= m_1x$ and $y= m_2x$ be α,

Since$ax^2 + by^2 + 2hxy = 0 = (y - m_1x).(y - m_2x)$

i.e. $y^2 - (m_1 + m_2).xy + m_1m_2x^2 = 0$………………………(1)

Also, $ax^2 + by^2 + 2hxy = 0$ or, $y^2 +\frac{a}{b} x^2 + \frac{2h}{b}xy = 0$ …..(2)

We compare the coefficients of equation (1) and (2) we get

$$m_1 +m_2 =\frac{-2h}{b}, \text{ and } \quad m_1m_2 =\frac{a}{b} \text{ ……………(3)}$$

then

$$\tan\alpha = \frac{m1-m2}{1+m1m2}=\frac{\sqrt{((m1+m2)^2-4m1m2)}}{1+m1m2} = \frac{\sqrt{((\frac{4h^2}{b})2-(\frac{4a}{b}))}}{1+(b/a)}$$

Therefore, $\tan\alpha = 2\,\frac{\sqrt{h^2-ab}}{a+b}$.

Then, $\alpha = \tan^{-1}(\frac{2\sqrt{(h^2-ab)}}{a+b})$………..(4)

The homogenous equation of second degree $ax^2 + by^2 + 2hxy = 0$ represent a pair of straight lines or two straight lines passing through the origin. The lines are real and distinct, coincident or imaginary according as $(h^2 - ab) > 0, = 0 \; or \; < 0$.

1.7 CONDITION FOR THE LINES TO BE PERPENDICULAR/ PARALLEL

Case (1): If the lines be perpendicular,then $\alpha=90^\circ$ Therefore,

$$\tan\alpha = \tan 90 = \infty =\frac{2\sqrt{(h^2-ab)}}{a+b}, \text{ i.e. } a+b = 0$$

i.e. [coefficient of x^2 + coefficient of y^2=o]

Case (2): If the lines be parallel ,then $\alpha=0$. Therefore,

$$\tan\alpha = \tan 0 = 0 =\frac{2\sqrt{(h^2-ab)}}{a+b}, \text{ then } \sqrt{(h^2 - ab)} = 0$$

then $h^2 = ab$ i.e. (½ coefficient of xy)²

= product of coefficient of x^2 and coefficient of y^2.

1.8 EQUATION OF ANY TWO PERPENDICULAR LINES THOUGH THE ORIGIN

If the lines represented by $ax^2 + 2hxy + by^2 = 0$ be perpendicular,

then $a + b = 0$ or $b = -a$. Hence the equation becomes $ax^2+2hxy-ay^2 = 0$.

i.e. $x^2 - y^2 + 2hxy/a = 0$ or $x^2 - y^2 + pxy = 0$. Where p is any constant.

Example 5: What curve does the equation $x^2 - 5xy + 4y^2 = 0$ represent?

Solution: Since, $x^2 - 5xy + 4y^2 = 0$

Or, $x^2 - xy - 4xy + 4y^2 = x(x - y) - 4y(x - y) = 0$

Or, $(x - 4y)(x - y) = 0$ Or, $x - 4y = 0$ or, $x - y = 0$

Which are straight lines, hence $x^2 - 5xy + 4y^2= 0$ represents a pair of straight lines.

Example 6: Find the angle between the pair of straight lines $x^2 + 4y^2 - 7xy = 0$.

Solution: Suppose α be the angle between the pair of straight lines then

$\tan\alpha = (2\sqrt{(h^2\text{-ab})})/((a+b))= 2\sqrt{\left(\left(\frac{-7}{2}\right)^2 - \frac{1\times 4}{1+4}\right)} = \frac{2\sqrt{(49-16/4)}}{5} = \frac{\sqrt{33}}{5}$

Therefore, $\alpha = \tan^{-1}\left(\frac{\sqrt{33}}{5}\right)$

Example 7: Find the equation of the pair of straight lines through the origin which are perpendicular to the lines represented by

$ax^2 + 2hxy + by^2 = 0$

Solution : If the lines represented by $ax^2 + 2hxy + by^2 = 0$ are

$y - m_1x = 0$ and $y- m_2x = 0$ then $m_1+m_2 = \frac{-2h}{b}$, $m_1m_2 = \frac{a}{b}$

the lines perpendicular to them and passing through origin will be

$y = (-1)/(m_1 x\ )$and $y =\frac{-1}{m_2 x}$.

Their combined equation is $(m_1y + x) (m_2y + x) = 0$

Or, $m_1 m_2 y^2 + (m_1 + m_2) xy + x^2 = 0$

Or, $(\frac{a}{b})y^2 + (\frac{-2h}{b})xy + x^2 = bx^2 - 2hxy + ay^2 = 0$

Example 8 : Find the equation of the pair of the straight lines through the origin which are perpendicular to the lines represented by

$2x^2 - 5xy + y^2 = 0$

Solution : since equation of pair of straight lines be $2x^2 - 5xy + y^2= 0$ ………….(1)

Suppose that equation of pair of straight lines be $y - m_1x = 0$

And $y - m_2x =0$

then their combined equation represented by equation (1) now its perpendicular

equations be $y = \frac{-1}{m_1}x$ and $y = (-1/m_2)x$, therefore, combined equation is

$(m_1y + x)(m_2y + x) = 0$

Therefore, $m_1m_2y^2 + (m_1 + m_2) xy + x^2 = 2y^2 + 5xy + x^2 = 0$

Equation of perpendicular lines be $2y^2 + 5xy + x^2 = 0$

Example 9: Prove that the product of the perpendiculars drawn from the

point (x_1, y_1) on the lines represented by

$ax^2 + 2hxy + by^2 = 0$ is $(ax_1^2 + 2hx_1y_1 + by_1^2)/[(a - b)^2 + 4b^2]^{1/2}$.

Proof: suppose that $ax^2 + 2hxy + by^2 = (y - m_1x)(y - m_2x)$

i.e. $m_1 + m_2 = -2h/b$ and $m_1m_2 = a/b$

If p_1 and p_2 be the perpendiculars to them from the point

(x_1, y_1) then $p_1p_2 = \frac{y_1-m_1x_1}{\sqrt{1+m_1^2}} \times \frac{y_1-m_2x_1}{\sqrt{1+m_2^2}}$

Therefore, $p_1p_2 = \frac{y_1^2+ m_1m_2x_1^2- (m_1+ m_2)x_1y_1}{\{1+ m_1^2+m_2^2+(m_1m_2)^2\}^{1/2}}$

$$= \frac{y_1^2-x_1y_1\left(-\frac{2h}{b}\right)+\frac{a}{b}x_1^2}{\{1+(m_1+m_2)^2-2(m_1m_2)+ m_1^2m_2^2\}^{1/2}}$$

$= (ax_1^2 + 2hx_1y_1 + by_1^2)/b\{1 + 4h^2/b^2 - 2(a/b) + (a/b)^2\}^{1/2}$

$= (ax_1^2 + 2hx_1y_1 + by_1^2)/\{(a - b)^2 + 4h^2\}^{½}$

Example 10: Prove that angle between the straight lines represented by

$(x^2 + y^2)\sin^2\alpha = (x\cos\theta - y\sin\theta)^2$ is 2α.

Proof: The given equation can be written as

$x^2(\sin^2\alpha - \cos^2\theta) + 2xy\sin\theta\cos\theta + y^2(\sin^2\alpha - \sin^2\theta) = 0$

This equation of the form $ax^2 + 2hxy + by^2 = 0$ and hence if φ be angle between them, then $\tan\varphi = 2\sqrt{(h^2 - ab)}/(a + b)$. Therefore,

$2\sqrt{(h^2 - ab)} = 2[\sin^2\theta\cos^2\theta - (\sin^2\alpha - \cos^2\theta)(\sin^2\alpha - \sin^2\theta)]^{½} = \sin2\alpha$

Since, $a + b = \sin^2\alpha - \cos^2\theta + \sin^2\alpha - \sin^2\theta = 2\sin^2\alpha - 1 = -\cos2\alpha$

Therefore, $\tan\varphi = \sin2\alpha/-\cos2\alpha = -\tan2\alpha$,

$\varphi = -2\alpha$ or $\varphi=2\alpha$.

Example 11: Find the condition that one of the lines given by the equation $ax^2 + 2hxy + by^2 = 0$ common to lines given by

$a'x^2 + 2h'xy + b'y^2 = 0$

Proof: suppose that $y = mx$ be a common line to both the pair then putting $y = mx$ in the two equations, we get

$ax^2 + 2hmx^2 + b'm^2x^2 = 0$ or, $bm^2 + 2hm + a = 0$ ………..(1)

$a'x^2 + 2h'mx^2 + b'm^2x^2 = 0$ or , $b'm^2 + 2h'm + a' = 0$……….(2)

$$\frac{m^2}{(a'h - a\,h')} = \frac{2m}{ab' - a'\,b} = \frac{1}{bh' - b'h}$$

$m = \frac{2(a'h - ah')}{ab' - a'b}$, and $m = \frac{ab' - a'b}{2(bh' - b'h)}$. So, $\frac{2(a'h - ah')}{ab' - a'b} = \frac{ab' - a'b}{2(bh' - b'h)}$

Therefore, $(ab' - a'b)\,(ab' - a'b) = 4(a'h - ah')\,(bh' - b'h)$

Example 12: Find the condition that one of the lines given by the equation $ax^2 + 2hxy + by^2 = 0$ be perpendicular to one of those lines given by $a'x^2 + 2h'xy + b'y^2 = 0$

Proof: Suppose that one of the line given by the first pair be $y = mx$, by given condition one of the line given by the second pair should be $y = \frac{-1}{m}x$.

Therefore, $bm^2 + 2hm + a = 0$ and $a'm^2 - 2h'm + b' = 0$

$$\frac{m^2}{2(hb' + h'b)} = \frac{m}{aa' - bb'} = \frac{1}{2(ha' + h'a)}$$

$$m = \frac{2(hb' + h'b)}{aa' - bb'} \text{ and, } m = \frac{aa' - bb'}{2(ha' + h'a)}.$$

$$\text{So, } \frac{2(hb' + h'b)}{aa' - bb'} = \frac{aa' - bb'}{2(ha' + h'a)}$$

Therefore, $(aa' - bb')^2 = 4(hb + h'b)(ha' + h'a)$

Example 13(a): Find the angle between the lines given by the equation $\lambda y^2 + (1 - \lambda^2)xy - \lambda x^2 = 0$

Solution: Since, $a + b = \lambda + (-\lambda) = 0$ hence, $\theta = 90$

(b): Find the angle between the pair of straight lines $y^2\sin^2\theta - xy\sin^2\theta + x^2(\cos^2\theta - 1) = 0$

Solution: Since, $a + b = \sin^2\theta + \cos^2\theta - 1 = 1 - 1 = 0$. Therefore, $\theta = 90$ (lines are perpendicular)

Note: If the represented by $ax^2 + 2hxy + by^2 = 0$ be perpendicular, then $a + b = 0$ or, $b = -a$, hence the equation becomes $ax^2 + 2hxy - ay^2 = 0$ or $x^2 + (\frac{2h}{b})xy - y^2 = 0$

Or, $x^2 + pxy - y^2 = 0$, where p is any constant.

1.9 EQUATION OF BISECTORS

Equation of pair of straight lines which passes through origin is $ax^2 + by^2 + 2hxy = 0$, if the lines represented by the given equation be $y = m_1x$ and $y = m_2x$

then $m_1 + m_2 = \frac{-2h}{b}$, $m_1m_2 = \frac{a}{b}$ equation of their bisectors are $\frac{y - m1x}{\sqrt{1 + m1^2}} = \frac{y - m2x}{\sqrt{1} + m2^2}$

$$\left(\frac{y - m1x}{\sqrt{(1 + m1^2)}} + \frac{y - m2x}{\sqrt{(1} + m2^2)}\right)\left(\frac{y - m1x}{\sqrt{(1} + m1^2)} - \frac{y - m2x}{\sqrt{(1 + m2^2)}}\right) = 0$$

Or, $\frac{(y - m1x)^2}{(1 + m1^2)} - \frac{(y - m2x)^2}{(1 + m2^2)} = 0$. By sloving this we get

$$-y^2(m_1 + m_2) + x^2(m_1 + m_2) - 2xy(1 - m_1m_2) = 0$$

$$\text{Or } (x^2 - y^2)(-2h/b) = 2xy(1 - a/b)$$

Or, $(x^2 - y^2)/(a-b) = xy/h$

Note : Since sum of the coefficients of x^2 and y^2 in the above equation is zero.

i.e. **a+b=0**,hence the bisectors are perpendicular.

1.10 GENERAL EQUATION OF SECOND DEGREE

General equation of second degree is $ax^2+by^2+2hxy+2gx+2fy+c = 0$, if this equation represents a pair of straight lines, suppose that general equation of straight lines be, $l_1x+m_1y+n_1 = 0$ and $l_2x+m_2y+n_2 = 0$ then product of these two lines

$(l_1x + m_1y + n_1)(l_2x + m_2y + n_2) = 0 \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c.$

Compairing the coefficients, we get

$l_1l_2 = a$, $m_1m_2 = b$, $n_1n_2 = c$, $l_1m_2 + l_2m_1 = 2h$, $m_1n_2 + m_2n_1 = 2f$, $n_1l_2 + n_2l_1 = 2g$.

or, $a(bc - f^2) - h(hc - gf) + g(hf - bg) = 0$

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0$$

that is $\mathbf{abc + 2fgh - af^2 - bg^2 - ch^2 = 0}$ this the required condition that $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represent a pair of general linear equations.

Note: Here a, b, c, stand for coefficients of x^2, y^2 and constant term respectively and f, g, h stand for half of the coefficients of y, x and xy.

Example14: Determine the equation of bisectors of the angle between the lines

$4x^2 - 16xy - 7y^2 = 0$

Solution: Since equation of angular bisector is $(x^2 - y^2)/a - b = xy/h$

Therefore, $(x^2 - y^2)/4 - (-7) = xy/-8$

Or, $8(x^2 - y^2) + 11xy = 0$

Example15: If $(a + b)^2 = 4h^2$, prove that one of the lines given by the equation

$ax^2 + 2hxy + by^2 = 0$ will bisect the angle between the coordinate axes.

Proof: The bisectors of the angle between the co-ordinate axes will make an angle 45 degree or 135 degree with the positive direction of x-axis and hence their equation are $y = \pm x$.

If $y = x$ belongs to $ax^2+2hxy+by^2 = 0$.Then $ax^2 + 2hx^2 + bx^2 = 0$

$\Rightarrow$ $a + b = -2h$. Similarly, if $y= -x$ belongs to $ax^2+2hxy+by^2 = 0$,

then $ax^2 - 2hx^2 + bx^2 = 0 \Rightarrow a+b = 2h$

Therefore, $a + b = \pm 2h$. Squaring of these we get $(a + b)^2 = 4h^2$.

Example16: Show that the line $y = mx$ bisects the angle between the lines

$ax^2 + 2hxy + by^2 = 0$ if $h(1 - m^2)= -m(a - b)$.

Proof: Since bisectors of the given pair of the lines is $(x^2 - y^2)/(a - b) = xy/-h$,

if $y = mx$ be one of the bisector then it satisfy the above relation so,

$(x^2 - m^2x^2)/(a -b) = x.mx/-h \Rightarrow (1 - m^2)/(a - b) = m/-h$

$\Rightarrow$ $h(1 - m^2) + m(a - b) = 0$

Example 17: Prove that the straight lines $ax^2 + 2hxy + by^2 = 0$ have the same pair of bisectors as those of the lines given by $a^2x^2 + 2h(a + b)xy + b^2y^2 = 0$

Proof: The bisector of the pair of the lines $a^2x^2 + 2hxy + b^2y^2 =0$ is given by

$\frac{x^2-y^2}{a^2-b^2} = \frac{xy}{h(a+b)}$ or, $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

but this is a equation of the bisectors of the lines given by the pair of the straight lines of the first equation.

Example 18: Prove that angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$

and one of the lines $ax^2 + hxy + by^2 + \lambda(x^2 + y^2) = 0$ is equal to the angle between

the other two lines of the system.

Proof: Since the two pairs have the bisectors $\frac{x^2-y^2}{(a+\lambda)-(b+\lambda)} = \frac{xy}{h}$

or, $\frac{x^2-y^2}{a+b} = \frac{xy}{h}$ and this is also the equation of bisector of first pair.

Examples19: If the pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ is such that

each pair bisects the angle between the other pair, prove that $pq = -1$.

Proof: Equation of the bisectors of $x^2 - 2pxy - y^2 = 0$ are $\frac{x^2-y^2}{1-(-1)} = \frac{xy}{-p}$

$\Rightarrow$ $x^2 - y^2 = \frac{-2xy}{p}$. -------------------- (1)

But the bisectors of first pair of lines are given by the second pair i.e,

$x^2 - y^2 = 2qxy$ --------------------- (2)

Compairing (1) and (2) we get $-\frac{1}{p} = q$ or, $pq = -1$

Example 20: For what value of λ does the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + \lambda = 0$ represents pair of straight lines. Find their equations.

Solution: Here, $a = 12$, $b = 2$, $c = \lambda$, $f = \frac{5}{2}$, $g = \frac{11}{2}$, $h = -5$. Putting these values in the condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

We get, $24\lambda + \frac{275}{2} - 75 - \frac{121}{2} - 25\lambda = 0 \Rightarrow \lambda = 2$.

Therefore, $12x^2-10xy+2y^2+11x-5y+2 = 0$.........(1)

represents a pair of straight lines and suppose that two straight lines be

$y = m_1x + c_1$, $y = m_2x + c_2$ then $(y - m_1x - c_1)(y - m_2x - c_2) = 0$

is same as equation (1) then

$y^2 + m_1m_2x^2 - (m_1 + m_2)xy - (m_1c_2 + m_2c_1)x - (c_1 + c_2)y + c_1c_2 = 0$ ......(2)

Compairing equation (1) and (2) we get

$m_1m_2 = \frac{12}{2}$, $m_1 + m_2 = 5$, $m_1c_2 + m_2c_1 = -\frac{11}{2}$, $c_1 + c_2 = \frac{5}{2}$, $c_1c_2 = 1$.

Therefore, $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$\Rightarrow 25 - 24 = 1$

Therefore, $m_1 - m_2 = \pm 1$, $m_1 + m_2 = 5$

$\Rightarrow m_1 = 3$ or 2 and $m_2 = 2$ or 3

Similarly, $(c_1 - c_2)^2 = (c_1 + c_2) - 4c_1c_2 = \frac{25}{4} - 4 \times 1 = \frac{9}{4}$

$\Rightarrow (c_1 - c_2)^2 = \frac{9}{4}$, $c_1 - c_2 = \pm\frac{3}{2}$, $c_1 + c_2 = \frac{5}{2}$

$\Rightarrow c_1 = 2$, or $\frac{1}{2}$, and $c_2 = \frac{1}{2}$, or 2.

Therefore, equation of straight lines be $y = 3x + 2$, and $y = 2x + \frac{1}{2}$

$\Rightarrow 3x - y + 2 = 0$ and $4x - 2y + 1 = 0$

Example 21: If $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ represents a pair of straight lines

then find the value of λ.

Solution: Condition for pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ then

$1\times \lambda \times 2+2\times(-\frac{5}{2})\times\frac{3}{2}\times(-\frac{3}{2}) - 1 \times (- \frac{5}{2})^2-\lambda(\frac{3}{2})^2 - 2\left(-\frac{3}{2}\right)^2 = 0$

$2\lambda+\frac{45}{4} - \frac{25}{4} - \frac{9\lambda}{4} - \frac{9}{2} = 0$

$-\frac{\lambda}{4} + \frac{1}{2} = 0 \Rightarrow \lambda = 2$

Example 22: If $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines

then find the value of λ .

Solution: Conditions for pair of straight lines is

Abc + 2fgh - af² - bg² - ch² = 0 then

$\lambda\times12\times-3+2\times(-8)\times(-5)-\lambda\times(-8)^2-12\times(-\frac{5}{2})^2 - (-3)(-5)^2 = 0$

or, $-36\lambda+80-64\lambda-\frac{12\times25}{4} - 75 = 0$

or, $-100\lambda = -150$ or, $\lambda = 2$

Example 23: Show that the equation $12x^2 - 10xy + 2y^2 + 11x - 5y + 2= 0$ represents a pair of straight lines. Find their equations.

Solution: Since condition for pair of straight lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$12\times2\times2 + (-5)\times\frac{11}{2}\times(-\frac{5}{2})-12\times(-\frac{5}{2})^2- 2\times(\frac{11}{2})^2- 2\times(-5)^2 = 0$

Suppose equation of two lines be $y = m_1x + c_1$ and $y = m_2x + c_2$

Then $(y- m_1x + c_1)(y - m_2x + c_2) = 0$

Or, $y^2 + m_1m_2x^2 - (m_1 + m_2)xy - (c_1 + c_2)y - (m_1c_2 + m_2c_1)x + c_1c_2 = 0$

Therefore, $m_1m_2 = 6$, $m_1 + m_2 = 5$, $c_1 + c_2 = \frac{5}{2}$, $c_1c_2 = 1$

Therefore, $(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = 25 - 24 = 1$

$\Rightarrow m_1 - m_2 = \pm 1$

$m_1 = 3$ or 2 and $m_2 = 2$ or 3

also, $(c_1 - c_2)^2 = (c_1 + c_2) - 4c_1c_2 = \frac{25}{4} - 4 \times 1 = \frac{9}{4}$

or, $c_1 - c_2 = \pm \frac{3}{2}$, $c_1 + c_2 = \frac{5}{2}$

or, $c_1 = 2$ or 1 and $c_2 = \frac{1}{2}$ or 2

Therefore, lines be $y = 3x + 2$ and $y = 2x + \frac{1}{2}$

Example 24: Prove that the point of intersection of the lines given by the equation $x^2 - 5xy + 4y^2 + x + 2y - 2 = 0$ is $(-\frac{6}{5}, -\frac{46}{40})$

Solution: Suppose that equation of two straight lines be $y = m_1x + c_1$ and $y = m_2x + c_2$,then $(y - m_1x + c_1)(y - m_2x + c_2) = 0$

$y^2 + m_1m_2x^2 - (m_1 + m_2)xy - (c_1 + c_2)y - (m_1c_2 + m_2c_1)x + c_1c_2 = 0$

Therefore, $m_1m_2 = \frac{1}{4}$, $m_1 + m_2 = \frac{5}{4}$, $c_1 + c_2 = -\frac{1}{2}$, $c_1c_2 = -\frac{1}{2}$

$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2 = (\frac{5}{4})^2 - 4\times\frac{1}{4} = \frac{25-16}{4} = \frac{9}{4}$

$m_1 - m_2 = \pm\frac{3}{2}$, $m_1 + m_2 = \frac{5}{4}$

$m_1 = \frac{11}{8}$, or $\frac{1}{8}$, $m_2 = \frac{1}{8}$ or ,$\frac{11}{8}$

Also, $(c_1 - c_2)^2 = (c_1 + c_2) - 4c_1c_2 = (-\frac{1}{2})^2 - 4\times-\frac{1}{2} = \frac{1}{4} + 2 = \frac{9}{4}$

$c_1 - c_2 = \pm\frac{3}{2}$, $c_1 + c_2 = -\frac{1}{2}$, $c_1 = \frac{1}{2}$ or -1 and $c_2 = -1$ or $\frac{1}{2}$

Therefore, equation of lines: $y = \frac{11}{8}x + \frac{1}{2}$, $y = \frac{1}{8}x - 1$

Or, $11x - 8y + 4 = 0$ and $x - 8y - 8 = 0$

Its intersection point is $(-\frac{6}{5}, -\frac{46}{40})$.

Example 25: Prove that the equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$

represents two parallel lines and find the distance between them.

Proof: Condition for parallel lines be $h^2 - ab = 0$

Therefore, $(4)^2 - 2\times8 = 0$, so lines be parallel to each other.

Also, $\frac{8}{2}x^2 + \frac{8}{2}xy + y^2 + \frac{26}{2}x + \frac{13}{2}y + \frac{15}{2} = 0$

Therefore, $y^2 + 4x^2 + 4xy + 13x + \frac{13}{2}y + \frac{15}{2} = 0$ gives two parallel straight lines.

Lines be the $2x+y+5 = 0$ and $2x+y+\frac{3}{2} = 0$. If p_1 and p_2 be their distances from origin, then the distance between them is

$P = p_1 \sim p_2 = \frac{5}{\sqrt{(4+1)}} \sim \frac{3/2}{\sqrt{(4+1)}} = \frac{7}{2\sqrt{5}}$

Check your progress

1. What is the condition for given lines $y = m_1 x + c_1$ and $Ax + By + C = 0$

   (a) to be parallel? (b) to be perpendicular?

2. Find the angle between the lines $2x + 3y = 7$ and $4x + 5y = 14$.

3. (a) Write the condition that the two lines

   $A_1x + B_1y + C_1 = 0$ and $A_2x + B_2y + C_2 = 0$

   (i) parallel.

   (ii) perpendicular.

   (b) Are the straight lines $x - 3y = 7$ and $2x - 6y - 16 = 0$, parallel?

   (c) Are the straight lines $x = y + 1$ and $x = -y + 1$ perpendicular?

4. Prove that the straight lines $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ have same pair of bisectors for all values of λ. Interpret the case $\lambda = -(a+b)$.

5. Show that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$ and one of the lines given by $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$ is equal to the angle between the other two lines of the system.

6. Show that the line $Ax + By + C = 0$ and the two lines $(Ax + By)^2 - 3(Ay - Bx)^2 = 0$ form an equilateral triangle.

7. Show that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines.

8. For what values of h does the equation $3x^2 + 2hxy - 3y^2 - 40x + 30y - 75 = 0$ represents two parallel lines ?

9. Find the equation of two straight lines passing through (1,1) and parallel to the straight lines $2x^2 + 5xy + 3y^2 + 2x - 1 = 0$.

10. Show that the four lines given by the equations $3x^2 + 8xy - 3y^2 = 0$ and $3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ form a square. Find the equations of the diagonals of the square.

1.11 POLAR COORDINATES

In the late 17th century the mathematician Bernoulli invented a coordinate system which is different from, but intimately related to, the cartesian system. This is the polar coordinate system, and was used extensively by **Newton**. Now, let us see what polar coordinates are.

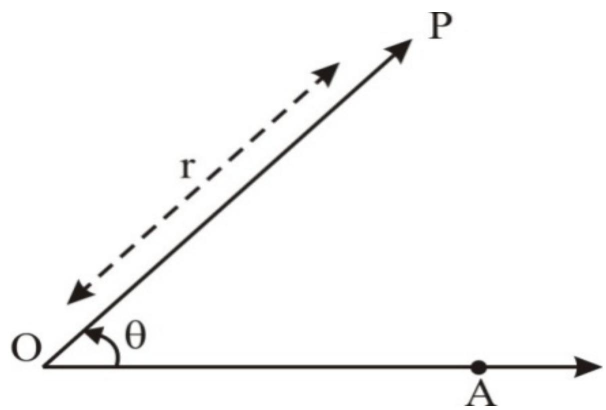


Fig. polar coordinate

To define them, we first fix a pole O and polar axis OA, as shown in Fig. 11. Then we can locate any point P in the plane, if we know the distance OP, say r, and the angle AOP, say θ radians. (Does this remind you of the geometric represent it by a pair (r, θ), where r is the "directed distance" of P from O and θ is ∠AOP, measured in radians in the anticlockwise direction. We use the term "directed distance" because r can be negative also. For instance, the point P in Fig. 1 can be represented by $\left(5, \frac{5\pi}{4}\right)$ or $\left(-5, -\frac{\pi}{4}\right)$. Note that by this method the point O corresponds to (r, θ), for any angle θ.

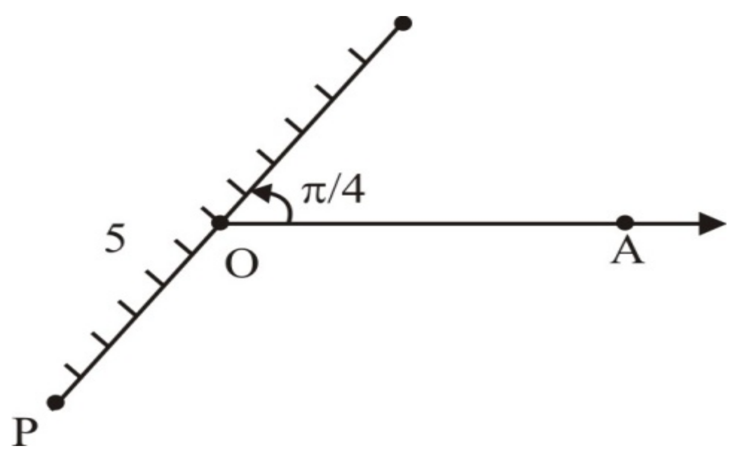


Fig.1 : P's polar coordinates are $\left(-5-\frac{\pi}{4}\right)$

Thus, for any point P we have a pair of real numbers (r, θ), for any angle θ. Thus, for any point P, we have a pair of real numbers (r, θ) that corresponds to it. They are called the polar coordinates.

Now, if we keep θ fixed, say θ = α, and let r take on all real values, we get the line OP (see Fig. 12), where ∠ AOP = α. Similarly, keeping r fixed, say r = a, and allowing θ to take all real values, the point P(r, θ) traces a circle to radius a, with centre at the pole (Fig. 14). Here note that a negative value of θ means that the angle has magnitude |θ|, but is taken in the clockwise direction. Thus, the point $\left(2, -\frac{\pi}{2}\right)$ is also represented by $\left(2, \frac{3\pi}{2}\right)$.

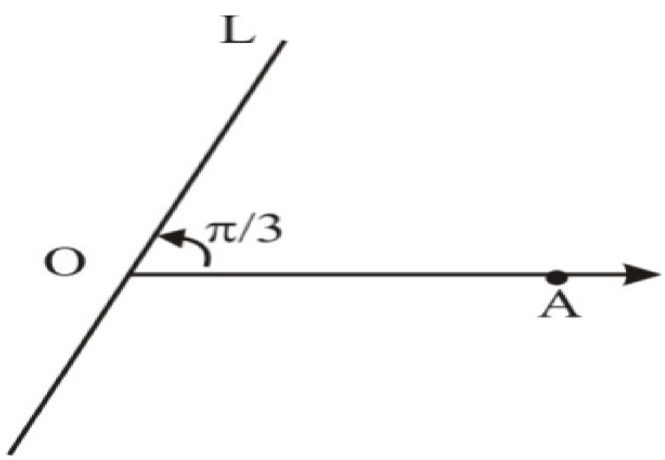


Fig.2 : The line L is given by θ = π/3.

As you have probably guessed, the cartesian and polar coordinates are very closely related

$$\left.\begin{array}{l} x = r\cos\theta, y = r\sin\theta, \text{or} \\ r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\dfrac{y}{x} \end{array}\right\}$$

Note that the origin and the pole are coinciding here. This is usually the situation. We use this relationship often while dealing with equations. The cartesian equation of the circle $x^2 + y^2 = 25$, reduces to the simple polar from r = 5. So we may prefer to use this similar form rather than the cartesian one. As θ is not mentioned, this means θ varies from 0 to 2π to 4π and so on.

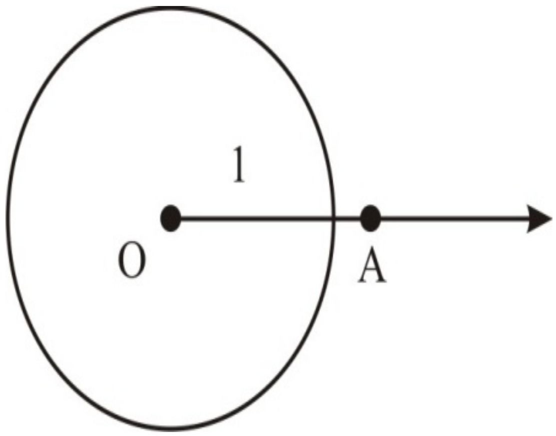


Fig.: circle r = 1

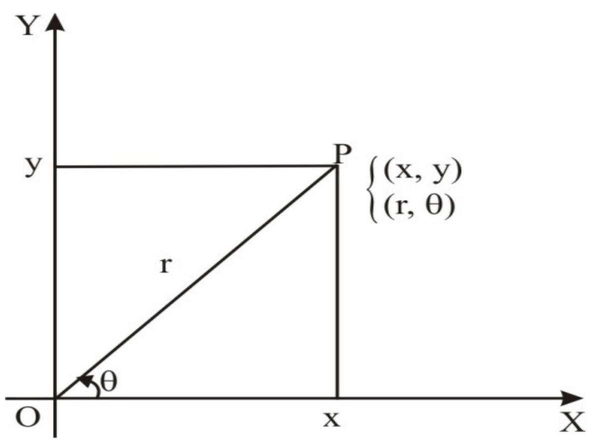


Fig. : Polar and Cartesian Coordinates.

Check your progress

1. Draw the graph of the curve r cos $\left(\theta - \frac{\pi}{4}\right)$ =0, as r and θ vary.

2. Find the cartesian forms of the equations

   (a) $r^2 = 3r \sin \theta$.

   (b) $r = a(1 - \cos \theta)$, where a is a constant.

1.12 POLAR EQUATION OF A CONIC WHEN THE FOCUS IS THE POLE

Let S is focus and ZM the directrix of the conic. Let SZ be perpendicular from S to the directrix, and let (r, θ) be the coordinate of a point P on the conic, S as a pole and SZ as initial line. From P draw PN

Fig. 1

and PM perpendiculars on SZ and ZM, and let e be the eccentricity of the conic and l be its semi- latus rectum SL. By the definition of the conic $SP|PM = e\ (constant)$

That is $, r = e.NZ = e\ (SZ - SN) = e\big(SL\big|e - SP \cos\theta\big)$

$$= l - er\ cos\ \theta$$

$Or, l|r = 1 + e\ cos\ \theta$. Which is the required polar equation.

Remark:1(a). The equation of the conic when the axis SZ is inclined at an angle α to the initial line is

$l|r = 1 + e \cos(\theta - \alpha),$

(b). The equation of the conic when the positive direction of the initial line is ZS instead of SZ, is $l|r = 1 - e\ cos\ \theta$.

1. If $e = 1,$ the conic is a parabola.
2. If $e < 1$, the conic is an ellipse.
3. If $e > 1$, the conic is a Hyperbola.
4. If $e = 0$, the conic is a circle.
5. If $e = \infty$, the conic is the pair of straight lines.

1.13 DIRECTRICES

The equation of the directrices of the conic

$l|r = 1 + e\ cos\ \theta$. If (r, θ)

be the coordinates of any point on the directrix ZM corresponding to the focus S, $r\ cos\ \theta = SZ = l|e$.

The equation of the directrix corresponding to the focus which is the pole, therefore, $l|r = e\ cos\ \theta$.

Fig. 2

Now to find the equation of the other directrix, let P' be a point (r, θ) on it and SZ' the perpendicular from S. Then,

$SZ' = SP' \cos(\pi - \theta) = -r\cos\theta,$

Now, $ZZ' = 2a|e \; and \; SZ = l|e.$

Hence, $SZ' = ZZ' - SZ = 2a|e - l|e$

$= 2l|\, e(1-e^2) - l|e = l(1+e^2)|e(1-e^2)$ Since,

$l = b^2\, |a = a(1-e^2).$

Equating the two values of $'$,

we get $-r\cos\theta = \; l(1+e^2)|e(1-e^2) \; or \; l|r$

$= -\, e(1-e^2)|(1+e^2)\cos\theta$

as the equation of the other directrix.

Example 1: Prove that the equations $l|r \; = 1 + \; e\cos\theta$ and $l|r \; = \; -1 + e\cos\theta$ represent the same conic.

Solution: The given equations are

$l|r \; = 1 + \; e\cos\theta$ ……….(1)

And $l|r \; = \; -1 + \; e\cos\theta.$ ………..(2)

We want to show that every point on the curve (1) also lies on the curve (2). Let $P(r_1, \theta_1)$ be any point on the curve (1) then,

$l|r_1 \; = 1 + \; e\cos\theta_1$…….(3)

Now also the coordinate of the point Pcan be expressed as $(-r_1, \pi + \theta_1)$ instead of (r_1, θ_1). This satisfies the equation (2)

$$l|(-r_1) \; = -1 + \; e\cos(\pi + \; \theta_1) - \; l|r_1$$

$= -1 - \; e\cos\theta_1 \; \Rightarrow l|r_1 \; = 1 + \; e\cos\theta_1.$

Which is same as (3). Thus every point on the curve (1) also lies on the curve (2). Similarly we can show that every point on the curve (2) also lies on the curve (1). Hence the both equations (1) and (2) represent the same curve.

Example 2: PSP' is the focal chord of the conic $l|r = 1 + e\cos\theta$. Prove that $1|SP + 1|SP' = 2|l$, where l is the semi- latus rectum.That is the semi-latus rectum is the harmonic mean between the segments of a focal chord.

Solution: Since equation of the conic is $|r = 1 + e\cos\theta$.

Let the chord PSP' make an angle α with the initial line. Then the vectorial angles of P and P' are α and $\pi + \alpha$ respectively.

From the equation of the conic $l|SP = 1 + e\cos\alpha$

and $l|SP' = 1 + e\cos(\alpha + \pi)$.

Adding these we get, $l|SP + l|SP' = 2$.

Therefore, $1|SP + 1|SP' = 2|l$.

Example 3: A circle is passing through the focus of a conic $l|r = 1 + e\cos\theta$ whose latus rectum is 2l meets the conic in four points whose distances from the foci are $r_{1,}r_{2,}r_{3,}$ and $r_{4,}$ Show that $1|r_1 + 1|r_2 + 1|r_3 + 1|r_4 = 2|l$.

Solution: We take the focus as pole and the axis of the conic as the initial line. The equation of the conic now be taken as

$l|r = 1 + e\cos\theta$……………(1)

The equation of the circle passing through the pole may be taken as $r = a\cos(\theta - \alpha)$ … … … … . (2)

where a is the diameter and α the angle which the diameter makes with the initial line. Eliminating θ between (1) and (2),

$$\{r - (acos\alpha)|e(1|r - 1)\}^2 = a^2(sin\alpha)^2\{1 - (1 - r)^2|(er)^2\}$$

Or, $e^2r^4 + 2r^3aecos\alpha + r^2(a^2 - 2aelcos\alpha - a^2e^2sin^2\alpha) - 2a^2lr + a^2l^2 = 0$ … … … … … … … . . (3)

If $r_{1,}r_{2,}r_{3,}$ and $r_{4,}$ be the distances from the point of the inter-section from the focus, then these are the roots of the equation (3).

Hence, $r_1r_2r_3 + r_1r_3r_4 + r_1r_2r_4 + r_2r_3r_4 = 2a^2l|e^2$ ……….(4)

And $r_1r_2r_3r_4 = a^2l^2|e^2$ … … … … . . (5)

Dividing (4) by (5), we get $1|r_1 + 1|r_2 + 1|r_3 + 1|r_4 = 2|l$

Example 4: Prove that the perpendicular chords of a rectangular hyperbola are equal.

Solution: Let PSP' and QSQ' be two perpendicular focal chords. Hence the vectorial angle of P is α, the the vectorial angle of Q is $(\pi|2 + \alpha)$,also vectorial angle of P' is $(\pi|2 + \alpha)$

We have,

$l|SP = 1 + e\cos\alpha$.....(1)

and $l|SP' = 1 + e\cos(\alpha + \pi)$........(2)

We have $PP' = SP + SP'$

$= l|\, 1 + e\cos\alpha + l\,|1 + e\cos(\alpha + \pi)$

$= l|\, 1 + e\cos\alpha + l\,|1 - e\cos\alpha$

$= 2l|1 - e^2(\cos\alpha)^2$

Therefore, $1|PP' = (1 - e^2(\cos\alpha)^2)|2l$ ....(3)

Similarly, we have

$1|QQ' = (1 - e^2(\cos(\pi|2 + \alpha))^2)|2l$

i.e. $1|QQ' = (1 - e^2(\sin\alpha)^2)|2l$.........(4)

In the case of rectangular hyperbola ,we have $e = \sqrt{2}$, therefore,

$PP' = 2l|1 - 2(\cos\alpha)^2 = 2l|\cos 2\alpha$

and $QQ' = 2l|1 - 2(\sin\alpha)^2 = 2l|\cos 2\alpha$. Hence, $PP' = QQ'$

Example 5: A point moves so that the sum of its distances from two fixed points S and S', is constant and equal to $2a$. Show that P lies on the conic$a(1 - e^2)|r = 1 - e\cos\theta$.

Referred to S as pole and SS' as the initial line, the SS' being equal to $2ae$.

Solution: Let the coordinates of P referred to S as pole and SS' as the initial line be (r, θ). Then, since $SP = r, S'P = 2a - r$.

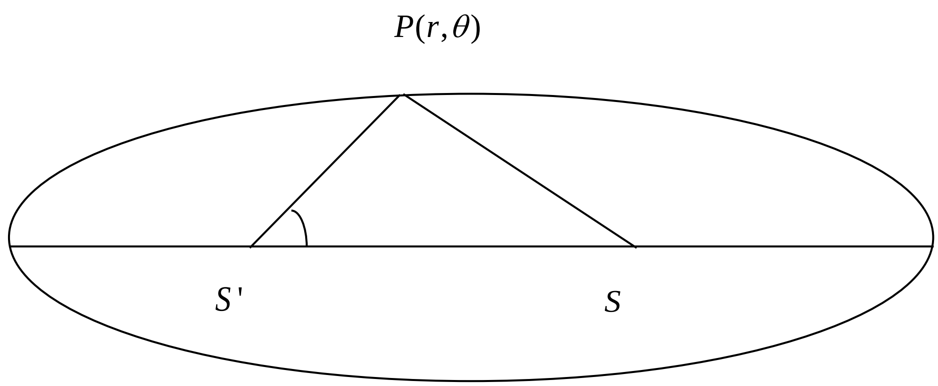
From the triangle ' , we have

$(S'P)^2 = (SP)^2 + SS'^2 - 2SP.SS'\cos\theta$

or, $(2a - r)^2 = r^2 + (2ae)^2 - 2r.2ae\cos\theta$

or, $a - r = ae^2 - er\cos\theta$.

This gives that $a(1 - e^2)|r = 1 - e\cos\theta$



Example-6 : A straight line drawn through the common focus S of a number

of conics meets them in the points P_1, P_2, P_3……. On it is taken a point Q such that the reciprocal of SQ is equal to the sum of the reciprocals of SP_1, SP_2, SP_3……… Prove that the locus of Q is a conic section whose focus is S and the reciprocal of whose latus-rectum is equal to the sum of the reciprocals of the latera recta of the given conics.

Solution: Suppose that general equation of a conic is

$l|r = 1 + e \cos \theta$………..(1)

Taking the common focus S as the pole and the common axis as the initial line, then the equations to the conics are

$l_n|r = 1 + e_n \cos \theta, \; n = 1, 2, 3, \ldots$…(2)

Suppose a straight line drawn through S make an angle β with the common axis of the conics.

Suppose this straight line meets the conic $l_n|r = 1 + e_n \cos \theta$,

at the point P_n, where n = 1, 2, 3,……..lie on the same straight line, therefore their vectorial angles are the same. Let (r_n, β) be the coordinates of the point P_n which lie on the conic

$l_n|r = 1 + e_n \cos \theta$ then $l_n|r_n = 1 + e_n \cos \beta, n = 1, 2, 3, \ldots$.

Suppose Q is the point (R, β) on this line. Then according to the question $1|SQ = \sum 1|SP_n$, or, $1|R = \sum 1|r_n$

Or, $1|R = \sum((1 + e_n \cos\beta)|l_n), n = 1, 2, 3, \ldots\ldots$

= $(1 + e_1 \cos\beta)|l_1 + (1 + e_2 \cos\beta)|l_2 \ldots$

$$= (1|l_1 + 1|l_2 + \ldots) + (e_1|l_1 + e_2|l_2 + \ldots)\cos\beta$$

=1|L+(1|K) cosβ, Where $1|L = 1|l_1 + 1|l_2 + \ldots$

Therefore, $L|R = 1 + E\cos\beta$ where $E = L|K$.

Hence, the locus of $Q(R, \beta)$ is $L|r = 1 + E \cos\theta$

This is the equation of a conic with focus S, semi-latus rectum L and eccentricity E.

Example 7: A chord of a conic subtends a constant angle at a focus of the conic. Show that the chord touches another conic.

Solution: Suppose that the equation of the conic whose focus is the pole, $l|r = 1 + e \cos \theta$………..(1)

Suppose a chord PQ of the conic (1) subtends a constant angle 2β at the focus S. Let $\alpha + \beta$ and $\alpha - \beta$ be the vectorial angles of the extremities of the chord PQ. Then the equation of the chord PQ is $l|r = e\cos\theta + sec\beta\cos(\theta - \alpha)$, or

$lcos\beta|r = e\cos\beta\cos\theta + \cos(\theta - \alpha)$…….(2)

Obviously the straight line (2)is the tangent to the conic

$lcos\beta|r = (e\cos\beta)\cos\theta$ at the point whose vectorial angle is α.

Example 8: Find the condition that the line $l|r = Acos\theta + Bsin\theta$ may be a tangent to the conic $l|r = ecos\,\theta$.

Solution: Suppose that the equation of the line is

$l|r = Acos\theta + Bsin\theta$ ……….(1)

is a tangent to the conic $l|r = 1 + ecos\,\theta$……..(2)

at the point whose vectorial angle is α. The equation of the tangent to (2) at the point α is

$l|r = e\cos\theta + \cos(\theta - \alpha)$,

$or, l|r = (e + cos\alpha)\cos\theta + \sin\theta\sin\alpha$, ……..(3)

The equation (1)and (3) should represent the same line. So, comparing the coefficients of $1|r$, $cos\theta$ and $sin\theta$, we have

$1 = (e + \cos\theta)|\, A = sin\theta|B$

Or, $cos\alpha = (A - e)$ and $sin\alpha = B$.

Squaring them and adding, we have

$(A - e)^2 + B^2 = 1$.

Which is the required condition.

Note : For tengents see 2.4.

Check your progress

(1) Prove that the equations $l|r = 1 - e\cos\theta$ and $l|r = -1 - e\cos\theta$ represent the same conic.

(2) If PSQ and PHR be two chords of an ellipse through the foci S and H, show that $PS|SQ + PH|HR$ is independent of the position of point P.

(3) If PSP' and QSQ' are two perpendicular focal chords of a conic; prove that $1|SP.SP' + 1|SQ.SQ'$ $is\ constant$.

(4) Show that the middle points of focal chords of a conic lie on another conic of the same kind.

(5) In any conic prove that the sum of reciprocals of two perpendicular focal chords is constant.

(6) PSP' is a focal chord of a conic. Prove that the locus of its middle point is a conic of the same kind as the original conic.

(7) A chord PQ of a conic whose eccentricity is e and semi- latus rectum l subtends a right angle at the focus S, show that

$$(1|SP - 1\,|l)^2 + (1|SQ - 1\,|l)^2 = (e\,|\,l)^2$$

(8) A point moves, so that the sum of its distances from two fixed points S and S' is constant and equal to $2a$. Show that P lies on the conic $a(1 - e^2)|r = 1 - ecos\theta$ refered to S as pole and SS' as initial line, SS' being equal to $2ae$.

(9) A circle of given radius passing through the focus S of a given conic intersects it in A, B, C, D: Show that $SA.SB.SC.SD$ is constant.

(10) Prove that the condition that the line $l|r = Acos\theta + Bsin\theta$ may touch the conic $l|r = 1 + e\,cos\,(\theta - \alpha)$ is $A^2 + B^2 - 2e(Acos\alpha + Bsin\alpha\,) + e^2 - 1 = 0$

1.14 EQUATION OF THE CHORD WHEN THE VECTORIAL ANGLES OF THE EXTREMITIES ARE GIVEN

Let the equation of the conic be$l|r = 1 + e\,cos\,\theta$.…………..(1)

Let the vectorial angles of the extrimities of the chord be

$(\alpha - \beta), (\alpha + \beta)$.

Since the general equation of a straight line on

$l|r = A\,cos\,(\theta - \alpha) + Bcos\,\theta$ ……………..(2)

It can easily seen by converting equation (2) in Cartesian co-ordinates. Suppose equation (2)be the equation of the given chord. Then it must pass through points on (1), whose vectorial angles are$(\alpha - \beta), and\ (\alpha + \beta)$.Putting $\theta = (\alpha - \beta)$ and $\theta = (\alpha + \beta)$ in (1)and (2), and equating the values of r, thus we get

$1 + e\,cos\,(\alpha - \beta) = A\,cos\,\beta + Bcos\,(\alpha - \beta)$,

and $1 + e\,cos\,(\alpha + \beta) = A\,cos\,\beta + Bcos\,(\alpha + \beta)$.

From these we have $A = sec\beta, B = e$. Substituting the values of A and B in (2), the required equation of the chord is

$\boldsymbol{l|r = sec\beta\,cos\,(\theta - \alpha) + ecos\,\theta}$

Note1: The equation of the chord of the conic

$l|r = 1 + e\,cos\,(\theta - \gamma)$

joining the points whose vectorial angles are

$(\alpha-\beta),(\alpha+\beta)$ is

$\boldsymbol{l|r = sec\beta\, cos\,(\theta - \alpha) + ecos(\,\theta - \gamma)}$.

Check your progress

(1) Show that the equation of the directrix of the conic $l|r = 1 + e\,cos\,\theta$ corresponding to the focus other than the pole is $l|r = \{(1 - e^2)|(1 + e^2)\}ecos\theta$.

(2) If the circle $r + 2acos\theta = 0$ cuts the conic $l|r = 1 + e\,cos\,(\theta - \alpha)$ in four points, find the equation in r which determines the distances of these four points from the pole. Show that if the algebraic sum of these four distances is equal to $2a$, the eccentricity is equal to $2cos\alpha$

(3) Prove that in a conic $l|r = 1 + e\,cos\,\theta$ the sum of the reciprocals of two perpendicular focal chords is constant**.**

Summary

(1) General equation of second degree is $ax^2+by^2+2hxy+2gx+2fy+c = 0$

(2) General equation of second degree is $ax^2+by^2+2hxy+2gx+2fy+c = 0$ represents a pair of straight lines if **abc + 2fgh - af²- bg²- ch² = 0** then

Case(1). $ab - h^2 = 0$, it represents a pair of straight lines.

Case(2.) $ab - h^2 \neq 0$, it represents a pair of intersecting straight lines.

Case(3) . $ab - h^2 < 0$, it represents a pair of real or imaginary straight lines.

Case(2.) $ab - h^2 > 0$, it represents a point.

Note(1) : Here a, b, c, stand for coefficients of x^2, y^2 and constant term respectively and f, g, h stand for half of the coefficients of y, x and xy.

Again, If **abc + 2fgh - af²- bg²- ch² ≠ 0** then

Case (1). $h = 0, a = b$, then it represent a circle.

Case (2). $ab - h^2 = 0$, it represents a parabola.

Case (3). $ab - h^2 > 0$, it represents an ellipse.

Case (4). $ab - h^2 < 0$, it represents a hyperbola.

Case (5). $ab - h^2 < 0 \; and \; a + b = 0$, it represents a rectangular hyperbola.

Note (2): If **abc + 2fgh - af²- bg²- ch² ≠ 0**in the above equation then it represents the **non- degenerate conic**.

(3) If **abc + 2fgh - af²- bg²- ch² = 0**in the above equation then it represents the **degenerate conic**.

(3) The homogenous equation of second degree $ax^2 + by^2 + 2hxy = 0$ represent a pair of straight lines or two straight lines passing through the origin then the angle between them is $\alpha = \tan^{-1}$- $\left(\frac{2\sqrt{(h^2-ab)}}{a+b}\right)$……….(4)

(4) The lines are real and distinct, coincident or imaginary according as $(h^2 - ab) > 0, = 0 \; or \; < 0.$

Case(1): If the lines be perpendicular,then $\alpha=90^{\circ}$ Therefore,

$\tan\alpha = \tan 90 = \infty = \frac{2\sqrt{(h^2-ab)}}{a+b}$, i.e. $a+b = 0$

i.e. [coefficient of x^2 + coefficient of y^2=o]

Case(2): If the lines be parallel ,then $\alpha=0$. Therefore,

$\tan\alpha = \tan 0 = 0 = \frac{2\sqrt{(h^2-ab)}}{a+b}$, then $\sqrt{(h^2 - ab)} = 0$

then $h^2 = ab$ i.e. (½ coefficient of xy)²

= product of coefficient of x^2 and coefficient of y^2.

(5) The required condition that $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represent a pair of general linear equations is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. i. e.

$$\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = 0$$

(6) The bisector of the pair of the lines $a^2x^2 + 2hxy + b^2y^2 = 0$ is given by $\frac{x^2-y^2}{a^2-b^2} = \frac{xy}{h(a+b)}$or, $\frac{x^2-y^2}{a-b} = \frac{xy}{h}$

7 (a). The equation of the conic when the axis SZ is inclined at an angle α to the initial line is $l|r = 1 + e\cos(\theta - \alpha)$,

(b). The equation of the conic when the positive direction of the initial line is ZS instead of SZ, is $l|r = 1 - e \cos \theta$.

(c). If $e = 1$, the conic is a parabola.

(d). If $e < 1$, the conic is an ellipse.

(e). If $e > 1$, the conic is a Hyperbola.

(f). If $e = 0$, the conic is a circle.

(g). If $e = \infty$, the conic is the pair of straight lines.

(8) The equation of the $directrix\ of\ the\ conic\ \ l|r = 1 + ecos\theta$ is

$$l|r = -e(1-e^2)|(1+e^2)cos\ \theta$$

(9) The equation of the chord of the conic $l|r = 1 + e \cos(\theta - \gamma)$

joining the points whose vectorial angles are $(\alpha - \beta), (\alpha + \beta)$ is

$$l|r = sec\beta \cos(\theta - \alpha) + ecos(\theta - \gamma)$$

UNIT-2 CURVE TRACING

Structure

2.1 INTRODUCTION

In this unit, our aim is to re-acquaint with tracing of conic and its different aspects of two dimensional geometry. The French philosopher mathematician **Rene Descartes** (1596--1650) was the first to realize that geometrical ideas can be translated into algebraic relations. The combination of Algebra and Plane Geometry came to be known as **Coordinate Geometry** or **Analytical Geometry**. A basic necessity for the study of Coordinate Geometry is thus, the introduction of a coordinate system and to define coordinates in the concerned space. We will briefly touch upon the equation of tangents at a point, equation of normals at a point of a conic. Next, we will talk about symmetry with respect to origin or a coordinate axis.

We have read about lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions. We have also seen that graphs of linear equations,

Which came out to be straight lines. Interestingly, the reverse problem Of the above is finding the equations of straight lines, under different conditions in a plane. The Analytical Geometry, more commonly called Coordinate Geometry, comes to our help in this regard.

In this unit we shall find equations asymptotes, polar, Auxiliary circle the point of intersection of two tangents and Director circle and try to solve the problem based on these.

2.2 OBJECTIVES

After studying this unit you should be able to find:

1. Tracing of conic and its related concepts
2. Equation of the tangent at the point whose vectorial angle is α
3. Equation of the normal at the point whose vectorial angle is α.
4. Asymptotes for different conics.
5. Equation of pole and Polar
6. Equation of the Auxiliary circle
7. The point of intersection of two tangents.
8. Equation of Director circle

2.3 TRACING OF A CONIC

The curves **(**a pair of a straight lines, a circle, a parabola, an ellipse and a hyperbola **)** which comes under the category of conic sections. It is derived from the fact that these curves were first obtained by cutting a cone in various ways.

Conic is the locus of a point which moves so that its distance from a fixed point (focus) is in a constant ratio to its perpendicular distance from a fixed straight line (directrix). The constant ratio is called eccentricity and it is denoted by e

The general equation of the second degree is is

$ax^2+by^2+2hxy+2gx+2fy+c = 0$……..(1)

We remove the term of xy form (1) then we have the following cases:

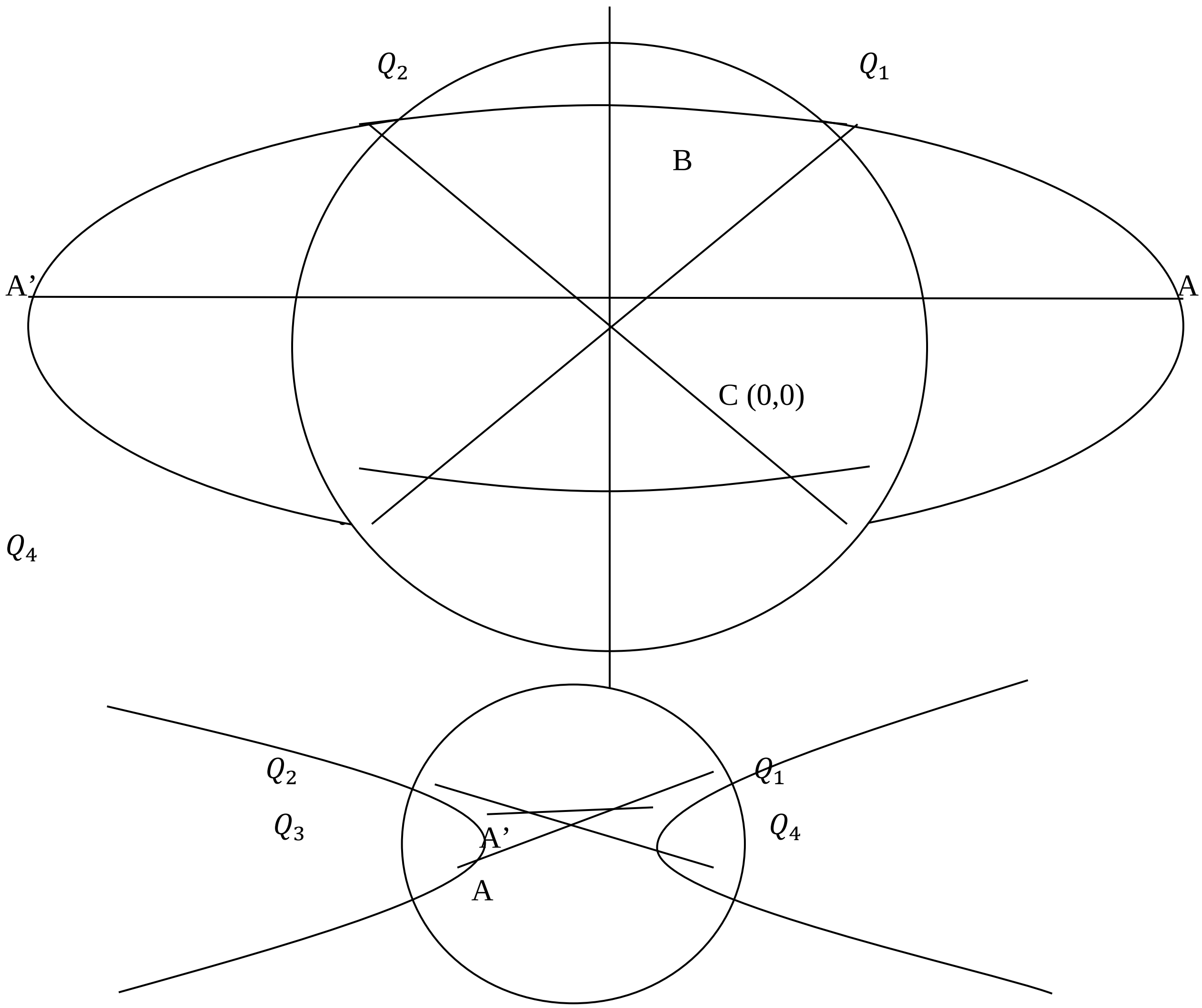
CaseI: Let $a \neq 0$ and $b \neq 0$, then equation (1) is written as

$$a(x^2 + 2gx|a + (g|a)^2) + b(y^2 + 2fy|b + (f|b)^2) - (g|a)^2) - (f|b)^2 + c = 0$$

Or, $a(x + g|a)^2 + b(y + f|b)^2 = (g|a)^2) + (f|b)^2 - c = K\ (say)$

Sifting the origin to $(-g|a, -f|b)$, then this equation becomes

$ax^2 + by^2 = K$……….(2)



(i) If $K = 0$, the equation (2) becomes $ax^2 + by^2 = 0$ and this represent a pair of straight lines. These straight lines are real if a and b are of the opposite signs and these lines are imaginary if a and b are of the same sign.

(ii) If $K \neq 0$, the equation (2) becomes $x^2|K|a + y^2|K|b = 1$. **……(3)**

If $K|a$ and $K|b$ are both positive, the equation (3) represents an ellipse which becomes a circle if in addition to being positive $K|a$ and $K|b$ are both equal.

Again the equation (3) represents a hyperbola if $K|a$ and $K|b$ are of opposite signs. If $K|a$ and $K|b$ are both negative, the equation (3) is said to represent an emaginary ellipse.

Case II : If one of a or b is zero while other is not zero. If we take $a = 0$ and $b \neq 0$ the the equation (1) will be by²+2hxy+2gx+2fy+c = 0

or, $(y+f|b)^2 = -(2g|b)x - c|b + (f|b)^2$ ………….(4)

If $g = 0$, then equation (4) represents two parallel straight lines, which are coincident if $f^2 - bc$ also is zero.

If $g \neq 0$, the equation (4) can be written as

$(y+f|b)^2 = -(2g|b)[x + c|2g + f^2\,|2bg]$

Shifting the origin to $(f^2\,|2bg - c|2g, -f|b)$, this equation becomes

$y^2 = -(2g|b)x$ which represents a parabola. Hence in each case the general equation of second degree represents a conic section.

Centre: The centre of a conic section is a point which bisects all those chords of the conic that passes through it. The general equation of the second degree namely

ax²+by²+2hxy+2gx+2fy+c = 0 will represent a conic with centre at the origin only if the coefficient of x = the coefficient of $y = 0$ I.e. only if $g = f = 0$. That is only if the first degree terms are absent from the equation of the conic. If the centre of the conic is to be at the origin, then for each point (x_1, y_1) on the conic, the point $(-x_1, -y_1)$ must also lie on the conic.

The coordinates of the centre of the conic ax²+by²+2hxy+2gx+2fy+c = 0 is $\left((hf-bg)\middle|(ab-h^2), (gh-af)|\,(ab-h^2)\right)$

To find the coordinates of the centre of a conic F(x, y) = ax²+by²+2hxy+2gx+2fy+c = 0. We have $\frac{\partial F}{\partial x} = 2(ax + hy + g)$, $\frac{\partial F}{\partial y} = 2(hx + by + f)$, for centre of the conic $F(x,y) = 0$ is obtained by solving the equations $ax + hy + g = 0$ and $hx + by + f = 0$

That is $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$

Example 1: Find the coordinates of the centre of the conic $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$.

Solution: Let $F(x,y) = 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

To find the coordinates of the centre we have $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$.

Therefore, $\frac{\partial F}{\partial x} = 28x - 4y - 44 = 0$ and $\frac{\partial F}{\partial y} = -4x + 22y - 58 = 0$

Solving these two equations we get$x|-150 = y| - 225 = 1| - 75$

So, $x = 2, y = 3$. The coordinates of the centre is $(2,3)$.

Example 2: Find the equation of the asymptotes of the conic $3x^2 - 2xy - 5y^2 + 7x - 9y = 0$ and find the equation of the conic which has the same asymptotes and which passes through the point $(2,2)$.

Solution: Since the equation of the asymptotes differs from the equation of the conic only by a constant term, therefore let the equation of the asymptotes be $3x^2 - 2xy - 5y^2 + 7x - 9y + \lambda = 0$….(1)

Where λ be a constant term. Equation (1) should represent a pair of straight lines, if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

Or, $3(-5)\lambda + 2(-9|2)(7|2)(-1) - 3(-9|2)^2 - (-5)(7|2)^2 - \lambda(-1)^2 = 0$, or, $\lambda = 2$.

The equation of the asymptotes is

$3x^2 - 2xy - 5y^2 + 7x - 9y + 2 = 0$…………(2)

Now let the equation of a conic having (2) for its asymptotes be

$3x^2 - 2xy - 5y^2 + 7x - 9y + 2 + \mu = 0$……….(3)

Where μ is a constant to be determined by the fact that the conic (3) is to pass through the point $(2, 2)$.

$3(4) - 2.2.2 - 5.4 + 7.2 - 9.2 + 2 + \mu = 0, or\ \mu = 18$Putting the value of μ in (3), the required equation of the conic is

$$3x^2 - 2xy - 5y^2 + 7x - 9y + 20 = 0$$

Example 3 (a) : Find the coordinates of its focus, axis, the vertex, the equation of the directrix and the length of its latus rectum of the parabola$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0$.

Solution: The second degree terms parabola$16x^2 - 24xy + 9y^2$ form a perfect square, therefore the given equation represents a parabola. Now we can write it as $(4x - 3y)^2 = 104x + 172y - 44$ …(1)

Now we introduce a new constant λ in both sides. So, we have

$(4x - 3y + \lambda)^2 = (104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 - 44$…(2)

Now we choose λ such that the lines $4x - 3y + \lambda = 0$ and

$(104 + 8\lambda)x + (172 - 6\lambda)y + \lambda^2 - 44 = 0$ are at right angles. For this we have $(4|3)\{-(104 + 8\lambda)|(172 - 6\lambda)\} = -1$

Or, $-4(104 + 8\lambda) = -3(172 - 6\lambda)$

Or, $50\lambda = 100$

Or, $\lambda = 2$

Putting this value of λ in (2), we have

$(4x - 3y + 2)^2 = 40(3x + 4y - 1)$

Or, $\{(4x - 3y + 2)|5\}^2 = 8\{(3x + 4y - 1)|5\}$………(3)

The equation (3) is of the standard form $Y^2 = 4pX$, where

$X = \{(3x + 4y - 1)|5\}$ and

$Y = (4x - 3y + 2)|5,\ 4p = 8$

(i). The axis of the parabola is $Y = 0$

$i.e. (4x - 3y + 2) = 0$

(ii). The tangent at the vertex is $X = 0$

$i.e. 3x + 4y - 1 = 0$

(iii). The vertex of the parabola is the point of the intersection of the lines $(4x - 3y + 2) = 0$ and $3x + 4y - 1 = 0$.

The coordinates of the vertex A are $(-1|5, 2|5)$

(iv). The length of the latus rectum is $4p = 8$

(v). The equation of the latus rectum is

$X = p\ i.e. (3x + 4y - 1)|5 = 2$

$i.e. 3x + 4y - 11 = 0$

(vi). The coordinates of the focus of the parabola are $(1, 2)$.

(vii). The equation of the directrix is given by $X = -p$

$. (3x + 4y - 1)|5 = -2\ i.e. 3x + 4y + 9 = 0.$

Example 3 (b) : Trace the conic $36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0$.

Solution: The given conic is

$F(x, y) = 36x^2 + 24xy + 29y^2 - 72x + 126y + 81 = 0.$

Hence, $a = 36,\ h = 12, b = 29.$ Since $h^2 = ab$. That is the second degree terms of the equation is not a perfect square, therefore the given equation represents a central conic. The coordinates of the centre are given by the equations

$\frac{\partial F}{\partial x} = 72x + 24y - 72 = 0, i.e.\ \ 3x + y - 3 = 0,$

$\frac{\partial F}{\partial y} = 24x + 58y + 126 = 0$ i.e. $12x + 29y + 63 = 0$

Solving these equations, coordinates of centre is $(2, -3)$

Now $c = gx_1 + fy_1 + c = (-36)(2) + (63)(-3) + 81 = -180$

Therefore the equation of the conic referred to the centre as origin is

$36x^2 + 24xy + 29y^2 - 180 = 0$

In the standard form of this equation is $1|5\, x^2 + 2|15\, xy + 29|180\, y^2 = 1$.

The squares of the lengths of the semi- axes are given by

$(A - 1|r^2)$(B - $1|r^2$)= H^2

Or, (1|5 -1$|r^2$)$(29|180 - 1|r^2) = (1|15)^2$

Or, $1|r^4 - (13|36)1|r^2 + 1|36 = 0$

Or, $r^4 - 13\,r^2 + 36 = 0$

Or, $({r_1}^2 - 9)({r_2}^2 - 4) = 0$

Or, ${r_1}^2 = 9,\ {r_2}^2 = 4$

The conic is an ellipse since both ${r_1}^2$ and ${r_2}^2$ are positive. The lengths of the major axis minor axis are $2r_1$ and $2r_2$ that is 6 and 4 respectively.

The equation of the major axis referred to the centre as origin is

$$(A - 1|{r_1}^2)x + Hy = 0,$$

Or, $(1|5 - 1|9)x + 1|15\,y = 0$

Or, $4x + 3y = 0$. Therefore, the equation of the major axis referred to the old coordinate axes is $4(x-2) + 3(y+3) = 3$

Or, $4x + 3y + 1 = 0$

The minor axis is the straight line perpendicular to the major axis and passing through the centre $(2, -3)$.So referred to the old coordinate axes the equation of the minor axis is $3(x-2) - 4(y+3) = 0$

Or, $3x - 4y - 18 = 0$

The points of intersection of the conic with the coordinate axes.

The given conic cuts the $x-$ axis in the points where $y = 0$ i.e.

$36x^2 - 72x + 81 = 0$, i. e. $4x^2 - 8x + 9 = 0$.

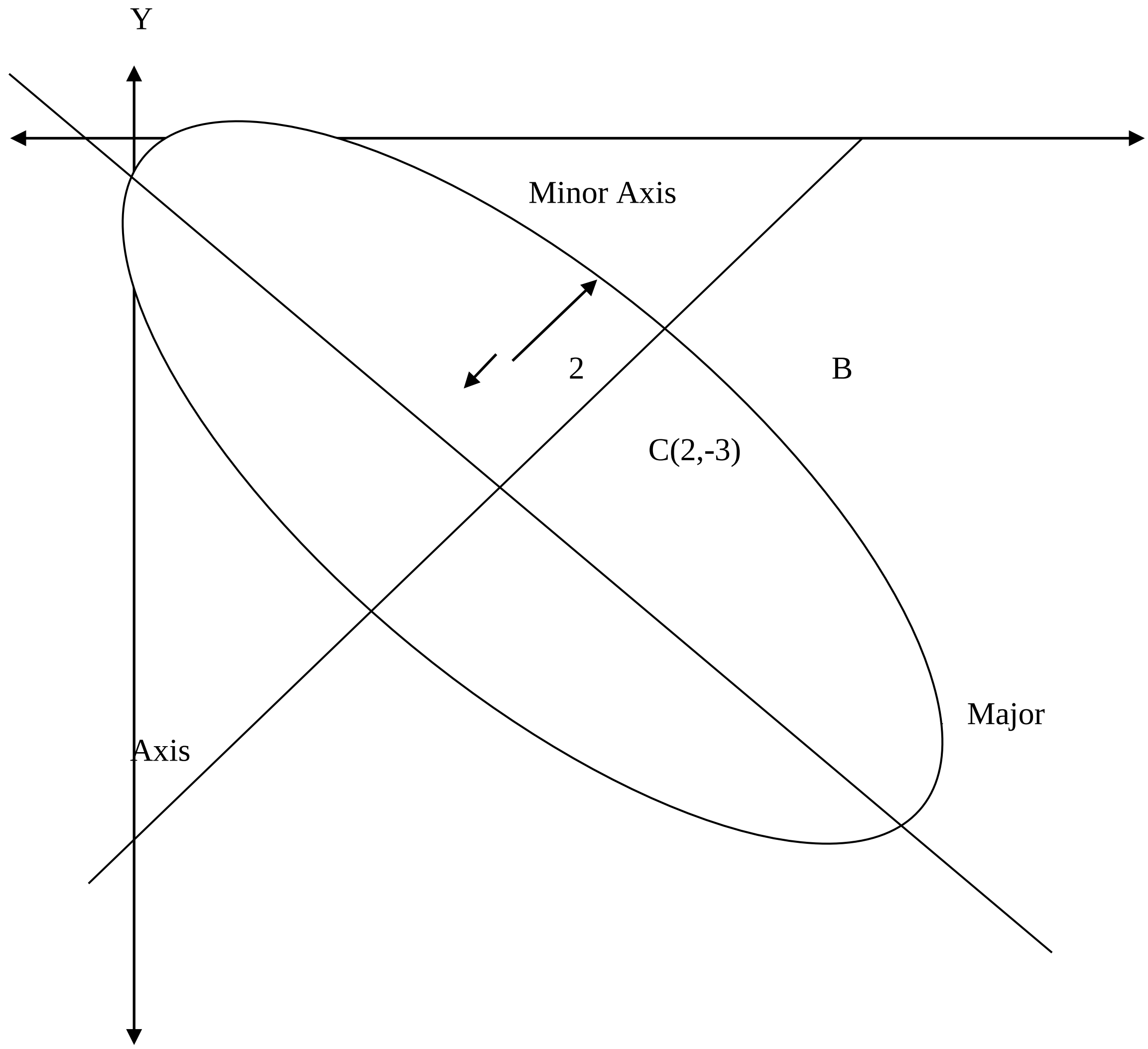
This gives the imaginary values of x because its discriminant

$b - 4ac = 64 - 4.4.9 = -ve$. Hence the given conic does not cut the $x-$ axis.

The given conic cuts the $y-$ axis in the points where $x = 0$.

Where $29y^2 + 126y + 81 = 0$. Where $y = -0.8\ and\ y = -3.6(nearly)$

Hence the shape of the given conic (which is an ellipse) is as shown in the figure. So, draw the major axis which passes through the centre $(2, -3)$and it cuts the x-axis at the point $(-1|4, 0)$.



Check your progress

$130ax - 60ay + 116a^2 = 0.$

(2) Find the lengths and the equations of the axes of the conic $5x^2 - 6xy + 5y^2 + 26x - 22y + 29 = 0$.

(3) Find the equation of the hyperbola whose asymptotes are parallel to $2x + 3y = 0$ and $3x + 2y = 0$, whose centre is at $(1, 2)$ and which passes through $(5, 3)$.

(4) Trace the curve $8x^2 - 4xy + 5y^2 - 16x - 14y + 17 = 0$. Find the coordinates of its foci and show that its axes lie along $2x - y - 1 = 0$ and $2x + 4y - 11 = 0$.

(5) Trace the curve $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$. Find the coordinates of its foci and the length of its latus rectum.

(6) Trace the curve $5x^2 + 4xy + 8y^2 - 12x - 12y = 0$. Find the coordinates of its foci and the length of its latus rectum.

(7) Trace the curve $11x^2 + 4xy + 14y^2 - 26x - 32y + 23 = 0$. Find the coordinates of its foci.

(8) Trace the curve $41x^2 + 24xy + 9y^2 - 130ax - 60ay + 116a^2 = 0$.

(9) Trace the hyperbola $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0$. Find the equations of its axes and asymptotes.

(10) Trace the curve $17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0$. Find its eccentricity, the equations of its axes, the coordinates of its foci and the equations of its directrices.

(11) Trace the parabola$9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0$. Find the coordinates of its focus, axis, the vertex, the equation of the directrix and the length of its latus rectum.

(12) Find the equation of the hyperbola whose asymptotes are parallel to $2x + 3y = 0$ and $3x + 2y = 0$, whose centre is at $(1, 2)$ and which passes through $(5, 3)$.

(13) Find the lengths and the equations of the axes of the conic $5x^2 - 6xy + 5y^2 + 26x - 22y + 29 = 0$

2.4 EQUATION OF THE TANGENT AT THE POINT WHOSE VECTORIAL ANGLE IS α

Suppose that equation of the conic is$l|r = 1 + e \cos \theta$……(1).

If the points on the conic (1) whose vectorial angles are

$(\alpha - \beta)$, and $(\alpha + \beta)$. Since the general equation of a straight line on $l|r = A \cos(\theta - \alpha) + B\cos\theta$ ……………..(2)

It can easily seen by converting equation (2) in Cartesian co-ordinates. Suppose equation (2)be the equation of the given chord. Then it must pass through points on (1), whose vectorial angles are$(\alpha - \beta), and\ (\alpha + \beta)$.Putting $\theta = (\alpha - \beta)$ and $\theta = (\alpha + \beta)$ in (1)and (2), and equating the values of r, thus we get

$1 + e \cos(\alpha - \beta) = A \cos\beta + B\cos(\alpha - \beta)$,

and $1 + e \cos(\alpha + \beta) = A \cos\beta + B\cos(\alpha + \beta)$.

From these we have $A = sec\beta, B = e$.

Substituting the values of A and B in (2), the required equation of the chord is

$\boldsymbol{l|r = sec\beta \cos(\theta - \alpha) + e\cos\theta}$

If the angles $(\alpha-\beta), and (\alpha+\beta)$ coincide, β becomes zero, and in this limiting position the chord becomes a tangent to the conic at the point whose vectorial angle is α. The equation of the tangent to the conic at the point whose vectorial angle is α is, therefore,

$\boldsymbol{l|r = cos}\,(\theta - \boldsymbol{\alpha}) + \boldsymbol{ecos\,\theta}$**………(1)**

Note1. If the equation of the conic is

$l|r = 1 + e\cos(\theta - \theta_1)$,

then the equation of the tangent at $'\alpha'$ is

$\boldsymbol{l|r = cos}\,(\theta - \boldsymbol{\alpha}) + \boldsymbol{ecos\,(\theta - \theta_1)}$

Note2: The equation of the tangent for the conic

$l|r = 1 + e\cos(\theta - \gamma)$ at the point "α" is

$\boldsymbol{l|r = cos}\,(\theta - \boldsymbol{\alpha}) + \boldsymbol{ecos(\theta - \gamma)}$.

Note: The slope of the tangent (1) is $(e + cos\alpha)|sin\alpha$**.**

2.5 EQUATION OF THE NORMAL AT THE POINT WHOSE VECTORIAL ANGLE IS $\boldsymbol{\alpha}$

Suppose that equation of the conic is$l|r = 1 + e\cos\theta$……(1)

Suppose that equation of the tangent at the point (r, α) on the conic (1) is $\boldsymbol{l}|r = \cos(\theta - \alpha) + e\cos\theta$ ………..(2)

The equation of the normal to the conic (1) which is perpendicular to thetangent of this conic is in the form

$A|r = \cos(\theta + \pi|2 - \alpha) + e\cos(\theta + \pi|2) \Rightarrow A|r = -\sin(\theta - \alpha) - e\,sin\theta$…………(3)

Now equation of the normal at the point (r', α) of the conic (1) which is perpendicular to the equation (3), that is equation (3) passes through the point (r', α), therefore, we get

$A|r' = -esin\alpha$…………(4).

Now from the equation of the conic,

$\boldsymbol{l}|r' = 1 + ecos\,\alpha$

Hence, from the equation (3), we get $A = -elsin\alpha|\,(1 + ecos\alpha)$

Substituting it in equation (2), the equation of the normal at the point whose vectorial angle is α is

$\boldsymbol{elsin\alpha|\,(1 + ecos\alpha)r = \sin(\theta - \alpha) + e\,sin\theta}$

Note3 : The equation of the normal for the conic

$l|r = 1 + e\cos(\theta - \gamma)$ at the point "α" is

$\boldsymbol{elsin\alpha}|\ (\mathbf{1} + \boldsymbol{ecos\alpha})\boldsymbol{r} = \ \mathbf{sin}(\boldsymbol{\theta} - \ \boldsymbol{\alpha}) + \boldsymbol{e}\ \boldsymbol{sin}(\boldsymbol{\theta} - \boldsymbol{\gamma})$.

Example 4: Chords of a conic subtended a constant angle 2α at the focus. Find the locus of the point where the chords are met by the internal bisector of the angle which they subtend at the focus.

Solution: Let the equation of the conic be$l|r = 1 + e\cos\theta$ and the vectorial angles of the extremities $(\beta - \alpha)$ and $(\beta + \alpha)$. This chord then subtends an angle 2α at the focus and its equation is

$l|r = sec\alpha\cos(\theta - \beta) + ecos\,\theta$……….(1)

$If\ (r', \theta')$ be the coordinates of the point where the chord is met by the internal bisector of the angle which it subtends at the focus, then $\theta' = \beta$………………..(2).

Since (r', θ') lies on (1),

$l|r' = sec\alpha\cos(\theta' - \beta) + ecos\,\theta'$…………(3)

From (2) and (3), we obtain that

$l|r' = sec\alpha + ecos\,\theta'$

or, $lcos\alpha|r' = 1 + ecos\alpha cos\,\theta'$

Hence the locus of (r', θ') is the conic

$lcos\alpha|r = 1 + ecos\alpha cos\,\theta$.

Example 5: If the normal at L, an extremity of the latus rectum of the conic $l|r = 1 + e\cos\theta$ meet the conic again at Q, show that $SQ = l(1 + 3e^2 + e^4)|(1 + e^2 - e^4)$.

Solution: The polar coordinates of the point L are $(l, (1|2)\,\pi)$, and the equation of the normal at L is

$\boldsymbol{l}|r(esin\pi|2|(1 + ecos\pi|2)) = \ sin(\theta - \pi|2) + e\ sin\theta$

Or, $le|r = esin\theta - cos\theta$ ……….(1)

Now we eliminating θ between (1) andthe equation of the conic,

$$\{el|r + (l - r)|er\}^2 = e^2\{1 - ((l - r)|er)^2\},$$

which gives on simplification

$(l - r)[(l - r).(1 + e^2) + 2le^2 + e^4(l + r)] = 0$.

The value $r = l$ corresponds to the point L. From the other factor we obtain $SQ = l(1 + 3e^2 + e^4)|(1 + e^2 - e^4)$

Example 6: If the normal at the points whose vectorial angles are α, β, γ on the parabola $l|r = 1 + e\cos\theta$ meet in a point (ρ, ϕ), show that $2\phi = \alpha + \beta + \gamma$.

Solution: The equation of the normal at a point on the parabola whose vectorial angle is$\theta_1$is$l\Big|r(sin\theta_1|(1+cos\theta_1)\Big)$

$= \sin(\theta - \theta_1) + sin\theta.$

If this passes through the point (ρ,ϕ),

then $l|\rho(sin\theta_1|(1+cos\theta_1)) = \sin(\phi - \theta_1) + sin\phi,$

or, $2l\left|2\rho(\sin(\theta_1|2)\cos(\theta_1|2)|\left(cos^2(\theta_1|2)\right)\right)$

$= \sin\phi(1+cos\theta_1) - sin\theta_1 cos\varphi$

$= \sin\phi\left(2cos^2(\theta_1|2)\right) - 2\sin(\theta_1|2)\,cos(\theta_1|2)cos\varphi$ or,

$l|\rho(tan^3(\theta_1|2) + (l|\rho + 2cos\phi)tan(\theta_1|2) - 2sin\phi = 0$

This is a cubic equation in $tan(\theta_1|2)$. If $\tan(\alpha|2)$, $\tan(\beta|2)$,

$\tan(\gamma|2)$ be three roots of this equation, we have

$\tan(\alpha|2 + \beta|2 + \gamma|2) =$

$(-2\rho sin\phi)l|1 - [(l+2\rho cos\phi)|l] = \tan\phi.$

Hence, $\tan(\alpha|2 + \beta|2 + \gamma|2) = \tan\phi.$

Therefore, $2\phi = \alpha+\beta+\gamma$

2.6 A SYMPTOTES

Suppose that equation of the conic is

$l|r = 1 + e\,cos\,\theta.$

The equation of the asymptotes of the hyperbola

$l|r = e\,cos\,\theta$ is $cos\,\theta = -1|e$

(Since the points at infinity on the conic $l|r = e\,cos\,\theta$

are given by $cos\,\theta = -1|e$).

Further we know that the asymptotes pass through the centre of the hyperbola.

Now the distance of the centre from the focus is $ae,$ where a is the semi-transverse axis of the hyperbola.

The length of perpendicular from S upon either asymptote is

$aesin\alpha = a\sqrt{e^2-1}$, where $cos\alpha = -1|e.$

The angle which this perpendicular makes with the initial line is $-(\pi|2-\alpha)$, or $(\pi|2-\alpha)$ depending upon which asymptote is taken.

$a\sqrt{e^2-1} = rcos(\theta - \alpha + \pi|2)$ and

$a\sqrt{e^2-1} = rcos(\theta + \alpha - \pi|2)$ these can be written as

$l|r = -\sqrt{e^2-1}\sin(\theta - \alpha)$ and

$l|r = -\sqrt{e^2-1}\sin(\theta + \alpha)$,

which is the required equation of the asymptotes of the conic, are the straight lines

$l|r = -l|r = (\sqrt{(e^2-1)}|e)\ (\sqrt{e^2-1}\cos\theta \pm \sin\theta)$.

Check your progress

(1) Find the condition that the line $l|r = A\cos\theta + B\sin\theta$ may be a tangent to the conic $l|r = 1 + e\cos\theta$.

(2) Prove that the equation of the locus of the foot of the perpendicular from the focus of a conic $l|r = 1 + e\cos\theta$ on any tangent to it is $r^2(e^2-1) - 2lercos\theta + l^2 = 0$. Discuss the particular case when $e = 1$.

(3) Prove that the condition that the line $l|r = Acos\theta + Bsin\theta$ may touch the conic $l|r = 1 + e\cos(\theta - \alpha)$ is $A^2 + B^2 - 2e(Acos\alpha + Bsin\alpha) + e^2 - 1 = 0$.

(4) Prove that the line $l|r = cos(\theta - \alpha) + \cos(\theta - \gamma)$ is tangent to the conic $l|r = 1 + e\cos(\theta - \gamma)$ at the point for which $\theta = \alpha$.

(5) PSP' is a focal chord of a conic; prove that the angle between the tangents at P and P' is $\tan^{-1}(2esin\,\alpha|\,1 - e^2)$ where α is the angle between the chord and the major axis.

(6) Prove that the exterior angle between any two tangents to a hyperbola is equal to half the difference of the vectorial angles of their point of contact.

(7) A focal chord PSP' of an ellipse is inclined at an angle α to the major axis. Show that the perpendicular from the focus on the tangent at P makes an angle $\tan^{-1}\{\sin\alpha\,|\,(e + cos\alpha)\}$ with the axis.

Example7: Two equal ellipses of eccentricity e, are placed with their axes at right angles and they have one focus S in common. If PQ be a common tangent, show that the angle PSQ is equal to $2sin^{-1}\,(e\,|\sqrt{2})$.

Solution: We take the common focus S as the pole and axis one ellipse as the initial line so that the axis of other ellipse makes an angle $\pi|2$ with the initial line.

Suppose that the equations of two ellipses be

$l|r = 1 + e \cos \theta$.........(1)

$l|r = 1 + e \cos (\theta - \pi|2)$

$l|r = 1 + e \sin \theta$.......(2)

It is given that PQ is a common tangent to the two ellipses. Let the vectorial angles of P, a point on (1). And Q, a point on (2), be α and β respectively. Therefore

$l|r = \cos (\theta - \alpha) + e\cos \theta)$

$l|r = \sin\alpha\sin\theta + (e + \cos\alpha)\cos \theta$.........(3)

and tangent to (2) at the point β $i.e.$

$l|r = \cos (\theta - \beta) + e\sin \theta$

$l|r = \cos \beta\cos\theta + (e + \sin\beta)\sin \theta$.........(4)

These tangents should be identical. Hence we comparing (3) and (4), we get

$1 = (\cos\alpha + e)|\ \cos\beta = \sin \alpha|\ (\sin\beta + e)$

Therefore, $\cos\alpha + e = \cos\beta$ or, $\cos\beta - \cos\alpha = e$

And $\sin\alpha = \sin \beta + e$ or, $\sin\alpha - \sin\beta = e$

Now squaring them and adding, we get $2 - 2(\cos\alpha \cos\beta + \sin\alpha \sin \beta) = 2e^2$

Or, $\cos(\alpha - \beta) = 1 - e^2$

Or, $1 - 2 \sin^2\{(\alpha - \beta)|2\} = 1 - e^2$

Or, $\sin^2\{(\alpha - \beta)|2\} = e^2|2$

Or, $\sin\{(\alpha - \beta)|2\} = e \mid \sqrt{2}$

Therefore, $(\alpha - \beta) = 2 \sin^{-1}(e \mid \sqrt{2})$.

Example 8: Prove that the portion of the tangent intercepted between the conic and the directrix subtends a right angle at the corresponding focus.

Solution: Let the equation of the conic referred to the focus S as the pole be $l|r = 1 + e \cos \theta$............(1)

The equation of the directrix corresponding to the focus S is

$l|r = e \cos \theta$...............(2)

Let the vectorial angle of any point P on the conic be α. The equation of the tangent at P is

$l|r = cos(\theta - \alpha) + ecos\,\theta$…………(3)

Now the vectorial angle θ of the point of intersection K of the tangent (3)andthe directrix (2)is is given by

$cos(\theta - \alpha) + ecos\,\theta = e\,cos\,\theta$

or, $cos(\theta - \alpha) = 0$, therefore, $(\theta - \alpha) = 9\,0^0$

the directrix subtends a right angle at the corresponding focus.

Example 9: If PSP' is a focal chord of the conic. Prove that the tangents at P and P' intersect on the directrix.

Solution: Let the equation of the conic be

$l|r = 1 + e\,cos\,\theta$………….(1)

Suppose that the focal chord $PSP'$is inclined at an angle α to the initial line so that the vectorial angles of P and P' are α and $\pi + \alpha$ respectively. The equations of the tangents at P and P' are

$l|r = cos(\theta - \alpha) + ecos\,\theta$…………(2)

$l|r = cos(\theta - (\pi + \alpha)) + ecos\,\theta$

$l|r = -cos(\theta - \alpha) + ecos\,\theta$…………(3)

The locus of the point of intersection of the tangents (2) and (3) is obtained by eliminating α between (2) and (3).

We adding (2) and (3),

$2l|r = 2ecos\,\theta$ i. e. $l|r = ecos\,\theta$

Which is the equation of the directrix. Hence tangents at P and P' intersect on the directrix.

Example 10: Two conics have a common focus; prove that two of their common chords pass through the intersection of their directrices.

Solution: Suppose that the equations of the two conics having a common focus be

$l|r = 1 + e_1\,cos\,\theta$………..(1)

$L|r = 1 + e_2 cos(\theta - \alpha)$……(2)

Equation of the directrices of these two conics be

$l|r = e_1\,cos\,\theta$………..(3)

$L|r = e_2 cos(\theta - \alpha)$……(4)

Now we changing (1)to the cartesian form, we have

$l = \sqrt{x^2 + y^2} + e_1 x$ or, $(l - e_1 x)^2 = (x^2 + y^2)$

Now we transform it to polar form, we have

$$(l|r - e_1 \cos\theta)^2 - 1 = 0 \ldots\ldots..(5)$$

The equation (5)is thus the polar equation of the conic (1)put in the form which when transformed to cartesian gives rational cartesian equation of the conic.

Similarly the equation (2) can be written as

$$(L|r - e_2 \cos(\theta - \alpha))^2 - 1 = 0 \ldots\ldots..(6)$$

Now any curve passing through the point of the intersection of the two conics (5) and (6) is given by

$$\{(l|r - e_1 \cos\theta)^2 - 1\} + \lambda\{(L|r - e_2 \cos(\theta - \alpha))^2 - 1\} = 0.(7)$$

Clearly if λ= -1, the equation (7) gives two lines, namely

$(l|r - e_1 \cos\theta) = \pm(l|r - e_1 \cos\theta)$

which clearly pass through the point of intersection of the two directrices

$(l|r - e_1 \cos\theta) = 0$ and $(l|r - e_1 \cos\theta)) = 0$

Since the straight lines passing through the points of intersection of the conics (1) and (2) are their common chords, therefore the common chords of (1) and (2) pass through the intersection of their directrices.

2.7 POLAR

Let $(r_{1,}\theta_1)$ be a given point on the conic $l|r = 1 + e\cos\theta$. We shall use the property that athe polar of a point is the chord of contact of tangents drawn from it to the conic

$l|r = 1 + e\cos\theta$.

If $(\alpha - \beta), (\alpha + \beta)$ be the vectorial angles of the points of contact, the equation of the chord of contact is

$l|r = \sec\beta \cos(\theta - \alpha) + e\cos\theta \ldots\ldots\ldots\ldots(1)$

Now equation of the tangent at $(\alpha - \beta)$ is

$l|r = \cos(\theta - \alpha + \beta) + e\cos\theta$

This passes through the point $(r_{1,}\theta_1)$, therefore,

$l|r_1 = \cos(\theta_1 - \alpha + \beta) + e\cos\theta_1 \ldots\ldots\ldots\ldots..(2)$

Similarly,equation of the tangent at $(\alpha + \beta)$ is

$l|r = \cos(\theta - \alpha - \beta) + e\cos\theta$

This passes through the point $(r_{1,}\theta_1)$, therefore,

$l|r_1 \ = \ cos\ (\theta_1 \ - \ \alpha - \ \beta) + ecos\ \theta_1$…………..(3)

From equation (2) and (3) we have

$cos\ (\theta_1 \ - \ \alpha + \ \beta)$ = $cos\ (\theta_1 \ - \ \alpha - \ \beta)$

That is $(\theta_1 \ - \ \alpha + \ \beta) = \pm(\theta_1 \ - \ \alpha - \ \beta)$

Since $\beta \neq 0$, then $(\theta_1 \ - \ \alpha + \ \beta) = -(\theta_1 \ - \ \alpha - \ \beta)$

That is, $\alpha = \ \theta_1$.

Substituting this value of α in equation (2) and (3),

we get $cos\beta = l|r_1 - ecos\ \theta_1$.

From the equation (1), the polar of the point $(r_{1,}\theta_1)$ is

$$(\boldsymbol{l|r - ecos\theta})(\boldsymbol{l|r_1 - ecos\ \theta_1}) = \mathbf{cos}(\boldsymbol{\theta - \theta_1})$$

***Remark*:1.** The pole of a line is the point of the intersection of the tangents at its extremities.

2. The polar of a point with respect to a given conic is the same as the chord of the contact of the tangents drawn from the point to the conic, but here the point must lie outside the conic**.**

Example 11: Show that the director circle of the conic

$l|r \ = \ 1 + \ e\ cos\ \theta$ is $r^2(1 - e^2) + 2elrcos\theta - 2l^2 = 0$.

Solution: The equations of the tangents at the points α, β of the given conic are

$l|r \ = \ \ cos\ (\theta \ - \ \alpha) + ecos\ \theta$ and

$l|r \ = \ \ cos\ (\theta - \ \beta) + ecos\ \theta$

If θ be the vectorial angle of the point where the tangents intersect each other,

$$cos\ (\theta - \ \alpha) = \ cos\ (\theta - \ \beta)$$

that is , $(\theta - \ \alpha) = \pm\ (\theta - \ \beta)$ Neglecting the positive sign,

$$\theta = (\alpha + \beta)|2 \dots \dots \dots \dots (1)$$

Substituting this value of θ in the equation of either tangent the radius vector r of the point of intersection can be written as

$$l|r \ = \ \ cos\ (\ \alpha - \ \beta)|2 + ecos\ (\alpha + \beta)|2 \ \text{…………..(2)}$$

Converting the equations of the tangents in coordinates, we see that they are at right angles if

$(cos\alpha + e)(cos\beta + e) + sin\alpha sin\beta = 0$

that is, if $e^2 + e(cos\alpha + cos\beta) + \cos(\alpha - \beta) = 0$

which can be written as

$$e^2 + e(\cos(\alpha+\beta)\,|2\cos(\alpha-\beta)|2 + 2\cos^2(\alpha-\beta)|2 - 1 = 0 \ldots..(3).$$

Eliminating α and β from equation (3) with the help of equations (1) and (2),

$$e^2 + 2ecos\theta(l|r - \ e\ cos\ \theta.) + 2(l|r - \ e\ cos\ \theta)^2 - 1 = 0$$

or, $r^2(1-e^2) + 2elrcos\theta - 2l^2 = 0$

Which is the equation of the circle.

Example 12: Prove that the two conics $l_1|r \ = \ 1 - \ e\ cos\ \theta$

and $l_2|r \ = \ 1 - \ e\ cos\ (\theta - \alpha)$ will touch one another if

$l_1{}^2(1 - e_2{}^2\ ) + \ l_2{}^2(1 - e_1{}^2\ ) = 2l_1l_2(1 - \ e_1e_2cos\alpha)$.

Solution: If the vectorial angle of the point of contact be θ', the equations of the tangents of the given conics are respectively

$$l_1|r \ = \ cos(\theta - \theta') - \ e\ cos\ \theta$$

and $l_2|r \ = \ \cos(\theta - \theta') - \ e\ cos\ (\theta - \alpha)$

that is, $l_1|r \ = (cos\theta - \ e_1)cos\theta + \ sin\theta sin\theta'$,

and $l_2|r \ = (\cos\theta' - e_2cos\alpha)cos\theta + (sin\theta' - \ e_2sin\alpha)\ sin\ \theta$

Comparing the coefficients,

$$l_2|l_1 = \ (\cos\theta' - e_2cos\alpha)|(cos\theta - \ e_1)$$

$$= \ (sin\theta' - \ e_2sin\alpha)|\ sin\theta'$$

from these we get, $sin\theta'(l_2 - \ l_1) = \ -e_2l_1sin\alpha$

And $cos\theta'(l_2 - \ l_1) = \ -e_1l_2 - e_2l_1cos\alpha.$

Now we squaring and adding them , we get

$l_1{}^2(1 - e_2{}^2\ ) + \ l_2{}^2(1 - e_1{}^2\ ) = 2l_1l_2(1 - \ e_1e_2cos\alpha)$.

Check your progress

(1) A conic is described having the same focus and eccentricity as the conic$l|r \ = \ 1 + \ e\ cos\ \theta$, and two conics touch at the point $\theta = \alpha$; prove that the length of its latus rectum is $2l(1 - \ e^2)|(e^2 + 2ecos\alpha + 1)$.

(2) P, Q, R are three points on the conic $l|r \ = \ 1 + \ e\ cos\ \theta$, the focus S beingthe pole; SP and SR meet the tangent at Q in M and N so that $SM \ = \ SN \ = \ l$. Prove that PR touches the conic$l|r \ = \ 1 +$

$2\,e\cos\theta$.

(3) The tangents at P and Q to a parabola meet at T. Show that $ST^2 = SP.SQ$

(4) Find the equation of the chord of the conic $l|r = 1 + e\cos\theta$ joining the points whose vectorial angles are $\pi|6$ and $\pi|3$.

(5) Tangents are drawn at the extremities of perpendicular focal radii of a conic. Show that the locus of their point of intersection is another conic having the same focus.

(6) In any conic prove that

(a) the tangents at the ends of a focal chord meet on the directrix.

(b) the portion of tangent intercepted between the curve and the directrix subtends a right angle at the corresponding focus.

(7) Two chords QP, PR of a conic subtendequal angles at the focus. Prove that the chord QR and the tangent at P intersect on the directrix.

(8) If the normals at three pointsof the parabola $r = a\,cosec^2(\theta|2)$, whose vectorial angles are α, β, γ meet in a point whose vectorial angle is ϕ, prove that $2\phi = \alpha + \beta + \gamma - \pi$.

(9) Prove that, if chords a conic subtend a constant angle at a focus, the tangents at the ends of the chords meet on a fixed conic and these chords will touch another fixed conic.

(10) A is the vertex of a conic, and AP a chord which meets the latus rectum in Q. A parallel chord $P'SQ'$ is drawn through the focus S. Show that the ratio $(AP.AQ)|(SQ'.SP')$ is constant.

(11) If tangent at any point of an ellipse makes an angle α with its major axis and an angle β with focal radius to the point of contact, show that $ecos\alpha = cos\beta$.

(12) Prove that two points on the conic $l|r = 1 + e\cos\theta$, whose vectorial angles are α and β respectively will be the extrimities of a diameter if $(e+1)|(e-1) = \tan\alpha|2\ \tan\beta|2$.

(13) Find the locus of the pole of a chord of the conic $l|r = 1 + e\cos\theta$, which subtends a constant angle2γ at the focus.

2.8 AUXILIARY CIRCLE

The locus of the foot of the perpendicular from the focus on any tangent to a conic (ellipse or hyperbola) is a circle, called the auxiliary circle of the conic.

Equation of the auxiliary circle of the conic

$l|r = 1 + e\cos\theta$. …..(1)

We take a point α on the conic(1). Equation of the tangent at the point α is $l|r = \cos(\theta - \alpha) + e\cos\theta$ ……..(2)

We change the equation (2) in Cartesian form, we have

$l = (\cos\alpha + e)x + \sin\alpha\, y$…….(3)

Equation of a line perpendicular to the (3) and passing through the focus is

$0 = \sin\alpha\, x - (\cos\alpha + e)y$

Changing it to the polar form, we have

$0 = \sin\alpha\, r\cos\theta - (\cos\alpha + e)r\sin\theta$

Or, $\sin(\theta - \alpha) = -e\sin\theta$………..(4)

Now the foot of the perpendicular from the focus $s$to the tangent (2) is the given by the intersection of (2) and (4), and hence its locus is obtained by eliminating the variable α between (2) and (4). Equations (2) and (4) may be written as

$l|r - e\cos\theta = \cos(\theta - \alpha)$ and $\sin(\theta - \alpha) = -e\sin\theta$

Squaring and adding these equations, we have

$$(e\sin(\theta))^2 + (l|r - e\cos\theta)^2 = 1$$

Or, $(l|r)^2 - 2(le)|r\cos\theta + e^2 - 1 = 0$

$(e^2 - 1)r^2 - 2ler\cos\theta + l^2 = 0.$

This is the required equation of the auxiliary circle.

Note: In the case of parabola $e = 1$, the equation of the tangent to the parabola $l|r = \cos(\theta - 0) + 1.\cos\theta$

at the vertex , which is the equation of the tangent to the parabola $l|r = 1 + \cos\theta$ at the vertex(i.e. at the point $\theta = 0$)

2.9 THE POINT OF INTERSECTION OF TWO TANGENTS

Suppose that equation of the conic is $l|r = 1 + e\cos\theta$………..(1)

The equation of the tangents at the points α and β are

$l|r = \cos(\theta - \alpha) + e\cos\theta$ ……..(2)

$l|r = \cos(\theta - \beta) + e\cos\theta$ ……..(3)

Now we substract equation (3) from (2), for finding of the points of intersection

$$\cos(\theta - \alpha) - \cos(\theta - \beta) = 0$$

$$\cos(\theta - \alpha) = \cos(\theta - \beta)$$

$$(\theta - \alpha) = \pm(\theta - \beta)$$

If we take the positive sign, we get $\alpha = \beta$ which is inadmissible. So, we take negative sign, we get

$(\theta - \alpha) = -(\theta - \beta)$ or, $\theta = (\alpha + \beta)|2$

Putting the value of θ in equation (2) or (3), we get

$l|r = cos\left((\alpha + \beta)|2 - \alpha\right) + ecos\,(\alpha + \beta)|2$

$= l|r = cos\left((\alpha - \beta)|2\right) + ecos\,(\alpha + \beta)|2$

If the point of intersection is (r', θ'), then we have

$$\theta' = (\alpha + \beta)|2$$

and $l|r' = cos\left((\alpha - \beta)|2\right) + ecos\,(\alpha + \beta)|2$

Note: In the case of parabola $e = 1$, the equation of the tangent to the parabola

$l|r' = cos\left((\alpha - \beta)|2\right) + cos\,(\alpha + \beta)|2$

$= 2cos\alpha|2cos\beta|2$

or, $r' = l|2sec\alpha|2sec\beta|2$ and $\theta' = (\alpha + \beta)|2$

2.10 DIRECTOR CIRCLE

The locus of the point of the intersection of two perpendicular tangents to a conic, is called the director circle of the conic.

Suppose that equation of the conic is $l|r = 1 + ecos\theta$………..(1)

The director circle of the conic (1) is the locus of the point of intersection of perpendicular tangents to the conic(1)

The equation of the tangents at the points α and β are

$l|r = cos\,(\theta - \alpha) + ecos\,\theta$ ……..(2)

$l|r = cos\,(\theta - \beta) + ecos\,\theta$ ……..(3) respectively

Now we substract equation (3) from (2), for finding of the points of intersection

$$\cos(\theta - \alpha) - \cos(\theta - \beta) = 0$$

$$\cos(\theta - \alpha) = \cos(\theta - \beta)$$

$$(\theta - \alpha) = \pm(\theta - \beta)$$

If we take the positive sign, we get $\alpha = \beta$ which is inadmissible. So, we take negative sign, we get

$(\theta - \alpha) = -(\theta - \beta)$ or, $\theta = (\alpha + \beta)|2$

Putting the value of θ in equation (2) or (3), we get

$l|r = cos\left((\alpha+\beta)|2-\alpha\right)+ecos\,(\alpha+\beta)|2$

$= l|r = cos\left((\alpha-\beta)|2\right)+ecos\,(\alpha+\beta)|2$

If the point of intersection is (r',θ'), of the tangents (2) and (3) then we have

$\theta' = (\alpha+\beta)|2$

and $l|r' = cos\left((\alpha-\beta)|2\right)+ecos\,(\alpha+\beta)|2$…..(4)

Changing the equation (2) of the tangent at the point α to Cartesian form, we have $l = (cos\alpha + e)x + sin\alpha\, y$

Therefore the slope of the tangent (2) is

$m_1 = -(\cos\alpha + e)|\sin\alpha$

Similarly, the slope of the tangent (3) is

$m_2 = -(\cos\beta + e)|\sin\beta.$

Since the tangents are perpendicular so, $m_1m_2 = -1$

Or, $[-(\cos\alpha + e)|\sin\alpha][-(\cos\beta + e)|\sin\beta] = -1$

Or, $(\cos\alpha\cos\beta + \sin\alpha\sin\beta) + e(\cos\alpha + cos\,\beta) + e^2 = 0$

Or,$cos(\alpha-\beta) + 2e\,cos((\alpha+\beta)|2)cos((\alpha-\beta)|2) + e^2 = 0$……….(5)

From (4), we have
$\theta' = (\alpha+\beta)|2$

and $l|r' - ecos\theta' = cos\left((\alpha-\beta)|2\right)$……….(6)

Eliminating α and β with the help of (5) and (6), we have

$2(l|r' - ecos\theta')^2 - 1 + 2ecos\theta'.(l|r' - ecos\theta') + e^2 = 0$

Or, $(1-e^2)r'^2 + 2ler'cos\theta' - 2l^2 = 0$

Therefore, the locus of the point (r',θ') is

$$(1-e^2)r^2 + 2lercos\theta - 2l^2 = 0$$

which is the required equation of the director circle.

Note: In the case of parabola $e = 1$, the equation of the director circle to the parabola becomes

$2lrcos\theta - l^2 = 0$ or, $l|r = cos\theta$

which is the equation of the directrix of the parabola

$l|r = 1 + cos\theta.$

Hence in the case of a parabola the locus of the point of intersection of perpendicular tangents is the directrix of the parabola.

Example 13: Show that the locus of the feet of perpendiculars from the focus S of a conic on chords subtending a constant angle 2γ at S is the circle whose polar equation referred to S as pole is $r^2(e^2 - sec^2\gamma) - 2lercos\theta + l^2 = 0$

where $2l$ is the latus rectum and e is the eccentricity of the conic.

Solution: Suppose that equation of the conic whose focus S as the pole be $l|r = 1 + cos\theta.$ ………(1)

Let PQ be the chord of (1) subtending an angle 2γ at the focus S. Let $(\alpha - \gamma)$ and $(\alpha + \gamma)$ be the vectorial angle of the extremities of the chord

PQ. Then the equation of the chord PQ is

$l|r = ecos\theta + sec\gamma\, cos(\theta - \alpha)$ …… (2)

$or, l|r = ecos\theta + sec\gamma\, cos\theta\, cos\alpha + sec\gamma\, sin\alpha\, sin\theta$

Or, $l = (e + sec\gamma\, cos\alpha)rcos\theta + (sec\, \gamma sin\alpha\,)rsin\theta$…..(3)

Equation of the perpendicular drawn from the focus S(pole as origin) to the line (3) is

$0 = (e + sec\gamma\, cos\alpha)rsin\theta - (sec\, \gamma sin\alpha\,)rcos\theta$

Or, $-esin\theta = \sec\gamma \sin(\theta - \alpha)$….. (4)

The foot of the perpendicular drawn from the focus $S$to the chord (2) is the point of the intersection of the lines (2) and (4)

The equation (2) can be written as

$(l|r - ecos\theta) = \sec\gamma \cos(\theta - \alpha)$ ……(5)

Squaring and adding equation (4) and (5), we get

$(esin\theta)^2 + (l|r - ecos\theta)^2 = sec^2\gamma)$

Or, $e^2 + l^2|r^2 - 2le|rcos\theta = \sec^2\gamma$

Or, $r^2(e^2 - \sec^2\gamma\,) - 2lercos\theta + l^2 = 0.$

Which is the required locus.

Check your progress

(1) If A, B, C be any three points on a parabola, and the tangents at these pointsform a triangle$A'B'C'$. Show that $SA.SB.SC = SA'.SB'.SC'$, S being the focus of the parabola**.**

(2) Find the equation of thwe circle circumscribing the triangle formed by tangents at three given points of a parabola.

(3) Prove that the centres of the four circles circumscribing the four triangles formed by the four tangents drawn to a parabola at points whose vectorial angles are $\alpha, \beta, \gamma, \delta$ lie on another circle which passes through the focus of the parabola.

(4) $P, Q\ R$are three points on the conic $l|r = 1 + cos\theta$. The focus S being the pole. The tangent at Q meets SP and SR in M and S so that $SM = SN = 1$. Prove that the chordPR touches the conic $l|r = 1 + 2ecos\theta$.

(5) If the tangents at any two points P and Q of a conic meet in a point $T$and if the chord PQ meets the directrix corresponding to S in a point K, prove that the angle KST is a right angle.

(6) Show that three normals can be drawn from a point (ρ, ϕ) to a parabola.

(7) Find the condition that the line $l|r = Acos\theta + Bsin\theta$ may be a tangent to the conic $l|r = 1 + ecos\theta$.

(8) Prove that the line $l|r = \cos(\theta - \alpha) + ecos(\theta - \gamma)$ is the thangent to the conic $l|r = 1 + ecos(\theta - \gamma)$ at the point for which $\theta = \alpha$.

Summary

(1) The general equation of the second degree is ax²+by²+2hxy+2gx+2fy+c = 0……..(1)

Let $a \neq 0$ and $b \neq 0$, then equation (1) is written as

$a(x^2 + 2gx|a + (g|a)^2) + b(y^2 + 2fy|b + (f|b)^2) - (g|a)^2) - (f|b)^2 + c = 0$

Or, $a(x + g|a)^2 + b(y + f|b)^2 = (g|a)^2) + (f|b)^2 - c = K\ (say)$

Sifting the origin to $(-g|a, -f|b)$, then this equation becomes

$ax^2 + by^2 = K$……….(2)

(i) If $K = 0$, the equation (2) becomes $ax^2 + by^2 = 0$ and this represent a pair of straight lines. These straight lines are real if a and b are of the opposite signs and these lines are imaginary if a and b are of the same sign.

(ii) If $K \neq 0$, the equation (2) becomes $x^2|K|a + y^2|K|b = 1$. ……(3)

If $K|a$ and $K|b$ are both positive, the equation (3) represents an ellipse which becomes a circle if in addition to being positive $K|a$ and $K|b$ are both equal.

Again the equation (3) represents a hyperbola if $K|a$ and $K|b$ are of opposite signs. If $K|a$ and $K|b$ are both negative, the equation (3) is said to represent an emaginary ellipse.

Case II: If one of a or b is zero while other is not zero. If we take $a = 0$ and $b \neq 0$ the the equation (1) will be

$by^2+2hxy+2gx+2fy+c = 0$

or, $(y + f|b)^2 = -(2g|b)x - c|b + (f|b)^2$ …………(4)

If $g = 0$, then equation (4) represents two parallel straight lines, which are coincident if $f^2 - bc$ also is zero.

If $g \neq 0$, the equation (4) can be written as

$(y + f|b)^2 = -(2g|b)[x + c|2g + f^2 |2bg]$

Shifting the origin to $(f^2 |2bg - c|2g, -f|b)$, this equation becomes $y^2 = -(2g|b)x$ which represents a parabola. Hence in each case the general equation of second degree represents a conic section.

(2) If the equation of the conic is $l|r = 1 + e\,cos\,(\theta)$, then the equation of the tangent at $'\alpha'$ is $l|r = cos\,(\theta - \alpha) + ecos\,(\theta)$

(3) If the equation of the conic is $l|r = 1 + e\,cos\,(\theta - \theta_1)$, then the equation of the tangent at $'\alpha'$ is $l|r = cos\,(\theta - \alpha) + ecos\,(\theta - \theta_1)$

Note 2: The equation of the tangent for the conic $l|r = 1 + e\,cos\,(\theta - \gamma)$ at the point "α" is $l|r = cos\,(\theta - \alpha) + ecos(\theta - \gamma)$.

Note 3: The slope of the tangent (1) is $(e + cos\alpha)|sin\alpha$**.**

(4) The equation of the normal at the point whose vectorial angle is α is $elsin\alpha|\,(1 + ecos\alpha)r = \sin(\theta - \alpha) + e\,sin\theta$

(5) The equation of the asymptotes of the conic $l|r = 1 + ecos\theta$, are the straight lines $l|r = -l|r = (\sqrt{(e^2 - 1)}|e)\,(\sqrt{e^2 - 1}\cos\theta \pm \sin\theta)$.

(6) The polar of the point (r_1, θ_1) of the conic $l|r = 1 + ecos\theta$ is

$$(l|r - ecos\theta)(l|r_1 - ecos\,\theta_1) = \cos(\theta - \theta_1)$$

***Remark*:1.** The pole of a line is the point of the intersection of the tangents at its extremities.

2. The polar of a point with respect to a given conic is the same as the chord of the contact of the tangents drawn from the point to the conic, but here the point must lie outside the conic**.**

3. The locus of the foot of the perpendicular from the focus on any tangent to a conic (ellipse or hyperbola) is a circle, called the auxiliary circle of the conic. The required equation of the auxiliary circle of the conic $l|r = 1 + ecos\theta$ is

$$(e^2 - 1)r^2 - 2lercos\theta + l^2 = 0.$$

(7) The equation of the director circle is the locus of the point (r', θ') on the conic $l|r = 1 + ecos\theta$ is $(1 - e^2)r^2 + 2lercos\theta - 2l^2 = 0$

Uttar Pradesh Rajarshi Tandon Open University

Bachelor of Science

UGMM-102

Analytical Geometry

BLOCK

2

SPHERE AND CYLINDER

UNIT-3

Geometry of Dimension

UNIT-4

Sphere

UNIT-5

Cylinder

DGB-021

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BLOCK INTRODUCTION

Unit-3 **Geometry of 3 Dimension :** Strainght line and plane, direction cosines and direction numbers, distance of a point from a line, various form of the equation of a plane, plane passing through three given points, angle between two lines and two planes, distance of a point from a plane, equation of line of intersection of two planes, intersection of line and plane-coplanar lines shortest distance between two skew lines.

Unit-4 **Sphere :** Equation of a sphere, Intersection of sphere and planes, Intersection of two sphere. Sphere passing through a circle, Inersection of a straight line and a sphere. Tangent planes, Polar planes, Plane of contact. Power of a point. Radical planes, Radical lines, Co-axel system of a sphere. Orthogonal system of sphere.

Unit-5 **Cylinder** : Equation of a cylinder with given base, Cylinder with given Axis parallel to co-ordinate axes. Enveloping cylinders, Right circular cylinder. Ruled surfaces, generating lines of a hyperboloid of one sheet and their simple properties.

UNIT-3 GEOMETRY OF 3 DIMENSION

Structure

3.1.23. Equation of a plane through a given line and parallel to an another line.

3.1.24. Foot of perpendicular and length of perpendicular from a point to a line.

3.1.25. coplanar lines

3.1.26. condition for the two lines to intersect(in symmetrical form)

3.1.27. condition for the two lines to intersect(in general form)

3.1.28. Equation of a straight line intersecting the two given (in symmetrical form)

3.1.29. Perpendicular distance of a point from a line and the coordinates of the foot of perpendicular.

3.1.30. To find the coordinates of the foot of the perpendicular

3.1.31. The shortest distance between any two non intersecting lines

3.1.32. Length and equation of the line of shortest distance

3.1.1 INTRODUCTION

In this unit, our aim is to re-acquaint with some essential elements of three dimensional geometry. The French philosopher mathematician **Rene Descartes** (1596--1650) was the first to realize that geometrical ideas can be translated into algebraic relations. The combination of Algebra and Plane Geometry came to be known as **Coordinate Geometry** or **Analytical Geometry**. A basic necessity for the study of Coordinate Geometry is thus, the introduction of a coordinate system and to define coordinates in the concerned space. We will briefly touch upon the distance formula and various ways of representing a plane and straight line algebraically. Next, we will talk about symmetry with respect to origin or a coordinate axis. Finally, we shall consider some ways in which a coordinate system can be transformed. This collection of topics may seem random to us .

We have read about planes and lines, angles and rectilinear figures in geometry. Recall that a line is the join of two points in a plane continuing endlessly in both directions .We have also seen that graphs of linear equations, which came out to be straight lines. Interestingly , there are problems of the above is finding the equations of straight lines ,under different conditions in a plane. The Analytical Geometry, more commonly called Coordinate

Geomatry, comes to our help in this regard.

In this unit we shall find equations of a straight Lines and planes in different forms and try to solve the problem based on those.

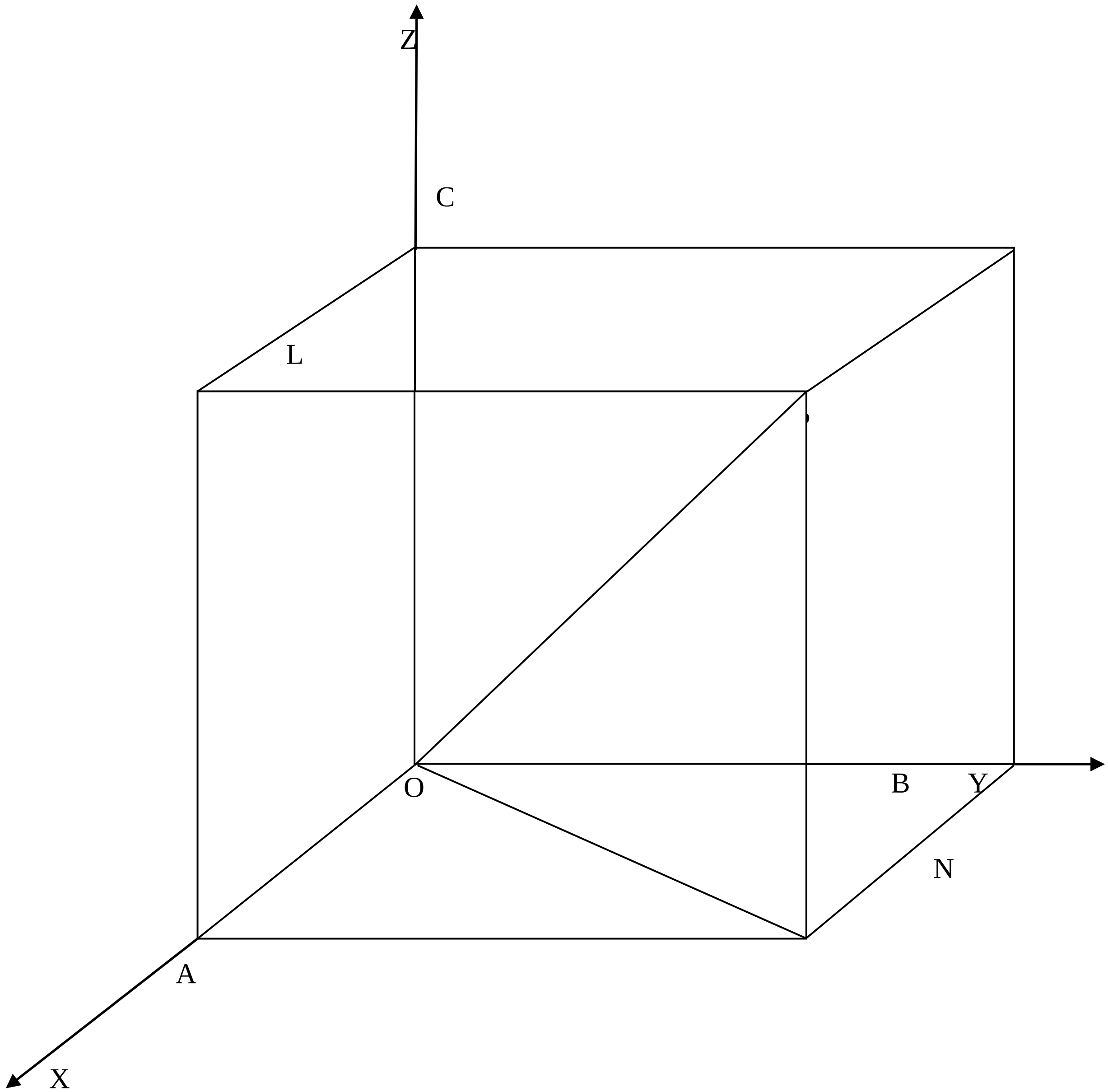
3.1.2 OBJECTIVES

After studying this unit you should be able to find:

1. Direction ratios and direction cosines of a line
2. Equation of a plane in different forms
3. Angle between two planes and condition for parallelism and perpendicular
4. Equation of a straight line in general form\ symmetrical form
5. Condition for parallelism\perpendicular of a line and a plane.
6. Equation of a plane through a given line (symmetrical form\general form) 7. Foot of perpendicular and length of perpendicular from a point to a line.
8. coplanar lines
9. condition for the two lines to intersect(in symmetrical form\general form).
10. Equation of a straight line intersecting the two given (in symmetrical form).
11. Perpendicular distance of a point from a line and the coordinates of the foot of perpendicular.
12. To find the coordinates of the foot of the perpendicular
13. The shortest distance between any two non intersecting lines.
14. Length and equation of the line of shortest distance.
15. The equation of the shortest distance.

3.1.3 COORDINATES OF A POINT IN SPACE

To fix the position of a point in space we required three concurrent lines which are not coplanar. Let $X'OX, Y'OY \ and \ Z'OZ$ be such straight lines whose positive directions are $X'OX, Y'O \ and \ Z'OZ$. Let P be a point in space and let planes parallel to the planes $YOZ, ZOX \ and \ XOY$ be drawn through P to meet the lines $X'X, Y'Y \ and \ Z'Z$in $A, B \ and \ C$, then position of P is Known when the segments OA, OB, OC are given in magnitude and sign. If $OA = x, OB = y \ and \ OC = z$ we say that (x, y, z) are the **Cartesian coordinates** of P.



The lines $X'OX, Y'OY$ and $Z'OZ$ are called the coordinate axes and the planes YOZ, ZOX and XOY are coordinate planes. The point O is called the origin.

3.1.4 DIRECTION COSINES OF A LINE

Let AB be a given straight line. We draw a line through O parallel to AB. The angles which AB makes with the coordinate axes are the same as those made by the parallel straight line. Denoting these angles by α, β and γ we say that $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the direction cosines of AB. The direction cosines of a line are usually denoted by the letters l, m and n.

Note: Quantities proportional to direction cosines of a given line are called **direction ratios**.

Theorem: If direction cosines of a given line are $l, m \; and \; n$ then $l^2 + m^2 + n^2 = 1$.

Proof: Let $l, m \; and \; n$ be the direction cosines of a given line. The direction cosines of OP which is drawn parallel to the given line are $l, m \; and \; n$. We draw PA perpendicular to OX. If (x, y, z) be the coordinates of P, then $OA = x$.

Let $OP = r$, and the angle POA be α, then from the right angled triangle AOP,

$AO|OP = cos\alpha$ that is $x|r = l$, or, $x = lr$.

Similarly, $y = mr$, and $z = nr$.

Now we squaring and adding them,

$$x^2 + y^2 + z^2 = r^2(l^2 + m^2 + n^2).$$

Since, $x^2 + y^2 + z^2 = r^2$.

Therefore, $l^2 + m^2 + n^2 = 1$.

3.1.5 DIRECTION COSINES OF THE LINES JOINING TWO GIVEN POINTS

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points in the space. Let $PQ = r$, then $r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$.

Transferring the origin to P, the axes remaining parallel to original axes, the coordinates of Q are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. If l, m, n be the direction cosines of PQ, we have from the preceding, $(x_2 - x_1|r = l, (y_2 - y_1)|r = m, and \; (z_2 - z_1)|r = n$. The direction cosines of the given line are thus proportional to the quantities $x_2 - x_1, y_2 - y_1, z_2 - z_1$, their actual values being

$$(x_2 - x_1)|r, (y_2 - y_1)|r \; and \; (z_2 - z_1)|r.$$

Example1: Show that the points $(1, 2, 3), (2, 1, 3)$ and $(3, 1, 2)$ are the vertices of an equilateral triangle.

Solution: Let $A(1, 2, 3), B(2, 1, 3)$ and $C(3, 1, 2)$ be the given points. Therefore, the distances are

$$AB = \sqrt{(2-1)^2 + (3-2)^2 + (1-3)^2} = \sqrt{6}$$

$$BC = \sqrt{(3-2)^2 + (1-3)^2 + (2-1)^2} = \sqrt{6}$$

And

$$CA = \sqrt{(3-1)^2 + (2-3)^2 + (1-2)^2} = \sqrt{6}$$

We can find that $AB = BC = CA$. Hence, the triangle ABC is equilateral.

Example2: Find the ratio in which $A(-2,4,5)$ and $B(3,-5,4)$ is divided by $YZ -$ plane.

Solution: Suppose that r:1 be the ratio in which YZ- plane divides the line joining $A(-2,4,5)$ and $B(3,-5,4)$, the point of division

$P = [(3r-2)|r+1, (-5r+4)|r+1, (4r+5)|r+1]$,

But the point P lies on YZ- plane. Therefore, the $X-$ co-ordinate $(3r-2)|r+1 = 0.$ that is $(3r-2) = 0$, therefore, $r = 2|3$.

Therefore, the ratio is $2:3$ internally.

Example3: Find the direction cosines of the line joining the points $(1,2,-3)$ *and* $(-2,3,1)$.

Solution: The directional cosines are proportional to $-2-1, 3-2, 1-(-3)$ that is $-3,1,4$. The actual direction cosines of the given line are

$$-3|\sqrt{9+1+16},\ 1|\sqrt{9+1+16},\ 4|\sqrt{9+1+16},$$

That is, $-3|\sqrt{26},\ 1|\sqrt{26}, 4|\sqrt{26}$.

3.1.6 PROJECTION OF A LINE SEGMENT

The projection of a given line AB on an another line CD is the segment $A'B'$ of CD where A' and B' are projections of A and B on CD.

Note :

1. To find the projection of AB on , we draw planes through points A and B which are perpendicular to CD intersecting CD in $A'$and B'. If θ is the angle between AB and CD, the length $A'B'$ of the projection is obviously $ABcos\theta$.

2. In determine the projection of one line on another line we must be taken regarding the sense of rotation.

3. For an actual angle of the projection is positive or negative according as the rotation is counter- clockwise or clockwise.

Angle between two lines: Suppose that $l, m, n, and\ l', m', n'$ be the direction cosines of two lines AB and CD. We want to find the angle between AB and CD in terms of their direction cosines.

We draw OP and OQ parallel to AB and CD respectively. Suppose angle between OP and OQ is θ which is same as the angle between AB and CD.The direction cosines of OP and OQ are $l, m, n \ and \ l', m', n'$ respectively.

Let the coordinates of P and Q be (x, y, z) and (x', y', z'). If $OQ = r'$, the projection of OQ on OP is $r'cos\theta$ which is equal to $lx' + my' + nz'$. Therefore,

$r'cos\theta = lx' + my' + nz'$

or, $cos\theta = l(x'|r') + m(y'|r') + n(z'|r')$

$= ll' + mm' + nn'$.

3.1.7 PLANE

A plane is a surface such that every straight line joining any two points on it lies wholly on it

Normal to a plane: A straight line which is perpendicular to every line lying in a plane is called a normal to that plane. It is also called a line perpendicular to that plane. All the normal to a plane are parallel lines.

Equation of a plane in general form: Equation of plane in normal form is $xcos\alpha + ycos\beta + zcos\gamma = p$

Hence, if l, m, n be the direction cosines of the normal to a plane directed from the origin to the plane and p be the length of the perpendicular from the originto the plane, then the equation of the plane is $\boldsymbol{lx + my + nz = p}$.

This is known as the **equation of a plane in normal form.**

3.1.8 GENERAL EQUATION OF A PLANE

The general equation of a plane is $ax + by + cz + d = 0$. That is every equation $ax + by + cz + d = 0$ of first degree in x, y, z always represents a plane and the coefficients a, b, c of x, y, z in this equation are direction ratios of normal to this plane.

The number of arbitrary constants in the general equation of the plane $ax + by + cz + d = 0$ or,

$a|dx + b|dy + c|dz = -1$. This equation show that there are three arbitrary constants namely $a|d, b|d, c|d$ in the equation of a plane. Therefore, the equation of a plane can be determined to satisfy the three conditions, each condition giving us the value of a constant.

$Note$: The equation of any plane passing through the origin is

$$ax + by + cz = 0$$

To reduce the general equation of the plane in normal form: Suppose that the general equation of a plane is

$$ax + by + cz + d = 0 \ldots\ldots (1)$$

If l, m, n are the direction cosines of the normal to the plane, then the equation of the plane in the normal form is

$$lx + my + nz = p \ldots\ldots\ldots. (2)$$

If (1) and (2) represent the same plane, then

$$l|a = m|b = n|c = p| - d = \pm\sqrt{(l^2 + m^2 + n^2)} \mid \sqrt{(a^2 + b^2 + c^2)}$$

$$= \pm 1 \mid \sqrt{(a^2 + b^2 + c^2)}$$

Where the same sign either positive or negative is to be chosen throughout.

$$l = \pm a \mid \sqrt{(a^2 + b^2 + c^2)}, m = \pm b \mid \sqrt{(a^2 + b^2 + c^2)}$$

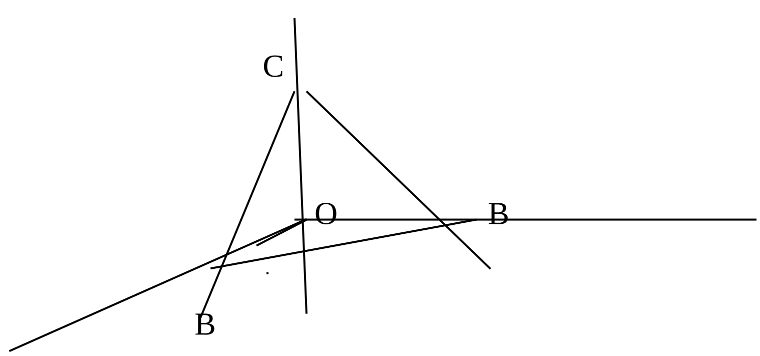
$$n = \pm c \mid \sqrt{(a^2 + b^2 + c^2)}, p = \pm d \mid \sqrt{(a^2 + b^2 + c^2)}$$

Substituting these values in equation (2), the normal form of the plane (1) is given by

$\pm ax \mid \sqrt{(a^2 + b^2 + c^2)} \pm by \mid \sqrt{(a^2 + b^2 + c^2)} \quad \pm cz \mid \sqrt{(a^2 + b^2 + c^2)}$ = $\pm d \mid \sqrt{(a^2 + b^2 + c^2)}$…….(3)

The sign of the equation (3) is so chosen that p is $\pm d \mid \sqrt{(a^2 + b^2 + c^2)}$ is always positive.

3.1.9 EQUATION OF A PLANE IN INTERCEPT FORM



Let O be the origin and let the plane meet the coordinate axes at the points A, B, C respectively such that $OA = a, OB = b$ and $OC = c$ with proper signs. Therefore the coordinates of the points A, B, C are

$A(a,0,0), B(0,b,0)$ and $C(0,0,c)$. Let the equation of the plane be $Ax + By + Cz + D = 0 \ldots\ldots(1)$

$Where\ D \neq 0$ because the plane does not pass through the origin $(0,0,0)$. Since the plane (1) passes through the points A(a, 0, 0), B(0, b, 0) and C(0, 0, c) therefore, $A = -D|a, B = -D|b$ and $C = -D|c$. Putting the values of A, B, C in (1), then the required equation of the plane is

$$(-D|a)\,x + (-D|b)y + (-D|c)z + D = 0$$
$$(-1|a)\,x + (-1|b)y + (-1|c)\,z + 1 = 0$$
$$x|a + y|b + z|c = 1$$

This is a equation of a plane in intercept form.

Note: The equation of $xy -$ plane is $z = 0$. The equation of $xz -$ plane is $y = 0$. The equation of $yz -$ plane is $x = 0$.

3.1.10 GENERAL EQUATION OF A PLANE THROUGH A GIVEN POINT AND PERPENDICULAR TO A GIVEN LINE

Suppose the coordinates of a point $P(x,y,z)$ on the plane. If the plane passes through the point $A(x_1, y_1, z_1)$, the line AP whose direction ratios are

$x - x_1, y - y_1, z - z_1$ lies in the plane. The direction ratios are

$x - x_1, y - y_1, z - z_1$ normal to the plane whose direction ratios are a, b, c. So, $a(x - x_1) + b\big(y - y_1\big) + c(z - z_1) = 0$.which is the required equation of the plane.

Remark: The equation of any plane passing through the origin is

$ax + by + cz = 0$, in which the coefficients of x, y, z i. e. a, b, c are direction ratios of the normal to the plane.

3.1.11 EQUATION OF A PLANE THROUGH THREE POINTS

Suppose the general equation of the plane is $ax + by + cz + d = 0\ldots\ldots\ldots(1)$

Since it passes through three points $A(x_1, y_1, z_1)$,

$B(x_2, y_2, z_2)$ and $C(\boldsymbol{x_3}, \boldsymbol{y_3}, \boldsymbol{z_3})$. So, we have

$ax_1 + by_1 + cz_1 + d = 0\ldots\ldots\ldots..(2)$

$ax_2 + by_2 + cz_2 + d = 0\ldots\ldots\ldots..(3)$

$ax_3 + by_3 + cz_3 + d = 0$………. (4)

$Eliminating\ a, b, c$ and d from the above equations (1), (2), (3) and (4) the equation of the plane is given by

$$\begin{bmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{bmatrix} = 0$$

$$x_3 \quad y_3 \quad z_3 \quad 1$$

Note :

1. The equation $yz -$ plane is $x = 0$.

2. The equation of $xz -$ plane is $y = 0$.

3. The equation of $z -$ coordinate of which each point lying on the $xy -$ plane is $z = 0$.

4. The equation of the plane parallel to the $yz -$ plane and at a distance 'a' from it. The $x -$ coordinate of each point on this plane is equal to 'a'. Hence the equation of the required plane is given by $x = a$

5. The equation of the plane parallel to the $xz -$ plane and at a distance 'b' from it. The $y -$ coordinate of each point on this plane is equal to 'b'. Hence the equation of the required plane is given by $y = b$.

6. The equation of the plane parallel to the $xy -$ plane and at a distance 'c' from it. The $z -$ coordinate of each point on this plane is equal to 'c'. Hence the equation of the required plane is given by $z = c$

7. Equation of the plane parallel $x -$ axis will be $by + cz + d = 0$.

8. Equation of the plane parallel $y -$ axis will be $ax + cz + d = 0$.

9. Equation of the plane parallel $z -$ axis will be $ax + by + d = 0$.

3.1.12 ANGLE BETWEEN TWO PLANES

The angle between two planes is defined as the angle between their normals drawn from any point to the planes. Suppose that equations of two planes be

$a_1x + b_1y + c_1z + d_1 = 0$……….(1)

$a_2x + b_2y + c_2z + d_2 = 0$……….(2)

$The\ direction\ ratios\ of$ the normal to the plane (1) are a_1, b_1, c_1 and $the\ direction\ ratios\ of$ the normal to the plane (2) are a_2, b_2, c_2. If θ is the angle between the planes (1) and (2) then θ be the angle between the normals whose direction ratios are a_1, b_1, c_1 and a_2, b_2, c_2.

$$cos\theta = (a_1a_2 + b_1b_2 + c_1c_2)|\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\ \sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}$$

For the acute angle between the two planes, $cos\theta$ is positive and for the obtuse angle it is negative. The numerical value of $cos\theta$ in both these cases is the same because, $\cos(\pi - \theta) = cos\theta$.

Note:

1. If the two planes are perpendicular, means their normals are perpendicular then

   $ll' + mm' + nn' = 0.$

   In the case of direction ratios of the planes,

   $$(a_1a_2 + b_1b_2 + c_1c_2) = 0.$$

2. If the two planes are parallel means their normals are parallel then $l|l' = m|m' = n|n'$.

   In the case of direction ratios of the planes,

   $(a_1|a_2 = b_1|b_2 = c_1|c_2)$

Remark: The equation of any plane parallel to the plane

$ax + by + cz + d = 0$ is $ax + by + cz + \lambda = 0$

3.1.13 PERPENDICULAR DISTANCE OF A POINT FROM THE PLANE

Suppose the equation of a plane is $ax + by + cz + d = 0$………(1).

Suppose there is a point $A(x_1, y_1, z_1)$. To find the length of the perpendicular from the point $A(x_1, y_1, z_1)$ to the plane (1) is

$$\pm(ax_1 + by_1 + cz_1 + d)|\sqrt{(a^2 + b^2 + c^2)}$$

Since the perpendicular distance of a point from the plane is always positive, therefore a positive or negative sign is to be attached before the radical according as $ax_1 + by_1 + cz_1 + d$ is positive or negative i. e. according as the point $A(x_1, y_1, z_1)$ lies on the same side or on the opposite side of the equation of the plane and thus $p = |(ax_1 + by_1 + cz_1 + d)|| \sqrt{(a^2 + b^2 + c^2)}$

Note1: If the equation of the plane is in the normal form

$lx + my + nz - p = 0$, the length p_1 of the perpendicular from the point $A(x_1, y_1, z_1)$ to the plane is given by

$$p_1 = lx_1 + my_1 + nz_1 - p,$$

For, in the case of $\sqrt{l^2 + m^2 + n^2} = 1$

Note2: For the distance between two parallel planes we find the perpendicular lengths of each planes from the origin and retain their signs. The algebraic difference of these two perpendicular distances is the distance between the given parallel planes. But while applying this method we should be careful that the coefficients of x in the two equations of the planes are of the same sign.

Example4: Find the perpendicular distance from the origin to the plane $2x + y + 2z = 3$. Also find the direction cosines of the normal to the plane.

Solution : The equation of the plane is $2x + y + 2z = 3$.

To reduce it into normal form by dividing it by $\sqrt{4 + 1 + 4} = \sqrt{9} = 3$, we get $\frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z = 1$ Hence the perpendicular distance of the plane from the origin is 1and direction cosines of the normal to the plane are $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$.

Example5 : The coordinates of a point A are $(2, 3, -5)$. Determine the equation to the plane through A at right angles to the line OA, where O is the origin.

Solution : Here the plane passes through the point $A(2, 3, -5)$ *and it* is perpendicular to the line OA. i.e. the line OA is normal to the plane.

The direction ratios of the line OA is $2 - 0, 3 - 0, -5 - 0$ $i.e. 2, 3, -5$

The plane passes through the point $(2,3, -5)$ so the equation of the plane is $2(x - 2) + 3(y - 3) - 5(z + 5) = 0$

$$\text{or } 2x + 3y - 5z - 38 = 0.$$

Example6 : Find the intercepts made on the coordinate axes by the plane $x - 3y + 2z = 9$

Solution: The equation of the given plane is $x - 3y + 2z = 9$ we divide each term by 9 on both sides we have

$x|9 + y| - 3 + z| (-9|2) = 1$. So, the intercept on x-axis is 9,

the intercept on y-axis is -3 and the intercept on z-axis is $-9|2$.

Example7: Find the equation of a plane passing through three points $A(0, -1, -1), B(4, 5, 1)$ and $C(3, 9, 4)$.

Solution: Equation of a plane passing through A is

$$a(x - 0) + b(y + 1) + c(z + 1) = 0$$

ax + $b(y + 1) + c(z + 1) = 0$………(1)

Also the plane (1) passes through the points $B(4, 5, 1)$ and $C(3, 9, 4)$, then we have

$a4 + b(5 + 1) + c(1 + 1) = 0$

$4a + 6b + 2c = 0$……….(2)

$a3 + b(9 + 1) + c(4 + 1) = 0$

$3a + 10b + 5c = 0 \dots\dots\dots\dots\dots\dots$ (3)

Now solving the equation (2)and (3), we get

$a|(30 - 20) = b\,|(6 - 20) = c\,|\,(40 - 18) = \lambda$

$= a = 10\lambda, b = -14\lambda\,, c = 22\lambda$

Putting the values of a, b, c in equation of the plane the we have

10λ x + (-14λ) $(y + 1) + 22\lambda(z + 1) = 0$

$10x - 14(y + 1) + 22(z + 1) = 0$

$5x - 7y + 11z + 4 = 0.$

***E*xample8**: Find the angle between the planes $2x - y + z = 7$ and $x + y + 2z = 9$.

***S*olution:** Suppose the angle between the planes be θ means θ be the angle between their normals whose direction ratios are $2, -1, 1$ and $1, 1, 2$

$\boldsymbol{cos}\theta = [(2)(1) + (-1)(1) + (1)(2)]|\sqrt{(2^2 + (-1)^2 + 1^2)}$
$\sqrt{(1^2 + (1)^2 + 2^2)} = 3\big|\sqrt{6}\sqrt{6} = 3\big|6 = 1|2.$

Or, $\theta = \pi|3$.

Hence, the acute angle between the given planes is $\theta = \pi|3$.

3.14 A PLANE THROUGH THE INTERSECTION OF TWO PLANES

Suppose that equation of two planes be

$P = a_1x + b_1y + c_1z + d_1 = 0$……….(1)

$Q = a_2x + b_2y + c_2z + d_2 = 0$……….(2)

Then $P + \lambda Q = 0$, represents a plane where λ is a parameter.

$P + \lambda Q = 0$ means

$a_1x + b_1y + c_1z + d_{1+} \lambda(a_2x + b_2y + c_2z + d_2) = 0$

$= (a_1 + \lambda\, a_2)x + (b_1 + \lambda\, b_2)y + (c_1 + \lambda\, c_2)z +$

$(d_1 + \lambda\, d_2) = 0$………..(3)

Equation (3) is of first degree in x, y, z so, it is an equation of a plane.

Remark 1: The axis of x is the line of intersection of the planes $y = 0$ and $z = 0$. So, the equation of any plane passing through x-axis is $y + \lambda\, z = 0$ where λ is a parameter, similarly any plane passing through y-axis is $z + \lambda\, x = 0$ and any plane passing through z-axis is $y + \lambda\, x = 0$ respectively.

Remark 2: A line is parallel to the plane: If the given line $(x - \alpha)|l = (y - \beta)|m = (z - \gamma)|n$ is parallel to the plane $ax + by + cz + d = 0$, then the line is parallel to the normal to the plane. So, we have $al + bm + cn = 0$

Remark 3: A line is perpendicular to the plane: If the given line $(x - \alpha)|l = (y - \beta)|m = (z - \gamma)|n$ is perpendicular to the plane $ax + by + cz + d = 0$, then the line is parallel to the normal to the plane. So, we have $a|l = b|m = c|n$

Example 9: Find the equation of the plane through the line of intersection of the planes $x + 2y + 3z + 5 = 0, x - 3y + z + 6 = 0$ and passes through the origin.

Solution : The equation of the plane through the line of intersection of the planes $(x + 2y + 3z + 5) + \lambda(x - 3y + z + 6) = 0$.

Since it passes through the origin $(0, 0, 0)$ then we get $5 + 6\lambda = 0$ $i.e. \lambda = -5|6$. Therefore the required equation of the plane is

$(x + 2y + 3z + 5) - 5|6(x - 3y + z + 6) = 0.$

$6(x + 2y + 3z + 5) - 5(x - 3y + z + 6) = 0$

$x + 27y + 13z = 0.$

Check your progress

1. Reduce the equation of the plane $x + 2y - 2z - 9 = 0$ to the normal form and hence to find the length of the perpendicular drawn from the origin to the given plane.

2. O is the origin and $A(a, b, c)$ is the point. Find the equation of the plane through A and right angle to OA.

3. $Find\ the\ equation\ of\ the\ plane\ perpendicular\ to$ the line segment from $A(-3, 3, 2)$ to $B(9, 5, 4)$ at the middle point of the segment.

4. $Find\ the\ intercepts\ made$ on the coordinates axes by the plane $x - + 2y - 2z = 9.$

5. A plane meets the coordinate axes in A, B, C such that the centroid of the triangle ABC is the point (p, q, r). Show that the equation of the plane is $x|p + y | q + z | r = 3.$

6. $Find\ the\ equation$ of the plane passing through the point $(1, 2, 1)$ and perpendicular to the line joining the points $(1, 4, 2)$ and $(2, 3, 5)$. Also find the perpendicular distance of the origin from the plane.

7. $Find\ the\ equation$ of the plane passing through the points (2, 2, -1), (3, 4, 2) and $(7, 0, 6)$.

8. Show that the four points $(0, -1, -1), (4, 5, 1)$ $(3, 9, 4)$ and $(-4, 4, 4)$ are coplanar.

9. Find the equation of the plane which is horizontal and passes through the point $(1, -2, -5)$.

10. Find the equation of the plane through the points $(1, -2, 2)$ and $(-3, 1, -2)$ and perpendicular to the plane $x + 2y - 3z = 5.$

11. Find the equation of the plane through the point $(1, 1, -1)$ and perpendicular to the planes $x + 2y + 3z - 7 = 0$ and $2x - 3y + 4z = 0.$

12. Find the equation of the plane through the point $(1, 3, 2)$ and parallel to the plane $3x - 2y + 2z + 33 = 0.$

13. Find the distance between the parallel planes $2x - y + 3z - 4 = 0$ and $6x - 3y + 9z + 13 = 0.$

14. Find the locus of a point, the sum of the squares of whose distances from the planes $x + y + z = 0, x - y = 0, x + y - 2z = 0$

is 7.

15. Find the equation of the plane through the line of intersection of the planes $x + 2y - 3z - 6 = 0$ and $4x + 3y - 2z - 2 = 0$ and passing through the origin.

16. Find the equation of the plane through the line of intersection of the planes $3x - 5y + 4z + 11 = 0$ and $2x - 7y + 4z - 3 = 0$ and passing through the point (-2, 1, 3).

17. Find the equation of the plane through the line of intersection of the planes $ax + by + cz + d = 0$ and $\alpha x + \beta y + \gamma z + \delta = 0$ and parallel to $x-$ axis.

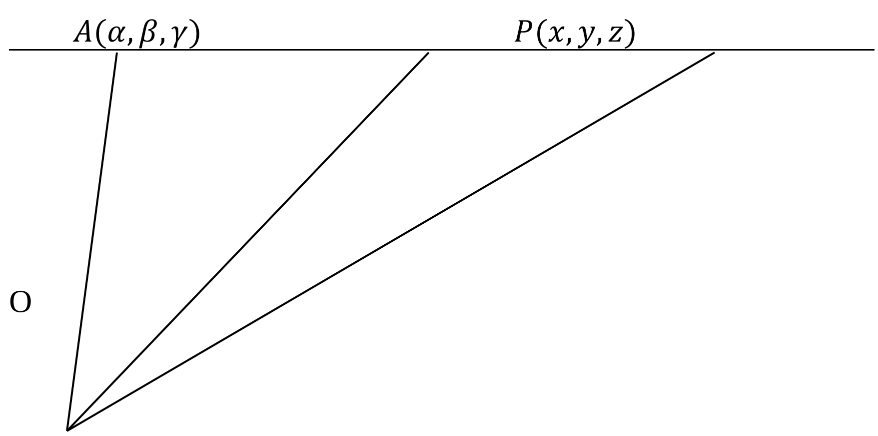
18. Prove that the equation $x^2 + 4y^2 + 4xy - z^2 = 0$ represents a pair of planes and find the angle between them.

3.1.15 EQUATION OF A STRAIGHT LINE (GENERAL FORM)

Every equation of the first degree in x, y, z represents a plane. Also, as two planes intersect in a line, therefore the two equations together represent that line. Thus
$ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ represent a straight line.

3.1.16 EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM

Equation of a straight line passing through a given point $A(\alpha, \beta, \gamma)$ and having direction cosines l, m, n. Suppose $P(x, y, z)$ be any point on a line such that $AP = r$. Now projection of AP *on the* x- axis,



we have $x - \alpha = lr,\ or, (x-\alpha)|l = r$. Similarly projections of AP on $y-$axis and $z-$ axis, we have $(y-\beta)|m = r$ and $(z-\gamma)|n = r$, therefore

$(x-\alpha)|l=(y-\beta)|m=(z-\gamma)|n$

This is equation of the straight line in the symmetrical form.

Note:

1. Equation of a straight line passing through a given point $A(\alpha,\beta,\gamma)$ and having direction cosines proportional to a,b,c is

$$(x-\alpha)|a=(y-\beta)|b=(z-\gamma)|c.$$

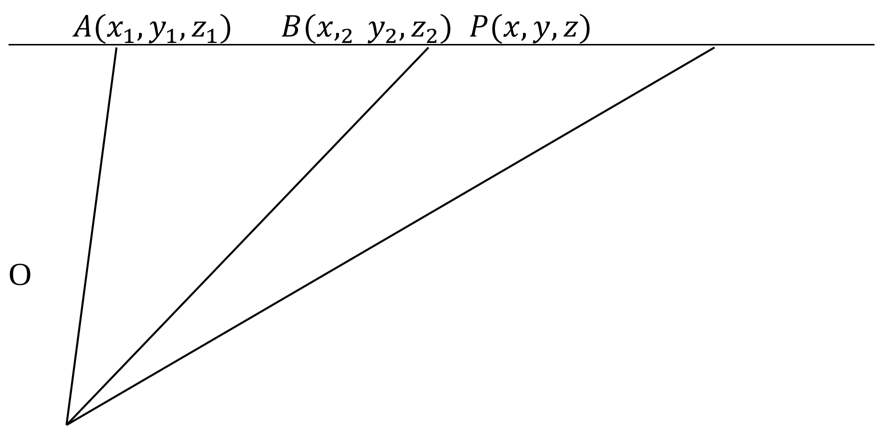
2. If any point $P(x,y,z)$ on this line then

$$(x-\alpha)|l=(y-\beta)|m=(z-\gamma)|n=r(say) \text{ is}$$

$(\alpha+lr,\beta+mr,\gamma+nr)$. It should be noted here that r is not the actual distance of any point $P(x,y,z)$ on the line from the given point $A(\alpha,\beta,\gamma)$.

3.1.17 EQUATION OF A STRAIGHT LINE PASSING THROUGH TWO GIVEN POINTS $P(x_1,y_1,z_1)$ AND $Q(x_2,y_2,z_2)$

The direction cosines of the line will be proportional to $x_1-x_2, y_1-y_2, z_1-z_2$ and it passes through $P(x_1,y_1,z_1)$ will be



$$(x-x_1)|(x_2-x_1)=(y-y_1)|(y_2-y_1)=(z-z_1)|(z_2-z_1)$$

If the equation of two lines are $(x-x_1)|a_1=(y-y_1)|b_1=(z-z_1)|c_1$ and $(x-x_1)|a_2=(y-y_1)|b_2=(z-z_1)|c_2$.

Here, we see that the direction ratio of both lines are a_1,b_1,c_1 and a_2,b_2,c_2 respectively, therefore,

$$cos\theta=(a_1a_2+b_1b_2+c_1c_2)|\sqrt{a_1^2+b_1^2+c_1^2}\ \sqrt{a_2^2+b_2^2+c_2^2}$$

Note :

1. If the lines are perpendicular then

$$ll' + mm' + nn' = 0.$$

In the case of direction ratio, $(a_1a_2 + b_1b_2 + c_1c_2) = 0$.

2. If the lines are parallel then $l|l' = m|m' = n|n'$.

In the case of direction ratio, $\left(a_1|a_2 = b_{1|}b_2 = c_{1|}c_2\right)$

3. Equation of a line passing through a point (x_1, y_1, z_1) and direction ratio are a, b, c is

$$(x - x_1)|a = (y - y_1)|b = (z - z_1)|c = \lambda$$

Therefore the general point on this line is

$x = x_1 + \lambda a,\ y = y_1 + \lambda b$ and $z = z_1 + \lambda c$.

4. Equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$(x - x_1)|(x_2 - x_1) = (y - y_1)|(y_2 - y_1) = (z - z_1)|(z_2 - z_1).$$

Example10: Find the equation of a line passing through the point (1, 2, -3) and its direction ratio are 2, 3, -4.

Solution: Equation of a line is

$$(x - x_1|a = (y - y_1)|b = (z - z_1)|c.$$

So, equation is $(x - 1)|2 = (y - 2)|3 = (z + 3)| - 4$.

Example11: Find the coordinate of the point of intersection of the line $(x + 1)|1 = (y + 3)|3 = (z - 2)|2$ with the plane$3x + 4y + 5z = 20$.

Solution: Since equation of the line is

$(x + 1)|1 = (y + 3)|3 = (z - 2)|2 = r$ (say)

that is coordinate of the point on the line is $(-1 + r, -3 + 3r, 2 + 2r)$. If this point lies on the plane $3x + 4y + 5z = 20, then$

$$3(r - 1) + 4(3r - 3) + 5(2r + 2) = 20.$$

$$25r = 25, i.e. r = 1$$

Putting the value of r, we get the coordinate of the point is $(0, 0, 4)$.

Example12: Find the equation of a line passing through two points

A(1, -2, 1) and B(3, -2, 0).

Solution: Equation of a line passing through two points $A(x_1,\ y_1,\ z_1)$ and $B(x_2,\ y_2,\ z_2)$ is

$$(x-\ x_1)|(x_2-x_1)=(y-\ y_1)|(y_2-y_1)=\ (z-\ z_1)|(z_2-z_1).$$

So, $(x-1)|(3-\ 1)\ =\ (y\ +2)|(-2\ +2)\ =\ (z-\ 1)|(0-1)$

$$(x-1)|2\ =\ (y+2)|0\ =\ (z-1)|-1$$

Example13: Find the equation of a line passing through the point

$(15,-7,-3)$ and parallel to the line

$(x-2)|3\ =\ (y-1)|1\ =\ (z-7)|9$

Solution: Equation of a line passing through the point $(15,-7,-3)$ and parallel to the line whose direction ratio are $3,1,9.$ So, the required equation is
$(x-15)|3\ =\ (y+7)|1\ =\ (z+3)|9.$

Example14: Find the distance of the point $(2,3,4)$ from the point where the line $(x-3)|1\ =\ (y-4)|2\ =\ (z-5)|2$ meets the plane

$x+\ y+\ z\ =\ 22.$

Solution: Any point on the above line is $(3\ +\ r,4+2r,5+2r).$ If it also lies on the plane $x+\ y+\ z\ =\ 22$, then

$3+r+4+2r+5+2r\ =\ 22$, or 5r = 10. Therefore, $r\ =\ 2.$

Putting the value of r we get the required coordinates of the point as

(5, 8, 9). So, the required distance

$=\sqrt{(5\ -2)^2\ +\ (8\ -\ 3)^2\ +(9\ -\ 4)^2}$

$=\ \sqrt{9\ +\ 25\ +\ 25}=\sqrt{59}.$

Example15: Show that the distance of the point of intersection of the line

$(x-2)|3\ =\ (y+1)|4\ =\ (z-2)|12.$

And the plane $x\ -\ y\ +\ z\ =\ 5$ from the point $(-1,-5,-10)$ is 13.

Solution: Equation of the given line are

$(x-2)|3\ =\ (y+1)|4\ =\ (z-2)|12=r$ (say)……………(1)

The coordinates of any point on the line (1) are

$(3r + 2, 4r - 1, 12r + 2)$. If this point lies on the plane

x- y + z = 5, we have

$3r + 2 - (4r - 1) + 12r + 2 = 5, or\ 11r = 0, or\ r = 0.$

Putting this value of r, the coordinates of the point of intersection of the line (1) and the given plane are (2, -1, 2).

Thus the required distance = the distance between the points (2, -1, 2) and (-1, -5, -10)

= $\sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$= \sqrt{9 + 16 + 144} = \sqrt{169 = 13}.$

Example16: Find the points in which the line

$(x+1)|-1 = (y-12)|5 = (z-7)|2$ cuts the surface $11x^2 - 5y^2 + z^2 = 0$

Solution: The equations of the given line are

$(x+1)|-1 = (y-12)|5 = (z-7)|2 = r$ (say)…………..(1)

The coordinates of any point on the line (1) are

$(-r-1, 5r+12, 2r+7)$. If this point lies on the given surface $11x^2 - 5y^2 + z^2 = 0$, we have

$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$, or

$r^2 + 5r + 6 = 0, or\ (r+2)(r+3) = 0, or\ r = -2, -3$

Putting this values of $r\ in\ (-r-1, 5r+12, 2r+7)$. The required points are of intersection are $(1,2,3)$ and $(2,-3,1)$,

Example17: Find the image of the point $(1,3,4)$ in the plane $2x - y + z + 3 = 0.$

Solution: The given plane is $2x - y + z + 3 = 0.$ …………(1)

The direction ratios of the line perpendicular to the given plane are $2, -1, 1.$

Let Q be the image of the given point $P(1,3,4)$ in the plane (1), then the line PQ is perpendicular to the plane (1). Equation of the line PQ passing through $P(1,3,4)$ and perpendicular to the plane (1).

$(x - 1)|2 = (y - 3)| - 1 = (z - 4)|1 = \lambda$ ……….. (2)

Coordinates of the point Q which is on the line (2) be

$(2\lambda + 1, -\lambda + 3, \lambda + 4)$ ……….. (3),

then the coordinates of the middle point N of PQ is

$((2\lambda + 1 + 1)|2, (-\lambda + 3 + 3)|2$,

$(\lambda + 4 + 4)|2) = (\lambda + 1, -\lambda|2 + 3, \lambda|2 + 4)$.

But this point N lies on the plane (1).

$$(2(\lambda + 1) + (-\lambda|2 + 3) + (\lambda|2 + 4) + 3) = 0$$

or, $3\lambda + 6 = 0$ $i.e. \lambda = -2$.

Putting this value in (3), the coordinate of Q (image of P) is in the given plane is $(-3, 5, 2)$.

3.1.18 GENERAL EQUATION OF THE STRAIGHT LINE IN SYMMETRICAL FORM

To transform the equations

$a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$

of a straight line to the symmetrical form. For this we are required to write the symmetrical form of the straight line given by the above equations (i). for this we must know the direction cosines or direction ratios of the line and (ii).The coordinates of a point on the line. To find these two we proceed as follows:

Step(1): To find the direction cosines or direction ratios of the line given by the above equation. Suppose l, m, n be the dirction cosines or direction ratios of the line. Since the line common to the both planes, therefore, it is perpendicular to the normals of both the planes.The direction ratios of the normals to the planes given by a_1, b_1, c_1 and a_2, b_2, c_2 respectively. Hence we have

$a_1l + b_1m + c_1n = 0$ and $a_2l + b_2m + c_2n = 0$

So. we have,

$l|(b_1c_2 - b_2c_1) = m|(c_1a_2 - c_2a_1) = n|(a_1b_2 - a_2b_1)$

Therefore direction cosines of the line are

$(b_1c_2 - b_2c_1)$, $(c_1a_2 - c_2a_1)$, $(a_1b_2 - a_2b_1)$.

Step(2) : To find the coordinates of a point on the line given by the above equations. We choose a point as the one where the line cuts the xy- plane (i. e. $z = 0$), provided the line is not parallel to the plane $z = 0$, provided $(a_1b_2 - a_2b_1) \neq 0$. Putting $z = 0$ in both equations of the planes we get, $a_1x + b_1y + d_1 = 0$, $a_2x + b_2y + d_2 = 0$

Solving these equations for x, y we get

$$x|(b_1d_2 - b_2d_1) = y|(d_1a_2 - d_2a_1) = 1|(a_1b_2 - a_2b_1)$$

Hence, the coordinates of a point on the line, where it cuts the plane $z = 0$ are

($(b_1d_2 - b_2d_1)|(a_1b_2 - a_2b_1), (d_1a_2 - d_2a_1)|(a_1b_2 - a_2b_1)$)

Hence the equation of a lie in the symmetrical form is
$\left(x - (b_1d_2 - b_2d_1)\middle|(a_1b_2 - a_2b_1)\right)|(b_1c_2 - b_2c_1)$

$= (y - (d_1a_2 - d_2a_1)|(a_1b_2 - a_2b_1))|(c_1a_2 - c_2a_1)$

$= (z - 0)|(a_1b_2 - a_2b_1).$

Note: If $(a_1b_2 - a_2b_1) = 0$, then instead of taking $z = 0$ we should we take the point where the line cuts $x = 0$ plane or $y = 0$ plane.

Example18: Find the symmetrical form of the equations of the line

$3x + 2y - z - 4 = 0, 4x + y - 2z + 3 = 0.$ and find its direction cosines.

Solution: The equations of the given line is

$3x + 2y - z - 4 = 0, 4x + y - 2z + 3 = 0$…………(1)

Suppose l, m, n are the direction cosines of the line (1). Since the line is common th the both planes, it is perpendicular to the normals to both the planes. Hence we have

$3l + 2m - n = 0,\ 4l + m - 2n = 0.$

Solving it we get,

$l|(-4 + 1) = m|(-4 + 6) = n\big|\left(3 - 8\right) or, l\ \big| -3 = m\ |2 = n\ | -5\ .$

Therefore the direction ratios of the line (1) are $-3, 2, -5$

The direction cosines l, m, n of the line (1) are given by

$l = 3\ |\ \sqrt{38}, m = 2\ |\ \sqrt{38}\ and\ n = -5\ |\ \sqrt{38}.$

Now to find the coordinates of a point on the line given by (1), we find the point where it meets the plane $z = 0$. Putting $z = 0$ in the given equation $3x + 2y - 4 = 0, 4x + y + 3 = 0$.

Solving these we get

$x \mid (6 + 4) = y \mid (-16 - 9) = 1 \mid (3 - 8)$

$or, x \mid 10 = y \mid -25 = 1 \mid -5$

$x = -2, y = 5$.

The line meets the plane $z = 0$ in the point $(-2, 5, 0)$ and direction ratios as $-3, 2, -5$.

Therefore the equations of the given line in symmetrical form are

$(x + 2) \mid -3 = (y - 5) \mid 2 = (z - 0) \mid -5$.

Example19: Find the angle between the lines

$x - 2y + z = 0, x + 2y + 2z = 0$ and

$x + 2y + 2z = 0, 3x + 9y + 5z = 0$.

Solution: suppose that a_1, b_1, c_1 be the direction ratios of the line of the intersection of the planes $x - 2y + z = 0,\ x + 2y + 2z = 0$

Since this line lies in both planes, therefore it is perpendicular to the normals of both these planes

$a_1 - 2b_1 + c_1 = 0$ and $a_1 + 2b_1 - 2c_1 = 0$.

Solving these we get $a_1 \mid 2 = b_1 \mid 3 = c_1 \mid 4$,

Therefore, the direction ratios of this line are $2, 3, 4$.

suppose that a_2, b_2, c_2 be the direction ratios of the line of the intersection of the planes $x + 2y + 2z = 0, 3x + 9y + 5z = 0$

Since this line lies in both planes, therefore it is perpendicular to the normals of both these planes

$a_2 + 2b_2 + 2c_2 = 0$ and $3a_2 + 9b_2 + 5c_2 = 0$.

Solving these we get $a_2 \mid 1 = b_2 \mid -2 = c_2 \mid 3$,

therefore, the direction ratios of this line are $1, -2, 3$.

If θ be the angle between the given lines, then

$cos\theta = (2.1 + 3.-2 + 4.3)\ |\sqrt{2^2 + 3^2 + 4^2}.\sqrt{1^2 + (-2)^2 + 3^2}$

$= 8\ |\ \sqrt{29}\sqrt{14}\ =\ 8\ |\ \sqrt{406}$

So, $\theta = cos^{-}(8\ |\ \sqrt{406})$

3.1.19 CONDITION OF PARALLELISM OF A LINE AND A PLANE

Suppose equation of a line is $(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$. Suppose equation of a plane is $ax + by + cz + d = 0$. If the line is parallel to the plane then this line must be perpendicular to the normal to this plane, so,

$al + bm + cn = 0$. Again the point (α,β,γ) should not lie on the plane. i.e. $a\alpha + b\beta + c\gamma + d \neq 0$.

Therefore, the required condition for parallel line to the given plane is

$al + bm + cn = 0$ ……………..(1)

$a\alpha + b\beta + c\gamma + d \neq 0$……………..(2)

3.1.20 CONDITION FOR PERPENDICULAR OF A LINE AND A PLANE

Suppose equation of a line is $(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$. Suppose equation of a plane is $ax + by + cz + d = 0$. If the line is perpendicular to the plane then this line must be parallel to the normal to this plane, so,

$$a|l = b|m = c|n$$

3.1.21 CONDITION FOR A LINE TO LIE IN A PLANE

Suppose equation of a line is $(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$. Suppose equation of a plane is $ax + by + cz + d = 0$. If the line lies in the plane then for all values of r the point $(\alpha + rl,\ \beta + rm, \gamma + rn)$ will lie on the given plane. So, $a(\alpha + rl) + b(\beta + rm) + c(\gamma + rn) + d = 0$.

$r(al + bm + cn) + (a\alpha + b\beta + c\gamma + d) = 0$ is true for all values of r. Therefore, the coefficient of $r = 0$ and the constant term $= 0$.

$(al + bm + cn) = 0$ and $(a\alpha + b\beta + c\gamma + d) = 0$

3.1.22 EQUATION OF A PLANE THROUGH A GIVEN LINE

Equation of the line is in the symmetrical form Equation of a plane through the given line

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$ is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)+d=0$ where $(al + bm + cn) = 0$.

The equations of the given line in symmetrical form are

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$. ………….(1)

The equation of any plane through (α,β,γ) is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$ ...…(2)

If it passes through the given line, its normal is perpendicular to the given line i. e. $(al + bm + cn) = 0$…………..(3)

From equation (2) and (3), the equation of any plane through the given line is $a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$,

where $(al + bm + cn) = 0$.

3.1.23 Equation of a plane through a given line and parallel to an another line

The equation of the plane through the line

$(x-\alpha)|l_1 = (y-\beta)|m_1 = (z-\gamma)|n_1$ and parallel to the line

$(x)|l_2 = (y)|m_2 = (z)|n_2$ is

$$\begin{bmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} = 0$$

Example20: Find the equation of a plane through the point (α',β',γ') and through the line whose equation is

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$.

Solution: The equations of the given line are

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$………….(1)

Equation of any plane through the line (1) is

$a(x-\alpha)+b(y-\beta)+c(z-\gamma)=0$………….(2)

Where $(al + bm + cn) = 0$………………..(3)

If the plane (2) passes through the point $(\alpha', \beta', \gamma')$ then from (2)

$a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) = 0$……………(4)

Eliminating a, b, c from (2), (4), (3), we get

$\begin{bmatrix} x-\alpha & y-\beta & z-\gamma \\ \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \end{bmatrix} = 0$ which is the required equation.

Example21: Find the equation of a plane which contains the two parallel lines $(x+1)|3 = (y-2)|2 = (z)|1$ and $(x-3)|3 = (y+4)|2 = (z-1)|1$

Solution: The equations two parallel lines are

$(x+1)|3 = (y-2)|2 = (z)|1$ ………….(1)

and $(x-3)|3 = (y+4)|2 = (z-1)|1$………..(2)

The equation of any plane through the line (1) is

$a(x+1) + b(y-2) + c(z) = 0$……..(3)

Where $3a + 2b + c = 0$ ……………….(4)

The line (2) also lies on the plane (3) if the point (3, -4, 1) lying on the line (2) also lies on the plane (3). Hence

$a(3 + 1) + b(-4 - 2) + c.1 = 0$

or, $4a - 6b + c = 0$ ………..(5)

Solving (4) and (5), we get $a|8 = b|1 = c|-26$

Putting these proportional values of a, b, c in (3) the required equation of the plane is $8(x + 1) + 1.(y - 2) - 26z = 0$

or, $8x + y - 26z + 6 = 0$.

Example22: Find the equation of a plane through the point $(\alpha', \beta', \gamma')$ and the line $(x - \alpha)|l = (y - \beta)|m = (z - \gamma)|n$.

Solution : Equation of any plane through the given line is $a(x - \alpha) + b(y - \beta) + c(z - \gamma) = 0$ …………(1)

Where, $al + bm + cn = 0$…………(2)

The plane (1) will pass through the point $(\alpha', \beta', \gamma')$ if

$a(\alpha' - \alpha) + b(\beta' - \beta) + c(\gamma' - \gamma) = 0$ …………(3)

The equation of the required plane will be obtained by eliminating a, b, c between the equations (1) (3) and (2). Hence eliminating the constants a, b, c between the above equations, the equation of the required plane is given by

$$\begin{bmatrix} x-\alpha & y-\beta & z-\gamma \\ \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \end{bmatrix} = 0$$

3.1.24 Foot of perpendicular and length of perpendicular from a point to a line

(a) In symmetrical form: Suppose that equation of a line in symmetrical form be

$(x - \alpha)|l = (y - \beta)|m = (z - \gamma)|n = r$(say)………(1)

The coordinate of any point N on the line is $(\alpha + lr, \beta + mr, \gamma + nr)$.

If N is the foot of the perpendicular from a given point $P(x_1, y_{1,} z_1)$ to the line (1), then the line PN is perpendicular to (1).

The direction ratios of the line PN are

$(\alpha + lr - x_1, \beta + mr - y_1, \gamma + nr - z_1)$……………(2)

$$l(\alpha + lr - x_1)m(\beta + mr - y_1) + n(\gamma + nr - z_1) = 0$$

$$r(l^2 + m^2 + n^2) = l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)$$

Or, $r = l(x_1 - \alpha) + m(y_1 - \beta) + n(z_1 - \gamma)|\,(l^2 + m^2 + n^2)$

Substituting the value of r in $(\alpha + lr, \beta + mr, \gamma + nr)$ and determine the coordinate of N, also, the foot of perpendicular and length of PN can be easily determined.

Equation of the perpendicular line from the point $P(x_1, y_{1,} z_1)$ to the line (1) are given by

$(\boldsymbol{x - x_1}) |\ (\boldsymbol{\alpha + lr - x_1})$ = $(\boldsymbol{y - y_1}) | (\boldsymbol{\beta + mr - y_1})$ = $(\boldsymbol{z - z_1}) |$ $(\boldsymbol{\gamma + nr - z_1})$

(b) **In general form:** The equations of the line in general form are

$ax + by + cz + d = 0$; $a'x + b'y + c'z + d' =$ 0…………..(1)

The perpendicular from a point $P(x_1, y_{1,} z_1)$ to the given line is the intersection of the two planes namely (i) the plane through the given point $P(x_1, y_{1,} z_1)$ and also through the line and (ii) the plane through the point $P(x_1, y_{1,} z_1)$ perpendicular to the given line

Now the equation of any plane through the line (1) is given by

$ax + by + cz + d + \lambda(a'x + b'y + c'z + d') = 0$……(2)

If it passes through the point $P(x_1, y_{1,} z_1)$, then

$$ax_1 + by_1 + cz_1 + d + \lambda(a'x_1 + b'y_1 + c'z_1 + d') = 0$$

Or, $\lambda = -(ax_1 + by_1 + cz_1 + d) |\ (a'x_1 + b'y_1 + c'z_1 + d')$

Putting this value in equation (2), we get

$(ax + by + cz + d) | (ax_1 + by_1 + cz_1 + d) = (a'x + b'y + c'z + d') | (a'x_1 + b'y_1 + c'z_1 + d')$……(3)

Also if l, m, n be the direction cosines of the given line (1), then we get

$al + bm + cn = 0$ and $a'l + b'm + c'n = 0$ Solving these we get $l |\ (bc' - b'c) = m |\ (ca' - a'c) = n |\ (ab' - a'b)$……..(4)

Now, we are to find the equation of the second plane which passes through P and is perpendicular to the line (1).

Since the plane is perpendicular to the line (1), therefore the direction cosines of its normal are proportional to l, m, n given by (4). Therefore the equation of the plane perpendicular to the line (1)and passing through the point $P(x_1, y_{1,} z_1)$ is

$l(x - x_1) + m(y - y_1) + n(z - z_1) = 0$…………(5)

Therefore the equations of the perpendicular line from the point $P(x_1, y_{1,} z_1)$ to the line (1) are given by the above equation (5).

3.1.25 COPLANAR LINES

Suppose that the equations of the given lines be

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$………………..(1)

And $(x-\alpha')|l' = (y-\beta')|m' = (z-\gamma')|n'$ ………(2)

If they intersect,, then they lie in a plane. If the lines are coplanar then they intersect and they must have a common point. Any point on the line (1) is $(\alpha+lr, \beta+mr, \gamma+nr)$ and any point on the line (2) is $(\alpha'+l'r', \beta'+m'r', \gamma'+n'r')$. Therefore,

$\alpha+lr = \alpha'+l'r',\ \beta+mr = \beta'+m'r', \gamma+nr = \gamma'+n'r'$. So,

$$\alpha-\alpha'+lr-l'r' = 0,$$

$\beta-\beta'+mr-m'r' = 0$ and $\gamma-\gamma'+nr-n'r' = 0$ Now we eliminating r and r' from these equations

$$\begin{bmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{bmatrix} = 0$$

3.1.26 CONDITION FOR THE TWO LINES TO INTERSECT (IN SYMMETRICAL FORM)

Suppose the equations of the given lines be

$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$………………..(1)

And $a_1x+b_1y+c_1z+d_1 = 0;\ a_2x+b_2y+c_2z+d_2 = 0$ ...(2)

Equation of any plane through the line (2) is

$a_1x+b_1y+c_1z+d_1+\lambda(a_2x+b_2y+c_2z+d_2) = 0$. Or,

$(a_1+\lambda a_2)x+(b_1+\lambda b_2)y+(c_1+\lambda c_2)z+(d_1+\lambda d_2) = 0$ ….(3)

If this plane is parallel to the line (1), then we have

$$(a_1+\lambda a_2)l+(b_1+\lambda l)m+(c_1+\lambda c_2)n = 0$$

Or, $(a_1l+b_1m+c_1n) = -\lambda(a_2l+b_2m+c_2n)$

Or, λ = - $(a_1l+b_1m+c_1n)|(a_2l+b_2m+c_2n)$……..(4)

Putting this value of λ in equation in (3) equation of the plane through the line (2) and parallel to the line (1) is given by

$$(a_1x + b_1y + c_1z + d_1)\,|(a_1l + b_1m + c_1n)$$

$= (a_2x + b_2y + c_2z + d_2)|\,(a_2l + b_2m + c_2n)$……(5)

If the line (1) lies on this plane then the point (α, β, γ) on the line (1) must satisfy (5) and so the condition for the lines (1) and (2) to be coplanar is $(a_1\alpha + b_1\beta + c_1\gamma + d_1)\,|(a_1l + b_1m + c_1n) = (a_2\alpha + b_2\beta + c_2\gamma + d_2)|\,(a_2l + b_2m + c_2n)$……..(6)

If the condition (6) is satisfied, the lines (1) and (2) are intersecting and the plane containing both the lines is given by the equation (5).

3.1.27 CONDITION FOR THE TWO LINES TO INTERSECT (IN GENERAL FORM)

Suppose the equations of the given lines be

$a_1x + b_1y + c_1z + d_1 = 0;\ \ a_2x + b_2y + c_2z + d_2 = 0$…..(1)

$a_3x + b_3y + c_3z + d_3 = 0;\ \ a_4x + b_4y + c_4z + d_4 = 0$……(2)

If these two lines are coplanar, then they intersect and let (α, β, γ) be the point of intersection. The coordinates of this point must satisfy the equations of these four planes representing the two lines. Therefore we have,

$$a_1\alpha + b_1\beta + c_1\gamma + d_1 = 0;\ a_2\alpha + b_2\beta + c_2\gamma + d_2 = 0$$

$$a_3\alpha + b_3\beta + c_3\gamma + d_3 = 0;\ \ a_4\alpha + b_4\beta + c_4\gamma + d_4 = 0$$

Now we eliminating $\alpha, \beta, and\ \gamma$ from these equations we find the required condition as

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} = 0$$

3.1.28 EQUATION OF A STRAIGHT LINE INTERSECTING TWO GIVEN STRAIGHT LINES (IN SYMMETRICAL FORM)

Suppose that equation of given lines be

$(x - \alpha)|l = (y - \beta)|m = (z - \gamma)|n = r$ …….(1)

And $(x - \alpha')|l' = (y - \beta')|m' = (z - \gamma')|n' = r'$ ….(2)

Any point on the line (1) is $P(\alpha + lr, \beta + mr, \gamma + nr)$ and any point on the line (2) is $Q(\alpha' + l'r', \beta' + m'r', \gamma' + n'r')$.

We are required to find the equations of a line which intersects the line (1)and the line (2). Suppose the required line intersect the lines (1) and (2) in the points P and Q respectively. The required line is one which joins the points P and Q.

Example: Find in symmetrical form the equations of the line $3x + 2y - z - 4 = 0; 4x + y - 2z + 3 = 0$. Also, find its direction cosines.

Solution: The equations of the given line in general form are

$3x + 2y - z - 4 = 0; 4x + y - 2z + 3 = 0$………..(1)

Let l, m, n are the direction cosines of the line (1). Since the line is common to the both the planes, it is perpendicular to the normals of the both the planes.

Hence we have, $3l + 2m - n = 0, 4l + m - 2n = 0$

Solving these, we get $|-3 = m|2 = n| - 5$. Therefore, the direction ratios of the line (1) are given by $-3, 2, -5$.

We have, $\sqrt{(-3)^2 + 2^2 + (-5)^2} = \sqrt{38}$.

Therefore, the direction ratios of the line (1) are given by

$l = -3\big|\sqrt{38},\ m = 2\big|\sqrt{38},\ n = -5|\sqrt{38}$

Now to find the coordinates of a point on the line given by (1), let us find the point where it meets the plane $z = 0$, Putting $z = 0$ in the equation given by (1), we have $3x + 2y - 4 = 0; 4x + y + 3 = 0$

Solving these we have $x|10 = y| - 25 = 1| - 5$

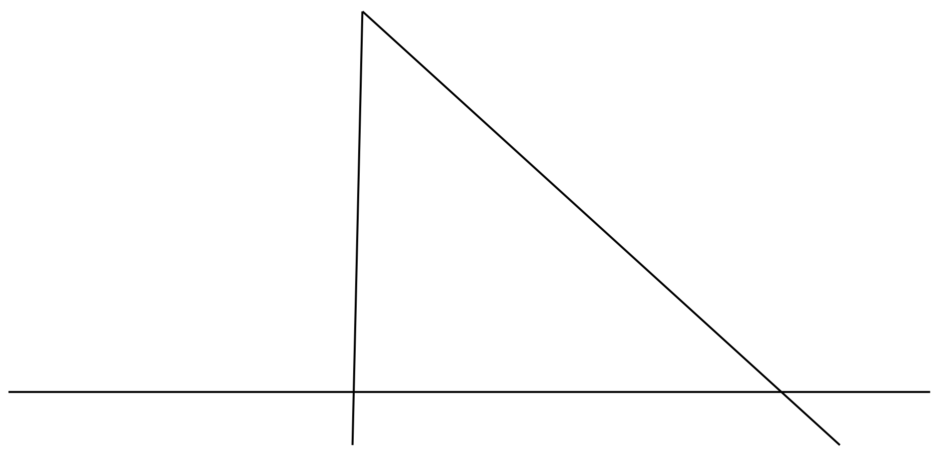
We get $x = -2, y = 5$.

Therefore the line meets the plane $z = 0$ in the point $(-2,\ 5, 0)$ and has direction ratios as $-3, 2, -5$. Therefore the equations of the given line in symmetrical form are $(x + 2)|-3 = (y - 5)|2 = (z - 0)| - 5$.

3.1.29 PERPENDICULAR DISTANCE OF A POINT FROM A LINE AND THE COORDINATES OF THE FOOT OF THE PERPENDICULAR

Let $P(x_1, y_1, z_1)$ be a given point and let

$\boldsymbol{P(x_1, y_1, z_1)}$



AB be a given line Let the equation of the line AB in the symmetrical form is

$(x-\alpha)|\, l = (y-\beta)|m = (z-\gamma)|n$………..(1)

Where l, m, n are direction cosines of (1). The line (1) is passing through the point (α, β, γ) . From P we draw PN perpendicular to AB. From the right angled triangle APN, we have

$$PN^2 = AP^2 - AN^2$$

Now AP = the distance between the points $A(\alpha, \beta, \gamma)$ and $P(x_1, y_1, z_1)$ = $\sqrt{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2}$ and

AN = projection of AP on AB $i.e.$ the projection of AP on a line whose direction cosines are l, m, n = $(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n$

$PN^2 = \{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\} - \{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\}^2$

$= \{(x_1-\alpha)^2 + (y_1-\beta)^2 + (z_1-\gamma)^2\}(l^2 + m^2 + n^2) - \{(x_1-\alpha)l + (y_1-\beta)m + (z_1-\gamma)n\}^2$

$= \{m(z_1-\gamma) - n(y_1-\beta)\}^2 + \{n(x_1-\alpha) - l(z_1-\gamma)\}^2 + \{l(y_1-\beta) - m(x_1-\alpha)\}^2$ by using Lagrange's identity.

3.1.30 TO FIND THE COORDINATES OF THE FOOT OF THE PERPENDICULAR

Since N, be the foot of the perpendicular, is a point on the line AB given by $(x-\alpha)|\, l = (y-\beta)|m = (z-\gamma)|n$………..(1). Its coordinates may be written as $(\alpha + lr, \beta + mr, \gamma + nr)$. The direction cosines of PN are $(\alpha + lr - x_1, \beta + mr - y_1, \gamma + nr - z_1)$, also PN is perpendicular to AB. Therefore,

$$(\alpha + lr - x_1).l + (\beta + mr - y_1)m + (\gamma + nr - z_1)n = 0$$

Or, $r(l^2 + m^2 + n^2) = (x_1 - \alpha).l + (y_1 - \beta)m + (z_1 - \gamma)n$

Or, $r = (x_1 - \alpha).l + (y_1 - \beta)m + (z_1 - \gamma)n$. Putting the value of r in $(\alpha + lr, \beta + mr, \gamma + nr)$ we get the coordinates of N.

Example23: From the point $P(1, 2, 3)$, PN is drawn perpendicular to the straight line $(x - 2)|3 = (y - 3)|4 = (z - 4)|5$. Find the distance PN, the equations to PN and coordinates of N.

Solution: The equations of the given line AB (say) are

$(x - 2)|3 = (y - 3)|4 = (z - 4)|5 = r$ (say)…..(1)

The line (1) is passing through the point $A(2, 3, 4)$. Since N, the foot of the perpendicular, is a point on the line (1), the coordinates of N may be written as $(3r + 2, 4r + 3, 5r + 4)$, therefore the direction ratios of PN are $(3r + 2 - 1,\ 4r + 3 - 2, 5r + 4 - 3)$ *i.e.* $(3r + 1, 4r + 1, 5r + 1)$. The direction ratios of the line AB whose equations are given by (1), are $3, 4, 5$. Since PN is perpendicular to AB, we have,

$3.(3r + 1) + 4.(4r + 1) + 5(5r + 1) = 0$, or $= -6|25$.

Putting the value of r in $(3r + 2, 4r + 3, 5r + 4)$, we get $N = (32|25, 51|25, 14|5)$, therefore,

PN = the distance between the points P and N

$$= \sqrt{\{(32|25 - 1)^2 + (51|25 - 2)^2 + (14|5 - 3)^2\}} = \sqrt{3}\,|\,5.$$

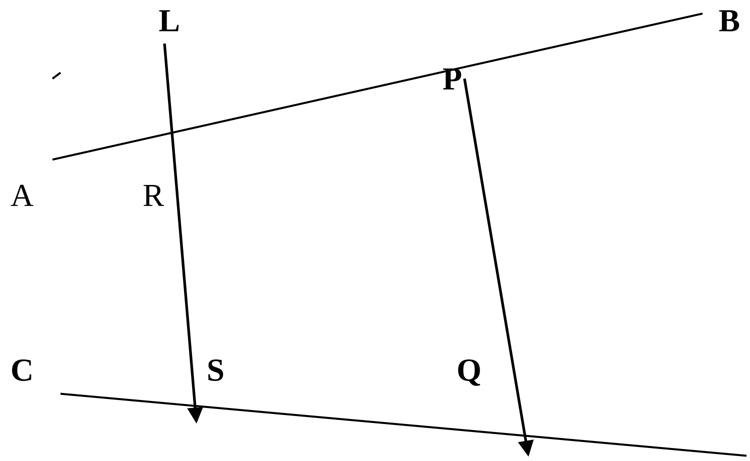
Putting the value of r in $(3r + 1, 4r + 1, 5r + 1)$, the direction ratios of PN are $7|25, 1|25, -5|25$ *i.e.* $7, 1, -5$. So, the equation to PN. Equation of a line passing through P(1, 2, 3) and having direction ratios $7, 1, -5$ are

$$(x - 1)|7 = (y - 2)|1 = (z - 3)| - 5.$$

Definition1.*Skew lines*: Those lines which do not intersect or the lines which do not lie in a plane.

Definition2. Shortest distance: The length of the line intercepted between two lines which is perpendicular to both is the shortest distance between them. The straight line which is perpendicular to each of the two skew lines is called the line of the shortest distance (S. D.).

3.1.31 THE SHORTEST DISTANCE BETWEEN ANY TWO NON- INTERSECTING LINES



Suppose that AB and CD be two non intersecting lines and LM a perpendicular line to both of them. RS is the portion of LM intercepted between AB and CD. We have to prove that RS is the shortest distance between AB and CD.

Let P and Q be any points on AB and CD respectively. RS is the projection of PQ on LM. If (θ) be the angle between PQ and LM, then $RS = PQcos\theta$ or, $RS\,|PQ = cos\theta$, since $cos\theta < 1$, therefore, $RS\,|PQ < 1$, i.e. RS is the shortest distance between these two lines.

3.1.32 LENGTH AND EQUATIONS OF THE LINE OF SHORTEST DISTANCE

(If the equations of the skew lines are in symmetrical form)

Suppose that the equations of two lines be

$(x-\ \alpha)|\ l = (y-\ \beta)|m = (z-\ \gamma)|n$…………(1)

$and\ (x-\ \alpha')|\ l' = (y-\ \beta')|m' = (z-\ \gamma')|n'$…………(2) Suppose λ, μ, ν be the direction cosines of the S. D. Since S.D. is perpendicular to each of the given lines, therefore,

$l\lambda + m\mu + n\nu = 0$ and $l'\lambda + m'\mu + n'\nu = 0$**,** therefore,

$\lambda|(mn'-\ m'n) =\ \mu|(nl'-\ n'l) =$
$\nu\ |(lm'-\ l'm) =$
$1\ |\sqrt{\{(mn'-\ m'n)^2 +\ (nl'-\ n'l)^2 + (lm'-\ l'm)^2}$ } = K(say)

Therefore, $\lambda =\ (mn'-\ m'n)K, \mu =\ (nl'-\ n'l)K$ and

$\nu =\ (lm'-\ l'm)K.$

If $P(\alpha, \beta, \gamma)$ be any point on the line (1) and $Q(\alpha', \beta', \gamma')$ be any point on the line (2), then the shortest distance will be the projection of the line PQ

joining these points on the line whose direction cosines are $\lambda, \mu, \nu, therefore,$

S.D. = $(\alpha - \alpha')\lambda + (\beta - \beta')\mu + (\gamma - \gamma')\nu$

$=(\alpha - \alpha')(mn' - m'n)K + (\beta - \beta')(nl' - n'l)K + (\gamma - \gamma')(lm' - l'm)K = \{(\alpha - \alpha')(mn' - m'n) + (\beta - \beta')(nl' - n'l)(\gamma - \gamma')(lm' - l'm)\}|\{\sqrt{\{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2}\}\}$

Equation of the plane containing the line (1) and the S.D. is

$$\begin{bmatrix} x-\alpha & y-\beta & z-\gamma \\ l & m & n \\ \lambda & \mu & \nu \end{bmatrix} = 0 \text{.......(3)}$$

Equation of the plane containing the line (1) and the S.D. is

$$\begin{bmatrix} x-\alpha' & y-\beta' & z-\gamma' \\ l' & m' & n' \\ \lambda & \mu & \nu \end{bmatrix} = 0 \text{........(4)}$$

Equations (3) and (4) taken together will represent the equations of the line of the shortest distance.

Note: If the lines are coplanar, the S. D. between them is zero, then

$$\begin{bmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{bmatrix} = 0$$

Two lines are coplanar if the shortest distance between them is zero.

Example24: Find the shortest distance between the lines

$(x - 1)|2 = (y - 2)| 3 = (z - 3)|4;$

$(x - 2)|3 = (y - 4)| 4 = (z - 5)|5.$

Also show that the equations of the shortest distance are

$11x + 2y - 7z + 6 = 0;\ 7x + y - 5z + 7 = 0.$

Solution: The given lines are

$(x - 1)|2 = (y - 2)| 3 = (z - 3)|4 = r_1\ (say)$.......(1)

$(x - 2)|3 = (y - 4)| 4 = (z - 5)|5 = r_2\ (say)$......(2)

Let l, m, n be the direction cosines of the line of S. D. Since it is perpendicular to both the given lines (1) and (2), therefore, we have

$$2l + 3m + 4n = 0;\ 3l + 4m + 5n = 0.$$

Solving these, we get
$l| - 1 = m|2 = n| - 1 = \sqrt{(l^2 + m^2 + n^2)}|\sqrt{((-1)^2 + (2)^2 + (-1)^2)}$

$$= 1 |\sqrt{6}$$

Therefore, the direction cosines of S.D. are $-1 |\sqrt{6},\ \ 2 |\sqrt{6}, -1 |\sqrt{6}$

Now $A(1, 2, 3)$ is a point on the line $(1)$and $B(2, 4, 5)$ is a point on the line (2).

The length of S. D. = the projection of join of A and B on the line whose direction cosines are $-1 |\sqrt{6},\ \ 2 |\sqrt{6}, -1 |\sqrt{6}$

= $-1 |(2-1) +\ 2 |(4-2) - 1 |\sqrt{6}(5-3) =\ 1 |\sqrt{6}$

The equation of S.D.: The equation of the plane through the line (1) and S. D. is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{bmatrix} = 0$$

Or, $11x\ +\ 2y - 7z\ +\ 6\ =\ 0 \ldots\ldots\ldots..(3)$

And the equation of the plane through the line (2) and S. D. is

$$\begin{bmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{bmatrix} = 0$$

Or, $7x\ +\ y - 5z\ +\ 7\ =\ 0\ \ldots\ldots(4)$

Therefore, from equations (3) and (4) the equations of S.D. are

$$11x\ +\ 2y - 7z\ +\ 6\ =\ 0;$$

$7x\ +\ y - 5z\ +\ 7\ =\ 0.$

Check your progress

(1)(a). The coordinates of two points A and B are $(-2, 2, 3)$ and $(13, -3, 13)$ respectively. A point P moves such that

$3PA\ = 2PB$. Find the locus of P.

1. (b). Show that the points $(0, 7, 10), (-1, 6, 6)$ and $(-4, 9, 6)$ form an isosceles right angled triangle.
2. Find the ratio in which the coordinate planes divide the line joining the points $(-2, 4, 7)$ and $(3, -5, 8)$.
3. If $\alpha, \beta\ and\ \gamma$ be the angles which a line makes with the coordinate axes, show that $sin^2\alpha +\ sin^2\beta + sin^2\gamma = 2$.
4. Show that the direction cosines of of the line which is equally inclined to the coordinate axes are $\pm 1|\sqrt{3}$, $\pm 1|\sqrt{3}$, $\pm 1|\sqrt{3}$.
5. Find the angle between the lines whose direction cosines are proportional to $1, 2, 4$ and $-2, 1, 5$.

6. Find the direction cosines of the line which is perpendicular to the lines with direction cosines proportional to $3, -1, 1$ and $-3, 2, 4$.

7. : Find the coordinate of the point I which the line $(x-2)|3 = (y+1)|4 = (z-2)|12$ meets the plane $x - 2y + z = 20$.

8. Show that the line joining the points $A(2,-3,-1)$ and $B(8,-1,2)$ has equations$(x-2)|6 = (y+3)|2 = (z+1)|3$. Find two points on the line whose distance from A is 14.

9. Find the equations of the straight lines through the point (a, b, c) which are

10. (i). parallel to z −axis (perpendicular to the $XY -$ plane) and (ii). Perpendicular to $Z -$ axis (parallel to $XY -$ plane).

11. Find the distance of the point $(1,3,4)$ from the plane $2x - y + z = 3$ measured parallel to the line $x|2 = y|-1 = z|-1$

12. Find the distance of the point $(1,-2, 3)$ from the plane $x - y + z = 5$ measured parallel to the line $x|2 = y|3 = z|-6$

13. Find the equations of the line through the point (x_1, y_1, z_1) at the right angles to the lines $x|l_1 = y|m_1 = z|n_1$ and $x|l_2 = y|m_2 = z|n_2$.

14. Find the coordinates of the foot of the perpendicular from the point $(2,3,7)$ to the plane $3x - y - z = 7$. Also find the the length of the perpendicular.

15. Find the equation of the plane through the point (α, β, γ) and (i). perpendicular to the straight line $(x - x_1)|l = (y - y_1)|m = (z - z_1)|n$. (ii). Parallel to the lines $x|l_1 = y|m_1 = z|n_1$ and $x|l_2 = y|m_2 = z|n_2$.

16. A variable plane makes intercepts on the coordinate axes the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to the plane is $(x^{-2} + y^{-2} + z^{-2})(x^2 + y^2 + z^2)^2 = k^2$

17. The planes $3x - y + z + 1 = 0, 5x + y + 3z = 0$ intersect in the line PQ. Find the equation of the plane through the point $(2,1,4)$ and perpendicular to PQ.

18. Find the equations of the line through the point $(1,2,3)$ parallel to the line $x - y + 2z - 5 = 0, 3x + y + z - 6 = 0$.

19. Find the equations of the line through the point $(1,2,3)$ parallel to the line $a_1x + b_1y + c_1z + d_1 = 0,\ a_2x + b_2y + c_2z + d_2 = 0$.

20. Prove that the lines $x = ay + b$; $z = cy + d$ and $x = a'x + b'$; $z = c'y + d'$ are perpendicular if $aa' + cc' + 1 = 0$.

21. Find the equation of the plane through the line $(x-\alpha)|l = (y - \beta)|m = (z-\gamma)|n$ and parallel to the line $(x-\alpha')|l' = (y - \beta')|m' = (z-\gamma')|n'$.

22. Find the equation of the plane through the line $P = a_1x + b_1y + c_1z + d_1 = 0,\ Q = a_2x + b_2y + c_2z + d_2 = 0$ and parallel to the line $x \mid l = y \mid m = z \mid n$.

23. Find the equation of the plane through the line $3x - 4y + 5z = 10,\ 2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

24. Find the equation of the plane through the points $(2, -1, 0), (3, -4, 5)$ and parallel to the line $3x = 2y = z$.

25. Find the equation of the plane through the point $(2, 1, 4)$ and perpendicular to the line of intersection of the planes $3x + 4y + 7z + 4 = 0$ and $x - y + 2z + 3 = 0$.

26. Find the equations of the perpendicular from the point

(3, -1, 11) to the line $)|2 = (y - 2)| 3 = (z - 3)|4$. Find also the coordinates of the foot of perpendicular and the length of the perpendicular.

27. Find the equation of the plane through the line$(x - 2)|2 = (y - 3)| 3 = (z - 4)|5$ and parallel to the coordinate axes.

28. Prove that the equation of the plane through the line $(x - 1)|3 = (y + 6)| 4 = (z + 1)|2$ and parallel to the line $(x - 2)|2 = (y - 1)| - 3 = (z + 4)|5$ is $25x - 11y - 17z - 109 = 0$ and show that the point $(2, 1, -4)$ lies on it.

29. Find the equation of the plane through the line $ax + by + cz = 0\,;\ a'x + b'y + c'z = 0$ and $\alpha x + \beta y + \gamma z = 0\,;\ \alpha' x + \beta' y + \gamma' z = 0$

30. Find the angle between the lines whose direction cosines are given by the equation $3l + m + n = 0$ and $6mn - 2nl + 5lm = 0$.

31. Find the angle between the lines whose direction cosines are given by the equation $l + m + n = 0$ and $2nl + 2lm - mn = 0$.

32. Show that the acute angle between the diagonals of a cube is $cos^{-1}(1|3)$.

33. If $A(3, 4, 5), B(4, 6, 3), C(-1, 2, 4)$ and $D(1, 0, 5)$ are the four points, find the projection of CD on AB.

34. Lines OP and OQ are drawn from O with direction cosines propostional to $1, -2, 1;\ 7, -6, 1$. Find the direction cosines of the normal to the plane OPQ.

35. Find the ratio in which the line segment joining the points $A(1, 2, 3)$ and $B(-4, 5, -2)$ is divided by the plane $x + 2y = 4 + z$.

36. Find the equation of the set of the points P such that its distance from the points $A(3, 4, -5)$ and $B(-2, 1, 4)$ are in the ratio $1: 2$.

37. Find the equation of a line which passes through a point $(2, -1, -1)$ parallel to the line $6x - 2 = 3y + 1 = 2z - 2$.

38. Find the coordinate of the point, where the through $(3, 4, 1)$ and $(5, 1, 6)$ meet the ZX-plane.

39. Find the equations of the perpendicular from the origin to the line $ax + by + cz + d = 0$; $a'x + b'y + c'z + d' = 0$

40. Find the distance of the point $P(3, 8, 2)$ from the line $(x - 1)|2 = (y - 3)|4 = (z - 2)|3$ measured to the plane $3x + 2y - 2z + 17 = 0$

41. Show that the lines$(x + 3)|2 = (y + 5)|3 = -(z - 2)|3$ and $(x + 1)|4 = (y + 1)|5 = -(z + 1)|1$ are coplanar. Find the equation of the plane containing them.

42. Prove that the lines$(x - 1)|2 = (y - 2)|3 = (z - 3)|4$ and $(x - 2)|3 = (y - 3)|4 = (z - 4)|5$ are coplanar. Find their point of intersection and the equation of the plane in which they lie.

43. Prove that the lines $3x - 5 = 4y - 9 = 3z$ and $x - 1 = 2y - 4 = 3z$ meet in a point and the equation of the plane in which they lie is $3x - 8y + 3z + 13 = 0$.

44. A line with direction cosines proportional to $2, 7, -5$ is drawn to intersect the lines $(x - 5)|3 = (y - 7)| -1 = (z + 2)|1$ and $(x + 3)| -3 = (y - 3)|2 = (z - 6)|4$. Find the coordinates of the points of intersection and the length intercepted on it.

45. Find the equations to the straight line drawn from the origin to intersect the lines $2x + 5y + 3z - 4 = 0: x - y - 5z - 6 = 0$. And $3x - y + 2z - 1 = 0: x + 2y - z - 2 = 0$.

46. A line with direction cosines proportional to $2, 1, 2$ meets each of the lines given by the equations $x = y + a = z: x + a = 2y = 2z$. Find the coordinates of each of the points of intersection.

47. Find the equations of the straight line through the origin and cutting each of the lines $(x - x_1)|l_1 = (y - y_1)|m_1 = (z - z_1)|n_1$ and $(x - x_2)|l_2 = (y - y_2)|m_2 = (z - z_2)|n_2$.

48. Find the equations of the straight line through the origin which will intersect both the lines

$$(x - 1)|1 = (y + 3)|4 = (z - 5)|3 \text{ and}$$

$$(x - 4)|2 = (y + 3)|3 + (z - 14)|4.$$

49. Find the equations of the perpendicular from $(1, 3, 7)$ on the line $x = 3 - 5t, y = 2 + 5t, z = -7 + 2t$.

50. Find the locus of a point whose distance from X- axis is twice its distance from the yz- plane.

51. Find the length of the perpendicular drawn from the origin to the line $x + 2y + 3z + 4 = 0;\ 2x + 3y + 4z + 5 = 0.$

52. Also find the equations of this perpendicular and the coordinates of the foot of the perpendicular.

53. Find the shortest distance between the lines $(x - 3)|1 = (y - 5) | -2 = (z - 7)|1 : (x + 1) | 7 = (y + 1)| -7 = (z + 1)|1.$ Find also its equations and the points in which it meets the given lines.

54. Find the shortest distance between the lines $(x - 3)|3 = (y - 8) | -1 = (z - 3)|1 : (x + 3) | -3 = (y + 7)|2 = (z - 6)|4.$ Find also its equations and the points in which it meets the given lines.

55. Find the shortest distance between the lines $(x - 1)|2 = (y - 2) | 3 = (z - 3)|4 : (x - 2) | 3 = (y - 3)|4 = (z - 4)|5.$ Hence show that the lines are coplanar.

56. Show that the shortest distance between the diagonals of a rectangular parallelepiped and its edges not meeting it are $bc| \sqrt{(b^2 + c^2)}$, $ca| \sqrt{(a^2 + c^2)}$, $ab| \sqrt{(b^2 + a^2)}$ where a, b, c are the lengths of the edges.

57. Find the length and equations of the shortest distance between the lines $3x - 9y + 5z = 0;\ x + y - z = 0$ and $6x + 8y + 3z - 13 = 0;\ x + 2y + z - 3 = 0.$

Summary

1. The general equation of the plane is $ax + by + cz + d = 0$

2. The equation of any plane passing through the origin is

$$ax + by + cz = 0$$

3. General equation of the plane in normal form:

   The general equation of the plane is $ax + by + cz + d = 0$….(1)

   Suppose the general equation of the normal form is

   $lx + my + nz = p \ldots\ldots\ldots(2)$ where

$$l|a = m|b = n|c = p| -d$$
$$= \pm\sqrt{(l^2 + m^2 + n^2)} \mid \sqrt{(a^2 + b^2 + c^2)}$$
$$= \pm 1 \mid \sqrt{(a^2 + b^2 + c^2)}$$

   Where the same sign either positive or negative is to be chosen throughout.

$l = \pm a \mid \sqrt{(a^2+b^2+c^2)}, m = \pm b \mid \sqrt{(a^2+b^2+c^2)}$

$n = \pm c \mid \sqrt{(a^2+b^2+c^2)}, p = \pm d \mid \sqrt{(a^2+b^2+c^2)}$

Substituting these values in equation (2), the normal form of the plane (1) is given by

$\pm cz \mid \sqrt{(a^2+b^2+c^2)} = \pm d \mid \sqrt{(a^2+b^2+c^2)}$…….(3)

The sign of the equation (3) is so chosen that p is $\pm d \mid \sqrt{(a^2+b^2+c^2)}$ is always positive.

4. If the lines$(x-\alpha)|l = (y-\beta)|m = (z-\gamma)|n$………………..(1)

   And $(x-\alpha')|l' = (y-\beta')|m' = (z-\gamma')|n'$ ………(2)

   (a) are perpendicular then $ll' + mm' + nn' = 0.$

   In the case of direction ratio, $(a_1a_2 + b_1b_2 + c_1c_2) = 0.$

   (b) If the lines are parallel then $l|l' = m|m' = n|n'.$

   In the case of direction ratio, $\left(a_1|a_2 = b_{1|}b_2 = c_{1|}c_2\right)$

   (c) Equation of a line passing through a point (x_1, y_1, z_1) and direction ratio are a, b, c is $(x - x_1)|a = (y - y_1)|b = (z - z_1)|c = \lambda$

   Therefore the general point on this line is

   $x = x_1 + \lambda a,\ y = y_1 + \lambda b$ and $z = z_1 + \lambda c.$

   (d) Equation of a line passing through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

   $(x - x_1)|(x_2 - x_1) = (y - y_1)|(y_2 - y_1) = (z - z_1)|(z_2 - z_1).$

5. Condition for parallel of a line and a plane $al + bm + cn = 0$

6. Condition for perpendicular of a line and a plane is $a|l = b|m = c|n$

7. Equation of a plane through a given line and parallel to an another line**:** Suppose the equation of the plane through the line

   $(x-\alpha)|l_1 = (y-\beta)|m_1 = (z-\gamma)|n_1$ and parallel to the line

   $(x)|l_2 = (y)|m_2 = (z)|n_2$ is

$$\begin{bmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{bmatrix} = 0$$

8. Equation of the perpendicular line from the point $P(x_1, y_{1,} z_1)$ to the line (1) are given by

   $(x - x_1)|\ (\alpha + lr - x_1) = (y - y_1)|(\beta + mr - y_1) = (z - z_1)|\ (\gamma + nr - z_1)$

9. Condition for the two lines to intersect (in general form)**:**Suppose the equations of the given lines be

$a_1x + b_1y + c_1z + d_1 = 0;\ \ a_2x + b_2y + c_2z + d_2 = 0$…..(1)

$a_3x + b_3y + c_3z + d_3 = 0;\ \ a_4x + b_4y + c_4z + d_4 = 0$……(2)

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix} = 0$$

10. COPLANAR LINES**:** Suppose that the equations of the given lines be

$(x-\alpha)|l\ =\ (y-\ \beta)|m\ =\ (z-\gamma)|n$………………..(1)

And $(x-\alpha')|l'\ =\ (y-\ \beta')|m'\ =\ (z-\gamma')|n'$ ………(2)

If they intersect,, then they lie in a plane(coplanar)if

$$\begin{bmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{bmatrix} = 0$$

.***Skew lines*****:** Those lines which do not intersect or the lines which do not lie in a plane.

Shortest distance: The length of the line intercepted between two lines which is perpendicular to both is the shortest distance between them. The straight line which is perpendicular to each of the two skew lines is called the line of the shortest distance (S. D.).

11. If the lines are coplanar, the S. D. between them is zero, then

$$\begin{bmatrix} \alpha'-\alpha & \beta'-\beta & \gamma'-\gamma \\ l & m & n \\ l' & m' & n' \end{bmatrix} = 0$$

Two lines are coplanar if the shortest distance between them is zero.

UNIT-4 THE SPHERE

Structure

$$x^2 + y^2 + z^2 = a^2 \text{ is } \left(\frac{la^2}{p}, \frac{ma^2}{p}, \frac{na^2}{p}\right)$$

4.1 INTRODUCTION

Definition (Sphere) 4.1:

In solid geometry, a sphere is the locus of all points equidistant from a fixed point. Fixed point is known as centre of the sphere and constant distance is known as the radius of the sphere.

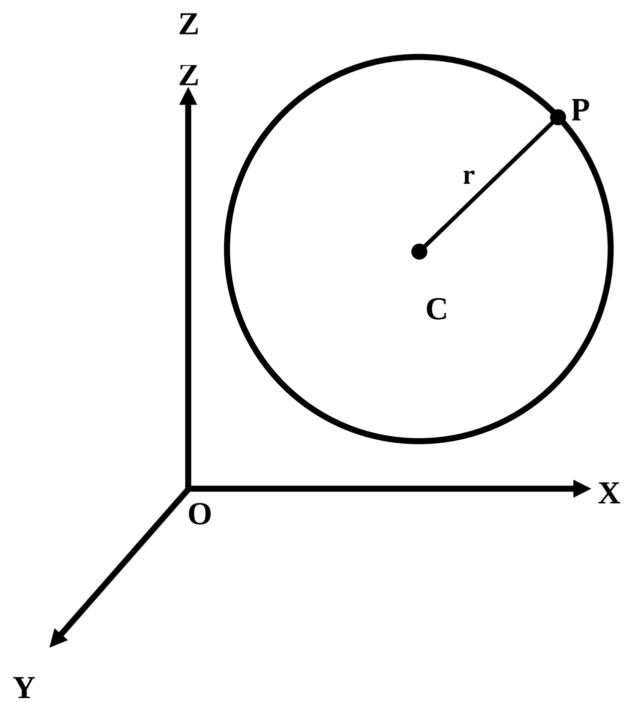


Figure 4.1

C is the centre of the sphere and $CP = r$ is the radius of the sphere.

Definition (Inside and Outside of a Sphere) 4.1:

A point is inside, outside, or on a sphere according as its distance from the center is less than, greater than, or equal to the radius of the sphere.

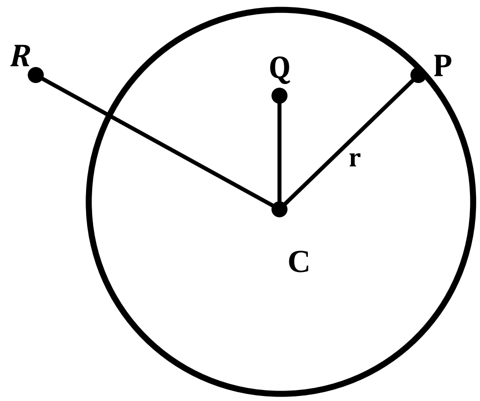


Figure 4.2

C is the centre of the sphere and r is radius.

$CP = r \Longrightarrow$ P lies on the sphere

$CQ < r \Longrightarrow$ Q lies inside the sphere

$CR > r \Longrightarrow$ R lies outside the sphere

Definition (Circle) 4.1:

Every section of a sphere made by a plane is a circle.

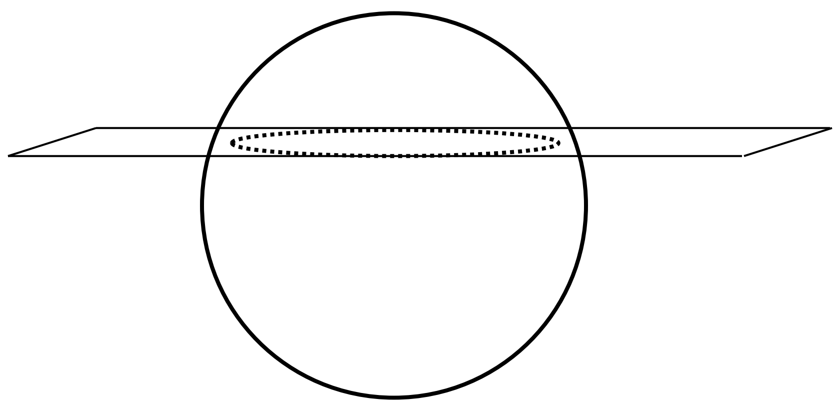


Figure 4.3

Definition (A Great Circle) 4.2:

A great circle of a sphere is a section made by a plane which passes through the center of the sphere.

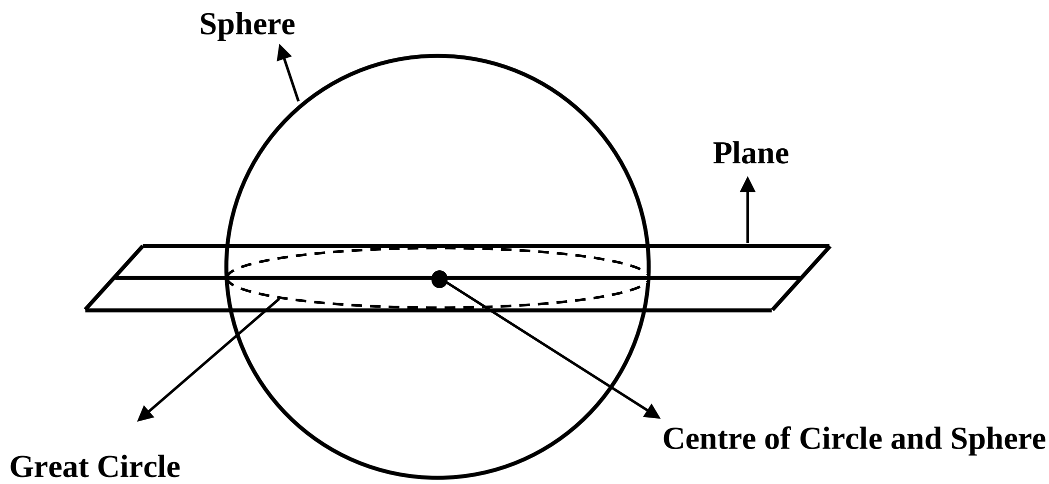


Figure 4.4

Definition (A Small Circle) 4.3:

A small circle of a sphere is a section made by a plane which does not pass through the center of the sphere.

4.2 OBJECTIVES

After reading this unit, you should be able to

- Understand the definition of sphere
- Understand that the point lies on the boundary, inside or outside the sphere.
- Understand the circle, great circle and small circle.
- Find the equation of a Sphere with centre at $C(u, v, w)$ and radius r
- Find the equation of a Sphere with centre at origin $O(0,0,0)$ and radius r
- Understand the general equation of the Sphere and determine its centre and radius
- Find the equation of the Sphere with a given diameter
- Find the equation of a circle and determine its centre and radius
- Understand great circle and find the equation of a sphere for which the circle is a great circle
- Show that the intersection of two sphere is a circle
- Find the equation of a sphere passing through a circle
- Understand the three possibilities that the line does not intersect the sphere or intersect the sphere at two point or it is tangent line
- Find the equation of tangent planes
- Find the condition of tangency
- Find the equation of plane of contact
- Find the pole and polar planes

- Show that the equation of the polar plane of a point $A(x_1, y_1, z_1)$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$
- Show that the equation of the polar plane of a point $A(x_1, y_1, z_1)$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$
- Show that the pole of the polar plane $lx + my + nz = p$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $\left(\frac{la^2}{p}, \frac{ma^2}{p}, \frac{na^2}{p}\right)$
- Find the condition that the two spheres are orthogonal
- Find the angle of intersection of two spheres
- Show that the two spheres are touch internally or externally and find their point of contact
- Find the length of a tangent and power of a point
- Find the radical plane of two spheres
- Find the radical axis(radical line) of three spheres
- Find the Coaxial system of spheres
- Find the limiting points of a co-axial system of spheres.

4.3 EQUATION OF A SPHERE WITH CENTRE AT C(u, v, w) AND RADIUS R

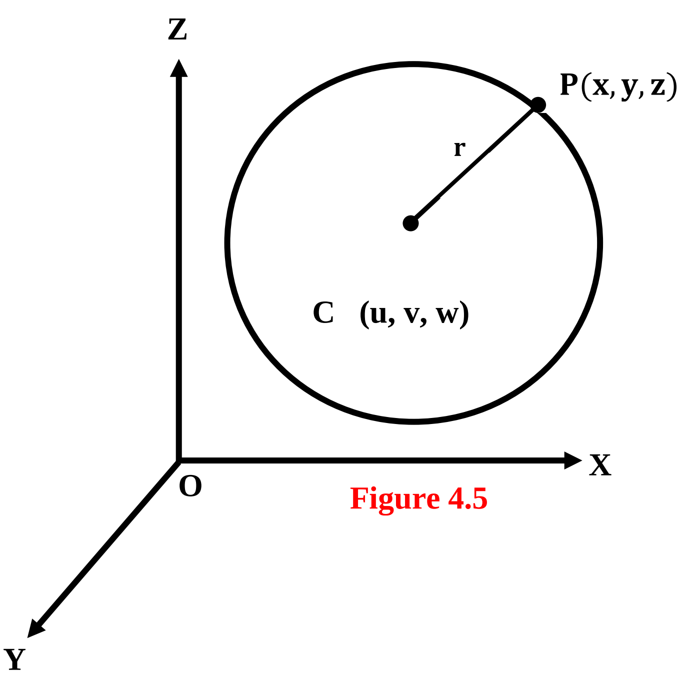


Figure 4.5

Let the centre of the sphere be the point $C(u, v, w)$ and its radius be r.

Let $P(x, y, z)$ be any point on the sphere

$\Longrightarrow \mathrm{CP} = \mathrm{r}$

$\Longrightarrow \sqrt{(x-u)^2 + (y-v)^2 + (z-w)^2} = \mathrm{r}$

$\Longrightarrow (x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$

4.4 EQUATION O F A SPHERE WITH CENTRE AT ORIGIN AND RADIOUS r

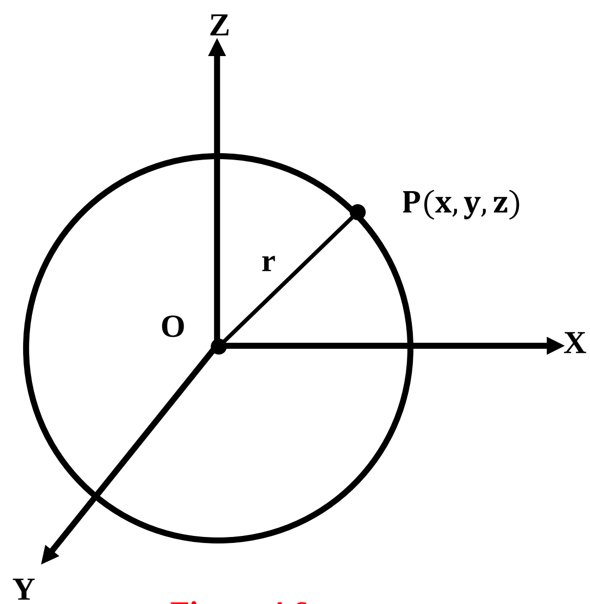


Figure 4.6

Let P(x, y, z) be any point on the sphere.

$\mathrm{OP} = \mathrm{r}$

$\Longrightarrow \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \mathrm{r}$

$\Longrightarrow \sqrt{x^2 + y^2 + z^2} = \mathrm{r}$

$\Longrightarrow x^2 + y^2 + z^2 = r^2$

4.5 GENERAL EQUATION OF THE SPHERE

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

$\Longrightarrow (x+u)^2 + (y+v)^2 + (z+w)^2 = \left(\sqrt{u^2 + v^2 + w^2 - d}\right)^2$

$\Longrightarrow \{x - (-u)\}^2 + \{y - (-v)\}^2 + \{z - (-w)\}^2 = \left(\sqrt{u^2 + v^2 + w^2 - d}\right)^2$

Centre is $(-u, -v, -w)$

$$\text{Radius} = \sqrt{u^2 + v^2 + w^2 - d}$$

4.6 EQUATION OF THE SPHERE WITH A GIVEN DIAMETER

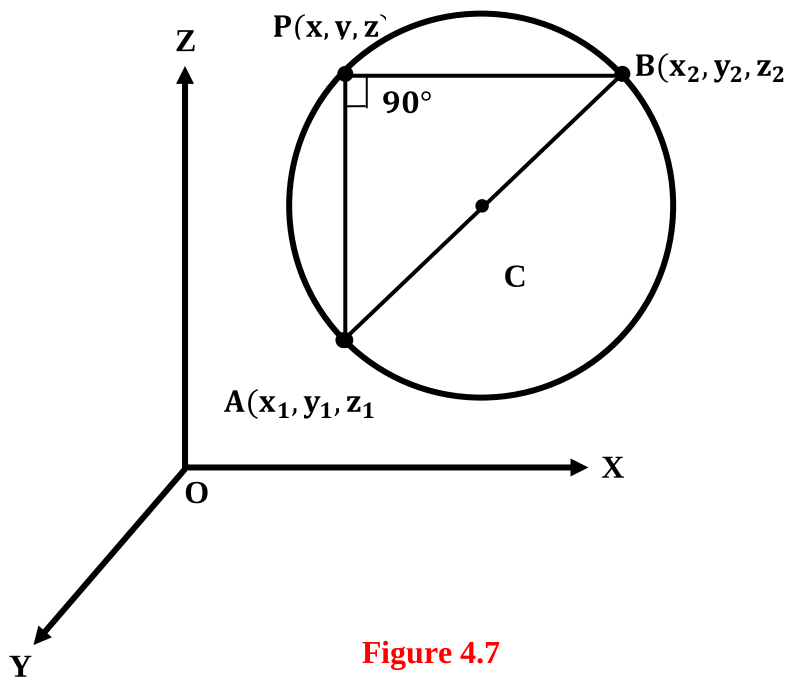


Figure 4.7

Direction ratios of AP are $x - x_1, y - y_1, z - z_1$

Direction ratios of BP are $x - x_2, y - y_2, z - z_2$

AP $\perp$ BP (angle in a semi circle)

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

Example 4.1: Find the equation of the sphere with centre at $(1,2,3)$ and radius 5.

Solution: Equation of a Sphere with centre at (u, v, w) and radius r is given by

$(x - u)^2 + (y - v)^2 + (z - w)^2 = r^2$

The required equation of the sphere is

$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$

$\Rightarrow x^2 + y^2 + z^2 - 2x - 4y - 6z - 11 = 0$

Example 4.2: Find the equation of the sphere with centre at $(0,0,4)$ and radius 4.

Solution: Equation of a Sphere with centre at (u, v, w) and radius r is given by

$$(x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$$

The required equation of the sphere is

$$(x-0)^2 + (y-0)^2 + (z-4)^2 = 16$$

$$\Rightarrow x^2 + y^2 + z^2 - 8z = 0$$

Example 4.3: Find the equation of the sphere whose centre is $(1,3,4)$ and which passes through the point $(-3,0,4)$.

Solution: Radius of the sphere $= \sqrt{(1+3)^2 + (3-0)^2 + (4-4)^2} = 5$

$$\text{Centre of the sphere} = (1,3,4)$$

The required equation of the sphere is

$$(x-1)^2 + (y-3)^2 + (z-4)^2 = 25$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 6y - 8z + 1 = 0$$

Example 4.4: Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 2x - 6y - 8z + 1 = 0$.

Solution: Equation of the given sphere is

$$x^2 + y^2 + z^2 - 2x - 6y - 8z + 1 = 0$$

$$\Rightarrow (x-1)^2 + (y-3)^2 + (z-4)^2 = (5)^2$$

$$\Rightarrow \text{Radius of the sphere} = 5 \text{ and Centre of the sphere} = (1,3,4)$$

Example 4.5: Find the equation of the sphere through the four points $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$.

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$
……… **(1)**

As the sphere **(1)** passes through the point $(0,0,0), (a,0,0), (0,b,0), (0,0,c)$

$\Rightarrow d = 0$

$a^2 + 2ua + d = 0 \Rightarrow u = -\frac{a}{2}$

$b^2 + 2vb + d = 0 \Rightarrow v = -\frac{b}{2}$

$c^2 + 2wc + d = 0 \Rightarrow w = -\frac{c}{2}$

The required equation of sphere is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \quad \text{(from eq. 1)}$$

Example 4.6: Find the equation of the sphere which passes through the points $(0,0,0), (a,0,0), (0,b,0)$ and whose centre lies on the plane $x + y + z = 0$

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$
……… **(1)**

As the sphere **(1)** passes through the point $(0,0,0), (a,0,0)$ *and* $(0,b,0)$

$\Rightarrow d = 0$

$a^2 + 2ua + d = 0 \Rightarrow u = -\frac{a}{2}$

$b^2 + 2vb + d = 0 \Rightarrow v = -\frac{b}{2}$

As the centre of the sphere $(-u, -v, -w)$ lies on the plane $x + y + z = 0$

$\Rightarrow -u - v - w = 0$

$\Rightarrow u + v + w = 0$

$\Rightarrow w = -u - v = \frac{a}{2} + \frac{b}{2}$

The required equation of sphere is given by

$$x^2 + y^2 + z^2 - ax - by + (a + b)z = 0$$

Example 4.7: Find the equation of the sphere circumscribing the tetrahedron whose faces are

$x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$

Solution:

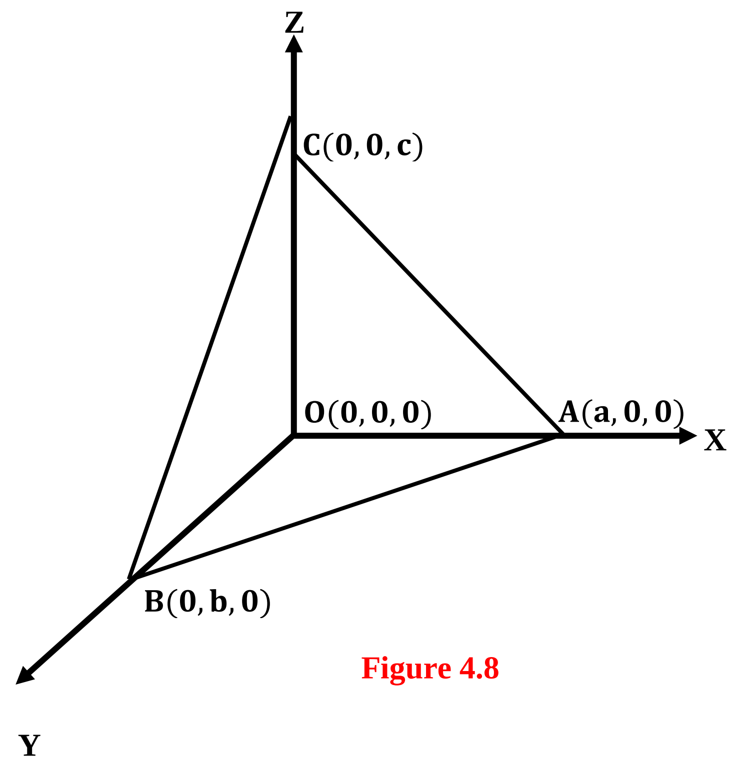


Figure 4.8

Equations of the given planes are

$x = 0$ ……...**(P_1)**

$y = 0$ ……... **(P_2)**

$z = 0$ ……...($\mathbf{P_3}$)

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ……... **(P_4)**

By solving **(P_1)**, **(P_2)** and **(P_3)**, we get the vertex $O(0,0,0)$

By solving **(P_2)**, **(P_3)** and **(P_4)**, we get the vertex $A(a,0,0)$

By solving **(P_3)**, **(P_1)** and **(P_4)**, we get the vertex $B(0,b,0)$

By solving **(P_1)**, **(P_2)** and **(P_4)**, we get the vertex $C(0,0,c)$

Therefore the sphere circumscribing the tetrahedron OABC is the sphere passing through the four points $O(0,0,0), A(a,0,0)$, $B(0,b,0)$ and $C(0,0,c)$ is given by

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Example 4.8: Find the equation of the sphere on the join of $(0,b,0)$ and $(0,0,c)$ as diameter.

Solution: The equation of the sphere on the join of $(0,b,0)$ and $(0,0,c)$ as diameter is given by

$$(x-0)(x-0) + (y-b)(y-0) + (z-0)(z-c) = 0$$

$$\Rightarrow x^2 + y^2 - by + z^2 - zc = 0$$

$\Rightarrow x^2 + y^2 + z^2 - by - zc = 0$

Example 4.9: Find the equation of the sphere with centre at (0,0,0) and touch the plane

$ax + by + cz + d = 0.$

Solution:

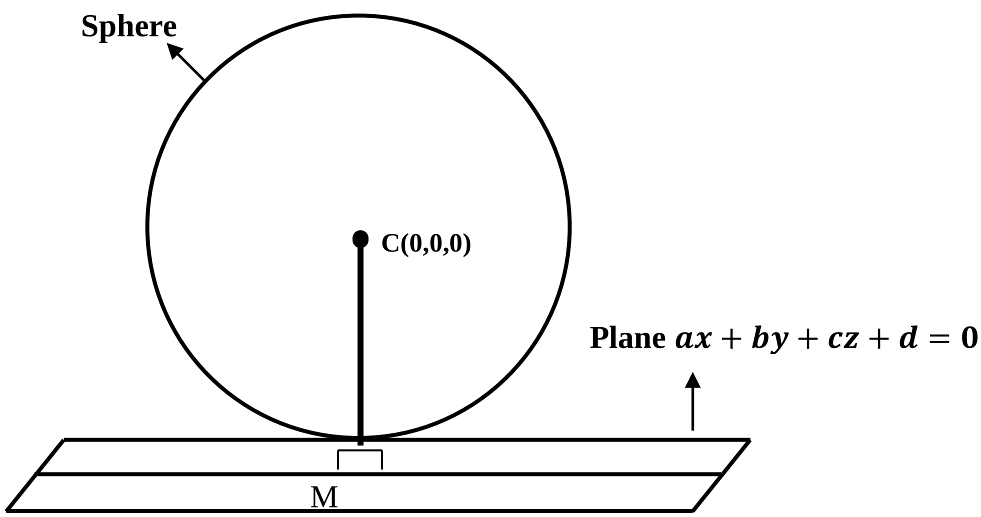


Figure 4.9

$$\text{Radius of sphere} = \text{CM} = \frac{a.0 + b.0 + c.0 + d}{\sqrt{a^2 + b^2 + c^2}} = \frac{d}{\sqrt{a^2 + b^2 + c^2}}$$

The required equation of sphere is given by

$$(x-0)^2 + (y-0)^2 + (z-0)^2 = \left(\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right)^2$$

$$\Rightarrow x^2 + y^2 + z^2 = \frac{d^2}{a^2 + b^2 + c^2}$$

Example 4.10: (i) Show that the point P(2,2,1) lies on the sphere $x^2 + y^2 + z^2 = 9$.

(ii) Show that the point Q(5,2,2) lies inside the sphere $x^2 + y^2 + z^2 - 6x + 4y + 4z - 32 = 0$.

(iii) Show that the point R(3,3,4) lies outside the sphere $x^2 + y^2 + z^2 + 2x + 2y - 4z - 19 = 0$.

Solution: (i) Equation of the given sphere

$x^2 + y^2 + z^2 = 9$

Radius of the given sphere $= 3$ and Centre $= O(0,0,0)$

$$\text{PO} = \sqrt{(2-0)^2 + (2-0)^2 + (1-0)^2} = \sqrt{9} = 3$$
$$= \text{Radius of the sphere}$$

Hence the point the point $P(2,2,1)$ lies on the sphere.

(ii) Equation of the given sphere

$$x^2 + y^2 + z^2 - 6x + 4y + 4z - 32 = 0$$

or
$$(x-3)^2 + (y+2)^2 + (z+2)^2 = 49$$

Radius of the given sphere $= 7$ and Centre $= O(3,-2,-2)$

$$\text{QO} = \sqrt{(5-3)^2 + (2+2)^2 + (2+2)^2} = \sqrt{36} = 6$$

$\text{QO} = 6 < 7(\text{Radius of the sphere})$

Hence the point the point $Q(5,2,2)$ lies inside the sphere.

(iii) Equation of the given sphere

$$x^2 + y^2 + z^2 + 2x + 2y - 4z - 19 = 0$$

or
$$(x+1)^2 + (y+1)^2 + (z-2)^2 = 25$$

Radius of the given sphere $= 5$ and Centre $= O(-1,-1,2)$

$$\text{RO} = \sqrt{(3+1)^2 + (3+1)^2 + (4-2)^2} = \sqrt{36} = 6$$

$\text{RO} = 6 > 5(\text{Radius of the sphere})$

Hence the point the point $R(3,3,4)$ lies outside the sphere.

Note: A point P(xq, y1, z1) lies on the sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Or outside the sphere or inside the sphere according as ${x_1}^2 + {y_1}^2 + {z_1}^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \; or > 0 \; or < 0$

Check Your Progress

1. Find the equation of the sphere with centre at $(0,0,0)$ and which passes through the point$(a, 0,0)$.

Ans. $x^2 + y^2 + z^2 = a^2$

2. Find the equation of the sphere with centre at $(2, -3, 4)$ and radius 5.

Ans. $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$

3. Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 4x + 6y + 2z + 5 = 0$.

Ans. Radius $= 3$ and Centre $= (2, -3, -1)$

4. Find the equation of the sphere on the join of $(2, -3, 1)$ and $(3, -1, 2)$ as diameter.

Ans. $x^2 + y^2 + z^2 - 5x + 4y - 3z + 11 = 0$

5. Find the equation of the sphere on the join of $(a, 0, 0)$ and $(0, b, 0)$ as diameter.

Ans. $x^2 + y^2 + z^2 - ax - by = 0$

6. Find the equation of the sphere with centre at (2,3,-4) and touch the plane $2x + 6y - 3z + 15 = 0$.

Ans. $x^2 + y^2 + z^2 - 4x - 6y + 8z - 20 = 0$

7. Find the equation of the sphere which passes through the points (4,1,0),(2,-3,4),(1,0,0) and touch the plane $2x + 2y - z = 11$.

Ans. $x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0$

8. Find the equation of the sphere which passes through the points (1,-3,4),(1,-5,2),(1,-3,0) and whose centre lies on the plane $x + y + z = 0$.

Ans. $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$

4.7 PLANE SECTION OF A SPHERE

Every section of a sphere S made by a plane P is a circle.

$$S \equiv (x-u)^2 + (y-v)^2 + (z-w)^2 = r^2$$

$$P \equiv lx + my + nz = p$$

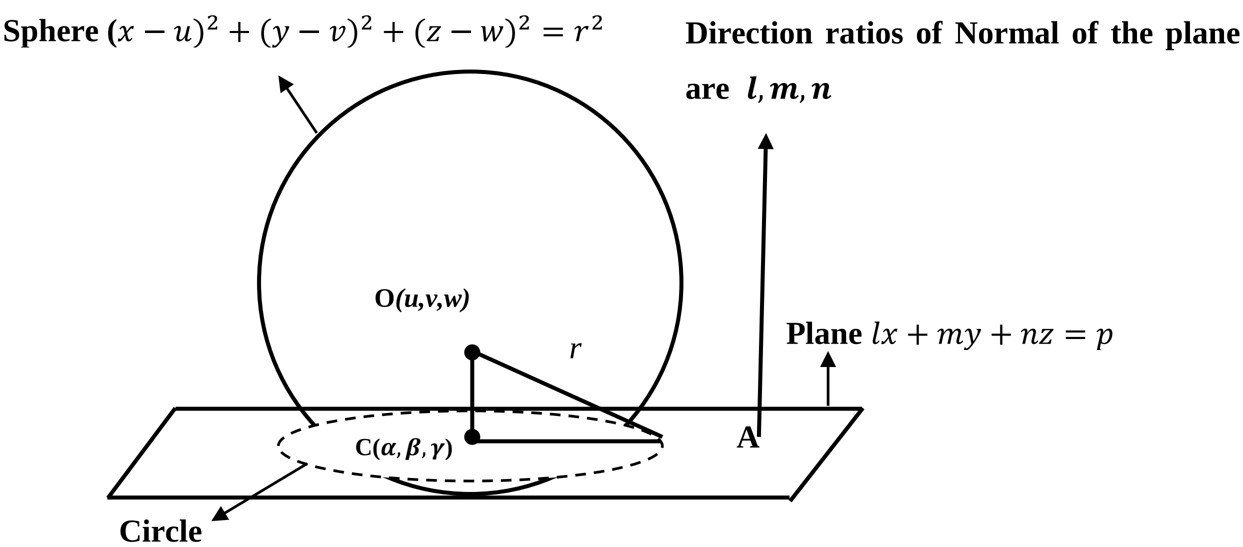


Figure 4.10

O(u,v,w) is the centre of the Sphere and $\mathrm{OA}=r$ is the radius of the Sphere

Let C(α,β,γ) is the centre of the circle and CA is the radius of the circle.

C(α,β,γ) must satisfied the equation of the plane $lx+my+nz=p$

i.e.$l\alpha+m\beta+n\gamma=p$ …….**(1)**

$\therefore$ Direction ratios of OC are $\alpha-u,\beta-u,\gamma-w$

And Direction ratios of normal of the plane are l,m,n

$\therefore$ OC is parallel to the normal of the plane

Hence, $\frac{\alpha-u}{l}=\frac{\beta-v}{m}=\frac{\gamma-w}{n}=\lambda(say)\Longrightarrow \quad \alpha=l\lambda+u,\beta=m\lambda+v\,,\gamma=n\lambda+w$

By putting the value of α,β,γ in equation **(1)**

$l(l\lambda+u)+m(m\lambda+v)+n(n\lambda+w)=p\Longrightarrow\lambda=\frac{p-lu-mv-nw}{l^2+m^2+n^2}$

By putting the value of $\lambda=\frac{p-lu-mv-nw}{l^2+m^2+n^2}$ in $\alpha=l\lambda+u \quad \beta=m\lambda+v \quad \gamma=n\lambda+w$ we get the coordinate of the centre of circle.

Now, $\mathrm{OC}=\sqrt{(\alpha-u)^2+(\beta-u)^2+(\gamma-w)^2}$

In Right Angle Triangle OCA,

$\mathrm{CA} = \sqrt{r^2 - \mathrm{OC}^2}$ is the radius of the circle.

4.8 GREAT CIRCLE

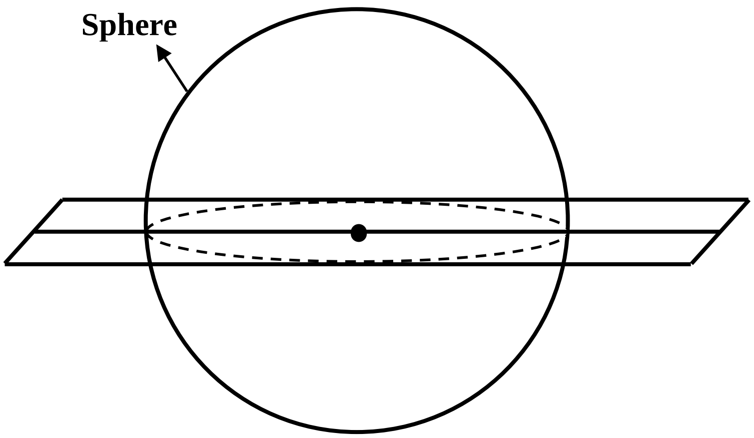


Figure 4.11

4.9 INTERSECTION OF TWO SPHERE

$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$

$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$

The intersection of two spheres $S_1 = 0$ and $S_2 = 0$ is a circle given by

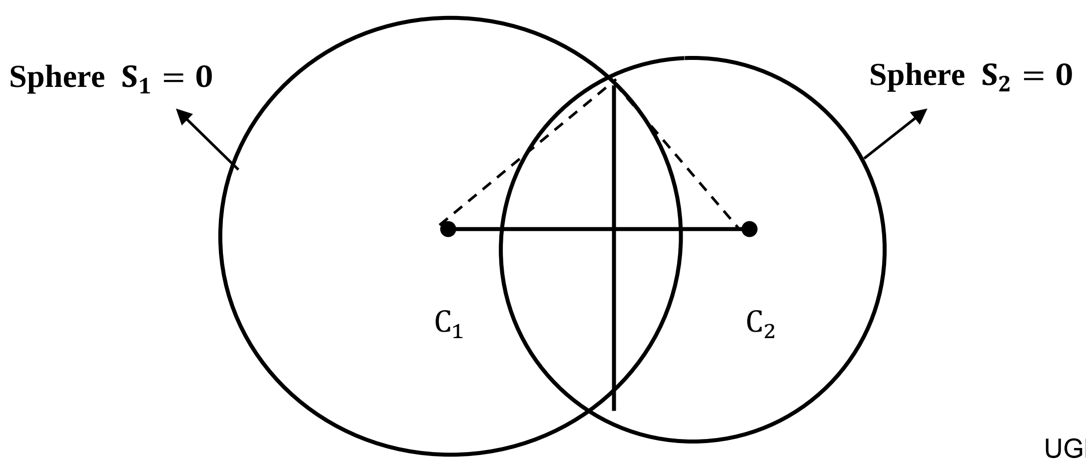
$S_1 = 0$ (Sphere)

$S_1 - S_2 = 0$(Plane)

or

$S_2 = 0$ (Sphere)

$S_1 - S_2 = 0$(Plane)



4.10 SPHERE PASSING THROUGH A CIRCLE

$$\left.\begin{aligned} S &\equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \\ P &\equiv ax + by + cz + d = 0 \end{aligned}\right\}$$

.......**(1)**

Equation of a sphere through the circle **(1)** is given by

$$S + \lambda P = 0$$

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \quad \ldots\ldots (2)$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

Equation of a sphere through $S_1 = 0$ and $S_2 = 0$ is given by

$$S_1 + \lambda\, S_2 = 0$$

Example 4.11: Find the radius of the circle $(x+1)^2 + (y+2)^2 + (z-6)^2 = 49$,

$3x + 5y + 4z + 9 = 0.$

Solution:

Sphere $(x+1)^2 + (y+2)^2 + (z-6)^2 = 49$

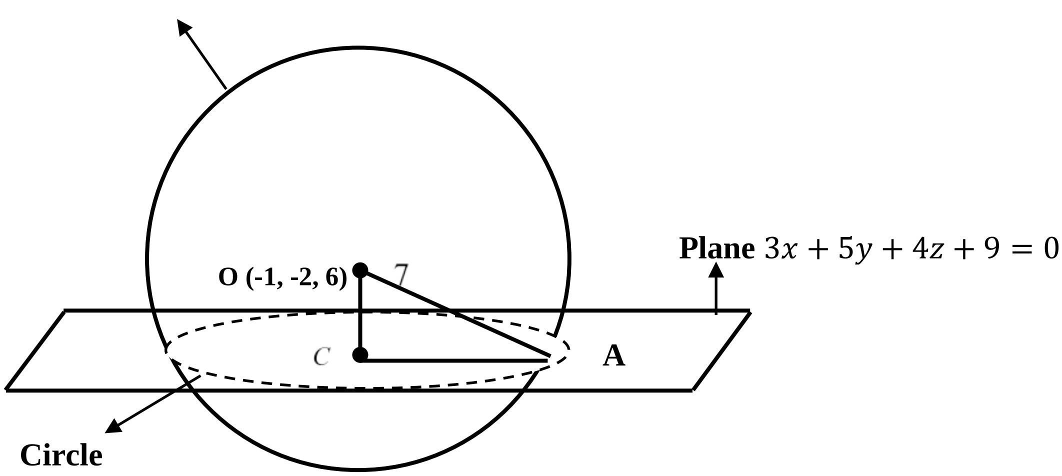


Figure 4.13

$$OC = \text{Length of the perpendicular from } O(-1,-2,6) \text{ to the plane } 3x + 5y + 4z + 9 = 0$$

$$= \frac{(3\times -1) + (5\times -2) + (4\times 6) + 9}{\sqrt{3^2 + 5^2 + 4^2}} = \frac{20}{\sqrt{50}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$\mathrm{OA} = \text{Radius of the sphere} = 7$

Now, in Right Angle Triangle OCA

$\mathrm{CA} = \sqrt{7^2 - \left(2\sqrt{2}\right)^2} = \sqrt{49 - 8} = \sqrt{41}$ is the radius of the circle.

Example 4.12: Find the centre and radius of the circle $(x-1)^2 + (y-2)^2 + (z-3)^2 = 25$,

$x + y + z = 2.$

Solution:

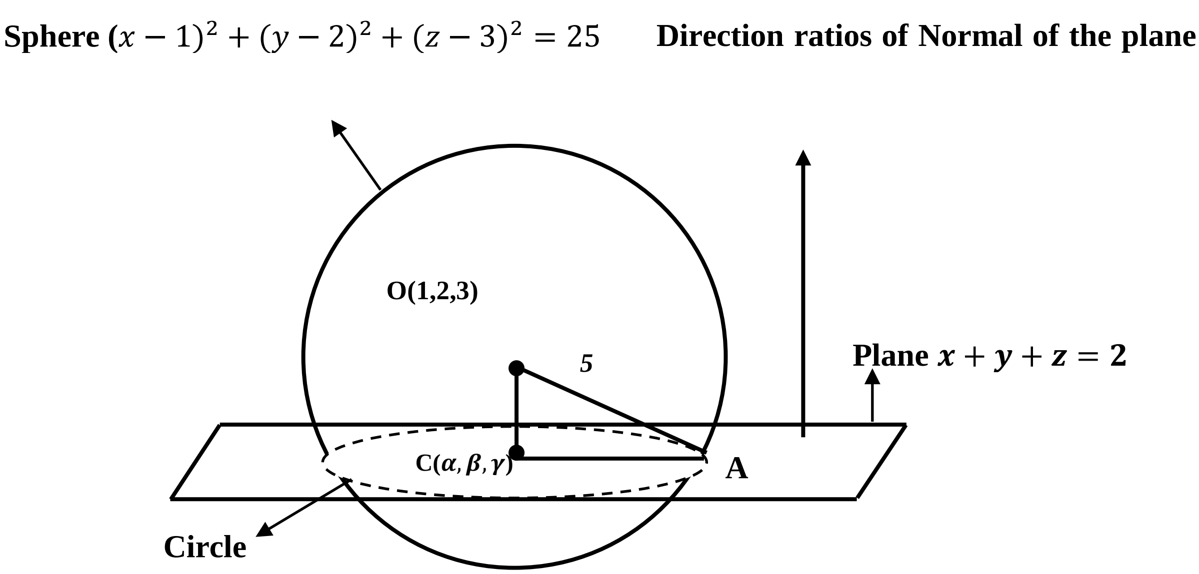


Figure 4.14

Radius of the given sphere $= \mathrm{OA} = 5$ and Centre $= \mathrm{O}(1,2,3)$

Let C(α, β, γ) is the centre of the circle and CA is the radius of the circle.

C(α, β, γ) must satisfied the equation of the plane $x + y + z = 2$

i.e.$\alpha + \beta + \gamma = 2$ …….**(1)**

Direction ratios of OC are $\alpha - 1, \beta - 2, \gamma - 3$

Direction ratios of normal of the plane are 1,1,1

OC is parallel to the normal of the plane

Hence, $\frac{\alpha-1}{1}=\frac{\beta-2}{1}=\frac{\gamma-3}{1}=\lambda(say) \Longrightarrow \quad \alpha=\lambda+1, \quad \beta=\lambda+2, \quad \gamma=\lambda+3$

By putting the value of α,β,γ in equation **(1)**

$\Longrightarrow \lambda+1+\lambda+2+\lambda+3=2 \Longrightarrow \lambda=-\frac{4}{3}$

By putting the value of $\lambda=-\frac{4}{3}$ we get the coordinate of the centre of circle

$$\alpha=-\frac{4}{3}\ +1\ \Longrightarrow \alpha=-\frac{1}{3}$$

$$\beta=-\frac{4}{3}\ +2 \Longrightarrow \beta=\frac{2}{3}$$

$$\beta=-\frac{4}{3}\ +3 \Longrightarrow \gamma=\frac{5}{3}$$

Now,

$$\text{OC}=\sqrt{\left(-\frac{1}{3}-1\right)^2+\left(\frac{2}{3}-2\right)^2+\left(\frac{5}{3}-3\right)^2}=\frac{4\sqrt{3}}{3}$$

Now, in Right Angle Triangle OCA

$$\text{CA}=\sqrt{5^2-\left(\frac{4\sqrt{3}}{3}\right)^2}=\sqrt{25-\frac{16}{3}}=\sqrt{\frac{59}{3}}$$ is the radius of the circle.

Example 4.13: Find the equation of the sphere through the circle $x^2+y^2+z^2=a^2$,

$x+y+z=0$ and the point (α,β,γ).

Solution: The equation of the sphere through the circle $x^2+y^2+z^2-a^2=0, x+y+z=0$ is given by

$(x^2+y^2+z^2-a^2)+\lambda(x+y+z)=0$ ……..**(S)**

As sphere **(S)** passes through the point (α,β,γ)

$\Longrightarrow (\alpha^2+\beta^2+\gamma^2-a^2)+\lambda(\alpha+\beta+\gamma)=0$

$\Longrightarrow \lambda=-\frac{\alpha^2+\beta^2+\gamma^2-a^2}{\alpha+\beta+\gamma}$

The required equation of the sphere is

$$(x^2+y^2+z^2-a^2)-\frac{(\alpha^2+\beta^2+\gamma^2-a^2)}{(\alpha+\beta+\gamma)}(x+y+z)=0$$

Example 4.14: Find the centre and radius of the circle $x^2+y^2+z^2-2y-4z-11=0$,

$x+2y+2z-15=0$.

Solution:

Sphere $(x-0)^2+(y-1)^2+(z-2)^2=16$

Direction ratios of Normal of the plane are $\mathbf{1,2,2}$

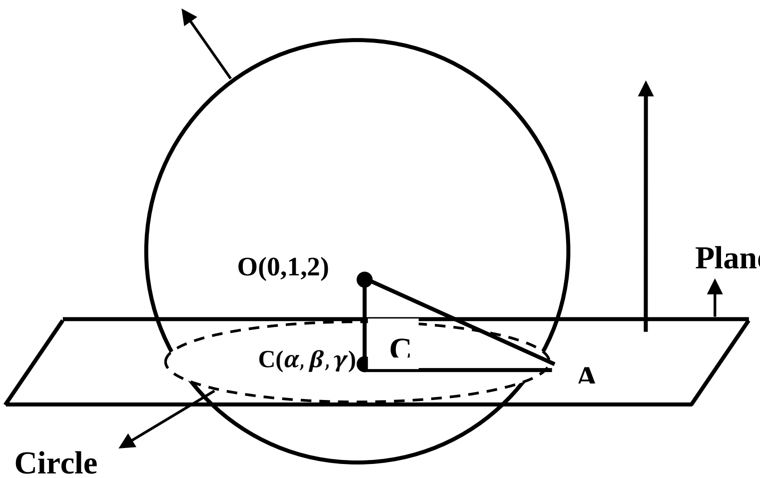


Figure 4.15

Radius of the given sphere $=$ OA $=4$ and Centre $=$ O$(0,1,2)$

Let C(α,β,γ) is the centre of the circle and CA is the radius of the circle.

C(α,β,γ) must satisfied the equation of the plane $x+2y+2z-15=0$

i.e. $\alpha+2\beta+2\gamma=15$ …….**(1)**

Direction ratios of OC are $\alpha-0,\beta-1,\gamma-2$

Direction ratios of normal of the plane are 1,2,2

OC is parallel to the normal of the plane

Hence, $\frac{\alpha}{1}=\frac{\beta-1}{2}=\frac{\gamma-2}{2}=\lambda(say)\Rightarrow \alpha=\lambda,\quad \beta=2\lambda+1,\quad \gamma=2\lambda+2$

By putting the value of α,β,γ in equation **(1)**

$\lambda + 4\lambda + 2 + 4\lambda + 4 = 15 \Rightarrow \lambda = 1$

By putting the value of $\lambda = 1$ we get the coordinate of the centre of circle

$\alpha = 1, \beta = 3, \gamma = 4$

Now,

OC $= \sqrt{(0-1)^2 + (1-3)^2 + (2-4)^2} = 3$

Now, in Right Angle Triangle OCA

CA $= \sqrt{4^2 - 3^2} = \sqrt{7}$ is the radius of the circle.

Example 4.15: Find the equation of a sphere for which the circle $x^2 + y^2 + z^2 = 16,\quad x + y + z = 6$ is a great circle.

Solution: The equation of the sphere through the circle

$x^2 + y^2 + z^2 - 16 = 0$, $x + y + z - 6 = 0$ is given by

$(x^2 + y^2 + z^2 - 16) + \lambda(x + y + z - 6) = 0$

$x^2 + y^2 + z^2 + \lambda x + \lambda y + \lambda z - 16 - 6\lambda = 0$ ……..**(S)**

The centre of the sphere **(S)** is $(-\frac{\lambda}{2}, -\frac{\lambda}{2}, -\frac{\lambda}{2})$

As the given circle is a great circle for the sphere **(S)**, then the centre of the sphere **(S)** lies on the plane $x + y + z = 6$

$\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 6$

$\Rightarrow \lambda = -4$

The required equation of the sphere is given by

$(x^2 + y^2 + z^2 - 16) - 4(x + y + z - 6) = 0$

$\Rightarrow x^2 + y^2 + z^2 - 4x - 4y - 4z + 8 = 0$

Example 4.16: Show that the equation of the circle whose centre is (1,2,3) and which lies on the sphere $x^2 + y^2 + z^2 = 16$ is $x^2 + y^2 + z^2 = 16,\quad x + 2y + 3z = 14$.

Solution:

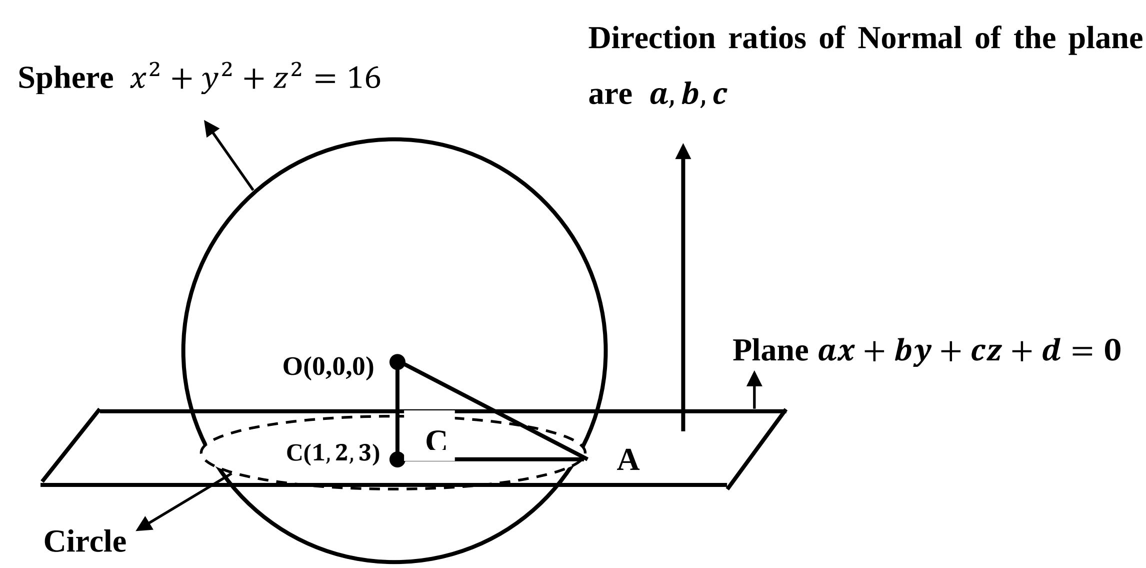


Figure 4.16

Let the equation of the circle through the sphere $x^2 + y^2 + z^2 = 16$ is

$x^2 + y^2 + z^2 = 16, \quad ax + by + cz + d = 0$

Centre of circle (1,2,3) must satisfied the equation of the plane $ax + by + cz + d = 0$

$\Longrightarrow a(x-1) + b(y-2) + c(z-3) = 0$ …….**(1)**

Direction ratios of OC are 1,2,3

Direction ratio of normal of the plane are a,b,c

OC is parallel to the normal of the plane

Hence, $\frac{a}{1} = \frac{b}{2} = \frac{c}{3} = \lambda(say) \Longrightarrow a = \lambda, \quad b = 2\lambda, \quad c = 3\lambda$

Putting the value of a,b,c in **(1)**

$\Longrightarrow \lambda(x-1) + 2\lambda(y-2) + 3\lambda(z-3) = 0$

$\Longrightarrow x + 2y + 3z = 14$

Hence the required equation of circle is

$x^2 + y^2 + z^2 = 16, \quad x + 2y + 3z = 14$

Example 4.17: Find the equation to the plane in which the circle of intersection of the spheres

$x^2+y^2+z^2-2x+4y-6z+12=0$ and $x^2+y^2+z^2+6x-7y-z-12=0$ lies. Find also the equation of the sphere through this circle and the point (1,1,1).

Solution:

$S_1 \equiv x^2+y^2+z^2-2x+4y-6z+12=0$ …... **(1)**

$S_2 \equiv x^2+y^2+z^2+6x-7y-z-12=0$ …... **(2)**

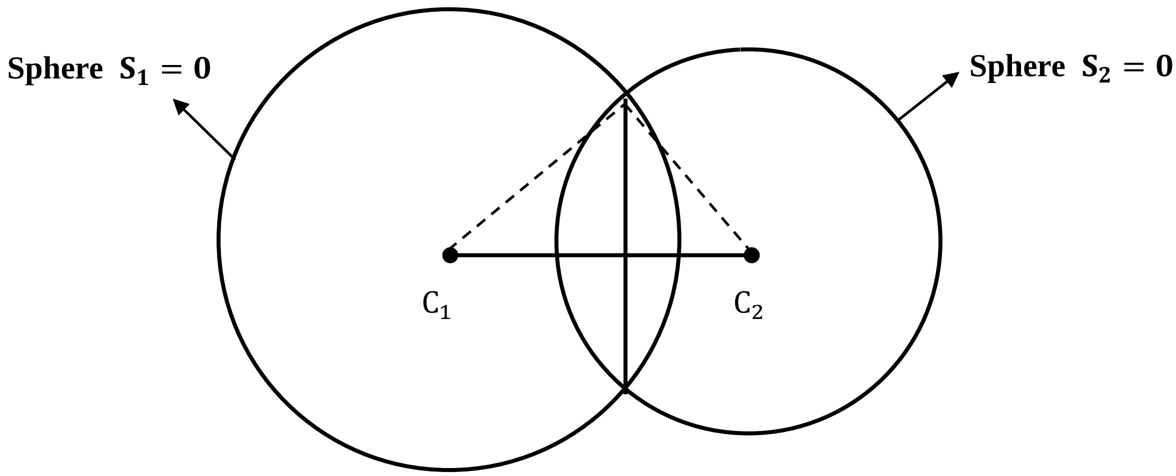


Figure 4.17

Equation of the plane in which the circle of intersection of the spheres lies is given by

$S_1 - S_2 = 0$

$\Longrightarrow (x^2+y^2+z^2-2x+4y-6z+12)-(x^2+y^2+z^2+6x-7y-z-12)=0$

$\Longrightarrow -8x+11y-5z+24=0$

$\Longrightarrow 8x-11y+5z-24=0$

Equation of the sphere through the circle of intersection of the spheres is given by

$(x^2+y^2+z^2-2x+4y-6z+12)+\lambda(8x-11y+5z-24)=0$ …….**(3)**

As the sphere **(3)** passes through the point (1,1,1)

$\Longrightarrow (1+1+1-2+4-6+12)+\lambda(8-11+5-24)=0$

$\Longrightarrow \lambda=\frac{1}{2}$

The required equation of the sphere is

$$(x^2+y^2+z^2-2x+4y-6z+12)+\frac{1}{2}\,(8x-11y+5z-24)=0$$

$\Longrightarrow 2x^2+2y^2+2z^2-4x+8y-12z+24+8x-11y+5z-24=0$

$\Longrightarrow 2x^2+2y^2+2z^2+4x-3y-7z=0$

Check Your Progress

1. Find the centre and radius of the circle

$(x-2)^2+(y-3)^2+(z-4)^2=36,\ 2x+6y+3z-6=0.$

Ans. Centre of the circle $=\left(\frac{6}{7},-\frac{3}{7},\frac{16}{7}\right)$, Radius of the circle $=2\sqrt{5}$

2. Find the equation of the sphere which passes through the point (α,β,γ) and the circle $x^2+y^2+z^2=a^2,\ x=0.$

Ans. $(x^2+y^2+z^2-a^2)\alpha-(\alpha^2+\beta^2+\gamma^2-a^2)x=0$

3. Find the equation of the sphere which passes through the point (α,β,γ) and the circle $x^2+y^2+z^2=a^2,\ y=0.$

Ans. $(x^2+y^2+z^2-a^2)\beta-(\alpha^2+\beta^2+\gamma^2-a^2)y=0.$

4. Find the equation of the sphere for which the circle

$x^2+y^2+z^2+7y-2z+2=0,\ 2x+3y+4z=8$ is a great circle.

Ans. $x^2+y^2+z^2-2x+4y-6z+10=0$

5. Find the equation to the plane in which the circle of intersection of the spheres $x^2+y^2+z^2+4x+6y+8z+10=0$ and $x^2+y^2+z^2+2x+4y+6z+8=0$ lies.

Ans. $x+y+z+1=0$

6. Prove that the circle $(x-2)^2+(y-3)^2+(z-4)^2=36$, $x-2y+2z=4$ is a great circle.

7. Show that the equation of the circle whose centre is $\left(\frac{6}{7},-\frac{3}{7},\frac{16}{7}\right)$

and which lies on the sphere $x^2 + y^2 + z^2 - 4x - 6y - 8z - 7 = 0$ is $x^2 + y^2 + z^2 - 4x - 6y - 8z - 7 = 0,\ 2x + 6y + 3z - 6 = 0$

8. Show that the equation to the circle whose centre is $\left(-\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\right)$ and which lies on the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 25$ is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 25,\ x + y + z = 2.$

4.11 INTERSECTION OF A STRAIGHT LINE AND A SPHERE

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ …………..(**S**)

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r(say)$ …………..(**L**)

Any point on line (L) is given by $(\alpha + lr, \beta + mr, \gamma + nr)$

If the line **(L)** intersect the sphere (**S**) the point $(\alpha + lr, \beta + mr, \gamma + nr)$ must satisfied its equation for some value of r.

$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 + 2u(\alpha + lr) + 2v(\beta + mr) + 2w(\gamma + nr) + d = 0 \quad r^2(l^2 + m^2 + n^2) + 2r\{l(\alpha + u) + m(\beta + v) + n(\gamma + w)\} + \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0$

This is quadratic equation in r .There are three possibilities

Case (i) The two roots are real and distinct then the line intersect the sphere at two point.

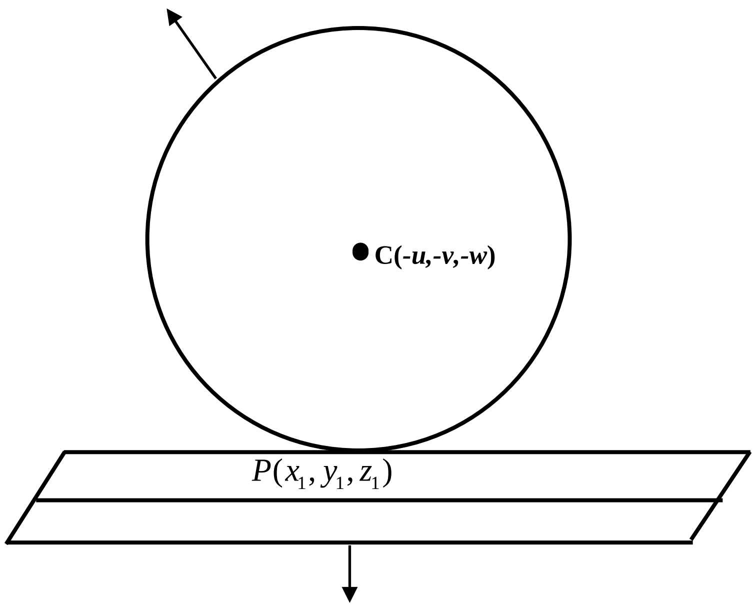
Case (ii) If both the roots are real and coincident then the line is a tangent line.

Case (iii) If the roots are imaginary then the line does not intersect the sphere.

4.12 TANGENT PLANES

The equation of Tangent plane of the Sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point $P(x_1, y_1, z_1)$ is given by $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0.$

Sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$



Tangent Plane $xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$

Figure 4.18

4.13 CONDITION OF TANGENCY

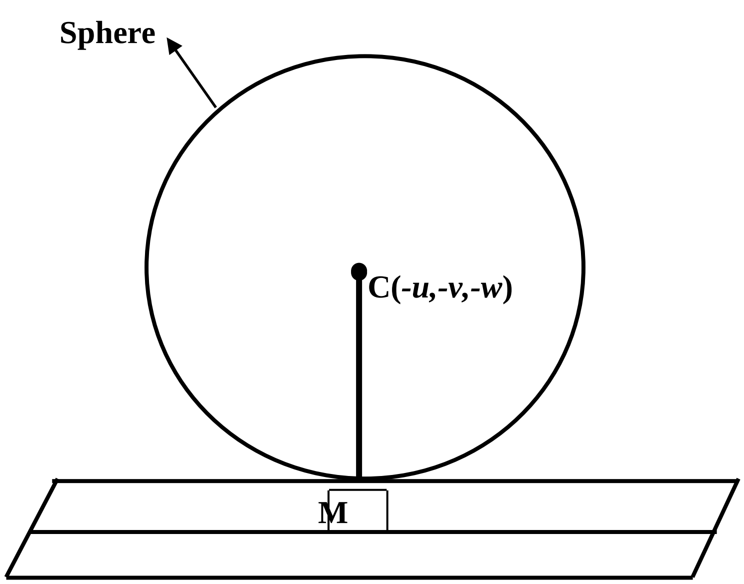


Figure 4.19

$$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$P \equiv lx + my + nz = p$

If the plane P is a tangent plane at the point M of the sphere S

Then,

Radius of the Sphere

$= \text{Length of the Perpendicular from } \boldsymbol{C(-u, -v, -w)} \text{ to the Plane } lx + my + nz - p = 0$

$$\Rightarrow \sqrt{u^2 + v^2 + w^2 - d} = \frac{|lu + mv + nw - p|}{\sqrt{l^2 + m^2 + n^2}}$$

$$\Rightarrow (lu + mv + nw - p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$$

Corollary 4.13.1: The condition that the plane $lx + my + nz = p$ touches the sphere
$x^2 + y^2 + z^2 = a^2$ is given by $p^2 = a^2(l^2 + m^2 + n^2)$.

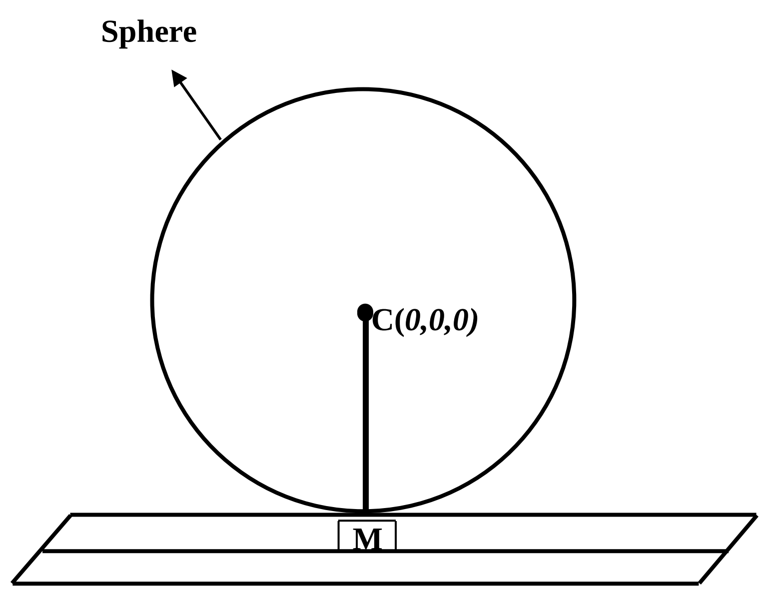


Figure 4.20

$S \equiv x^2 + y^2 + z^2 = a^2$

$P \equiv lx + my + nz = p$

If the plane P is a tangent plane at the point M of the sphere S

Then,

Radius of the Sphere
$=$ Length of the Perpendicular from C(0,0,0)to the Plane $lx + my + nz - p = 0$

$$\Rightarrow a = \frac{|-p|}{\sqrt{l^2 + m^2 + n^2}}$$

$$\Rightarrow p^2 = a^2(l^2 + m^2 + n^2)$$

Example 4.18: Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere

$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and find the point of contact.

Solution:

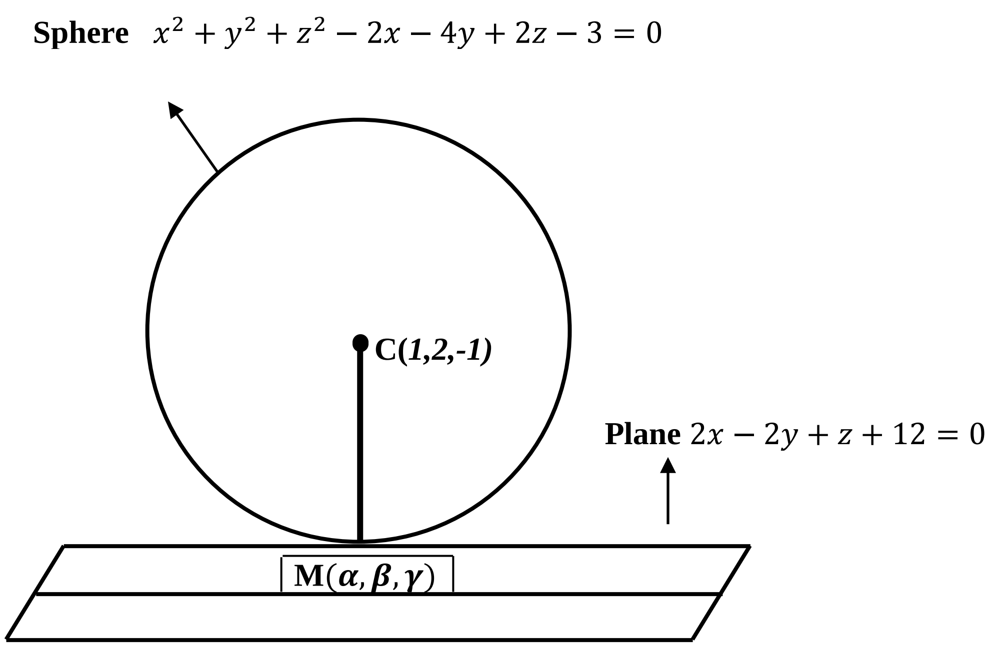


Figure 4.21

The equation of the given sphere is

$$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$$

Centre of the sphere $= \mathrm{C}(1,2,-1)$

Radius of the Sphere $= \sqrt{(1)^2 + (2)^2 + (-1)^2 - (-3)} = 3$

Length of the Perpendicular from $\mathrm{C}(1,2,-1)$ to the Plane $2\mathrm{x} - 2\mathrm{y} + \mathrm{z} + 12 = 0$

$$= \frac{|2 \times 1 - 2 \times 2 - 1 + 12|}{\sqrt{(2)^2 + (-2)^2 + (1)^2}} = \frac{9}{3} = 3 = \text{Radius of the sphere}$$

$\Longrightarrow$ Plane touch the sphere

Let $M(\alpha, \beta, \gamma)$ be the point of contact

Direction ratios of CM are $\alpha - 1, \quad \beta - 2, \quad \gamma + 1$

Direction ratios of normal of the plane are 2,-2,1

CM parallel to the normal to the plane

$$\Longrightarrow \frac{\alpha - 1}{2} = \frac{\beta - 2}{-2} = \frac{\gamma + 1}{1} = \lambda(say)$$

$$\Longrightarrow \alpha = 2\lambda + 1, \quad \beta = -2\lambda + 2, \quad \gamma = \lambda - 1$$

$M(\alpha, \beta, \gamma)$ must satisfied the equation of the plane $2x - 2y + z + 12 = 0$

$$\Longrightarrow 2\alpha - 2\beta + \gamma + 12 = 0$$

$\Longrightarrow 2\alpha - 2\beta + \gamma + 12 = 0$
………..**(1)**

Putting the value of α, β, γ in equation **(1)**

$$\Longrightarrow 2(2\lambda + 1) - 2(-2\lambda + 2) + (\lambda - 1) + 12 = 0$$

$$\Longrightarrow 4\lambda + 2 + 4\lambda - 4 + \lambda - 1 + 12 = 0$$

$$\Longrightarrow \lambda = -1$$

The required point of contact is M(-1,4,-2)

Example 4.19 : Find the equation of the tangent plane of the sphere

$x^2 + y^2 + z^2 - 2x - 4y + 2z = 0$ at origin O(0,0,0).

Solution : Since the equation of the tangent plane at (α, β, γ) is given by

$$\alpha x + \beta y + \gamma z - (x + \alpha) - 2(y + \beta) + (z + \gamma) = 0$$

Hence the equation of the tangent plane at O(0,0,0) is

$$0x + 0y + 0z - (x + 0) - 2(y + 0) + (z + 0) = 0$$

$$\Longrightarrow -x - 2y + z = 0$$
$$\Longrightarrow x + 2y - z = 0$$

Example 4.20 : Find the equation of the tangent planes of the sphere

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$ which are parallel to the plane $x - y + z = 0$.

Solution:

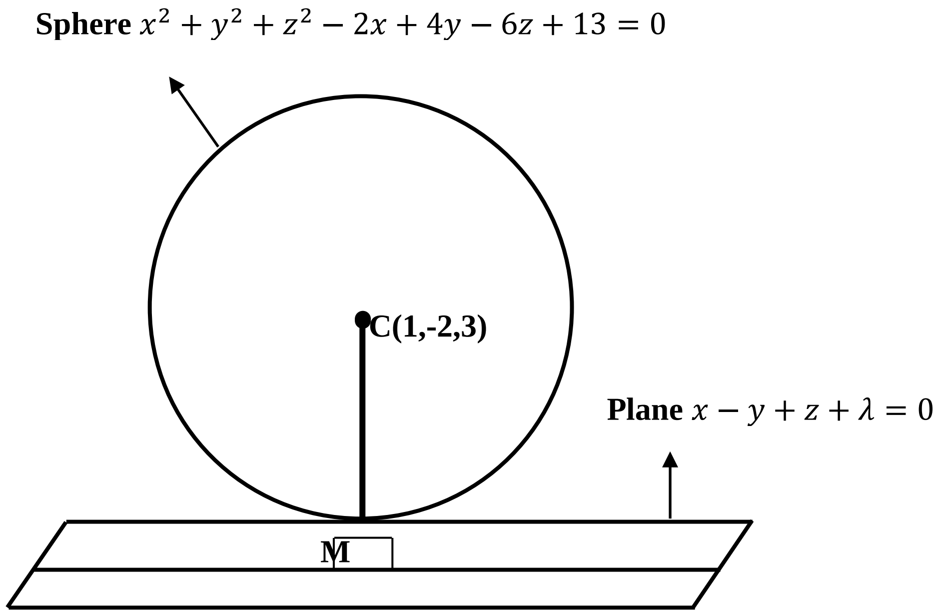


Figure 4.22

The equation of the given sphere is

$x^2 + y^2 + z^2 - 2x + 4y - 6z + 13 = 0$

Centre of the sphere $=\ \mathrm{C}(1, -2,3)$

Radius of the Sphere $= 3\sqrt{(1)^2 + (-2)^2 + (3)^2 - (13)} = 1$

Any plane parallel to $x - y + z = 0$ is given by

$x - y + z + \lambda = 0$ ……. **(1)**

If plane (1) is the tangent plane of the given sphere then

Radius of the sphere

$=$ Length of the Perpendicular from $\mathrm{C}(1, -2,3)$to the Plane $x - y + z + \lambda = 0$

$$\Rightarrow 1 = \pm \frac{1 + 2 + 3 + \lambda}{\sqrt{(1)^2 + (-1)^2 + (1)^2}} = \pm \frac{6 + \lambda}{\sqrt{3}}$$

$\Rightarrow \lambda = -6 \pm \sqrt{3}$

The required tangent planes are

$x - y + z - 6 \pm \sqrt{3} = 0$

Example 4.21: Find the equation of the tangent planes of the sphere at point of intersection of the sphere $x^2 + y^2 + z^2 = 49$ and the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{6}$.

Solution: Equation of the given line is

$\frac{x}{3} = \frac{y}{2} = \frac{z}{6} = \lambda(\text{say})$
……..**(1)**

Any point on the line is given by $(3\lambda, 2\lambda, 6\lambda)$

If the line **(1)** intersect the sphere then the point $(3\lambda, 2\lambda, 6\lambda)$ satisfied the equation of the sphere for some value of λ.

$\Longrightarrow 9\lambda^2 + 4\lambda^2 + 36\lambda^2 = 49$

$\Longrightarrow \lambda = \pm 1$

The point of intersection are $(3, 2,6)$ and $(-3, -2, -6)$

Equation of tangent plane at $(3, 2,6)$ is given by

$3x + 2y + 6z = 49$

Equation of tangent plane at $(-3, -2, -6)$is given by

$-3x - 2y - 6z = 49$

Example 4.22: Show that the line $\frac{x-1}{2} = \frac{y}{1} = \frac{z}{2}$ intersect the sphere $x^2 + y^2 + z^2 + 2x + 2y + z = 0$ at points $(-1, -1, -2)$ and $\left(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$.

Solution: Equation of the given line is

$\frac{x-1}{2} = \frac{y}{1} = \frac{z}{2} = \lambda(\text{say})$
……..**(1)**

Any point on the line is given by $(2\lambda + 1, \lambda, 2\lambda)$

If the line **(1)** intersects the sphere then the point $(2\lambda + 1, \lambda, 2\lambda)$satisfied the equation of the sphere for some value of λ.

$\Longrightarrow 4\lambda^2 + 4\lambda + 1 + \lambda^2 + 4\lambda^2 + 4\lambda + 2 + 2\lambda + 2\lambda = 0$

$\Longrightarrow 9\lambda^2 + 12\lambda + 3 = 0$

$\Rightarrow 3\lambda^2 + 4\lambda + 1 = 0$

$\Rightarrow \lambda = -1, -\frac{1}{3}$

Hence the required point of intersection are $(-1,-1,-2)$ and $\left(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$.

Example 4.23: Find the equation of the sphere whose centre at origin and which touch the line

$\frac{x-1}{2} = \frac{y}{1} = \frac{z}{2}$.

Solution :

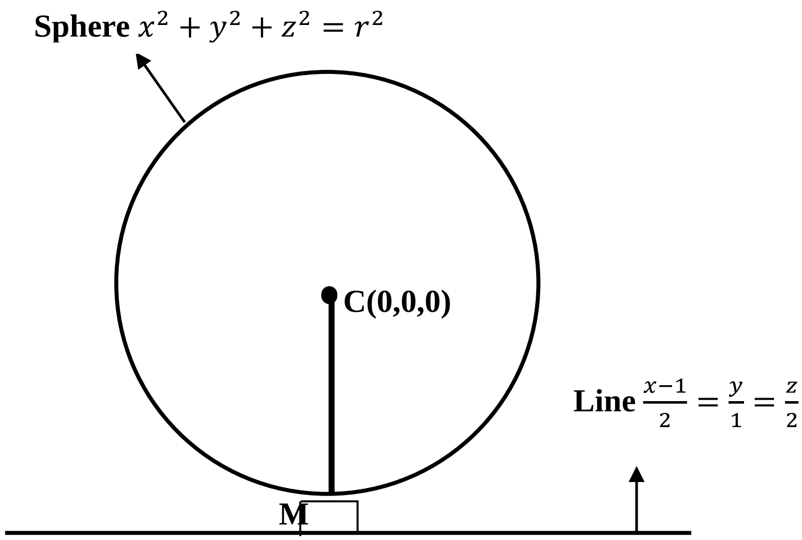


Figure 4.23

The equation of the sphere with centre at origin is given by

$x^2 + y^2 + z^2 = r^2$

The equation of the given line

$\frac{x-1}{2} = \frac{y}{1} = \frac{z}{2} = \lambda(\text{say})$
.... **(1)**

Any point on the line **(1)** is given by $(2\lambda + 1, \lambda, 2\lambda)$

Let $(2\lambda + 1, \lambda, 2\lambda)$ is the point of contact M then

Direction ratios of CM are $2\lambda + 1, \quad \lambda, \quad 2\lambda$

CM is perpendicular to the given line

$\Rightarrow 2(2\lambda + 1) + 1(\lambda) + 2(2\lambda) = 0$

$$\Longrightarrow 4\lambda + 2 + \lambda + 4\lambda = 0$$

$$\Longrightarrow \lambda = -\frac{2}{9}$$

Now the point of contact is $M(\frac{5}{9}, -\frac{2}{9}, -\frac{4}{9})$

Radius of the sphere $= r = CM = \sqrt{\left(\frac{5}{9}\right)^2 + \left(-\frac{2}{9}\right)^2 + \left(-\frac{4}{9}\right)^2} = \frac{\sqrt{5}}{3}$

The required equation of the sphere is

$$x^2 + y^2 + z^2 = \left(\frac{\sqrt{5}}{3}\right)^2$$

$$\Longrightarrow x^2 + y^2 + z^2 = \frac{5}{9}$$

Example 4.24: Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which passes through the line $48-3x = 2y + 30 = 3z$.

Solution:

Sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$

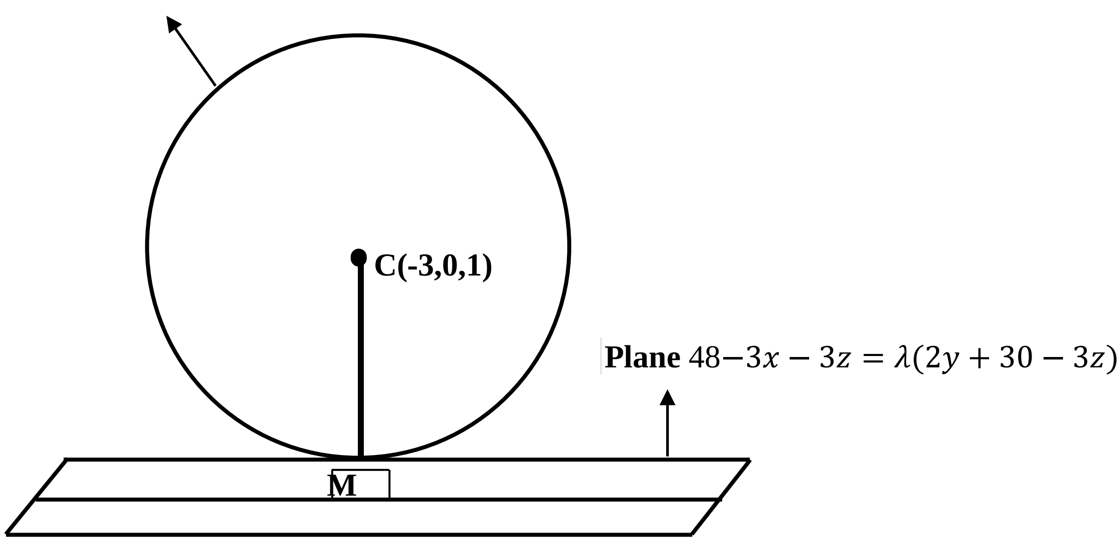


Figure 4.24

Equation of the given line is

$48-3x = 2y + 30 = 3z$
…… **(1)**

Equation of the plane through the line **(1)** is given by

$48-3x - 3z = \lambda(2y + 30 - 3z)$

$\Longrightarrow\ -3x - 2\lambda y + (3\lambda - 3)z - 30\lambda + 48 = 0$
…… **(P)**

If the plane **(P)** is a tangent plane of the sphere **S**

Then,

Radius of the Sphere

= Length of the Perpendicular from $\mathbf{C(-3, 0, 1)}$ to the Plane

$$3 = \frac{9 + 3\lambda - 3 - 30\lambda + 48}{\sqrt{(-3)^2 + (-2\lambda)^2 + (3\lambda - 3)^2}}$$

$$\Longrightarrow 3 = \frac{54 - 27\lambda}{\sqrt{18 + 13\lambda^2 - 18\lambda}}$$

$$\Longrightarrow 1 = \frac{18 - 9\lambda}{\sqrt{18 + 13\lambda^2 - 18\lambda}}$$

$\Longrightarrow 13\lambda^2 - 18\lambda + 18 = (18 - 9\lambda)^2$

$\Longrightarrow 13\lambda^2 - 18\lambda + 18 = 324 + 81\lambda^2 - 324\,\lambda$

$\Longrightarrow 306 + 68\lambda^2 - 306\lambda = 0$

$\Longrightarrow 2\lambda^2 - 9\lambda + 9 = 0$

$\Longrightarrow 2\lambda^2 - 9\lambda + 9 = 0$

$\Longrightarrow\ \lambda = 3, \dfrac{3}{2}$

The required equations of the tangent planes are

$-9x - 6y + 6z - 42 = 0$

$3x + 2y - 2z + 14 = 0$

Example 4.25: Find the equations of the tangent planes to the sphere

$(x-2)^2 + (y-1)^2 + (z-1)^2 = 1$ which passes through the x-axis.

Solution:

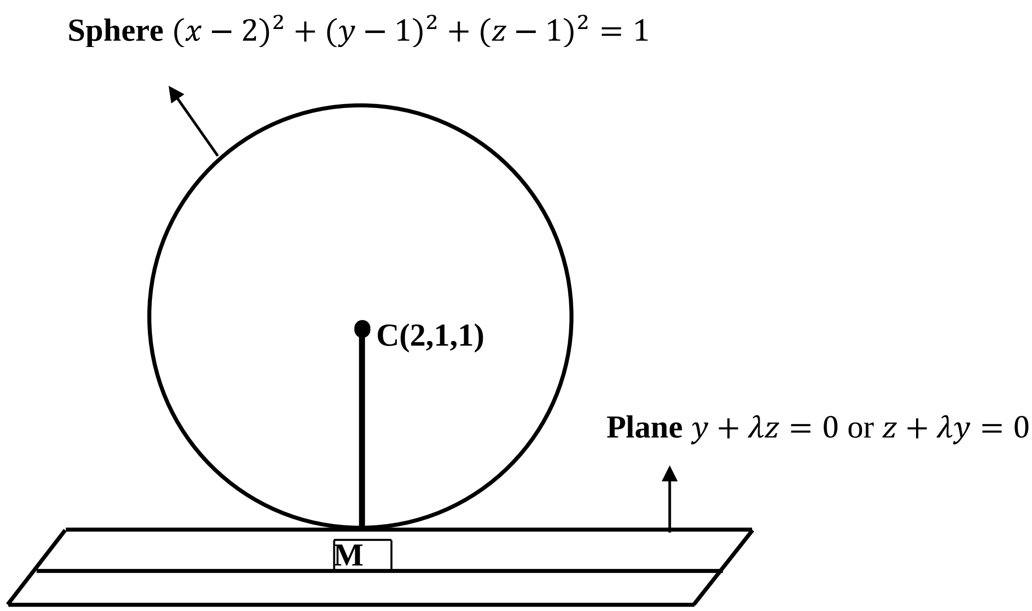


Figure 4.25

The equation of the given sphere is

$S \equiv (x-2)^2+(y-3)^2+(z-4)^2=1$

Equation of x-axis is given by

$y=0$ and $z=0$
…… **(1)**

Equation of the plane through the line **(1)** is given by

$y+\lambda z=0$
…… **(P)**

If the plane (**P)** is a tangent plane of the sphere S

Then,

Radius of the Sphere

$= \text{Length of the Perpendicular from } \mathbf{C}(\mathbf{2},\mathbf{1},\mathbf{1}) \text{ to the Plane}$

$$1=\frac{1+\lambda}{\sqrt{1+\lambda^2}}$$

$\Longrightarrow 1+\lambda^2=(1+\lambda)^2 \Longrightarrow \lambda=0$

The required equations of the tangent plane is $y=0$

In the similar way if we consider the plane through the line **(1)** in the form

$z + \lambda y = 0$

Then the equation of the tangent plane is $z = 0$

Hence the required tangents planes through the x-axis are $y = 0$ and $= 0$.

Example 4.26: Find the equations of the tangent planes to the sphere

$(x-2)^2 + (y-3)^2 + (z-4)^2 = 1$ which passes through the x-axis.

Solution:

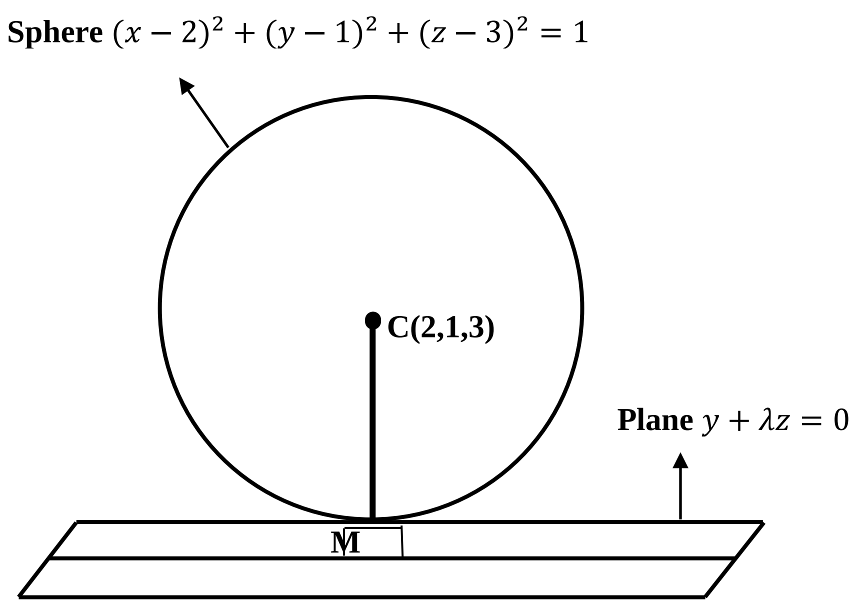


Figure 4.26

Equation of x-axis is given by

$\mathrm{y} = 0$ and $\mathrm{z} = 0$
…… **(1)**

Equation of the plane through the line **(1)** is given by

$y + \lambda z = 0$
…… **(P)**

If the plane **(P)** is a tangent plane of the sphere S

Then,

Radius of the Sphere

$=$ Length of the Perpendicular from $\mathbf{C(2,1,3)}$ to the Plane

$$1 = \frac{1+3\lambda}{\sqrt{1+\lambda^2}}$$

$$\Longrightarrow 1+\lambda^2 = (1+3\lambda)^2$$

$$\Longrightarrow 1+\lambda^2 = 1+9\lambda^2+6\lambda$$

$$\Longrightarrow \lambda(8\lambda+6) = 0$$

$$\Longrightarrow \lambda = 0, -\frac{3}{4}$$

The required equations of the tangent planes are

$y = 0,\ 4y - 3z = 0$

4.14 PLANE OF CONTACT

The plane of contact is the locus of the point of contact of the tangent plane which passes through a given point (not on the sphere)

To find the equation of the plane of contact of tangent plane through the point $Q(\alpha,\beta,\gamma)$ to the sphere $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

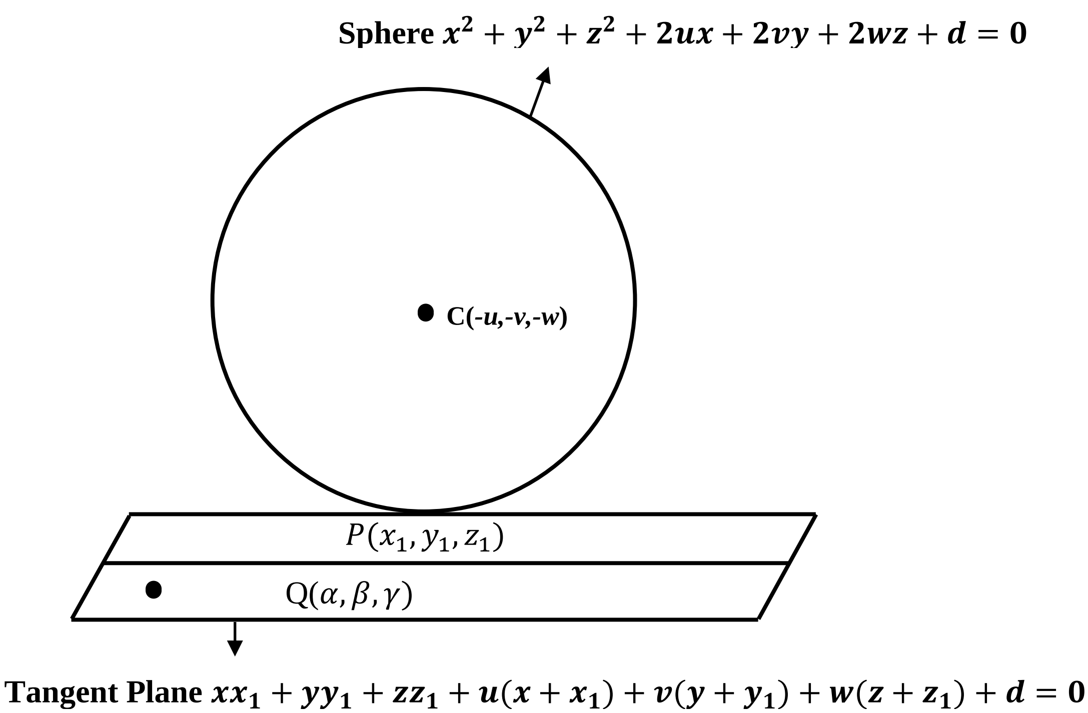


Figure 4.27

Equation of tangent plane through the point $P(x_1, y_1, z_1)$ of the sphere is given by

$xx_1 + yy_1 + zz_1 + u(x + x_1) + v(y + y_1) + w(z + z_1) + d = 0$……(P)

If the plane (P) passes through the point Q(α, β, γ) external to the sphere, then we have

$\alpha x_1 + \beta y_1 + \gamma z_1 + u(\alpha + x_1) + v(\beta + y_1) + w(\gamma + z_1) + d = 0$

Hence the locus of P $(x_1 \to x, y_1 \to y, z_1 \to z)$ is

$$\alpha x + \beta y + \gamma z + u(\alpha + x) + v(\beta + y) + w(\gamma + z) + d = 0$$

4.15 POLE AND POLAR PLANES

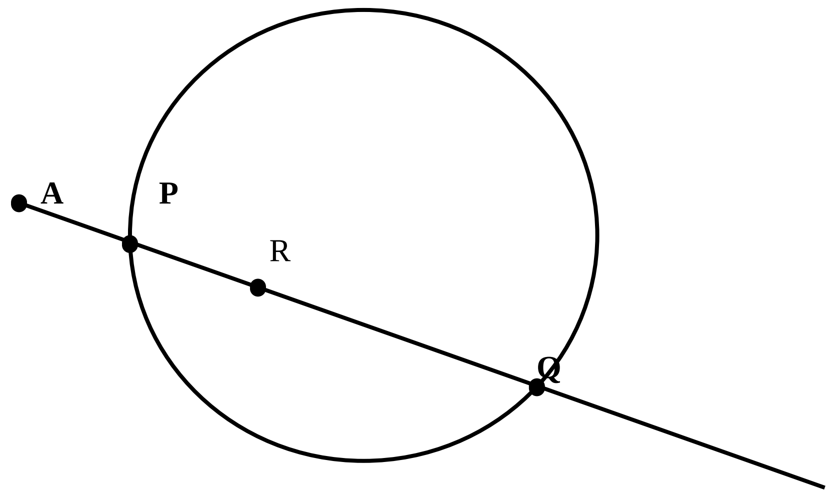


Figure 4.28

Consider a line through a fixed point A to intersect a given Sphere in the point P and Q. Take a point R on this line in such way that AR is harmonic mean of AP and AQ

i.e. $\frac{1}{\text{AP}} + \frac{1}{\text{AQ}} = \frac{2}{\text{AR}}$

The locus of the point R is called the **Polar Plane**. The fixed point A is called the pole of the polar plane.

4.16 THE EQUATION OF THE POLAR PLANE OF POINT A (x_1, y_1, z_1) WITH RESPECT TO THE SPHERE $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$

Let the equation of a line passes through the point $A(x_1, y_1, z_1)$ with direction cosines l,m,n is given by $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ ……..**(1)**

Any general point on the line is $(x_1 + lr, y_1 + mr, z_1 + nr)$

If the line intersect the sphere then the point $(x_1 + lr, y_1 + mr, z_1 + nr)$ satisfied the equation of sphere for some value of r

$\Longrightarrow (x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 = a^2$

$\Longrightarrow r^2 + 2r(x_1 l + y_1 m + z_1 n) + x_1^2 + y_1^2 + z_1^2 - a^2 = 0$

It is a quadratic equation in r give two roots say r_1 and r_2

$\Longrightarrow r_1 + r_2 = -2(x_1 l + y_1 m + z_1 n)$ and $r_1 r_2 = x_1^2 + y_1^2 + z_1^2 - a^2$

Now let the two point on sphere are P$(x_1 + lr_1, y_1 + mr_1, z_1 + nr_1)$ and Q$(x_1 + lr_2, y_1 + mr_2, z_1 + nr_2)$

Now, $AP = r_1$ and $AQ = r_2$

Now, by definition of Polar Plane

$$\frac{1}{AP} + \frac{1}{AQ} = \frac{2}{AR} \Longrightarrow \frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{AR} \Longrightarrow \frac{r_1 + r_2}{r_1 r_2} = \frac{2}{AR}$$

$$\Longrightarrow \frac{-2(x_1 l + y_1 m + z_1 n)}{x_1^2 + y_1^2 + z_1^2 - a^2} = \frac{2}{AR}$$

$\Longrightarrow x_1^2 + y_1^2 + z_1^2 - a^2 + \text{AR}(x_1 l + y_1 m + z_1 n) = 0$ …….**(2)**

Now, let the coordinate of the point R be (x,y,z)

Then, $AR = r$ (By equation (**1**) $x - x_1 = lr, y - y_1 = mr, z - z_1 = nr$ and $l^2 + m^2 + n^2 = 1$)

Now, by equation **(2)**

$x_1^2 + y_1^2 + z_1^2 - a^2 + r(x_1 l + y_1 m + z_1 n) = 0$ …… **(3)**

By Equation **(1)**

$$x = x_1 + lr \Longrightarrow l = \frac{x - x_1}{r}$$

$$y = y_1 + mr \Longrightarrow m = \frac{y - y_1}{r}$$

$$z = z_1 + nr \Longrightarrow n = \frac{z - z_1}{r}$$

Putting the value of l,m,n in equation **(3)**

$$x_1^2 + y_1^2 + z_1^2 - a^2 + r\left\{x_1\left(\frac{x - x_1}{r}\right) + y_1\left(\frac{y - y_1}{r}\right) + z_1\left(\frac{z - z_1}{r}\right)\right\} = 0$$

$$\Rightarrow x_1^2 + y_1^2 + z_1^2 - a^2 + \{x_1(x - x_1) + y_1(y - y_1) + z_1(z - z_1)\} = 0$$

$$\Rightarrow xx_1 + yy_1 + zz_1 = a^2$$

4.17 THE POLE OF THE POLAR $\mathbf{lx + my + nz = p}$ WITH RESPECT TO THE SPHERE $\mathbf{x^2 + y^2 + z^2 = a^2}$ is $\left(\frac{\mathbf{la^2}}{\mathbf{p}}, \frac{\mathbf{ma^2}}{\mathbf{p}}, \frac{\mathbf{na^2}}{\mathbf{p}}\right)$

Let $A(x_1, y_1, z_1)$ be the required pole. The equation of the polar plane of a point $A(x_1, y_1, z_1)$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$.

As $lx + my + nz = p$ and $xx_1 + yy_1 + zz_1 = a^2$ represent same polar plane.

Therefore,

$$\frac{l}{x_1} = \frac{m}{y_1} = \frac{n}{z_1} = \frac{p}{a^2} \Rightarrow x_1 = \frac{la^2}{p}, y_1 = \frac{ma^2}{p}, z_1 = \frac{na^2}{p}$$

4.18 THE POLAR LINE OF $\mathbf{\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}}$ WITH RESPECT TO THE SPHERE IS GIVEN BY $\mathbf{\alpha x + \beta y + \gamma z - a^2 = 0 = lx + my + nz}$

Any point on the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$ is given by $lr + \alpha, mr + \beta, nr + \gamma$

Polar plane of the point $(lr + \alpha, mr + \beta, nr + \gamma)$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is

$$x(lr + \alpha) + y(mr + \beta) + z(nr + \gamma) = a^2$$

$$\Rightarrow (\alpha x + \beta y + \gamma z - a^2) + r(lx + my + nz) = 0$$

This plane for all values of r passes through the line

$$\alpha x + \beta y + \gamma z - a^2 = 0 = lx + my + nz$$

4.19 ORTHOGONAL SYSTEM OF SPHERE

Two Spheres are said to intersect orthogonally if their angle of intersection is right angle.

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

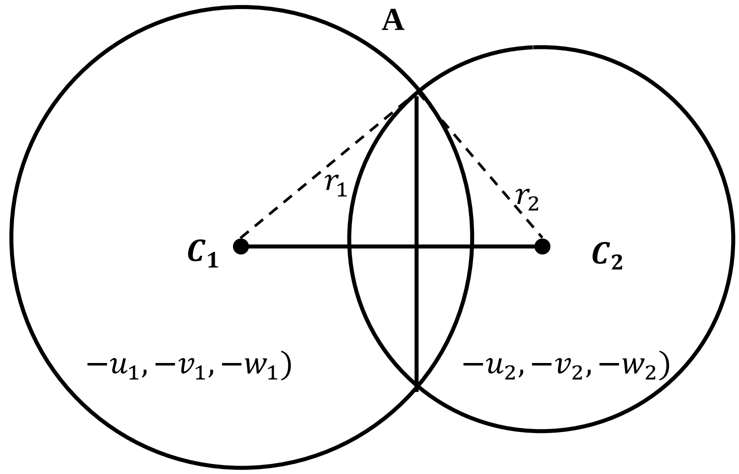


Figure 4.29

If the Sphere S_1 and S_2 intersect orthogonally then $\angle C_1AC_2 = 90°$

Now, in right angle triangle ΔC_1AC_2

$(C_1C_2)^2 = (C_1A)^2 + (C_2A)^2$

$\Longrightarrow (C_1C_2)^2 = (r_1)^2 + (r_2)^2$

$$\Longrightarrow (u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = ({u_1}^2 + {v_1}^2 + {w_1}^2 - d_1) + ({u_2}^2 + {v_2}^2 + {w_2}^2 - d_2)$$

$\Longrightarrow 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

4.20 TOUCHING SPHERES

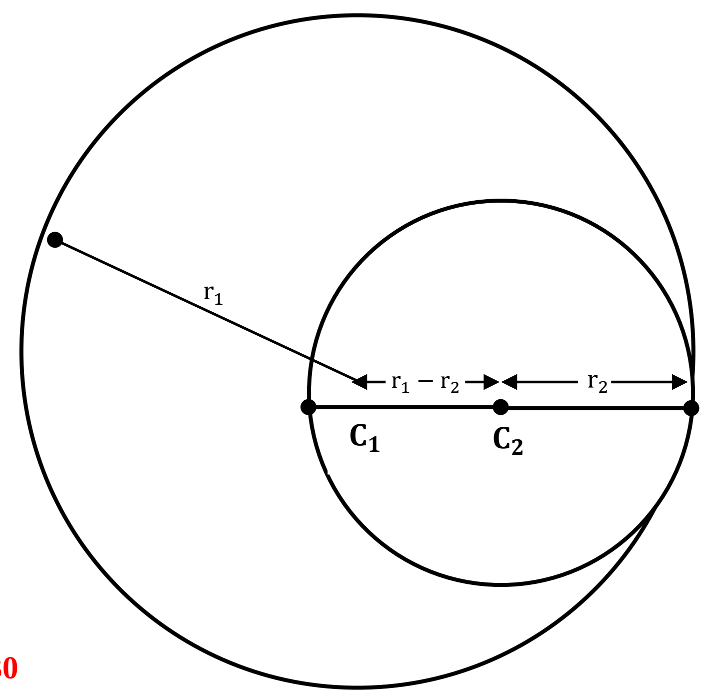


Figure 4.30

Touch Internally if $\mathbf{C_1C_2 = r_1 - r_2}$

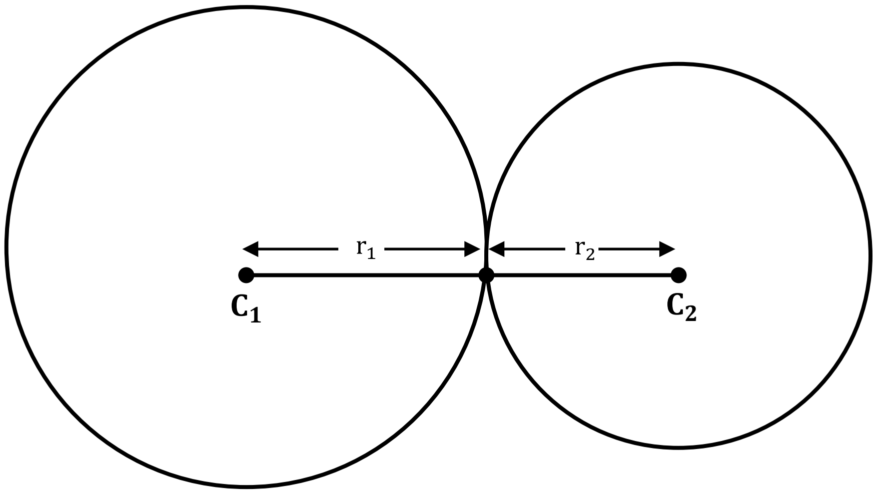


Figure 4.31

Touch Externally if $\mathbf{C_1C_2 = r_1 + r_2}$

Example 4.27: Show that the polar line of $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ with respect to the sphere

$x^2 + y^2 + z^2 = 9$ is given by $x + 2y + 3z - 9 = 0$, $2x + 3y + 4z = 0$.

Solution: Equation of the given line

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r$$

Any point on the line is given by $2r + 1, 3r + 2, 4r + 3$

Polar plane of the point $(2r + 1, 3r + 2, 4r + 3)$ with respect to the sphere $x^2 + y^2 + z^2 = 9$ is

$$x(2r + 1) + y(3r + 2) + z(4r + 3) = 9$$

$$\Longrightarrow (x + 2y + 3z - 9) + r(2x + 3y + 4z) = 0$$

This plane for all values of r passes through the line

$$x + 2y + 3z - 9 = 0; 2x + 3y + 4z = 0$$

This is the required equation of polar line of the given line.

Example 4.28: Show that the spheres $x^2 + y^2 + z^2 = 4$; $(x-4)^2 + (y-2)^2 + (z-4)^2 = 16$ touch externally and find point of contact.

Solution :

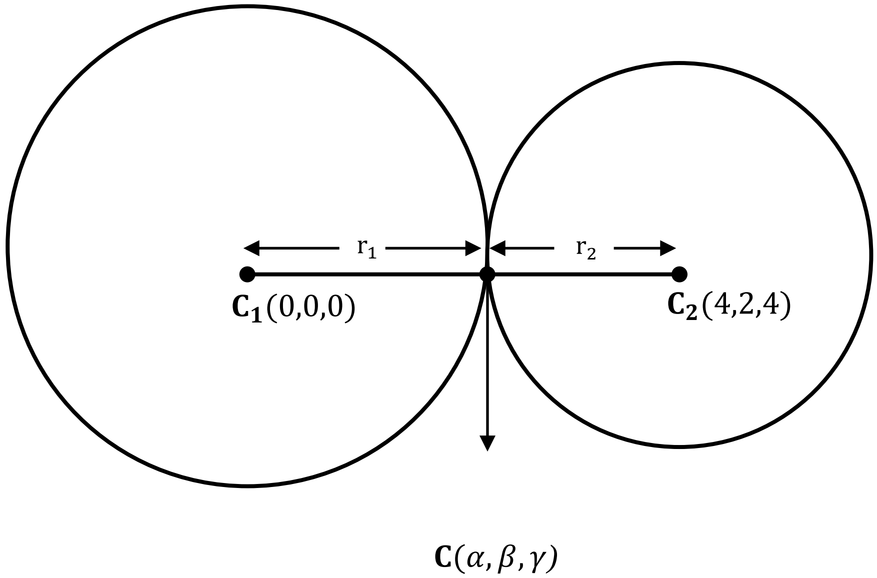


Figure 4.32

Equation of the first given sphere is

$S_1 \equiv x^2 + y^2 + z^2 = 4$

Radius of the sphere $r_1 = 2$

Centre of the sphere $= C_1(0,0,0)$

Equation of the second given sphere is

$(x-4)^2 + (y-2)^2 + (z-4)^2 = 16$

$S_2 \equiv x^2 + y^2 + z^2 - 8x - 4y - 8z + 20 = 0$

Radius of the sphere $r_2 = 4$

Centre of the sphere $= C_2(4,2,4)$

$C_1C_2 = \sqrt{(4-0)^2 + (2-0)^2 + (4-0)^2} = 6$

$r_1 + r_2 = 2 + 4 = 6$

$C_1C_2 = r_1 + r_2$

Hence the spheres touch externally.

Let $C(\alpha, \beta, \gamma)$ be the point of contact.

Hence,

$$\frac{\alpha - 0}{4 - 0} = \frac{\beta - 0}{2 - 0} = \frac{\gamma - 0}{4 - 0} = \lambda(say)$$

$\Rightarrow \alpha = 4\lambda, \beta = 2\lambda, \gamma = 4\lambda$

As $C(\alpha, \beta, \gamma)$ be the common point for both sphere

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 4$
……. **(1)**

$\alpha^2 + \beta^2 + \gamma^2 - 8\alpha - 4\beta - 8\gamma + 20 = 0$
……. **(2)**

By solving equation **(1)** and **(2)** we get

$-8\alpha - 4\beta - 8\gamma + 24 = 0$

$\Rightarrow 2\alpha + \beta + 2\gamma - 6 = 0$
……. **(3)**

By putting the value of α, β, γ in equation **(3)**

$8\lambda + 2\lambda + 8\lambda - 6 = 0$

$\Rightarrow 18\lambda = 6$

$\Rightarrow \lambda = \dfrac{1}{3}$

The required point of contact is $\left(\frac{4}{3}, \frac{2}{3}, \frac{4}{3}\right)$

Example 4.29: Show that the spheres

$x^2 + y^2 + z^2 = 25$ and $x^2 + y^2 + z^2 - 18x - 24y - 40z + 225 = 0$

touch externally and their point of contact is $\left(\frac{9}{5}, \frac{12}{5}, 4\right)$.

Solution:

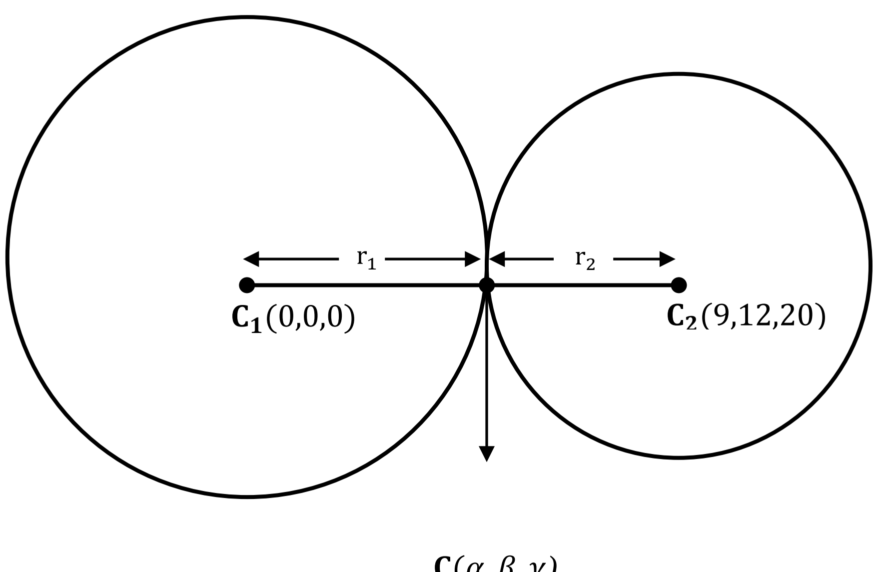


Figure 4.33

$x^2 + y^2 + z^2 = 25$ ....(S_1)

Radius of the sphere $r_1 = 5$

Centre of the sphere $= C_1(0,0,0)$

$x^2 + y^2 + z^2 - 18x - 24y - 40z + 225 = 0$

$(x-9)^2 + (y-12)^2 + (z-20)^2 = 400$ ....(S_2)

Radius of the sphere $r_2 = 20$

Centre of the sphere $= C_2(9,12,20)$

$C_1C_2 = \sqrt{(9-0)^2 + (12-0)^2 + (20-0)^2} = 25$

$r_1 + r_2 = 5 + 20 = 25$

$C_1C_2 = r_1 + r_2$

Hence the spheres touch externally.

Let $C(\alpha, \beta, \gamma)$ be the point of contact.

Hence,

$$\frac{\alpha - 0}{9 - 0} = \frac{\beta - 0}{12 - 0} = \frac{\gamma - 0}{20 - 0} = \lambda(say)$$

$\Longrightarrow \alpha = 9\lambda, \beta = 12\lambda, \gamma = 20\lambda$

As $C(\alpha, \beta, \gamma)$ be the common point for both sphere

$\Longrightarrow \alpha^2 + \beta^2 + \gamma^2 = 25$

$\alpha^2 + \beta^2 + \gamma^2 - 18\alpha - 24\beta - 40\gamma + 225 = 0$

By solving both the equation

we get,

$-18\alpha - 24\beta - 40\gamma + 250 = 0$

$\Longrightarrow 9\alpha + 12\beta + 20\gamma - 125 = 0$

By putting the value of α, β, γ

$81\lambda + 144\lambda + 400\lambda - 125 = 0$

$\Longrightarrow 625\lambda = 125$

$$\Longrightarrow \lambda = \frac{1}{5}$$

The required point of contact is $\left(\frac{9}{5}, \frac{12}{5}, 4\right)$

Example 4.30: Two spheres of radii r_1 and r_2 cut orthogonally prove that the radius of the common circle is $\frac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$.

Solution:

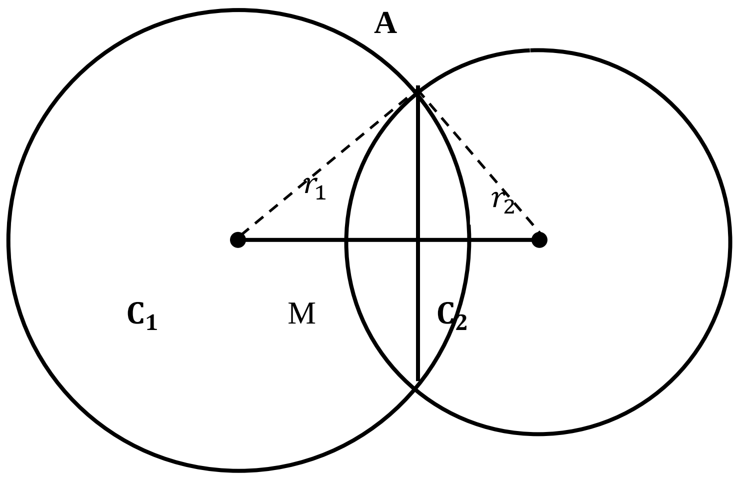


Figure 4.34

If the Spheres S_1 and S_2 intersect orthogonally then $\angle C_1AC_2 = 90°$

$\mathrm{C_1C_2} = \sqrt{r_1^2 + r_2^2}$

Area of the triangle $\Delta C_1AC_2 = \frac{1}{2}r_1r_2$

Let $AM = r$ be the radius of common circle

Then Area of the triangle $\Delta C_1AC_2 = \frac{1}{2}r\,(C_1C_2) = \frac{1}{2}r\sqrt{r_1^2 + r_2^2}$

By comparing the area of ΔC_1AC_2

$\frac{1}{2}r_1r_2 = \frac{1}{2}r\sqrt{r_1^2 + r_2^2}$

$\Longrightarrow r = \frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$

Example 4.31: Find the angle of intersection of the spheres $(x-1)^2 + (y-2)^2 + (z-3)^2 = 4$

and $(x-3)^2 + (y-1)^2 + (z+1)^2 = 9$.

Solution: The equations of the given spheres are

$S_1 \equiv (x-3)^2 + (y-1)^2 + (z+1)^2 = 9$

$S_2 \equiv (x-1)^2 + (y-2)^2 + (z-3)^2 = 4$

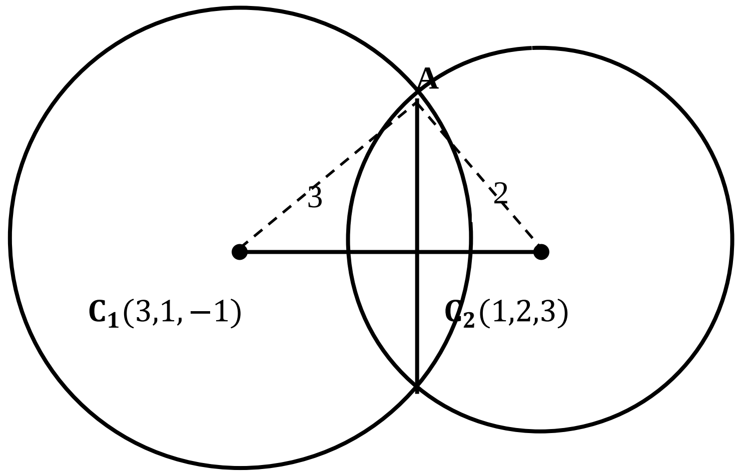


Figure 4.35

Let $\angle C_1AC_2 = \theta$ be the angle of intersection of the Spheres S_1 and S_2 .

Now,

$$C_1C_2 = \sqrt{(3-1)^2 + (1-2)^2 + (-1-3)^2} = \sqrt{21}$$

In ΔC_1AC_2

$$\cos\theta = \frac{3^2 + 2^2 - \left(\sqrt{21}\right)^2}{2 \times 3 \times 2} = -\frac{2}{3}$$

$$\Rightarrow \cos\theta = \frac{2}{3} \text{ (taking the accute angle)}$$

$$\Rightarrow \theta = cos^{-1}\left(\frac{2}{3}\right)$$

Example 4.32: Show that the two spheres $(x-0)^2 + (y+3)^2 + (z+1)^2 = 2$ and

$(x+3)^2 + (y+4)^2 + (z+2)^2 = 9$ are orthogonal.

Solution: The equations of the given spheres are

$$(x-0)^2 + (y+3)^2 + (z+1)^2 = \left(\sqrt{2}\right)^2$$

$$(x+3)^2 + (y+4)^2 + (z+2)^2 = (3)^2$$

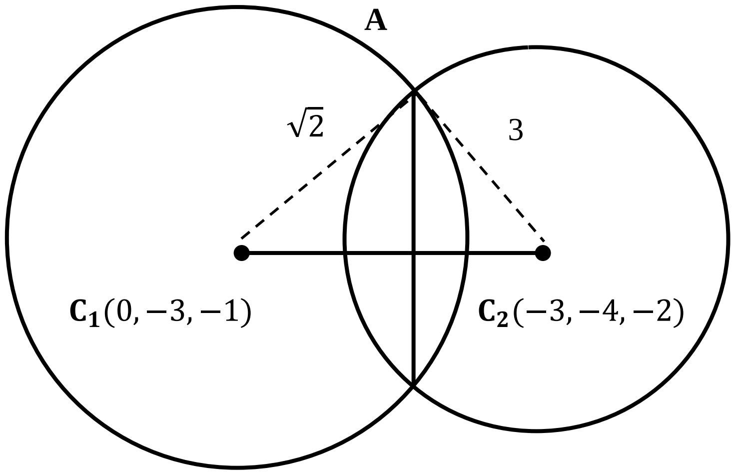


Figure 4.36

$C_1C_2 = \sqrt{(0+3)^2 + (-3+4)^2 + (-1+2)^2} = \sqrt{11}$

$(C_1C_2)^2 = (C_1A)^2 + (C_2A)^2$

$\Longrightarrow$ The spheres are orthogonal

Example 4.33: Show that the two spheres $x^2 + y^2 + z^2 + 2x + 2y + 1 = 0$ and

$x^2 + y^2 + z^2 + 4y - 2z + 3 = 0$ are orthogonal. Find their plane of intersection.

Solution: The equations of the given spheres are

$S_1 \equiv x^2 + y^2 + z^2 + 2x + 2y + 1 = 0$

$u_1 = 1, v_1 = 1, w_1 = 0, d_1 = 1$

$S_2 \equiv x^2 + y^2 + z^2 + 4y - 2z + 3 = 0$

$u_2 = 0, v_2 = 2, w_2 = -1, d_2 = 3$

Two spheres are orthogonal if

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = 0 + 4 + 0 = 4$

$d_1 + d_2 = 1 + 3 = 4$

$\Longrightarrow 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$

Hence the spheres S_1 and S_2 are orthogonal.

Plane of intersection is given by

$S_1 - S_2 = 0 \Longrightarrow 2x - 2y + 2z - 2 = 0$

$\Longrightarrow x - y + z - 1 = 0$

Example 4.34: Find the equation of the sphere that passes through the two points (0,0,0), (0,2,0) and cuts orthogonally the two spheres

$x^2 + y^2 + z^2 + 2\text{x} - 25 = 0$ and $x^2 + y^2 + z^2 - 4\text{z} - 8 = 0$.

Solution: Let the equation of the sphere be

$x^2 + y^2 + z^2 + 2\text{ux} + 2\text{vy} + 2\text{wz} + \text{d} = 0$
…….**(S)**

Sphere **(S)** passes through the points (0,0,0) and (0,2,0)

$\Longrightarrow \text{d} = 0$ and $\text{v} = -1$

Sphere **(S)** cuts the spheres $x^2 + y^2 + z^2 + 2\text{x} - 25 = 0$ orthogonally

By applying the condition of orthogonality

$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2 \Longrightarrow 2u(1) = 0 - 25 \Longrightarrow u = -\frac{25}{2}$

Again, sphere **(S)** cuts the sphere $x^2 + y^2 + z^2 - 4\text{z} - 8 = 0$ orthogonally

$\Longrightarrow 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2 \Longrightarrow 2w(-2) = 0 - 8 \Longrightarrow w = 2$

The required equation of the sphere is

$x^2 + y^2 + z^2 - 25\text{x} - 2\text{y} + 4\text{z} = 0$

Check Your Progress

1. Prove that the polar plane of any point on the line $\frac{x}{2} = \frac{y-1}{3} = \frac{z+3}{4}$ with respect to the sphere $x^2 + y^2 + z^2 = 1$ passes through the line $\frac{2x+3}{13} = \frac{y-1}{-3} = \frac{z}{-1}$.

2. Find the equation of the tangent plane of the sphere $x^2 + y^2 + z^2 = 9$ at $(1, -2, 2)$.

Ans. $x - 2y + 2z = 9$.

3. Show that the spheres $x^2 + y^2 + z^2 + 2\text{x} - 25 = 0$ and $x^2 +$

$y^2 + z^2 - 25x - 2y + 4z = 0$ are orthogonal.

4. Show that the spheres $x^2 + y^2 + z^2 - 4z - 8 = 0$ and $x^2 + y^2 + z^2 - 25x - 2y + 4z = 0$ are orthogonal.

5. Show that the spheres $x^2 + y^2 + z^2 + 6y + 2z + 8 = 0$ and $x^2 + y^2 + z^2 + 6x + 8y + 4z + 20 = 0$ are orthogonal. Find their plane of intersection.

Ans. $3x + y + z + 6 = 0.$

6. Find the angle of intersection of two intersecting spheres $x^2 + y^2 + z^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + z^2 + 4y - 2z + 2 = 0$.

Ans. $\theta = cos^{-1}\left(\frac{1}{2\sqrt{3}}\right)$

7. Find the angle of intersection of two intersecting spheres $x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$ and $x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$.

Ans. $\theta = cos^{-1}\left(\left|\frac{2u_1u_2 + 2v_1v_2 + 2w_1w_2 - d_1 - d_2}{2\sqrt{u_1^2 + v_1^2 + w_1^2 - d_1}\sqrt{u_2^2 + v_2^2 + w_2^2 - d_2}}\right|\right)$

4.21 THE LENGTH OF THE TANGENT AND POWER OF A POINT

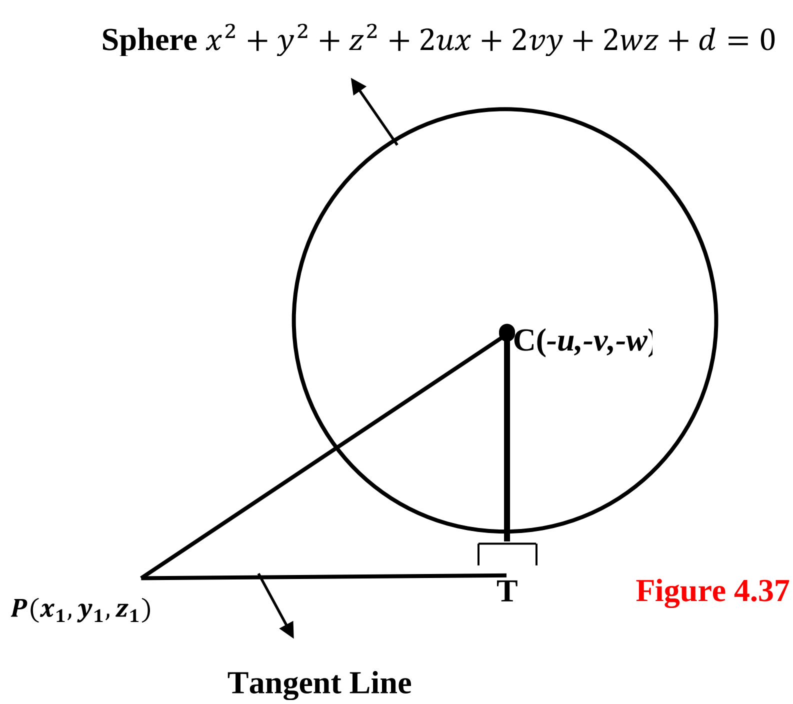


Figure 4.37

$\text{Centre of Sphere} = C(-u, -v, -w)$

$\text{Radius} = CT = \sqrt{u^2 + v^2 + w^2 - d}$

T is the point of contact.

$CT \perp PT$

$PT^2 = PC^2 - CT^2$

$PT^2 = \{(x_1 + u)^2 + (y_1 + v)^2 + (z_1 + w)^2\} - (u^2 + v^2 + w^2 - d)$

$PT^2 = x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d$

PT^2 is also known as the **Power of the point** P with respect to the given sphere.

4.22 THE RADICAL PLANE OF TWO SPHERES

The locus of a point whose powers with respect to two given spheres are the same is called the radical plane of the two spheres.

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

Let $P(\alpha, \beta, \gamma)$ be any point.

The power of the point $P(\alpha, \beta, \gamma)$ with respect to the sphere $S_1 = 0$ is given by

$$(PT_1)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2u_1\alpha + 2v_1\beta + 2w_1\gamma + d_1$$

and the power of the point $P(\alpha, \beta, \gamma)$ with respect to the sphere $S_2 = 0$ is given by

$$(PT_2)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2u_2\alpha + 2v_2\beta + 2w_2\gamma + d_2$$

For radical plane

$$(PT_1)^2 = (PT_2)^2$$

$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + 2u_1\alpha + 2v_1\beta + 2w_1\gamma + d_1 = \alpha^2 + \beta^2 + \gamma^2 + 2u_2\alpha + 2v_2\beta + 2w_2\gamma + d_2$

$\Rightarrow 2u_1\alpha + 2v_1\beta + 2w_1\gamma + d_1 = 2u_2\alpha + 2v_2\beta + 2w_2\gamma + d_2$

$\Rightarrow 2\alpha(u_1 - u_2) + 2\beta(v_1 - v_2) + 2\gamma(w_1 - w_2) + (d_1 - d_2) = 0$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$2x(u_1 - u_2) + 2y(v_1 - v_2) + 2z(w_1 - w_2) + (d_1 - d_2) = 0$

or $S_1 - S_2 = 0$

This is the required equation of radical plane of two spheres.

4.23 THE RADICAL AXIS (RADICAL LINE) OF THREE SPHERES

The radical planes of three spheres taken two at a time pass through a common line which is said to be the radical axis (or radical line) of the three spheres.

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0 \qquad \text{..... (1)}$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0 \qquad \text{..... (2)}$$

$$S_3 \equiv x^2 + y^2 + z^2 + 2u_3x + 2v_3y + 2w_3z + d_3 = 0 \qquad \text{..... (3)}$$

Radical plane of S_1 and S_2 is given by $S_1 - S_2 = 0$ ......($\mathbf{P_1}$)

Radical plane of S_2 and S_3 is given by $S_2 - S_3 = 0$ ...($\mathbf{P_2}$)

Radical plane of S_3 and S_1 is given by $S_3 - S_1 = 0$ ....($\mathbf{P_3}$)

Plane **($\mathbf{P_1}$)** and **($\mathbf{P_2}$)** represent a line $S_1 = S_2 = S_3$

Plane **($\mathbf{P_2}$)** and **($\mathbf{P_3}$)** represent a line $S_1 = S_2 = S_3$

Plane **($\mathbf{P_3}$)** and **($\mathbf{P_1}$)** represent a line $S_1 = S_2 = S_3$

Clearly these three planes pass through the line $S_1 = S_2 = S_3$ which is the equation of the radical line (or radical axis) of three given spheres.

4.24 COAXIAL SYSTEM OF SPHERES

A family of spheres is called a coaxial system of spheres if for all the spheres any two of them have the same radical plane.

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0$$

The radical plane of the spheres $S_1 = 0$ and $S_2 = 0$ is given by

$S_1 - S_2 = 0$

The equation of the co-axial system of spheres determined by the spheres $S_1 = 0$ and $S_2 = 0$ is given by the following different three ways

$S_1 + \lambda(S_1 - S_2) = 0$ or $S_2 + \mu(S_1 - S_2) = 0$ or $S_1 + \nu S_2 = 0$.

4.25 LIMITING POINTS OF A CO-AXIAL SYSTEM OF SPHERES

The centres of the spheres of a co-axial system which have zero radius are called limiting points of the co-axial system.

Example 4.35: Find the length of the tangent and power of the point P(6,6,5) with respect to the sphere

$x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0.$

Solution:

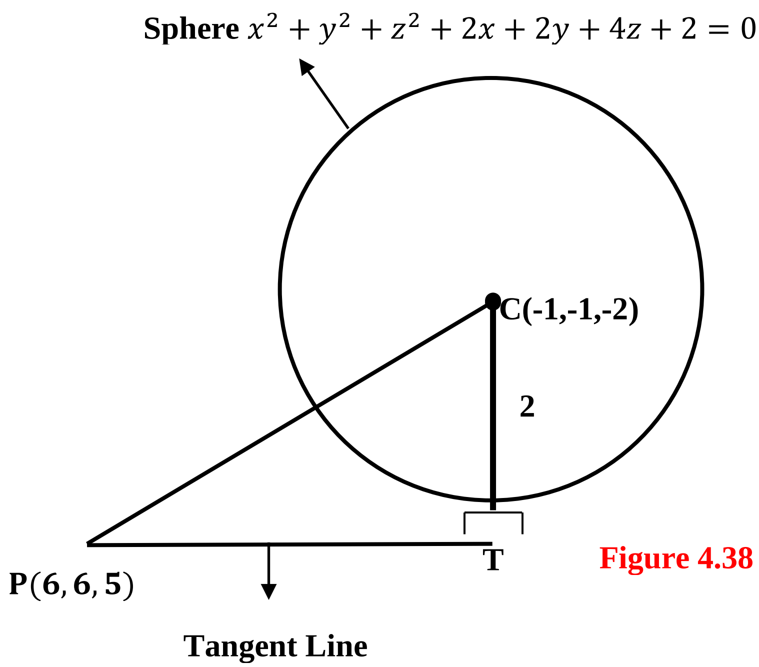


Figure 4.38

Equation of the given sphere

$$x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$$

Centre of Sphere $= \mathrm{C}(-1, -1, -2)$

Radius $= \mathrm{CT} = \sqrt{(-1)^2 + (-1)^2 + (-2)^2 - 2} = 2$

T is the point of contact.

$\mathrm{CT} \perp \mathrm{PT}$

$\mathrm{PT}^2 = \mathrm{PC}^2 - \mathrm{CT}^2$

$\mathrm{PT}^2 = \{(6+1)^2 + (6+1)^2 + (5+2)^2\} - (2)^2$

$\mathrm{PT}^2 = 143$

$\mathrm{PT} = \sqrt{143}$

Hence the length of the tangent is $\mathrm{PT} = \sqrt{143}$ and power of the point P(6,6,5) with respect to the given sphere is $\mathrm{PT}^2 = 143$.

Example 4.36: Find the radical plane of the spheres $x^2 + y^2 + z^2 + 4x + 6y + 7z + 8 = 0$ and $x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$.

Solution: Equation of the given spheres are

$$S_1 \equiv x^2 + y^2 + z^2 + 4x + 6y + 7z + 8 = 0$$

and $S_2 \equiv x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$

Radical plane of the given spheres is given by

$\Rightarrow S_1 - S_2 = 0$

$$\Rightarrow (x^2 + y^2 + z^2 + 4x + 6y + 7z + 8) - (x^2 + y^2 + z^2 + 2x + 2y + 4z + 2) = 0$$

$\Rightarrow 2x + 4y + 3z + 6 = 0$

Example 4.37: Find the equation of the radical axis of the spheres

$S_1 \equiv x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$

$S_2 \equiv x^2 + y^2 + z^2 + 4x + 4z + 4 = 0$

$S_3 \equiv x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0$

Solution: The radical plane of the spheres $S_1 = 0$ and $S_2 = 0$ is given by

$S_1 - S_2 = 0$

$\Rightarrow -2x + 2y - 2z - 2 = 0$

$\Rightarrow x - y + z + 1 = 0$

Again the radical plane of the spheres $S_1 = 0$ and $S_3 = 0$ is given by

$S_1 - S_3 = 0$

$\Rightarrow x - 4y + 6z + 4 = 0$

The equation of required radical axes is given by

$x - y + z + 1 = 0; x - 4y + 6z + 4 = 0$

Example 4.38: Find the limiting points of the co-axial system of spheres determined by the spheres

$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0, x^2 + y^2 + z^2 - 6y - 6z + 6 = 0.$

Solution: The equations of the given spheres are

$S_1 \equiv x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$

$S_2 \equiv x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$

The radical plane of the spheres $S_1 = 0$ and $S_2 = 0$ is given by

$S_1 - S_2 = 0$

$\Rightarrow 3x + 3y + 6z = 0$

$\Longrightarrow x + y + 2z = 0$

The equation of radical plane is

$P \equiv x + y + 2z = 0$

The equation of the co-axial system of spheres determined by the spheres $S_1 = 0$ and $S_2 = 0$ is given by

$S_1 + \lambda P = 0$

$\Longrightarrow (x^2 + y^2 + z^2 + 3x - 3y + 6) + \lambda(x + y + 2z) = 0$

$\Longrightarrow x^2 + y^2 + z^2 + (3 + \lambda)x + (-3 + \lambda)y + 2\lambda z + 6 = 0$

Centre of the sphere is $\left(\frac{-3-\lambda}{2}, \frac{3-\lambda}{2}, -\lambda\right)$

$$\text{Radius} = \sqrt{\left(\frac{-3-\lambda}{2}\right)^2 + \left(\frac{3-\lambda}{2}\right)^2 + (-\lambda)^2 - 6}$$

$$= \frac{1}{2}\sqrt{(3+\lambda)^2 + (3-\lambda)^2 + 4\lambda^2 - 24} = \frac{1}{2}\sqrt{6\lambda^2 - 6}$$

If the radius of the sphere is zero, then

$$\frac{1}{2}\sqrt{6\lambda^2 - 6} = 0$$

$\Longrightarrow 6\lambda^2 - 6 = 0$
$\Longrightarrow \lambda^2 - 1 = 0$
$\Longrightarrow \lambda = 1, -1$
Putting $\lambda = 1$ in the co-ordinates of centre of the sphere, we get $(-2,1,-1)$
Again putting $\lambda = -1$ in the co-ordinates of centre of the sphere, we get $(-1,2,1)$
The required limiting points of the co-axial system of spheres are $(-2,1,-1)$ and $(-1,2,1)$.

4.26 SUMMARY

We conclude with summarizing what we have covered in this unit

- The definition of sphere
- A point lies on the boundary, inside or outside the sphere.
- Circle, Great circle and Small circle.
- Equation of a Sphere with centre at $C(u, v, w)$ and radius r
- Equation of a Sphere with centre at origin $O(0,0,0)$ and radius r
- General equation of the Sphere and determine its centre and radius

- The equation of the Sphere with a given diameter
- The equation of a circle and determine its centre and radius
- The great circle and find the equation of a sphere for which the circle is a great circle
- Intersection of two sphere
- Equation of a sphere passing through a circle
- A line does not intersect the sphere or intersect the sphere at two point or it is tangent line
- Equation of tangent planes
- Condition of tangency
- Equation of plane of contact
- Pole and polar planes
- The equation of the polar plane of a point A(x_1, y_1, z_1) with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$
- The equation of the polar plane of a point A(x_1, y_1, z_1) with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $xx_1 + yy_1 + zz_1 = a^2$
- The pole of the polar plane $lx + my + nz = p$ with respect to the sphere $x^2 + y^2 + z^2 = a^2$ is $\left(\frac{la^2}{p}, \frac{ma^2}{p}, \frac{na^2}{p}\right)$
- Condition that the two spheres are orthogonal
- Angle of intersection of two spheres
- The two spheres are touch internally or externally and find their point of contact
- The length of a tangent and power of a point
- The radical plane of two spheres
- The radical axis(radical line) of three spheres
- The Coaxial system of spheres
- The limiting points of a co-axial system of spheres

4.27 TERMINAL QUESTIONS

1. Find the equation of the sphere with centre at $(2,3,4)$ and which passes through the point$(1,2,8)$.

Ans. $(x-2)^2 + (y-3)^2 + (z-4)^2 = 18$

2. Find the equation of the sphere with centre at $(1,2,3)$ and radius 6.

Ans. $(x-1)^2 + (y-2)^2 + (z-3)^2 = 36$

3. Find the centre and radius of the sphere $x^2 + y^2 + z^2 - 4\text{x} - 6\text{y} - 2\text{z} + 5 = 0$.

Ans. Radius $= 3$ and Centre $= (2,3,1)$

4. Find the equation of the sphere on the join of $(2,4,6)$ and $(-2,-4,-6)$ as diameter.

Ans. $x^2 + y^2 + z^2 - 56 = 0$

5. Find the equation of the sphere with centre at (α, β, γ) and which touch the plane

$ax + by + cz + d = 0.$

Ans. $(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = \frac{(a\alpha + b\beta + c\gamma + d)^2}{a^2+b^2+c^2}$

6. Find the equation of the sphere with centre at $(0,0,0)$ and which passes through the point$(0, b, 0)$.

Ans. $x^2 + y^2 + z^2 = b^2$

7. Find the equation of the sphere with centre at $(0,0,0)$ and which passes through the point $(0,0,c)$.

Ans. $x^2 + y^2 + z^2 = c^2$

7. Find the equation of the sphere on the join of $(a, 0,0)$ and $(0,0,c)$ as diameter.

Ans. $x^2 + y^2 + z^2 - ax - cz = 0$

8. **(i)** Show that the point $P(2,1,2)$ lies on the sphere $x^2 + y^2 + z^2 = 9$.

(ii) Show that the point $Q(1,1,-4)$ lies inside the sphere $x^2 + y^2 + z^2 - 6x + 4y + 4z - 32 = 0$.

(iii) Show that the point $R(4,4,7)$ lies outside the sphere $x^2 + y^2 + z^2 + 2x + 2y - 4z - 19 = 0$.

9. Find the centre and radius of the circle $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$, $x + y + z - 3 = 0$.

Ans. Centre of the circle $= (0,1,2)$, Radius of the circle $= \sqrt{13}$

10. Find the equation of the sphere which passes through the point (α, β, γ) and the circle $x^2 + y^2 + z^2 = a^2, z = 0$.

Ans. $(x^2 + y^2 + z^2 - a^2)\gamma - (\alpha^2 + \beta^2 + \gamma^2 - a^2)z = 0$.

11. Find the equation of the sphere for which the circle

$x^2 + y^2 + z^2 + 2x - 8 = 0, 2x + 2y + z + 8 = 0$ is a great circle.

Ans. $x^2 + y^2 + z^2 + \frac{14x}{3} + \frac{8y}{3} + \frac{4z}{3} + \frac{8}{3} = 0$

12. Find the equation to the plane in which the circle of intersection of the spheres

$x^2 + y^2 + z^2 + 7x + 16y + 9z + 18 = 0$ and $x^2 + y^2 + z^2 + 4x + y + 3z + 8 = 0$ lies.

Ans. $3x + 15y + 6z + 10 = 0$

13. Prove that the circle $(x-4)^2 + (y+2)^2 + (z-2)^2 = 36$, $x - 2y + 2z = 12$ is a great circle.

14. Show that the equation of the circle whose centre is $(0,1,2)$ and which lies on the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$ is $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16,\ x + y + z - 3 = 0$.

15. Show that the line $x = y = z$ intersect the sphere $x^2 + y^2 + z^2 = a^2$ at the point $(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}})$ and $(-\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}}, -\frac{a}{\sqrt{3}})$.

16. Show that the line $\frac{x+2}{2} = \frac{y-3}{2} = \frac{z-3}{-1}$ is the tangent line of the sphere $x^2 + y^2 + z^2 + 2x - 2y - 2z - 6 = 0$ and find the point of contact.

Ans. $(-2,3,3)$

17. Find the equation of the tangent plane of the sphere $x^2 + y^2 + z^2 + 2x - 2y - 2z - 6 = 0$ at the point $(-2,3,3)$.

Ans. $x - 2y - 2z + 14 = 0$

18. Show that the plane $x - 2y - 2z + 14 = 0$ is a tangent plane of the sphere

$x^2 + y^2 + z^2 + 2x - 2y - 2z - 6 = 0$ and find the point of contact..

Ans. $(-2,3,3)$

19. Show that the plane $y = 0$ touches the sphere $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$ and find the point of contact.

Ans. Point of Contact (2,0,2).

20. Show that the plane $z = 0$ touches the sphere $(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$ and find the point of contact.

Ans. Point of Contact (2,2,0).

21. Show that the plane $2x - y - 22 = 16$ touches the sphere $x^2 + y^2 + z^2 - 4x + 2y + 2z - 3 = 0$, and find the point of contact.

Ans. $(\mathbf{4}, -\mathbf{2}, -\mathbf{3})$

22. Show that the spheres $x^2 + y^2 + z^2 + 2x + 4y + 6z + 7 = 0$ and $x^2 + y^2 + z^2 - 4x - 2y + 8z + 9 = 0$ are orthogonal.

23. Find the radical plane of the spheres $x^2 + y^2 + z^2 + 7x + 9y + 7z + 8 = 0$ and $x^2 + y^2 + z^2 + 2x + 2y + 4z + 2 = 0$.

Ans. $5x + 7y + 3z + 6 = 0$.

24. Find the equation of the radical axis of the spheres

$S_1 \equiv x^2 + y^2 + z^2 + 4x + 2y + 2z + 2 = 0$

$S_2 \equiv x^2 + y^2 + z^2 + 2x + y + z + 4 = 0$

$$S_3 \equiv x^2 + y^2 + z^2 + x + 3y - 4z - 2 = 0$$

Ans. $2x + y + z - 2 = 0 = 3x - y + 6z + 4$

4.28 FURTHER READINGS

1. Analytical Solid Geometry by Shanti Narayan and P.K. Mittal, Published by S. Chand & Company Ltd. 7th Edition.

2. A text book of Mathematics for BA/B.Sc Vol 1, by V Krishna Murthy & Others, Published by S. Chand & Company, New Delhi.

3. A text Book of Analytical Geometry of Three Dimensions, by P.K. Jain and Khaleel Ahmed, Published by Wiley Eastern Ltd., 1999.

4. Co-ordinate Geometry of two and three dimensions by P. Balasubrahmanyam, K.Y. Subrahmanyam, G.R. Venkataraman published by Tata-MC Gran-Hill Publishers Company Ltd., New Delhi.

5. Plane and solid Geometry by C.A. Hart, Published by Forgotten Books 2013.

6. The Project Gutenberg EBook of Solid Geometry with Problems and Applications by H. E. Slaught and N. J. Lennes, 2009

UNIT-5 CYLINDER

Structure

5.1 INTRODUCTION

Definition (Cylindrical Surface) 5.1:

A cylindrical surface is a surface generated by a moving straight line that continually intersects a fixed curve and remains parallel to a fixed straight line not coplanar with the given curve. The moving line is the generator, and the generator in any one of its positions is an element of the surface.

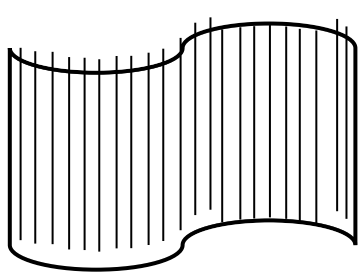


Figure 5.1 (Cylindrical Surface)

Definition (Cylinder) 5.2:

A solid bounded by a cylindrical surface and two parallel plane sections cutting all its elements is called a cylinder.

Definition (Right Section) 5.3:

A right section of a cylinder is made by a plane cutting each of its elements at right angles.

Definition (Circular Cylinders) 5.4:

A circular cylinder is one whose right section is a circle.

The radius of a circular cylinder is the radius of its right section.

Definition (Right Cylinder) 5.5:

A right cylinder is a cylinder whose elements are perpendicular to the bases **(Fig.5.2).**

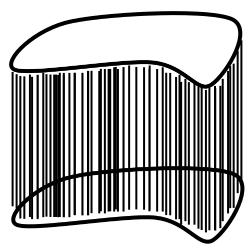


Fig. 5.2(Right Cylinder)

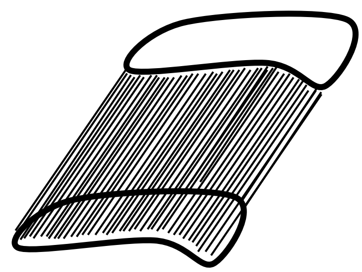


Fig. 5.3 (Oblique Cylinder)

Definition (Oblique Cylinder) 5.6:

An oblique cylinder is a cylinder whose elements are not perpendicular to the bases **(Fig.5.3).**

Definition (Right Circular Cylinder) 5.7:

A right circular cylinder is a right cylinder whose base is a circle.

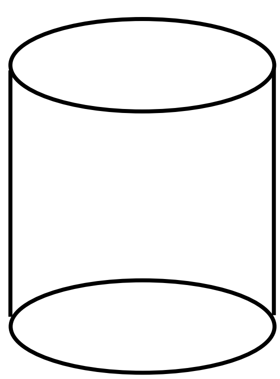


Fig. 5.4 (Right Circular Cylinder)

Definition (Oblique Circular Cylinder) 5.8:

An oblique circular cylinder is an oblique cylinder whose base is a circle.

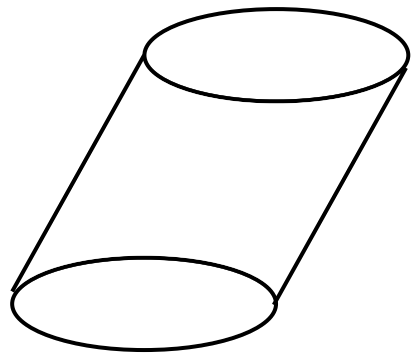


Fig. 5.5 (Oblique Circular Cylinder)

Note 5.1: If a right section of a cylinder is a circle then the cylinder is a circular cylinder. In more generality, if a right section of a cylinder is a conic section (parabola, ellipse, hyperbola) then the solid cylinder is said to be parabolic, elliptic and hyperbolic respectively.

Definition (Axis of a Cylinder) 5.9:

The line passing through the centers of two right sections of a circular cylinder is the axis of the cylinder.

A right circular cylinder may be generated by revolving a rectangle about one of its sides as an axis.

5.2 OBJECTIVES

After reading this unit, you should be able to

- Understand the quadratic equation in x, y, z.
- Understand the cylindrical surface, Cylinder, Right Cylinder, Oblique Cylinder, Right Circular Cylinder and Oblique Circular Cylinder.
- Find the equation of a cylinder with given base and generators are parallel to a fixed line
- Find the equation of a cylinder with given base and generators are parallel to a co-ordinate axis i.e. x-axis, y-axis and z-axis.
- Define Enveloping Cylinder

- Find the equation of the enveloping cylinder to the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of the enveloping cylinder to the surface $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of the enveloping cylinder to the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of a right circular cylinder of radius a whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l,m,n* are the direction cosines.

- Find the equation of a right circular cylinder of radius a whose axis is the line

  $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l,m,n* are the direction ratios.

- Understand ruled surface

- Understand Hyperboloid of one sheet

5.3 QUADRATIC EQUATION

The general equation of the second degree in *x, y, z is given by*

$$\boldsymbol{ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0}$$

with at least one of the coefficients *a, b, c, f, g* or h of the second-degree terms being non-zero.

5.3.1 EQUATION OF CONIC WITH THE INTERSECTION OF yz-PLANE, zx-PLANE AND xy-PLANE

(i) Intersection with the yz-plane ($x = 0$)

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0, x = 0$$

or

$$by^2 + cz^2 + 2fyz + 2vy + 2wz + d = 0, x = 0$$

(ii) Intersection with the zx-plane ($y = 0$)

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0, y = 0$$

or

$$ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0, y = 0$$

(iii) Intersection with the xy-plane ($z = 0$)

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0, z = 0$$

or

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0, z = 0$$

5.4 CYLINDER

Definition (**Cylinder**) **5.10**: A cylinder is a surface generated by a variable line which moves parallel to a fixed line and intersects a given curve or a touches a given surface.

The moving line is called **generator** and the curve which it intersect, is called the **guiding curve**.

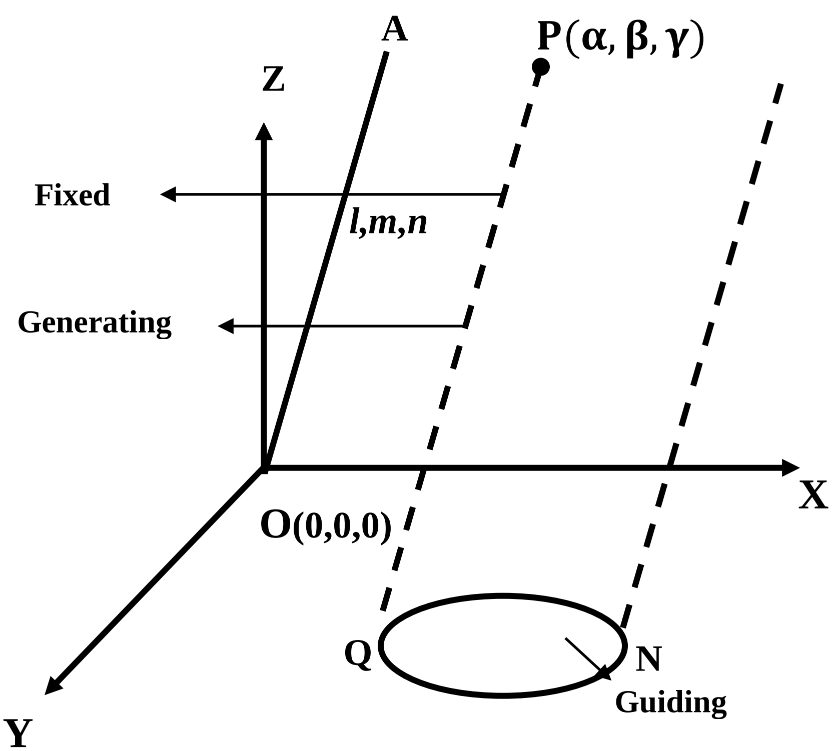


Figure: 5.6

5.5 EQUATION OF A CYLINDER WITH GIVEN BASE AND GENERATORS PARALLEL TO A FIXED LINE

Equation of the fixed line OA, passing through the origin O(0,0,0) with direction cosine *l,mn* is given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

…………………(5.1)

Also let the given conic QMN is given by

$$ax^2+by^2+2hxy+2ux+2vy+d=0, \qquad z=0$$

…………………(5.2)

Let $P(\alpha,\beta,\gamma)$ be any point on the surface of cylinder. The equation of generating line through the point $P(\alpha,\beta,\gamma)$ and parallel to the fixed line OA is given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

…………………(5.3)

Let the generating line **(5.3)** meets the plane $z=0$ in Q

Putting $z=0$ in **(5.3)** we get the coordinate of the point Q

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$$

As PQ is the generating line of the cylinder, the coordinate point $Q\left(\alpha-\frac{l\gamma}{n},\beta-\frac{m\gamma}{n},0\right)$ must satisfied the equation of conic **(5.2)**

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+b\left(\beta-\frac{m\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+2u\left(\alpha-\frac{l\gamma}{n}\right)+2v\left(\beta-\frac{m\gamma}{n}\right)+d=0$$

$$\Longrightarrow a(n\alpha-l\gamma)^2+b(n\beta-m\gamma)^2+2h(n\alpha-l\gamma)(n\beta-m\gamma)$$
$$+2un(n\alpha-l\gamma)+2vn(n\beta-m\gamma)+dn^2=0$$

Hence the locus of P $(\alpha\to x,\beta\to y,\gamma\to z)$ is

$$a(nx-lz)^2+b(ny-mz)^2+2h(nx-lz)(ny-mz)$$
$$+2un(nx-lz)+2vn(ny-mz)+dn^2=0$$

This is the required equation of cone.

5.6 EQUATION OF A CYLINDER WITH GIVEN BASE AND GENERATORS PARALLEL TO A CO-ORDINATE AXIS

Case (i) Generator of the cylinder parallel to the x-axis

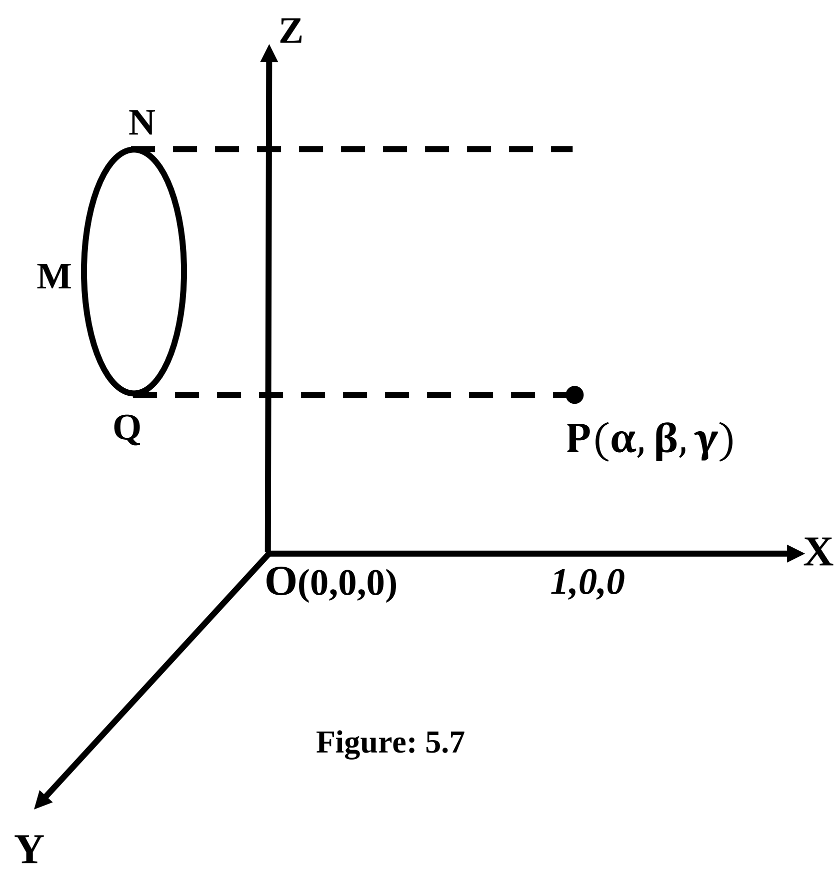


Figure: 5.7

Equation of x-axis is given by

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} \qquad \textbf{…………………(5.4)}$$

Also let the given conic QMN is given by

$$by^2 + cz^2 + 2fyz + 2vy + 2wz + d = 0, \qquad x = 0 \qquad \textbf{…………………(5.5)}$$

Let $P(\alpha, \beta, \gamma)$ be any point on the surface of cylinder. The equation of generating line through the point $P(\alpha, \beta, \gamma)$ and parallel to the fixed line (x-axis) is given by

$$\frac{x-\alpha}{1}=\frac{y-\beta}{0}=\frac{z-\gamma}{0}$$

…………………(5.6)

Let the generating line **(5.6)** meets the plane $x=0$ in Q

Putting $x=0$ in **(5.6)** we get the coordinate of the point Q

$$\frac{0-\alpha}{1}=\frac{y-\beta}{0}=\frac{z-\gamma}{0}$$

$$(0,\beta,\gamma)$$

As PQ is the generating line of the cylinder, the coordinate point $Q(0,\beta,\gamma)$ must satisfied the equation of conic **(5.5)**

$$b\beta^2+c\gamma^2+2f\beta\gamma+2v\beta+2w\gamma+d=0$$

Hence the locus of P $(\alpha\to x,\beta\to y,\gamma\to z)$ is

$$by^2+cz^2+2fyz+2vy+2wz+d=0$$

This is the required equation of cone.

Case (ii) Generator of the cylinder parallel to the y-axis

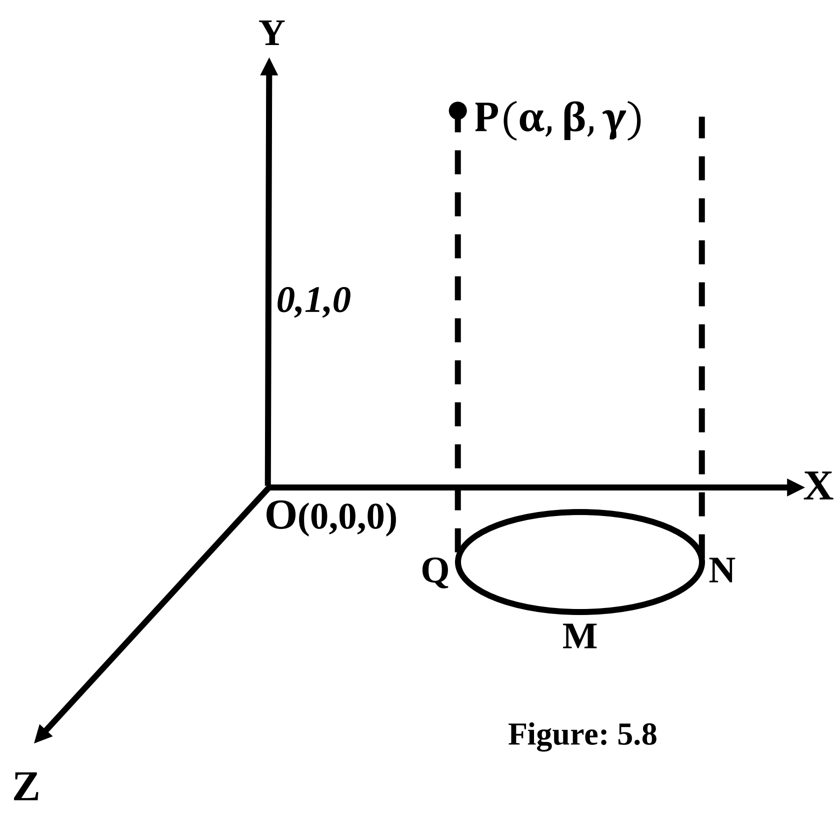


Figure: 5.8

Equation of y-axis is given by

$$\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$$

…………………(5.7)

Also let the given conic QMN is given by

$$ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0, \qquad y = 0$$
…………………(5.8)

Let $P(\alpha, \beta, \gamma)$ be any point on the surface of cylinder. The equation of generating line through the point $P(\alpha, \beta, \gamma)$ and parallel to the fixed line (x-axis) is given by

$$\frac{x-\alpha}{0} = \frac{y-\beta}{1} = \frac{z-\gamma}{0}$$
…………………(5.9)

Let the generating line **(5.9)** meets the plane $y = 0$ in Q

Putting $y = 0$ in **(5.9)** we get the coordinate of the point Q

$$\frac{x-\alpha}{0} = \frac{0-\beta}{1} = \frac{z-\gamma}{0}$$

$$(\alpha, 0, \gamma)$$

As PQ is the generating line of the cylinder, the coordinate point $Q(\alpha, 0, \gamma)$ must satisfied the equation of conic **(5.8)**

$$a\alpha^2 + c\gamma^2 + 2g\gamma\alpha + 2u\alpha + 2w\gamma + d = 0$$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$$ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0$$

This is the required equation of cone.

Case (iii) Generator of the cylinder parallel to the z-axis

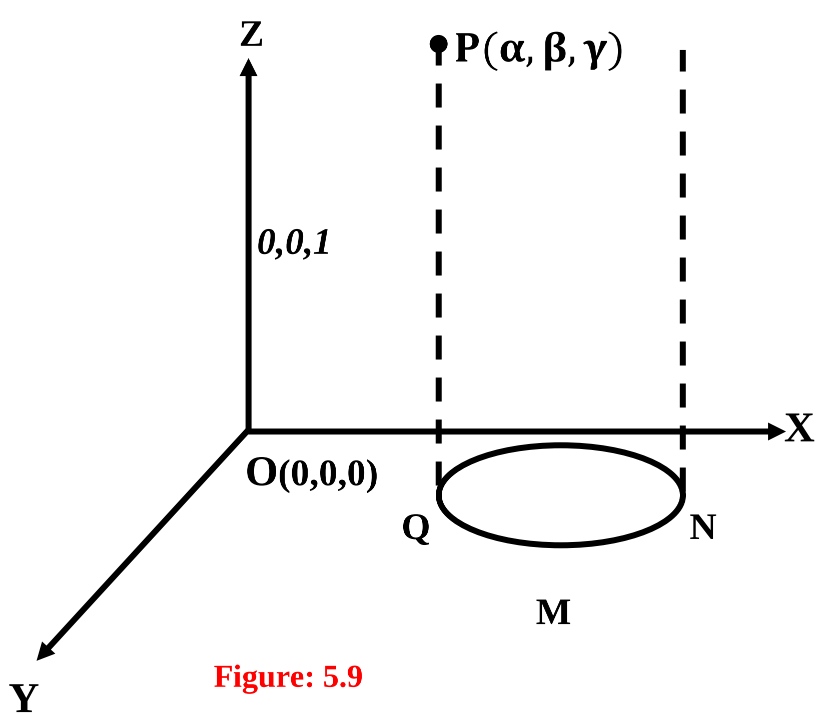


Figure: 5.9

Equation of z-axis is given by

$$\frac{\boldsymbol{x}}{\mathbf{0}} = \frac{\boldsymbol{y}}{\mathbf{0}} = \frac{\boldsymbol{z}}{\mathbf{1}}$$ **…………………(5.10)**

Also let the given conic QMN is given by

$$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0, \quad z = 0$$ **…………………(5.11)**

Let $P(\alpha, \beta, \gamma)$ be any point on the surface of cylinder. The equation of generating line through the point $P(\alpha, \beta, \gamma)$ and parallel to the fixed line (z-axis) is given by

$$\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{z-\gamma}{1}$$ **…………………(5.12)**

Let the generating line **(5.12)** meets the plane $z = 0$ in Q

Putting $z = 0$ in **(5.12)** we get the coordinate of the point Q

$$\frac{x-\alpha}{0} = \frac{y-\beta}{0} = \frac{0-\gamma}{1}$$

$$(\alpha, \beta, 0)$$

As PQ is the generating line of the cylinder, the coordinate point $Q(\alpha, \beta, 0)$ must satisfied the equation of conic **(5.11)**

$a\alpha^2 + b\beta^2 + 2h\alpha\beta + 2u\alpha + 2v\beta + d = 0$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$

This is the required equation of cone.

Note 5.2:

(i) The general equation in x and y i.e. $f(x, y) = ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$ represents a cylinder whose generators are parallel to z-axis.

In other words $f(x, y) = ax^2 + by^2 + 2hxy + 2ux + 2vy + d = 0$ represents a cylinder passing through the conic $f(x, y) = ax^2 + by^2 +$

$2hxy + 2ux + 2vy + d = 0, z = 0$ with generators parallel to z-axis.

(ii) The general equation in y and z i.e. $f(y,z) = by^2 + cz^2 + 2fyz + 2vy + 2wz + d = 0$

represents a cylinder whose generators are parallel to x-axis.

In other words $f(y,z) = by^2 + cz^2 + 2fyz + 2vy + 2wz + d = 0$ represents a cylinder passing through the conic $f(y,z) = by^2 + cz^2 + 2fyz + 2vy + 2wz + d = 0, x = 0$ with generators parallel to x-axis.

(iii) The general equation in z and x i.e. $f(z,x) = ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0$ represents a cylinder whose generators are parallel to y-axis.

In other words $f(z,x) = ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0$ represents a cylinder passing through the conic $f(z,x) = ax^2 + cz^2 + 2gzx + 2ux + 2wz + d = 0, y = 0$ with generators parallel to y-axis.

Example 5.1: Find the equation of the cylinder with generators parallel to the x-axis and passing through the circle $x^2 + y^2 + z^2 = 9, 2x = y + z$.

Solution: Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder.

Eq^n generator of the cylinder passing through the point $P(\alpha, \beta, \gamma)$ and parallel to the x-axis, is given by

$$\frac{x-\alpha}{1} = \frac{y-\beta}{0} = \frac{z-\gamma}{0} = r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(5.13)}$$

An arbitrary point on the line is given by $(\alpha + r, \beta, \gamma)$

As the cylinder passing through the circle $x^2 + y^2 + z^2 = 9, 2x = y + z$, the generator of cylinder also passing through the circle.

Let $(\alpha + r, \beta, \gamma)$ point satisfy the equation of circle

i.e.

$$(\alpha + r)^2 + \beta^2 + \gamma^2 = 9 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(5.14)}$$

$$2(\alpha + r) = \beta + \gamma \qquad \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(5.15)}$$

By equation **(5.15)**

$$r = \frac{\beta + \gamma}{2} - \alpha$$

Putting the value of $r = \frac{\beta+\gamma}{2} - \alpha$ in equation **(5.14)**, we get

$$\left(\frac{\beta+\gamma}{2}\right)^2 + \beta^2 + \gamma^2 = 9$$

$$\Longrightarrow \beta^2 + \gamma^2 + 2\beta\gamma + 4\beta^2 + 4\gamma^2 = 36$$

$$\Longrightarrow 5\beta^2 + 5\gamma^2 + 2\beta\gamma = 36$$

Taking locus P $(\alpha \to x, \beta \to y, \gamma \to z)$

$$5y^2 + 5z^2 + 2yz = 36$$

This is the required equation of cylinder.

Example 5.2:

Find the equation of right circular cylinder passing through the circle

$x^2 + y^2 + z^2 = 9, 2x = y + z.$

Solution:

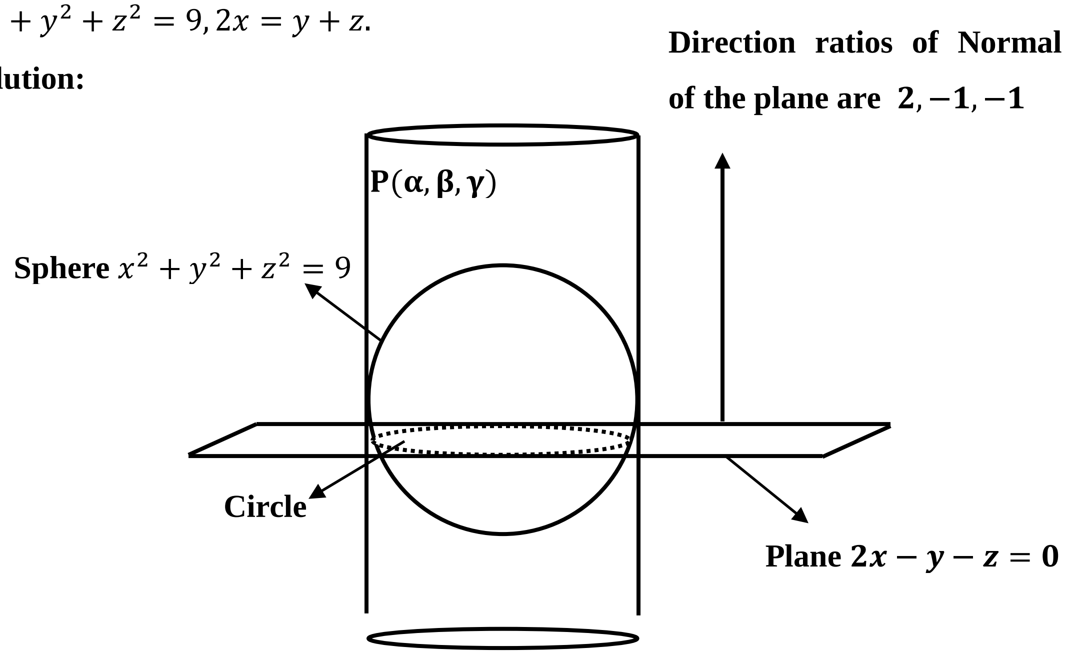


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Figure: 5.10

Given equation of circle

$$x^2 + y^2 + z^2 = 9$$

$$2x - y - z = 0$$

Direction ratios of normal of the plane $(2x - y - z = 0)$ are 2,-1,-1.

The generators of right circular cylinder are parallel to normal of the plane

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder.

Eq^n generator of the cylinder passing through the point $P(\alpha, \beta, \gamma)$ is given by

$$\frac{x-\alpha}{2} = \frac{y-\beta}{-1} = \frac{z-\gamma}{-1} = r$$

………………..**(5.16)**

An arbitrary point on the line is given by $(\alpha + 2r, \beta - r, \gamma - r)$

As the cylinder passing through the circle $x^2 + y^2 + z^2 = 9, 2x = y + z$, the generator of cylinder also passing through the circle.

Let $(\alpha + 2r, \beta - r, \gamma - r)$ point satisfy the equation of circle

i.e.

$$(\alpha + 2r)^2 + (\beta - r)^2 + (\gamma - r)^2 = 9$$
…………………..**(5.17)**

$$2(\alpha + 2r) = (\beta - r) + (\gamma - r)$$
………………….**(5.18)**

By equation **(5.18)**

$6r = \beta + \gamma - 2\alpha$

$$\Rightarrow r = \frac{\beta + \gamma - 2\alpha}{6}$$

Putting the value of $r = \frac{\beta+\gamma-2\alpha}{6}$ in equation **(5.17)**, we get

$$\left(\alpha + \frac{2\beta + 2\gamma - 4\alpha}{6}\right)^2 + \left(\beta - \frac{\beta+\gamma-2\alpha}{6}\right)^2 + \left(\gamma - \frac{\beta+\gamma-2\alpha}{6}\right)^2 = 9$$

$$\Rightarrow (6\alpha + 2\beta + 2\gamma - 4\alpha)^2 + (6\beta - \beta - \gamma + 2\alpha)^2 + (6\gamma - \beta - \gamma + 2\alpha)^2 = 9 \times 36$$

$$\Rightarrow (2\alpha + 2\beta + 2\gamma)^2 + (6\beta - \beta - \gamma + 2\alpha)^2 + (6\gamma - \beta - \gamma + 2\alpha)^2 = 9 \times 36$$

$$\Rightarrow 4(\alpha + \beta + \gamma)^2 + (5\beta - \gamma + 2\alpha)^2 + (2\alpha - \beta + 5\gamma)^2 = 9 \times 36$$

$$\Rightarrow 4(\alpha + \beta + \gamma)^2 + (2\alpha + 5\beta - \gamma)^2 + (2\alpha - \beta + 5\gamma)^2 = 324$$

$$\Rightarrow 4\alpha^2 + 4\beta^2 + 4\gamma^2 + 8\alpha\beta + 8\beta\gamma + 8\gamma\alpha + 4\alpha^2 + 25\beta^2 + \gamma^2 + 20\alpha\beta - 10\beta\gamma - 4\gamma\alpha + 4\alpha^2 + \beta^2 + 25\gamma^2 - 4\alpha\beta - 10\beta\gamma + 20\gamma\alpha = 324$$

$$\Rightarrow 12\alpha^2 + 30\beta^2 + 30\gamma^2 + 24\alpha\beta - 12\beta\gamma + 24\gamma\alpha = 324$$

$$\Rightarrow 2\alpha^2 + 5\beta^2 + 5\gamma^2 + 4\alpha\beta - 2\beta\gamma + 4\gamma\alpha = 54$$

Taking locus P $(\alpha \to x, \beta \to y, \gamma \to z)$

$$2x^2 + 5y^2 + 5z^2 + 4xy - 2yz + 4zx = 54$$

This is the required equation of cylinder.

Example 5.3:

Find the equation to the right circular cylinder for its base the circle

$x^2 + y^2 + z^2 = 9, x - y + z = 3.$

Solution:

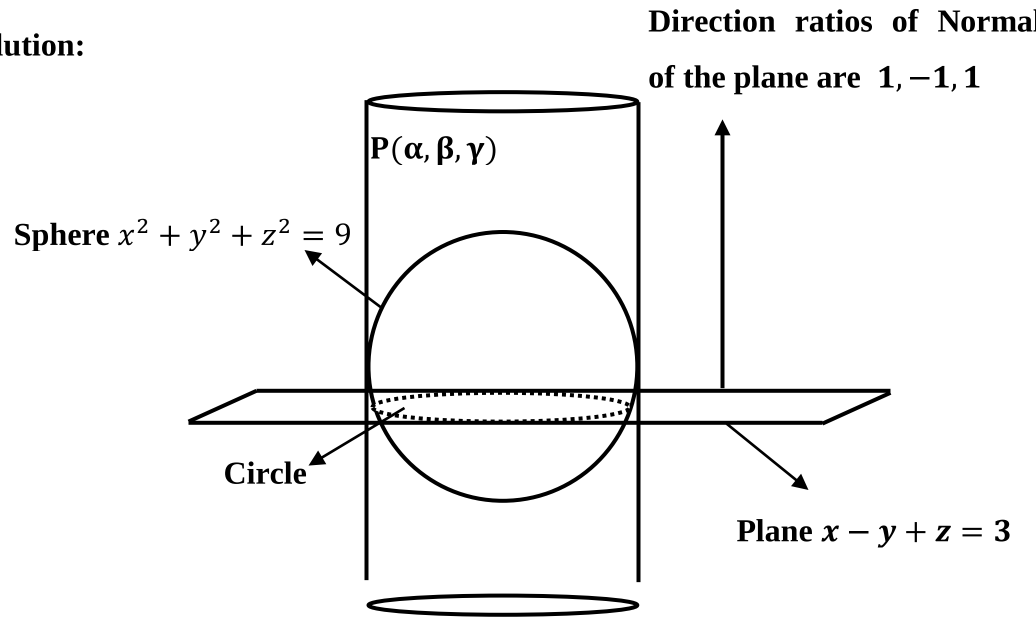

Figure: 5.11

Given equation of circle

$$x^2 + y^2 + z^2 = 9$$

$$x - y + z = 3$$

Direction ratios of normal of the plane ($x - y + z = 3$) are 1,-1,1.

The generators of right circular cylinder are parallel to normal of the plane

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder.

Eq^n generator of the cylinder passing through the point $P(\alpha, \beta, \gamma)$ is given by

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-1} = \frac{z-\gamma}{1} = r$$
………………..**(5.19)**

An arbitrary point on the line is given by $(\alpha + r, \beta - r, \gamma + r)$

As the cylinder passing through the circle $x^2 + y^2 + z^2 = 9, x - y + z = 3$, the generator of cylinder also passing through the circle.

Let $(\alpha + r, \beta - r, \gamma + r)$ point satisfy the equation of circle

i.e.

$$(\alpha + r)^2 + (\beta - r)^2 + (\gamma + r)^2 = 9$$
…………………..**(5.20)**

$$(\alpha + r) - (\beta - r) + (\gamma + r) = 3$$
………………….(5**.21**)

By equation (**5.21**)

$3r = 3 + \beta - \gamma - \alpha$

$$\Rightarrow r = \frac{3 + \beta - \gamma - \alpha}{3}$$

Putting the value of $r = \frac{3+\beta-\gamma-\alpha}{3}$ in equation **(5.20)**, we get

$$\left(\alpha + \frac{3+\beta-\gamma-\alpha}{3}\right)^2 + (\beta - \frac{3+\beta-\gamma-\alpha}{3})^2 + (\gamma + \frac{3+\beta-\gamma-\alpha}{3})^2 = 9$$

$$\Rightarrow (3\alpha + 3 + \beta - \gamma - \alpha)^2 + (3\beta - 3 - \beta + \gamma + \alpha)^2 + (3\gamma + 3 + \beta - \gamma - \alpha)^2 = 9 \times 9$$

$$\Rightarrow (2\alpha + \beta - \gamma + 3)^2 + (\alpha + 2\beta + \gamma - 3)^2 + (-\alpha + \beta + 2\gamma + 3)^2 = 81$$

$$\Rightarrow (2\alpha + \beta - \gamma + 3)^2 + (\alpha + 2\beta + \gamma - 3)^2 + (-\alpha + \beta + 2\gamma + 3)^2 = 81$$

$$\begin{aligned}\Rightarrow 4\alpha^2 + \beta^2 + \gamma^2 + 9 + 4\alpha\beta - 4\alpha\gamma + 12\alpha - 2\beta\gamma + 6\beta - 6\gamma + \alpha^2 \\ + 4\beta^2 + \gamma^2 + 9 + 4\alpha\beta + 2 - 6\alpha + 4\beta\gamma - 12\beta - 6\gamma + \alpha^2 \\ + \beta^2 + 4\gamma^2 + 9 - 2\alpha\beta - 4\alpha\gamma - 6\alpha + 4\beta\gamma + 6\beta + 12\gamma \\ = 81\end{aligned}$$

$$\Rightarrow 6\alpha^2 + 6\beta^2 + 6\gamma^2 + 6\alpha\beta - 6\alpha\gamma + 6\beta\gamma = 54$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + \alpha\beta - \alpha\gamma + \beta\gamma = 9$$

Taking locus P ($\alpha \to x, \beta \to y, \gamma \to z$)

$$x^2 + y^2 + z^2 + xy - zx + yz = 9$$

This is the required equation of cylinder.

Check Your Progress

1. Find the equation of the cylinder with the generators parallel to x-axis and passing through the curve $ax^2 + by^2 = 2z,\ lx + my + nz = p$.

$$\boldsymbol{Ans.\ a\left(\frac{p-my-nz}{l}\right)^2 + by^2 = 2z\ .}$$

2. Find the equation of the cylinder with the generators parallel to y-axis and passing through the curve $ax^2 + by^2 = 2z,\ lx + my + nz = p$.

Ans. $\boldsymbol{ax^2 + b\left(\frac{p-lx-nz}{m}\right)^2 = 2z}$.

3. Find the equation of the cylinder with the generators parallel to z-axis and passing through the curve $ax^2 + by^2 = 2z,\ lx + my + nz = p$.

Ans. $\boldsymbol{n(ax^2 + by^2) + 2(lx + my - p) = 0}$.

5.7 ENVELOPING CYLINDER

Definition (Enveloping Cylinder) 5.11: It is the cylinder whose generators are parallel to a fixed line and touch a given surface.

In other words, Enveloping cylinder of a surface is the locus of the tangent lines to the surface which are parallel to a given line.

5.7.1 EQUATION OF THE ENVELOPING CYLINDER TO THE SPHERE $\boldsymbol{x^2 + y^2 + z^2 = a^2}$ WHOSE GENERATORS ARE PARALLEL TO THE LINE

$$\boldsymbol{\frac{x}{l} = \frac{y}{m} = \frac{z}{n}}.$$

Let P(α, β, γ) be any point on the enveloping cylinder. The generating line through the point P(α, β, γ) is given by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$
…………..**(5.22)**

Any general point on the generating line **(5.22)** is given by

$$\mathrm{Q}(\alpha + lr, \beta + mr, \gamma + nr)$$

If Q$(\alpha + lr, \beta + mr, \gamma + nr)$ be the point of intersection of **(5.22)** and the sphere

$x^2 + y^2 + z^2 = a^2$, the coordinate of Q must satisfy the equation of sphere.

Therefore,

$$(\alpha + lr)^2 + (\beta + mr)^2 + (\gamma + nr)^2 = a^2$$

$\Longrightarrow r^2(l^2+m^2+n^2)+r(2\alpha l+2\beta m+2\gamma n)+(\alpha^2+\beta^2+\gamma^2-a^2)=0$ ……….**(5.23)**

Since **(5.22)** is the tangent of sphere, roots of the equation **(5.23)** must be equal.

Therefore,

$B^2-4AC=0$

$\Longrightarrow (2\alpha l+2\beta m+2\gamma n)^2-4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)=0$

$\Longrightarrow (\alpha l+\beta m+\gamma n)^2-(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)=0$

Hence the locus of P $(\alpha\to x,\beta\to y,\gamma\to z)$ is

$(lx+my+nz)^2-(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)=0$

$\Longrightarrow (\boldsymbol{lx}+\boldsymbol{my}+\boldsymbol{nz})^{\mathbf{2}}=(\boldsymbol{l^2}+\boldsymbol{m^2}+\boldsymbol{n^2})(\boldsymbol{x^2}+\boldsymbol{y^2}+\boldsymbol{z^2}-\boldsymbol{a^2})$

This is the required equation of enveloping cylinder.

5.7.2 EQUATION OF THE ENVELOPING CYLINDER TO THE SURFACE $\boldsymbol{ax^2+by^2+cz^2=1}$ WHOSE GENERATORS ARE PARALLEL TO THE LINE

$$\frac{\boldsymbol{x}}{\boldsymbol{l}}=\frac{\boldsymbol{y}}{\boldsymbol{m}}=\frac{\boldsymbol{z}}{\boldsymbol{n}}.$$

Let P(α, β, γ) be any point on the enveloping cylinder. The generating line through the point P(α, β, γ) is given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

………..**(5.24)**

Any general point on the generating line **(5.24)** is given by

$$Q(\alpha+lr,\beta+mr,\gamma+nr)$$

If $Q(\alpha+lr,\beta+mr,\gamma+nr)$ be the point of intersection of **(5.24)** and the $ax^2+by^2+cz^2=1$, the coordinate of Q must satisfy the equation of surface.

Therefore,

$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma+nr)^2=1$

$\Longrightarrow r^2(al^2+bm^2+cn^2)+r(2a\alpha l+2b\beta m+2c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ ….**(5.25)**

Since **(5.24)** is the tangent of sphere, roots of the equation (**5.25**) must be equal.

Therefore,

$B^2 - 4AC = 0$

$\Longrightarrow (2a\alpha l + 2b\beta m + 2c\gamma n)^2 - 4(al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

$\Longrightarrow (a\alpha l + b\beta m + c\gamma n)^2 - (al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$(alx + bmy + cnz)^2 - (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1) = 0$

$\Longrightarrow \boldsymbol{(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)}$

This is the required equation of enveloping cylinder.

5.7.3 EQUATION OF THE ENVELOPING CYLINDER TO THE SURFACE $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$ WHOSE GENERATORS ARE PARALLEL TO THE LINE $\boldsymbol{\frac{x}{l} = \frac{y}{m} = \frac{z}{n}}$.

Let P(α, β, γ) be any point on the enveloping cylinder. The generating line through the point P(α, β, γ) is given by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \quad \text{……………….. } \mathbf{(5.26)}$$

Any general point on the generating line **(5.26)** is given by

$$Q(\alpha + lr, \beta + mr, \gamma + nr)$$

If $Q(\alpha + lr, \beta + mr, \gamma + nr)$ be the point of intersection of **(5.26)** and the surface $\boldsymbol{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1}$, the coordinate of Q must satisfy the equation of surface.

Therefore,

$$\frac{(\alpha + lr)^2}{a^2} + \frac{(\beta + mr)^2}{b^2} + \frac{(\gamma + nr)^2}{c^2} = 1$$

$\Longrightarrow r^2\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right) + r\left(\frac{2\alpha l}{a^2} + \frac{2\beta m}{b^2} + \frac{2\gamma n}{c^2}\right) + \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = 0$…….**(5.27)**

Since **(5.26)** is the tangent of surface, roots of the equation **(5.27)** must be equal.

Therefore,

$B^2 - 4AC = 0$

$$\Rightarrow \left(\frac{2\alpha l}{a^2} + \frac{2\beta m}{b^2} + \frac{2\gamma n}{c^2}\right)^2 - 4\left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = 0$$

$$\Rightarrow \left(\frac{\alpha l}{a^2} + \frac{\beta m}{b^2} + \frac{\gamma n}{c^2}\right)^2 - \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} - 1\right) = 0$$

Hence the locus of P ($\alpha \to x, \beta \to y, \gamma \to z$) is

$$\left(\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2}\right)^2 - \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0$$

$$\Rightarrow \boldsymbol{\left(\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2}\right)^2 = \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)}$$

This is the required equation of enveloping cylinder.

Example 5.4: Find the enveloping cylinder of the sphere $x^2 + y^2 + z^2 - x - y + 2z - 2 = 0$ having the generators parallel to the line $x = y = z$.

Solution: Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder. The generating line through the point $P(\alpha, \beta, \gamma)$ is given by

$$x - \alpha = y - \beta = z - \gamma = r \quad \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(5.28)}$$

Any general point on the generating line **(5.28)** is given by

$$Q(\alpha + r, \beta + r, \gamma + r)$$

If $Q(\alpha + r, \beta + r, \gamma + r)$ be the point of intersection of **(5.28)** and the sphere $x^2 + y^2 + z^2 - x - y + 2z - 2 = 0$, the coordinate of Q must satisfy the equation of sphere.

Therefore,

$$(\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - (\alpha + r) - (\beta + r) + 2(\gamma + r) - 2 = 0$$

$\Rightarrow 3r^2 + r(2\alpha + 2\beta + 2\gamma) + (\alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta + 2\gamma - 2) = 0$…….**(5.29)**

Since **(5.28)** is the tangent of surface, roots of the equation **(5.29)** must be equal.

Therefore,

$$B^2 - 4AC = 0$$

$$\Rightarrow (2\alpha + 2\beta + 2\gamma)^2 - 4 \times 3 \times (\alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta + 2\gamma - 2) = 0$$

$$\Rightarrow (\alpha + \beta + \gamma)^2 - 3(\alpha^2 + \beta^2 + \gamma^2 - \alpha - \beta + 2\gamma - 2) = 0$$

$$\Rightarrow \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha - 3\alpha^2 - 3\beta^2 - 3\gamma^2 + 3\alpha + 3\beta - 6\gamma + 6 = 0$$

$\Rightarrow -2\alpha^2 - 2\beta^2 - 2\gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha + 3\alpha + 3\beta - 6\gamma + 6 = 0$

$\Rightarrow 2\alpha^2 + 2\beta^2 + 2\gamma^2 - 2\alpha\beta - 2\beta\gamma - 2\gamma\alpha - 3\alpha - 3\beta + 6\gamma - 6 = 0$

Hence the locus of P ($\alpha \to x, \beta \to y, \gamma \to z$) is

$2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2zx - 3x - 3y + 6z - 6 = 0$

This is the required equation of enveloping cylinder.

Example 5.5: Find the enveloping cylinder of the surface $x^2 + 2y^2 + z^2 - 2 = 0$ having the generators parallel to the line $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z-4}{1}$.

Solution: Let $P(\alpha, \beta, \gamma)$ be any point on the enveloping cylinder. The generating line through the point $P(\alpha, \beta, \gamma)$ is given by

$$\frac{x-\alpha}{2} = \frac{y-\beta}{3} = \frac{z-\gamma}{1} = r \quad \ldots\ldots\ldots\ldots\ldots\ldots\textbf{(5.30)}$$

Any general point on the generating line **(5.30)** is given by

$$Q(\alpha + 2r, \beta + 3r, \gamma + r)$$

If $Q(\alpha + 2r, \beta + 3r, \gamma + r)$ be the point of intersection of **(5.30)** and the surface

$x^2 + 2y^2 + z^2 - 2 = 0$, the coordinate of Q must satisfy the equation of surface.

Therefore,

$(\alpha + 2r)^2 + 2(\beta + 3r)^2 + (\gamma + r)^2 - 2 = 0$

$\Rightarrow 23r^2 + r(4\alpha + 12\beta + 2\gamma) + (\alpha^2 + 2\beta^2 + \gamma^2 - 2) = 0 \ldots\ldots\textbf{(5.31)}$

Since **(5.30)** is the tangent of surface, roots of the equation **(5.31)** must be equal.

Therefore,

$B^2 - 4AC = 0$

$\Rightarrow (4\alpha + 12\beta + 2\gamma)^2 - 4 \times 23 \times (\alpha^2 + 2\beta^2 + \gamma^2 - 2) = 0$

$\Rightarrow (2\alpha + 6\beta + \gamma)^2 - 23(\alpha^2 + 2\beta^2 + \gamma^2 - 2) = 0$

$\Rightarrow 4\alpha^2 + 36\beta^2 + \gamma^2 + 24\alpha\beta + 12\beta\gamma + 4\gamma\alpha - 23\alpha^2 - 46\beta^2 - 23\gamma^2 + 46 = 0$

$\Rightarrow -19\alpha^2 - 10\beta^2 - 22\gamma^2 + 24\alpha\beta + 12\beta\gamma + 4\gamma\alpha + 46 = 0$

$\Rightarrow 19\alpha^2 + 10\beta^2 + 22\gamma^2 - 24\alpha\beta - 12\beta\gamma - 4\gamma\alpha - 46 = 0$

Hence the locus of P ($\alpha \to x, \beta \to y, \gamma \to z$) is

$19x^2 + 10y^2 + 22z^2 - 24xy - 12yz - 4zx - 46 = 0$

This is the required equation of enveloping cylinder.

Check Your Progress

1. Find the enveloping cylinder of the surface $ax^2 + by^2 = 2z$ having the generators parallel to the x-axis.

$\boldsymbol{Ans.\ by^2 = 2z}$ **.**

2. Find the enveloping cylinder of the surface $ax^2 + by^2 = 2z$ having the generators parallel to the y-axis.

$\boldsymbol{Ans.\ ax^2 = 2z}$ **.**

5.8 EQUATION OF A RIGHT-CIRCULAR CYLINDER

To find the equation of a right circular cylinder of radius a whose axis is the line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l*,*m*,*n* are the direction cosines.

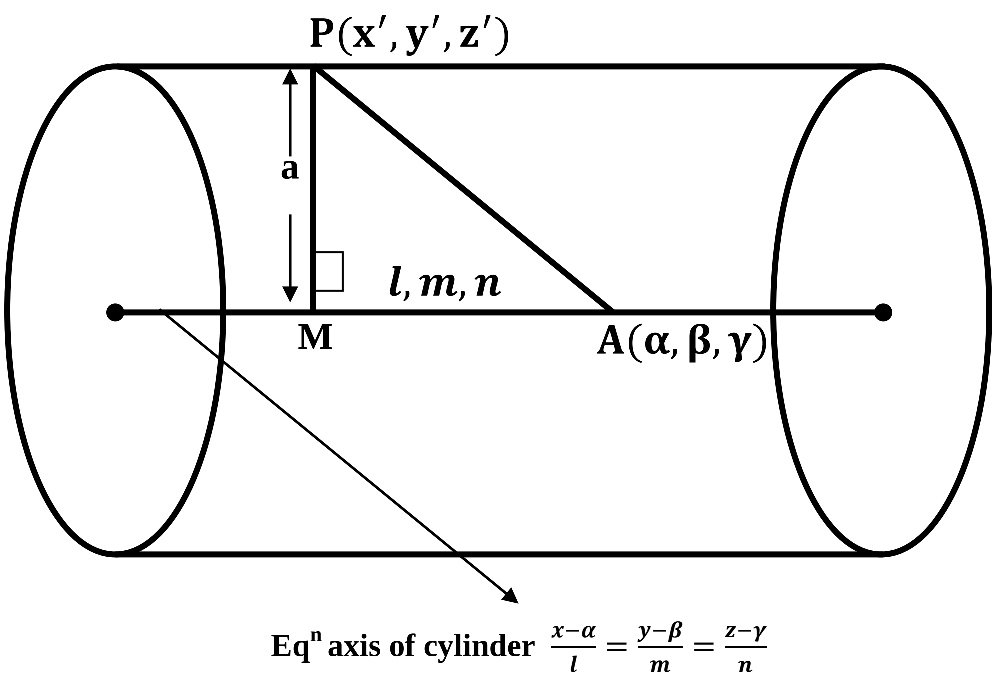

Figure: 5.12

AM is the projection of AP on axis of cylinder.

Therefore, AM $= (x' - \alpha)l + (y' - \beta)m + (z' - \gamma)n$

In Right angled triangle PMA

$$\mathrm{PM}^2 = \mathrm{PA}^2 - \mathrm{AM}^2$$

$$\Rightarrow \mathrm{a}^2 = (\mathrm{x}' - \alpha)^2 + (\mathrm{y}' - \beta)^2 + (\mathrm{z}' - \gamma)^2 - \{(x' - \alpha)l + (y' - \beta)m + (z' - \gamma)n\}^2$$

As, $l^2 + m^2 + n^2 = 1$, we have

$$\mathrm{a}^2 = \{(\mathrm{x}' - \alpha)^2 + (\mathrm{y}' - \beta)^2 + (\mathrm{z}' - \gamma)^2\}(l^2 + m^2 + n^2) - \{(x' - \alpha)l + (y' - \beta)m + (z' - \gamma)n\}^2$$

By Lagrange's identity

$$\mathrm{a}^2 = \{(x' - \alpha)m - (y' - \beta)l\}^2 + \{(y' - \beta)n - (z' - \gamma)m\}^2 + \{(z' - \gamma)l - (x' - \alpha)n\}^2$$

Hence the locus of P ($\alpha \to x, \beta \to y, \gamma \to z$) is

$$\mathrm{a}^2 = \{(x - \alpha)m - (y - \beta)l\}^2 + \{(y - \beta)n - (z - \gamma)m\}^2 + \{(z - \gamma)l - (x - \alpha)n\}^2$$

or

$$\mathrm{a}^2 = \begin{vmatrix} x - \alpha & y' - \beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y - \beta & z - \gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z - \gamma & x - \alpha \\ n & l \end{vmatrix}^2$$

Note 5.3: Equation of a right circular cylinder of radius a whose axis is the line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l*,*m*,*n* are the direction ratios is given by

$$\mathrm{a}^2(l^2 + m^2 + n^2) = \{(x - \alpha)m - (y - \beta)l\}^2 + \{(y - \beta)n - (z - \gamma)m\}^2 + \{(z - \gamma)l - (x - \alpha)n\}^2$$

or

$$\mathrm{a}^2(l^2 + m^2 + n^2) = \begin{vmatrix} x - \alpha & y - \beta \\ l & m \end{vmatrix}^2 + \begin{vmatrix} y - \beta & z - \gamma \\ m & n \end{vmatrix}^2 + \begin{vmatrix} z - \gamma & x - \alpha \\ n & l \end{vmatrix}^2$$

Example 5.6:

Find the equation to the right circular cylinder of radius 3 and its axis is the line

$$\frac{x - 1}{2} = \frac{y - 3}{2} = \frac{z - 5}{-1}$$

Solution :

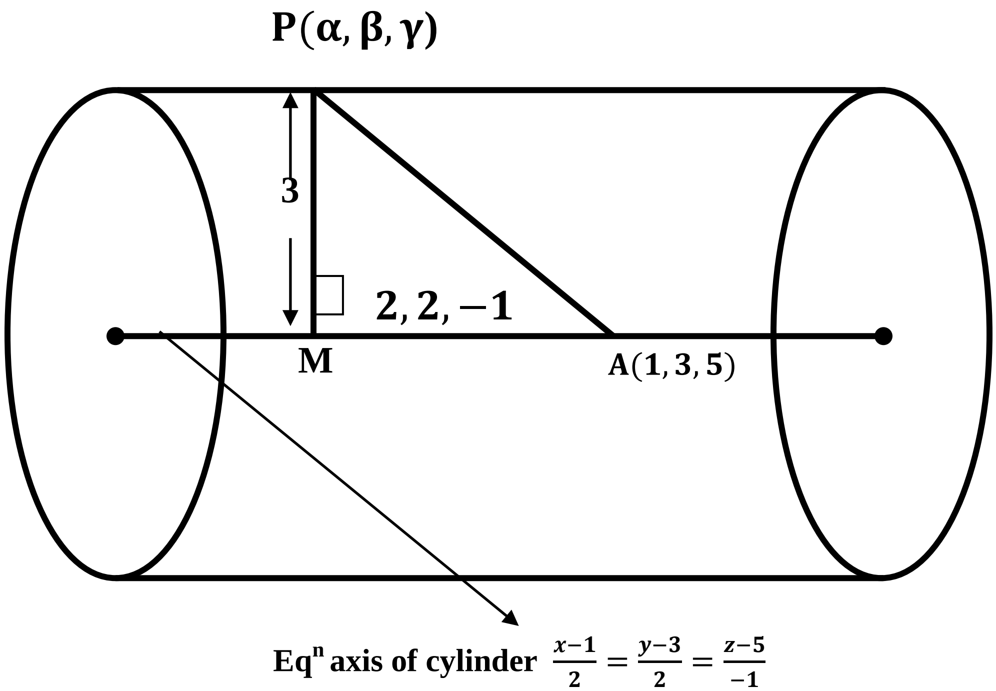

Figure: 5.13

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder

As shown in the figure

$PM = 3$ and $AP^2 = (\alpha - 1)^2 + (\beta - 3)^2 + (\gamma - 5)^2$

Now, $MA =$ Projection of AP on the axis $= \frac{2(\alpha-1)+2(\beta-3)-1(\gamma-5)}{\sqrt{2^2+2^2+(-1)^2}} = \frac{2\alpha+2\beta-\gamma-3}{3}$

In right angled triangle ΔPMA,

$AP^2 - MA^2 = 9$

$$\Rightarrow (\alpha - 1)^2 + (\beta - 3)^2 + (\gamma - 5)^2 - \left(\frac{2\alpha + 2\beta - \gamma - 3}{3}\right)^2 = 9$$

$$\Rightarrow 9\{(\alpha - 1)^2 + (\beta - 3)^2 + (\gamma - 5)^2\} - (2\alpha + 2\beta - \gamma - 3)^2 = 81$$

$$\Rightarrow 9\{\alpha^2 + 1 - 2\alpha + \beta^2 + 9 - 6\beta + \gamma^2 + 25 - 10\gamma\} - (4\alpha^2 + 4\beta^2 + \gamma^2 + 9 + 8\alpha\beta - 4\alpha\gamma - 12\alpha - 4\beta\gamma - 12\beta - 6\gamma) = 81$$

$$\Rightarrow 9\{\alpha^2 + \beta^2 + \gamma^2 - 2\alpha - 6\beta - 10\gamma + 35\} - (4\alpha^2 + 4\beta^2 + \gamma^2 + 8\alpha\beta - 4\beta\gamma - 4\alpha\gamma - 12\alpha - 12\beta - 6\gamma + 9) = 81$$

$$\Rightarrow 5\alpha^2 + 5\beta^2 + 8\gamma^2 - 6\alpha - 42\beta - 84\gamma - 8\alpha\beta + 4\beta\gamma + 4\alpha\gamma + 306 = 81$$

$$\Rightarrow 5\alpha^2 + 5\beta^2 + 8\gamma^2 - 6\alpha - 42\beta - 84\gamma - 8\alpha\beta + 4\beta\gamma + 4\alpha\gamma + 225 = 0$$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$$5x^2 + 5y^2 + 8z^2 - 8xy + 4yz + 4zx - 6x - 42y - 84z + 225 = 0$$

This is the required equation of cylinder.

Example 5.7:

Find the equation to the right circular cylinder of radius 2 and its axis is the line

$$\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-2}{5}$$

Solution:

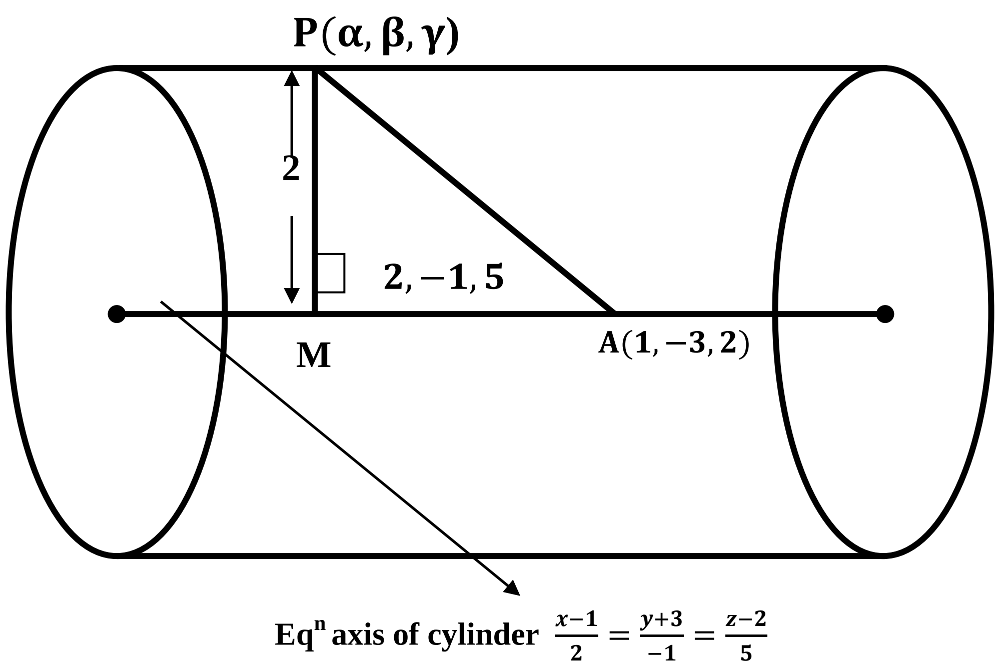

Figure: 5.14

Let $P(\alpha, \beta, \gamma)$ be any point on the cylinder

As shown in the figure

$PM = 2$ and $AP^2 = (\alpha - 1)^2 + (\beta + 3)^2 + (\gamma - 2)^2$

Now, $\quad MA = \text{Projection of AP on the axis} = \frac{2(\alpha-1)-1(\beta+3)+5(\gamma-2)}{\sqrt{2^2+(-1)^2+(5)^2}} = \frac{2\alpha-\beta+5\gamma-15}{\sqrt{30}}$

In right angled triangle ΔPMA,

$AP^2 - MA^2 = PM^2$

$$\Rightarrow (\alpha-1)^2 + (\beta+3)^2 + (\gamma-2)^2 - \left(\frac{2\alpha-\beta+5\gamma-15}{\sqrt{30}}\right)^2 = 4$$

$$\Rightarrow 30\{(\alpha-1)^2 + (\beta+3)^2 + (\gamma-2)^2\} - (2\alpha-\beta+5\gamma-15)^2 = 120$$

$$\Rightarrow 30\{\alpha^2 + 1 - 2\alpha + \beta^2 + 9 + 6\beta + \gamma^2 + 4 - 4\gamma\} - (4\alpha^2 + \beta^2 + 25\gamma^2 + 225 - 4\alpha\beta + 20\alpha\gamma - 60\alpha - 10\beta\gamma + 30\beta - 150\gamma) = 120$$

$$\Rightarrow 30\{\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 6\beta - 4\gamma + 14\} - (4\alpha^2 + \beta^2 + 25\gamma^2 - 4\alpha\beta - 10\beta\gamma + 20\alpha\gamma - 60\alpha + 30\beta - 150\gamma + 225) = 120$$

$$\Rightarrow 26\alpha^2 + 29\beta^2 + 5\gamma^2 + 150\beta + 30\gamma + 4\alpha\beta + 10\beta\gamma - 20\alpha\gamma + 195 = 120$$

$$\Rightarrow 26\alpha^2 + 29\beta^2 + 5\gamma^2 + 4\alpha\beta + 10\beta\gamma - 20\alpha\gamma + 150\beta + 30\gamma + 75 = 0$$

Hence the locus of P $(\alpha \to x, \beta \to y, \gamma \to z)$ is

$$26x^2 + 29y^2 + 5z^2 + 4xy + 10yz - 20zx + 150y + 30z + 75 = 0$$

This is the required equation of cylinder

Example 5.8 :

Find the equation to the right circular cylinder of radius 2 and having as axis the line

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$$

Solution: The required equation of right circular cylinder is given by

$$2^2(2^2+1^2+2^2) = \begin{vmatrix} x-1 & y-2 \\ 2 & 1 \end{vmatrix}^2 + \begin{vmatrix} y-2 & z-3 \\ 1 & 2 \end{vmatrix}^2 + \begin{vmatrix} z-3 & x-1 \\ 2 & 2 \end{vmatrix}^2$$

$$4(9) = [\{x-2y+3\}^2 + \{2y-z-1\}^2 + \{2z-2x-4\}^2]$$

$$36 = x^2 + 4y^2 + 9 - 4xy - 12y + 6x + 4y^2 + z^2 + 1 - 4yz + 2z - 4y + 4z^2 + 4x^2 + 16 - 8zx + 16x - 16z$$

$$5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z - 10 = 0.$$

Example 5.9: Find the equation of right circular cylinder of radius 2 and its axis passes through the point (1,2,3) and direction ratios are 2,−3,6.

Solution: Equation axis of right circular cylinder is given by

$$\frac{x-1}{2}=\frac{y-2}{-3}=\frac{z-3}{6}$$

The required equation of cylinder is given by

$$2^2\{2^2+(-3)^2+6^2\} = \begin{vmatrix} x-1 & y-2 \\ 2 & -3 \end{vmatrix}^2 + \begin{vmatrix} y-2 & z-3 \\ -3 & 6 \end{vmatrix}^2 + \begin{vmatrix} z-3 & x-1 \\ 6 & 2 \end{vmatrix}^2$$

$$196=(-3\mathrm{x}-2\mathrm{y}+7)^2+(6\mathrm{y}+3\mathrm{z}-21)^2+(2\mathrm{z}-6\mathrm{x})^2$$

$$196=9\mathrm{x}^2+4\mathrm{y}^2+49+12\mathrm{xy}-28\mathrm{y}-42\mathrm{x}+36\mathrm{y}^2+9\mathrm{z}^2+441 + 36\mathrm{yz}-126\mathrm{z}-252\mathrm{y}+4\mathrm{z}^2+36\mathrm{x}^2-24\mathrm{zx}$$

$$45\mathrm{x}^2+40\mathrm{y}^2+13\mathrm{z}^2+12\mathrm{xy}+36\mathrm{yz}-24\mathrm{zx}-42\mathrm{x}-280\mathrm{y}-126\mathrm{z} + 294=0$$

Check Your Progress

1. Prove that equation of right circular cylinder of radius 'r' and axis is the x-axis is $y^2+z^2=r^2$.
2. Prove that equation of right circular cylinder of radius 'r' and axis is the y-axis is $x^2+z^2=r^2$.
3. Prove that equation of right circular cylinder of radius 'r' and axis is the z-axis is $x^2+y^2=r^2$.

5.9 RULED SURFACE

Definition (Ruled Surface) 5.12: In geometry, a surface S is ruled if through every point of S there is a straight line that lies on S.

It therefore has a parameterization of the form

$$\mathbf{x}(u,v)=\mathbf{b}(u)+v\boldsymbol{\delta}(u)$$

where **b** is called the base curve and **δ** is director curve. The straight lines themselves are called rulings.

Example 5.10: Parameterization of Hyperboloid of one Sheet

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

$x=a\ (cosu-v\ sinu),\ y=b\ (sinu+v\ cosu),\ z=c\ v$

$$\begin{bmatrix} a\ (cosu-v\ sinu) \\ b\ (sinu+v\ cosu) \\ c\ v \end{bmatrix}=\begin{bmatrix} a\ cosu \\ b\ sinu \\ 0 \end{bmatrix}+v\begin{bmatrix} -a\ sinu \\ b\ cosu \\ c \end{bmatrix}$$

and

$x=a\ (cosu+v\ sinu),\ y=b\ (sinu-v\ cosu),\ z=c\ v$

$$\begin{bmatrix} a\ (cosu+v\ sinu) \\ b\ (sinu-v\ cosu) \\ c\ v \end{bmatrix}=\begin{bmatrix} a\ cosu \\ b\ sinu \\ 0 \end{bmatrix}+v\begin{bmatrix} a\ sinu \\ -b\ cosu \\ c \end{bmatrix}$$

are two parameterizations of Hyperboloid of one sheet.

Definition (Doubly Ruled Surface) 5.13:

A surface that contains two families of rulings is known as doubly ruled surface.

The plane, Hyperbolic Paraboloid and Hyperboloid of one Sheet are doubly ruled surface.

5.10 HYPERBOLOID OF ONE SHEET

The standard equation of a hyperboloid of one sheet is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad \text{……………}\textbf{(5.32)}$$

(P1) Origin is the centre of Hyperboloid of one sheet

$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}=r$

A general point on line is (lr, mr, nr)

$\frac{(lr)^2}{a^2}+\frac{(mr)^2}{b^2}-\frac{(nr)^2}{c^2}=1 \Rightarrow r=\pm\frac{1}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}$

A line passing through the origin cut the surface **(5.32)** at two points

P$\left(\frac{l}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}, \frac{m}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}, \frac{n}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}\right)$ and Q$\left(\frac{-l}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}, \frac{-m}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}, \frac{-n}{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}}}\right)$

Obviously origin is the middle point of the Chord PQ.

Hence, the origin is the centre of the surface as every chord passing through the origin is bisected at the origin.

In other words, If $P(\alpha, \beta, \gamma)$ be any point on the surface **(5.32)** then the point $Q(-\alpha, -\beta, -\gamma)$ will also lie on the surface. This shows that the origin O(0,0,0) is the middle point of chord PQ. This shows that all the chord of the surface which passes through the origin have their middle point at the origin. Hence the surface **(5.32)** has a centre at origin.

(P2) The intercepts of the hyperboloid of one sheet with the x, y, z-axes.

(i) If the surface **(5.32)** meets the x-axis, put $y = 0 \; and \; z = 0$

We get, $\frac{x^2}{a^2} + \frac{0}{b^2} - \frac{0}{c^2} = 1$

$\Rightarrow \frac{x^2}{a^2} = 1$

$\Rightarrow x = a, -a$

Hence the surface **(5.32)** meets the x-axis at the points $\boldsymbol{A(a, 0, 0)}$ **and** $\boldsymbol{A'(-a, 0, 0)}$**.**

(ii) If the surface **(5.32)** meets the y-axis, put $x = 0 \; and \; z = 0$

we get, $\frac{0}{a^2} + \frac{y^2}{b^2} - \frac{0}{c^2} = 1$

$\Rightarrow \frac{y^2}{b^2} = 1 \Rightarrow y = b, -b$

Hence the surface **(5.32)** meets the y-axis at the points $\boldsymbol{B(0, b, 0)}$ **and** $\boldsymbol{B'(0, -b, 0)}$**.**

(iii) If the surface **(5.32)** meets the z-axis, put $x = 0 \; and \; y = 0$

we get, $\frac{0}{a^2} + \frac{0}{b^2} - \frac{z^2}{c^2} = 1$

$\Rightarrow -\frac{z^2}{c^2} = 1$

$\Rightarrow z^2 = -c^2$

which admits no real solution for real c.

Hence the surface **(5.32)** does not meet the z-axis.

(P3) The **traces** of the Hyperboloid of one sheet are ellipses in the xy-plane

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for $z=0(xy-plane)$

hyperbolas in xz-plane

$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ for $y=0(xz-plane)$

hyperbolas in xz-plane

$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ for $x=0(yz-plane)$

(P4) The Hyperboloid of one sheet is symmetrical about the three co-ordinate planes. These are the principal planes and the co-ordinate axes are the principal axes of the Hyperboloid of one sheet.

(P5) The section of the Hyperboloid of one sheet by the plane $z=k$ is the ellipse given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{K^2}{c^2}, z=k$$

Thus it is an ellipse whose centre is on z-axis.

For, $k=0$, it is called the principal ellipse.

The section of the Hyperboloid of one sheet by the plane $y=k$ is the hyperbola given by

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{k^2}{b^2}, y=k$$

The section of the Hyperboloid of one sheet by the plane $x=k$ is the hyperbola given by

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{k^2}{a^2}, x=k$$

(P6) Hyperboloid of one sheet is not a bounded surface.

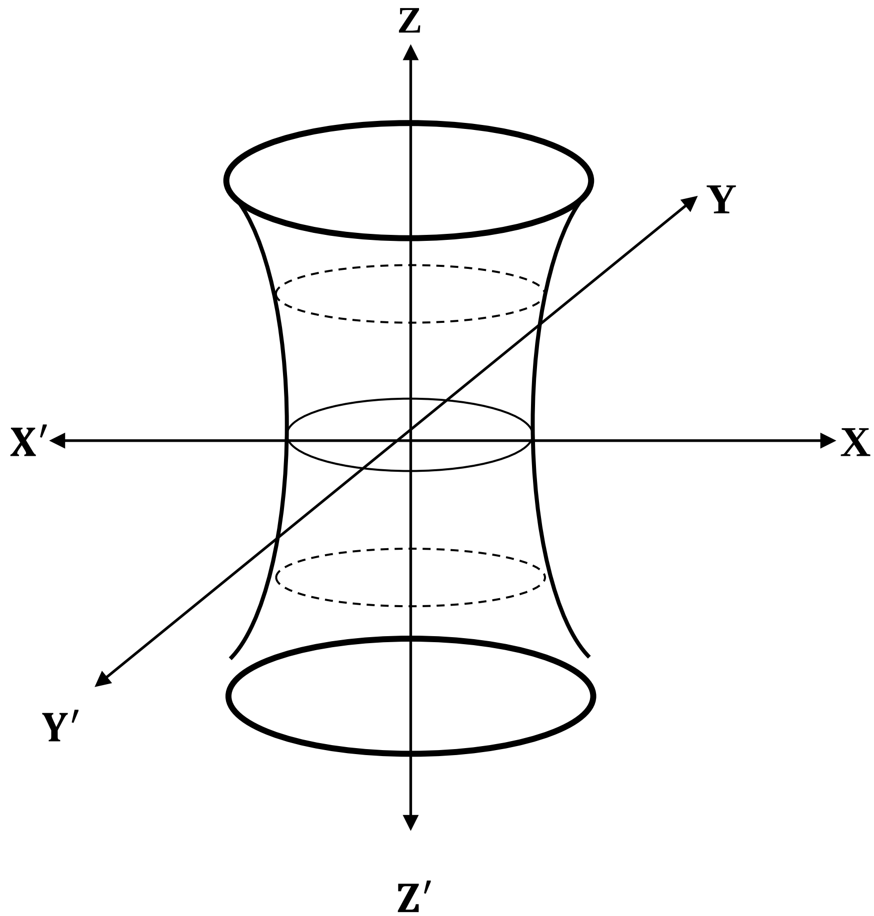

Figure: 5.15

5.11 SUMMARY

We conclude with summarizing what we have covered in this unit.

- Quadratic equation in x,y,z.
- Definition of cylindrical surface, Cylinder, Right Cylinder, Oblique Cylinder, Right Circular Cylinder and Oblique Circular Cylinder.
- Find the equation of a cylinder with given base and generators are parallel to a fixed line
- Find the equation of a cylinder with given base and generators are parallel to a co-ordinate axis i.e. x-axis, y-axis and z-axis.
- Define Enveloping Cylinder
- Find the equation of the enveloping cylinder to the sphere $x^2 + y^2 + z^2 = a^2$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of the enveloping cylinder to the surface $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of the enveloping cylinder to the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

- Find the equation of a right circular cylinder of radius a whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l*,*m*,*n* are the direction cosines.

- Find the equation of a right circular cylinder of radius a whose axis is the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$, where *l*,*m*,*n* are the direction ratios

- Ruled surface

- Hyperboloid of one sheet

5.12 TERMINAL QUESTIONS

1. Find the equation of the cylinder with the generators parallel to x-axis and passing through the circle $x^2 + y^2 + z^2 = a^2,\ lx + my + nz = p$.

$\boldsymbol{Ans.}\ \boldsymbol{\left(\frac{p-my-nz}{l}\right)^2 + y^2 + z^2 = a^2.}$

2. Find the equation of the cylinder with the generators parallel to y-axis and passing through the circle $x^2 + y^2 + z^2 = a^2,\ lx + my + nz = p$.

$\boldsymbol{Ans.}\ \boldsymbol{x^2 + \left(\frac{p-lx-nz}{m}\right)^2 + z^2 = a^2.}$

3. Find the equation of the cylinder with the generators parallel to z-axis and passing through the circle $x^2 + y^2 + z^2 = a^2,\ lx + my + nz = p$.

$\boldsymbol{Ans.}\ \boldsymbol{x^2 + y^2 + \left(\frac{p-lx-my}{n}\right)^2 = a^2.}$

4. Prove that equation of the cylinder passing through the curve $x^2 + y^2 = 1, z = 0$ with the generators parallel to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

is $\boldsymbol{\left(x - \frac{z}{3}\right)^2 + \left(y - \frac{2z}{3}\right)^2 = 0.}$

5. Find the enveloping cylinder of the surface $ax^2 + by^2 + cz^2 = 1$ having the generators parallel to the x-axis.

$\boldsymbol{Ans.\ by^2 + cz^2 = 1.}$

6. Find the enveloping cylinder of the surface $ax^2 + by^2 + cz^2 = 1$ having the generators parallel to the y-axis.

$\boldsymbol{Ans.\ ax^2 + cz^2 = 1\ .}$

7. Find the enveloping cylinder of the surface $ax^2 + by^2 + cz^2 = 1$ having the generators parallel to the z-axis.

$\boldsymbol{Ans.\ ax^2 + by^2 = 1.}$

8. Find the equation of right circular cylinder of radius 2 and its axis is the line

$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$.

$\boldsymbol{Ans.}$ $\boldsymbol{5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8zx + 22x - 16y - 14z - 10 = 0}$.

9. Find the equation of right circular cylinder of radius 2 and its axis passes through the point (1,0,0) and its direction ratios are 2,1,3.

$\boldsymbol{Ans.}$ $\boldsymbol{10x^2 + 13y^2 + 5z^2 - 4xy - 6yz - 12zx - 20x + 4y + 12z - 10 = 0}$.

5.13 FURTHER READINGS

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Uttar Pradesh Rajarshi Tandon Open University

Bachelor of Science

UGMM-102

Analytical Geometry

BLOCK

3

CONES AND CENTRAL CONICOIDS

DGB-021

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BLOCK INTRODUCTION

Unit-6 **Cones :** Equation of a cone with a given base, Intersection of a cone and a plane passing through the vertex of cone, tangent plane, reciprocal cone, Enveloping cone, right circular cone.

Unit-7 **Central Conicoids-I :** Standard equation of a Central conicoid, ellipsoid, hyperboloid of one sheet and two sheets, tangent planes, tangent lines, polar planes and polar lines.

Unit-8 **Central Conicoids-II :** Enveloping cones and cylinders section with a given centres. Diametral plane conjugate diameters, normal drawn from a given point.

UNIT-6 THE CONE

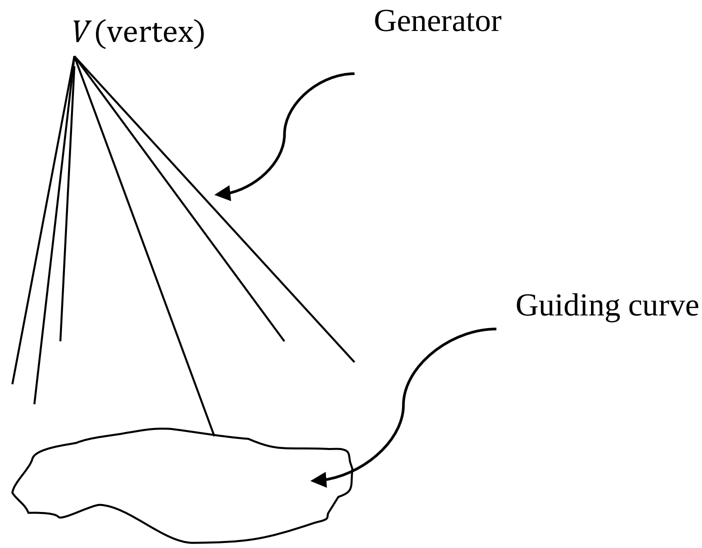
Structure

6.1 INTRODUCTION

You have studied the mensuration of solids earlier in school where you calculated the volume and the surface area of some specific kind of cones, i.e. the solid right circular cones. In this unit, we shall study the general definition of a cone and derive its equation. In the next unit, you will come to know that the surface of a cone is a particular case of some more generalized surfaces called conicoids. We shall begin this unit by first defining a cone and deriving the equation of a cone whose vertex is the origin. Then we shall obtain the condition for the general equation of second degree to represent a cone, equation of a cone with given conic for base and the angle between the lines in which a plane cuts a cone. We shall also discuss whether a cone could have three mutually perpendicular generators. The concepts like tangent lines, tangent planes and the condition of tangency are important as you have already seen in case of sphere. We shall discuss these concepts for the surface of a cone. We shall also study reciprocal cone, enveloping cone and right circular cone in this unit. Let us begin with the definition of the surface cone-

Definition: A cone is a surface generated by a moving straight line passing through a fixed point and intersecting a given curve or touching a given surface.

The fixed point is called the **vertex** and the given curve (or surface) is called the **guiding curve (**or **guiding surface)**. The variable straight line is called the **generator** (or the **generating line**) of the cone.



A cone which is represented by an equation of second degree is called a **quadratic** (or **quadric**) cone. Any straight line other than the generators intersects a quadric cone in two points.

6.2 OBJECTIVES

After reading this unit, you should be able to

- Define a cone
- Obtain the equation of a cone with vertex as origin
- Derive the condition for the general equation of second degree to represent a cone

- Obtain the equation of a cone with a given conic for base
- Understand how a plane through the vertex cuts the cone and to determine the angle between the lines in which a plane cuts a cone
- Find the condition when a cone has three mutually perpendicular generators
- Discuss the tangent line and the tangent plane
- obtain the condition of tangency
- derive the equations of reciprocal cone, enveloping cone and a right circular cone

6.3 EQUATION OF A CONE WITH VERTEX AS ORIGIN

Let us obtain the equation of a quadric cone which has origin as its vertex. Let the quadric cone be represented by the following equation of second degree

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy$$
$$+2wz + d = 0 \quad \ldots\ldots(1)$$

Let $P(x',y',z')$ be any point on the cone. Then the equations of the generator joining the origin $O(0,0,0)$ and $P(x',y',z')$ are

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = r \text{ (say)} \quad \ldots\ldots \ (2)$$

This generator OP lies wholly on the cone represented by (1). Any point Q on this generator may be given by (rx',ry',rz'). The points $P(x',y',z')$ and $Q(rx',ry',rz')$ must satisfy equation (1). Hence we have

$$F(x',y',z') = ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2ux'$$
$$+2vy' + 2wz' + d = 0 \quad \ldots.(3)$$

and $F(rx',ry',rz')$
$$= r^2(ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y')$$
$$+2r(ux' + vy' + wz') + d = 0 \quad \ldots\ldots.(4)$$

Equation (4) is an identity as it is true for all values of r. Therefore the coefficients of r^2, r and the constant term must vanish separately, i.e.

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0 \ \ldots\ldots.(5)$$

$$ux' + vy' + wz' = 0 \ldots\ldots.(6)$$

and $d = 0$.

Equation (6) shows that if u, v and w are not all zero, then the locus of $P(x', y', z')$ will be the plane $ux + vy + wz = 0$. But this is against our assumption that the point P lies on the cone. Therefore we must have $u = 0, v = 0, w = 0$. Also we have $d = 0$, hence equation (1) gives

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \ldots\ldots (7)$$

This is a homogeneous equation of the second degree and represents a quadric cone with vertex as origin.

Conversely, if we are given a general homogeneous equation of second degree in x, y and z, then we can show that this represents a quadric cone with vertex as origin.

Suppose we are given equation (7). If a point $P(x', y', z')$ lies on the surface represented by (7), then

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0$$

Therefore, for all values of r, we have

$$a(rx')^2 + b(ry')^2 + c(rz')^2 + 2f(ry')(rz') + 2g(rz')(rx')$$
$$+2h(rx')(ry') = 0$$

This shows that if $P(x', y', z')$ lies on (7), then (rx', ry', rz') also lies on it. Therefore all the points on the line OP lies on the surface given by (7). Thus equation (7) represents a surface which is generated by the lines passing through the origin, i.e. equation (7) represents a cone with origin as its vertex. Therefore *every homogeneous equation of second degree always represents a quadric cone with vertex at the origin*.

You can check for yourself that a homogeneous equation of any degree represents a cone through the origin.

Deduction Let the equation of a cone with vertex as origin be given by

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

Let the equations of a generator be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} (= r)$$

Any point on this line will be (lr, mr, nr). These coordinates must satisfy the equation $F(x, y, z) = 0$ for all values of r i.e. $F(lr, mr, nr) = 0$. Therefore

$$al^2r^2 + bm^2r^2 + cn^2r^2 + 2fmnr^2 + 2gnlr^2 + 2hlmr^2 = 0$$

$$\text{or} \quad al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$$

which is the same as $F(l, m, n) = 0$. Thus the direction cosines of the generator satisfy the homogeneous equation of the cone with vertex at the origin.

6.4 ILLUSTRATIVE EXAMPLES

Example 6.4.1 Find the equation of the cone whose vertex is $(0,0,0)$ and which passes through the curve given by $lx + my + nz = p, ax^2 + by^2 = 2z$

Solution Remember if the guiding curve is given by two equations in which one equation is of first degree, then the equation of the cone is obtained by making the other equation homogeneous with the help of the first equation. So here we have

$$ax^2 + by^2 = 2z \quad \dots\dots(8)$$

$$\text{and } lx + my + nz = p \quad \text{or} \quad \frac{lx + my + nz}{p} = 1 \quad \dots.(9)$$

Making equation (8) homogeneous with the help of (9), we get the required equation of the cone with the vertex at origin as

$$ax^2 + by^2 = 2z\left(\frac{lx + my + nz}{p}\right)$$

or

$$p(ax^2 + by^2) = 2z(lx + my + nz)$$

Example 6.4.2 Find the equation of the cone whose vertex is $(0,0,0)$ and which passes through the circle given by $x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0, x - y + z = 2$

Solution We have

$$x^2 + y^2 + z^2 + x - 2y + 3z - 4 = 0 \quad \dots\dots..(10)$$

$$x - y + z = 2 \quad \text{or} \quad \frac{x - y + z}{2} = 1 \quad \dots\dots.(11)$$

Making equation (10) homogeneous with the help of (11) we have

$$x^2+y^2+z^2+(x-2y+3z)\left(\frac{x-y+z}{2}\right)-4\left(\frac{x-y+z}{2}\right)^2=0$$

or $$x^2+2y^2+3z^2+xy-yz=0$$

Example 6.4.3 Find the equation of the cone passing through the coordinate axes.

Solution Let the equation of the cone be given by

$$F(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

Since the dc's of the coordinate axes are 1,0,0; 0,1,0 and 0,0,1, hence they must satisfy the equation $F(x,y,z)=0$. Therefore

$$a.1+b.0+c.0+2f.0.0+2g.0.1+2h.1.0=0$$

$$a.0+b.1+c.0+2f.1.0+2g.0.0+2h.0.1=0$$

$$a.0+b.0+c.1+2f.0.1+2g.1.0+2h.0.0=0$$

Which give $a=0, b=0, c=0$. Hence the required equation of the cone becomes

$$0+0+0+2fyz+2gzx+2hxy=0$$

or $$fyz+gzx+hxy=0$$

6.5 CONDITION FOR THE GENERAL EQUATION OF SECOND DEGREE TO REPRESENT A CONE

Let a cone be represented by the following equation of second degree

$$F(x,y,z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy$$
$$+2wz+d=0 \quad \ldots\ldots(12)$$

Let (α,β,γ) be the vertex of the cone. Shifting the origin to the vertex (α,β,γ), the equation of the cone becomes

$$a(x+\alpha)^2+b(y+\beta)^2+c(z+\gamma)^2+2f(y+\beta)(z+\gamma)$$
$$+2g(z+\gamma)(x+\alpha)+2h(x+\alpha)(y+\beta)+2u(x+\alpha)$$
$$+2v(y+\beta)+2w(z+\gamma)+d=0$$

or $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2x(a\alpha+h\beta+g\gamma+u)$

$$+2y(h\alpha + b\beta + f\gamma + v) + 2z(g\alpha + f\beta + c\gamma + w)$$
$$+(a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d) = 0$$

…….

(13) Now (13) represents a cone with vertex at the origin and therefore it must be a homogeneous equation of second degree in x, y, z. Hence the coefficients of x, y, z and the absolute term must vanish separately, i.e.

$$a\alpha + h\beta + g\gamma + u = 0 \quad \dots\dots..(14)$$

$$h\alpha + b\beta + f\gamma + v = 0 \quad \dots\dots..(15)$$

$$g\alpha + f\beta + c\gamma + w = 0 \quad \dots\dots..(16)$$

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 2v\beta + 2w\gamma + d = 0$$

…..... (17)

Equation (17) can be written as

$$\alpha(a\alpha + h\beta + g\gamma + u) + \beta(h\alpha + b\beta + f\gamma + v) + \gamma(g\alpha + f\beta + c\gamma + w)$$
$$+(u\alpha + v\beta + w\gamma + d) = 0$$

Using (14), (15) and (16) we get

$$u\alpha + v\beta + w\gamma + d = 0 \quad \dots\dots..(18)$$

Eliminating α, β, γ between (14), (15), (16) and (18), we have

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \quad \dots\dots(19)$$

This is the required condition that equation (12) represents a cone.

The coordinates (α, β, γ) of the vertex can be obtained by solving any three of the equations (14), (15), (16) and (18).

There is an easy way to obtain the coordinates of the vertex. First make equation (12) homogeneous by introducing a fourth variable ‘t’. Let

$$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt$$
$$+2wzt + dt^2 = 0$$

Then you will observe that

$$\frac{\partial F}{\partial x} = 0 \Rightarrow ax + hy + gz + ut = 0$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow hx + by + fz + vt = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow gx + fy + cz + wt = 0$$

and

$$\frac{\partial F}{\partial t} = 0 \Rightarrow ux + vy + wz + td = 0$$

Putting $t = 1$, we get the equations

$$ax + hy + gz + u = 0, hx + by + fz + v = 0 gx + fy + cz + w = 0, ux + vy + wz + d = 0$$

If $x = \alpha, y = \beta, z = \gamma$ is the solution of any three of the above equations, then this solution must satisfy the remaining fourth equation also.

So the working rule for solving numerical problems is simple.

1. Make the equation homogeneous by introducing a fourth variable 't'

2. Solve the equations $\left(\frac{\partial F}{\partial x}\right)_{t=1} = 0, \left(\frac{\partial F}{\partial y}\right)_{t=1} = 0, \left(\frac{\partial F}{\partial z}\right)_{t=1} = 0$

3. If the solution thus obtained satisfies $\left(\frac{\partial F}{\partial t}\right)_{t=1} = 0$, then this solution gives the vertex of the cone represented by $F(x, y, z) = 0$.

6.6 EQUATION OF A CONE WITH GIVEN CONIC FOR BASE

Suppose we are given the following conic in xy-plane

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0 \ldots\ldots (20)$$

Now we shall find the equation of a cone whose vertex is given as (α, β, γ) and base the conic given by (20).

The equations of any straight line through the vertex (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots\ldots (21)$$

This line (21) meets the plane $z = 0$ (xy-plane) at the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}\ , z=0$$

$$i.e.\ \ x=\alpha-\frac{l\gamma}{n}, y=\beta-\frac{m\gamma}{n}, z=0$$

Hence the point is $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$.

If equations (21) represent the generator of the cone with base given by (20), this point must lie on (20). i.e.

$$a\left(\alpha-\frac{l\gamma}{n}\right)^2+2h\left(\alpha-\frac{l\gamma}{n}\right)\left(\beta-\frac{m\gamma}{n}\right)+b\left(\beta-\frac{m\gamma}{n}\right)^2+2g\left(\alpha-\frac{l\gamma}{n}\right)$$

$$+2f\left(\beta-\frac{m\gamma}{n}\right)+c=0 \qquad \ldots\ldots(22)$$

The required equation of the cone, i.e. the locus of the line (21) can be obtained by eliminating l, m, n between (21) and (22).

From (21), we have

$$\frac{l}{n}=\frac{x-\alpha}{z-\gamma}, \qquad \frac{m}{n}=\frac{y-\beta}{z-\gamma}$$

Substituting these values in (22), we have

$$\begin{aligned} a\left\{\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\}^2+2h\left\{\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\}\left\{\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\} \\ +b\left\{\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\}^2+2g\left\{\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right\} \\ +2f\left\{\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right\}+c=0 \end{aligned}$$

After simplification we get

$$a(\alpha z-\gamma x)^2+2h(\alpha z-x\gamma)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$

$$+2g(\alpha z-x\gamma)(z-\gamma)+2f(\beta z-y\gamma)(z-\gamma)+c(z-\gamma)^2=0$$

This is the required equation of the cone with vertex (α, β, γ) and base given by (20).

6.7 ILLUSTRATIVE EXAMPLES

Example 6.7.1 Prove that

$$ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$$

represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$

Solution Let (α,β,γ) be the vertex of the cone. Shifting the origin to the vertex (α,β,γ), the equation of the cone becomes

$$a(x+\alpha)^2+b(y+\beta)^2+c(z+\gamma)^2+2u(x+\alpha)+2v(y+\beta)+2w(z+\gamma)+d=0$$

or $ax^2+by^2+cz^2+2x(a\alpha+u)+2y(b\beta+v)+2z(c\gamma+w)$

$$+(a\alpha^2+b\beta^2+c\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0 \ldots.(23)$$

If this equation represents a cone, it must be a homogeneous equation of second degree in x,y,z. Hence the coefficients of x,y,z and the absolute term must vanish separately, i.e.

$$a\alpha+u=0, b\beta+v=0, c\gamma+w=0 \quad \ldots\ldots(24)$$

and $$a\alpha^2+b\beta^2+c\gamma^2+2u\alpha+2v\beta+2w\gamma+d=0 \ldots..(25)$$

From (24) we have

$$\alpha=-\frac{u}{a}, \qquad \beta=-\frac{v}{b}, \qquad \gamma=-\frac{w}{c}$$

Substituting these values in (25), we get

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}-\frac{2u^2}{a}-\frac{2v^2}{b}-\frac{2w^2}{c}+d=0$$

or $$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$$

Example 6.7.2 Find the equation of a cone whose vertex is the point (α,β,γ) and whose generating lines pass through the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$$

Solution Any line through (α,β,γ) is given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots\ldots(26)$$

This line (26) meets the plane $z=0$ at the point given by

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}\ , z=0$$

$$i.e.\ \ x=\alpha-\frac{l\gamma}{n}, y=\beta-\frac{m\gamma}{n}, z=0$$

Hence the point is $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$.

If equations (26) represent the generator of the cone, this point must lie on

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$$

Hence we have

$$\frac{\left(\alpha-\frac{l\gamma}{n}\right)^2}{a^2}+\frac{\left(\beta-\frac{m\gamma}{n}\right)^2}{b^2}=1$$

or $b^2(\alpha n-\gamma l)^2+a^2(\beta n-\gamma m)^2=a^2b^2n^2 \ \ldots..(27)$

Eliminating l, m, n between (26) and (27) we get

$$b^2\{\alpha(z-\gamma)-\gamma(x-\alpha)\}^2+a^2\{\beta(z-\gamma)-\gamma(y-\beta)\}^2= a^2b^2(z-\gamma)^2$$

or $b^2(\alpha z-\gamma x)^2+a^2(\beta z-\gamma y)^2=a^2b^2(z-\gamma)^2$

This is the required equation of the cone.

Example 6.7.3 Prove that the equation

$$4x^2-y^2+2z^2-3yz+2xy+12x-11y+6z+4=0$$

represents a cone. Find the coordinates of its vertex.

Solution Let us make the equation homogeneous by introducing variable t. Let

$$F(x,y,z,t)=4x^2-y^2+2z^2-3yz+2xy+12xt-11yt+6zt+4t^2=0$$

Now

$$\frac{\partial F}{\partial x}=8x+2y+12t, \qquad \frac{\partial F}{\partial y}=-2y+2x-3z-11t,$$

$$\frac{\partial F}{\partial z}=4z-3y+6t, \qquad \frac{\partial F}{\partial t}=12x-11y+6z+8t$$

Putting $t=1$ and equating to zero, we get

$$8x + 2y + 12 = 0 \ldots\ldots (28)$$

$$-2y + 2x - 3z - 11 = 0 \ldots\ldots (29)$$

$$4z - 3y + 6 = 0 \ \ldots\ldots (30)$$

$$12x - 11y + 6z + 8 = 0 \ \ldots\ldots (31)$$

Solving (28), (29) and (30) we get $x = -1, y = -2, z = -3$.

You can see that this solution satisfies (31) as

$$12(-1) - 11(-2) + 6(-3) + 8 = 0$$

Hence the given equation represents a cone with vertex $(-1, -2, -3)$.

Example 6.7.4 Show that the lines drawn through the point (α, β, γ) whose d.c.'s satisfy the relation $al^2 + bm^2 + cn^2 = 0$ generate the cone

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

Solution Equations of any line through (α, β, γ) with dc's l, m, n are given by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots (32)$$

It is given that the dc's satisfy the equation

$$al^2 + bm^2 + cn^2 = 0 \ldots\ldots (33)$$

Eliminating l, m, n between (32) and (33), we get the equation of the cone as

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0$$

Example 6.7.5 The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the coordinate axes in A, B, C. Prove that the equation to the cone generated by lines through O to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{a}{c} + \frac{c}{a}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

Solution First of all we shall find the equations of the circle through A, B, C. Obviously the coordinates of A, B, C are $(a, 0,0), (0, b, 0), (0,0, c)$ respectively. From your study of sphere you know that the equation of the sphere through O, A, B, C is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Therefore the equations of the circle ABC are given by

$$x^2 + y^2 + z^2 - ax - by - cz = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \ldots\ldots(33)$$

We make the first equation homogeneous with the help of the second equation as follows

$$x^2 + y^2 + z^2 - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

$$\text{i.e. } yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{a}{c} + \frac{c}{a}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

Example 6.7.6 Find the equation of the cone whose vertex is (1,2,3) and guiding curve is the circle $x^2 + y^2 + z^2 = 4, x + y + z = 1$

Solution Any straight line through (1,2,3) can be given by

$$\frac{x-1}{l} = \frac{y-2}{m} = \frac{z-3}{n} = r \text{ (say)} \ldots\ldots(34)$$

The equations of the circle are

$$x^2 + y^2 + z^2 = 4, x + y + z = 1 \quad \ldots\ldots(35)$$

Any point on (34) is $(lr + 1, mr + 2, nr + 3)$. If the line (34) meets the circle (35) in this point, then

$$(lr + 1)^2 + (mr + 2)^2 + (nr + 3)^2 = 4 \quad \ldots\ldots(36)$$

$$(lr + 1) + (mr + 2) + (nr + 3) = 1 \quad \ldots\ldots(37)$$

From (37) we have

$$r = \frac{-5}{l + m + n}$$

Putting this value in (36) we get

$$\left(\frac{-5l}{l + m + n} + 1\right)^2 + \left(\frac{-5m}{l + m + n} + 2\right)^2 + \left(\frac{-5n}{l + m + n} + 3\right)^2 = 4$$

$$\text{or } (m + n - 4l)^2 + (2l - 3m + 2n)^2 + (3l + 3m - 2n)^2 = 4(l + m + n)^2 \quad \ldots(38)$$

Eliminating l, m, n between (34) and (38), i.e. replacing l, m, n in (38) by $(x - 1), (y - 2)$ and $(z - 3)$ respectively, we have

$$(y + z - 4x - 1)^2 + (2x - 3y + 2z - 2)^2 + (3x + 3y - 2z - 3)^2$$

$$= 4(x + y + z - 6)^2$$

$$\text{or} \quad 5x^2 + 3y^2 + z^2 + 6yz - 4zx - 2xy + 6x + 8y + 10z - 26 = 0$$

This is the required equation of the cone.

Example 6.7.7 Show that cone of the second degree can be found to pass through two sets of rectangular axes through the same origin.

Solution Let the first system of rectangular axes consist of the coordinate axes and the second one consist of rectangular axes with dc's l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3.

We know that the equation of the cone through the coordinate axes is given by

$$fyz + gzx + hxy = 0 \quad \ldots\ldots(39)$$

Now the dc's of the lines must satisfy the homogeneous equation (39). For the first two lines with dc's l_1, m_1, n_1 and l_2, m_2, n_2, we have

$$fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \quad \ldots\ldots(40)$$

$$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0 \quad \ldots\ldots(41)$$

Adding (40) and (41), we get

$$f(m_1n_1 + m_2n_2) + g(n_1l_1 + n_2l_2) + h(l_1m_1 + l_2m_2) = 0 \quad \ldots\ldots(42)$$

Since the axes are rectangular, we have

$l_1m_1 + l_2m_2 + l_3m_3 = 0,\ m_1n_1 + m_2n_2 + m_3n_3 = 0,\ n_1l_1 + n_2l_2 + n_3l_3 = 0$

$\Rightarrow l_1m_1 + l_2m_2 = -l_3m_3,\ m_1n_1 + m_2n_2 = -m_3n_3,\ n_1l_1 + n_2l_2 = -n_3l_3$

Hence (42) becomes

$$-fl_3m_3 - gm_3n_3 - hn_3l_3 = 0$$

$$\text{or} \quad fl_3m_3 + gm_3n_3 + hn_3l_3 = 0$$

This shows that the third axis with dc's l_3, m_3, n_3 also lies on the cone. Hence the result.

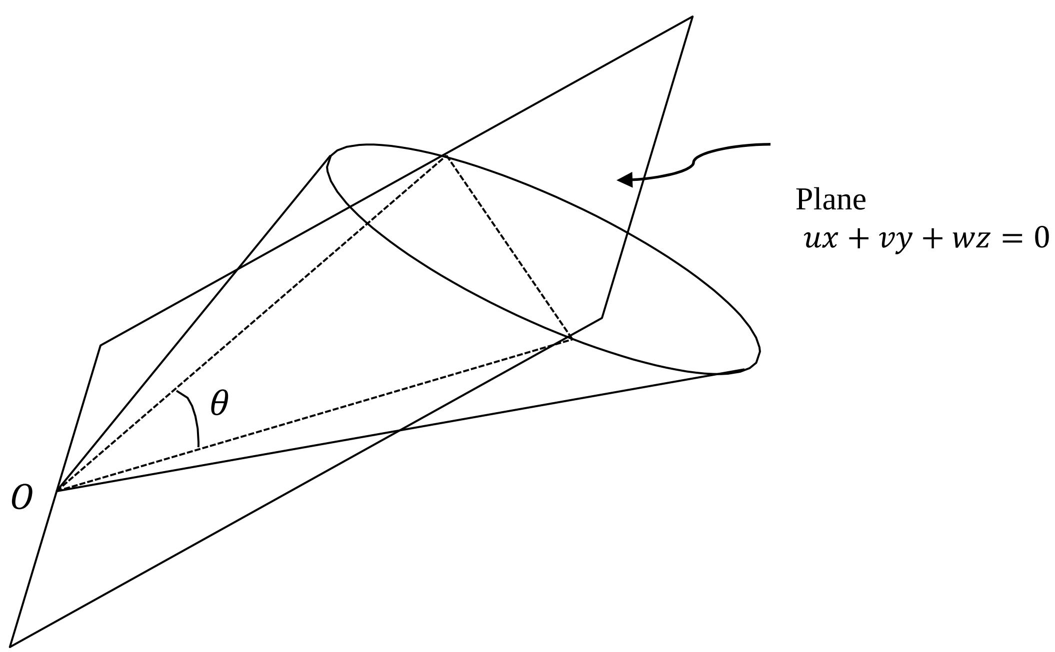
6.8 THE ANGEL BETWEEN THE LINES IN WHICH A PLANE CUTS A CONE

We shall find the angle between the lines in which the plane

$$ux + vy + wz = 0 \quad \dots\dots(43)$$

cuts the cone

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \dots\dots(44)$$



Since the cone (44) has the vertex at the origin and the plane (43) also passes through the origin, hence the line of intersection will pass through the origin. The equations of this line of intersection can be given as

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r \text{ (say)} \quad \dots\dots(45)$$

Any point on this line will be (lr, mr, nr). This point must lie on the plane (43) and the cone (44). Hence

$$ulr + vmr + wnr = 0 \text{ or } ul + vm + wn = 0 \dots\dots(46)$$

$$\text{and } r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) = 0$$

$$\text{i.e.} \quad al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \dots\dots(47)$$

From (46) we have $l = -\frac{(vm+wn)}{u}$. Putting this value in (47) we get

$$a\frac{(vm + wn)^2}{u^2} + bm^2 + cn^2 + 2fmn - 2gn\frac{(vm + wn)}{u} - 2hm\frac{(vm + wn)}{u} = 0$$

$$\text{or} \quad (av^2 + bu^2 - 2huv)\left(\frac{m}{n}\right)^2 + 2(avw - uvg - uhw + fu^2)\left(\frac{m}{n}\right)$$

$$+(aw^2 + cu^2 - 2guw) = 0 \quad \ldots\ldots (48)$$

This is a quadratic equation in (m/n) and it shows that the plane (43) cuts the cone (44) in two lines. Let l_1, m_1, n_1 and l_2, m_2, n_2 be the dc's of these lines. Thus the equation (48) will have two roots- (m_1/n_1) and (m_2/n_2) (say). Therefore

$$\frac{m_1}{n_1} + \frac{m_2}{n_2} = -\frac{2(avw - uvg - uhw + fu^2)}{(av^2 + bu^2 - 2huv)} \quad \ldots\ldots (49)$$

$$\text{and} \qquad \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{(aw^2 + cu^2 - 2guw)}{(av^2 + bu^2 - 2huv)}$$

$$\text{or} \qquad \frac{m_1 m_2}{(aw^2 + cu^2 - 2guw)} = \frac{n_1 n_2}{(av^2 + bu^2 - 2huv)}$$

By symmetry we can write

$$\frac{l_1 l_2}{(cv^2 + bw^2 - 2fvw)} = \frac{m_1 m_2}{(aw^2 + cu^2 - 2guw)}$$
$$= \frac{n_1 n_2}{(bu^2 + av^2 - 2huv)} \quad \ldots (50)$$

From (49) we have

$$\frac{m_1 n_2 + m_2 n_1}{n_1 n_2} = -\frac{2(avw - uvg - uhw + fu^2)}{(av^2 + bu^2 - 2huv)}$$

$$\text{or} \quad \frac{m_1 n_2 + m_2 n_1}{-2(avw - uvg - uhw + fu^2)} = \frac{n_1 n_2}{(av^2 + bu^2 - 2huv)}$$

$$= \frac{m_1 m_2}{(aw^2 + cu^2 - 2guw)} = \lambda \text{ (say), using (50)}$$

Now $(m_1 n_2 - m_2 n_1)^2 = (m_1 n_2 + m_2 n_1)^2 - 4 m_1 m_2 n_1 n_2$

$$= 4\lambda^2 (avw - uvg - uhw + fu^2)^2$$
$$- 4\lambda^2 (aw^2 + cu^2 - 2guw)(av^2 + bu^2 - 2huv)$$

$$= 4\lambda^2 w^2 P^2$$

Where $P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$

Hence we have $m_1n_2 - m_2n_1 = \pm 2\lambda wP$. Similarly $n_1l_2 - n_2l_1 = \pm 2\lambda uP$ and $l_1m_2 - l_2m_1 = \pm 2\lambda vP$

If θ is the angle between the lines in which the plane (43) cuts the cone (44), then

$$\tan\theta = \frac{\sqrt{(n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2}$$

$$= \frac{\sqrt{4\lambda^2u^2P^2 + 4\lambda^2v^2P^2 + 4\lambda^2w^2P^2}}{\lambda(cv^2 + bw^2 - 2fvw) + \lambda(aw^2 + cu^2 - 2guw) + \lambda(bu^2 + av^2 - 2huv)}$$

$$\text{or} \quad \tan\theta = \frac{2P(u^2 + v^2 + w^2)^{1/2}}{(a + b + c)(u^2 + v^2 + w^2) - F(u, v, w)}$$

Corollary 1. Condition of perpendicularity

If the lines are perpendicular, we have $\theta = \frac{\pi}{2}$ and hence

$$\tan\theta = \infty$$

$$\Longrightarrow (a + b + c)(u^2 + v^2 + w^2) - F(u, v, w) = 0$$

$$\Longrightarrow (a + b + c)(u^2 + v^2 + w^2) = F(u, v, w)$$

Corollary 2. Condition of tangency of a plane and a cone

The plane (43) touches the cone (44), if the lines are coincident.

i.e. $\theta = 0$ and hence

$$\tan\theta = 0$$

$$\Longrightarrow 2P(u^2 + v^2 + w^2)^{1/2} = 0$$

$$\Longrightarrow P = 0$$

$$\Longrightarrow \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0$$

$$\Longrightarrow Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$$

Where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h respectively in the determinant

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\text{i.e. } A = bc - f^2, B = ca - g^2, C = ab - h^2$$

$$F = gh - af, G = hf - bg, H = fg - ch$$

6.9 THREE MUTUALLY PERPENDICULAR GENERATORS OF A CONE

Let the equation of the cone be given as

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \;\ldots\ldots(51)$$

Suppose the plane $ux + vy + wz = 0$ cuts the cone (51) in perpendicular generators. Then from the condition of perpendicularity we have

$$(a + b + c)(u^2 + v^2 + w^2) - F(u, v, w) = 0 \;\ldots.(52)$$

Now the equations of the normal to the plane through the origin (i.e. vertex of the cone) are

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w} \;\ldots\ldots(53)$$

This line will be a generator of the cone if it lies on the surface of the cone, i.e. the dc's of the line satisfies the equation of the cone

$$F(u, v, w) = 0$$

Hence from (52) we have

$$(a + b + c)(u^2 + v^2 + w^2) = 0$$

$$\Rightarrow a + b + c = 0 \quad \text{since } u^2 + v^2 + w^2 \neq 0$$

Therefore the condition that the cone (51) may have three perpendicular generators is

$$a + b + c = 0$$

Conversely, if the above condition is satisfied and if the normal (53) lies on the cone (51), then $F(u, v, w) = 0$ and hence the condition (52) gets satisfied for values of u, v, w. Therefore a plane which is perpendicular to a generator cuts the cone in two mutually perpendicular generators. Hence many sets of three mutually perpendicular generators exist.

6.10 ILLUSTRATIVE EXAMPLES

Example 6.10.1 Find the equations of the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Find also the angle between the lines of section.

Solution: The equation of the cone is

$$4x^2 - y^2 + 3z^2 = 0 \quad \ldots..(54)$$

and the given plane is

$$2x + y - z = 0 \quad \ldots\ldots(55)$$

Let the plane (55) cut the cone (54) in the lines given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Since the line lies on the plane and the cone, hence

$$4l^2 - m^2 + 3n^2 = 0 \quad \ldots(56)$$

$$\text{and} \quad 2l + m - n = 0 \ldots.(57)$$

From (57), we have $\quad n = 2l + m \quad \ldots(58)$

Substituting this value in (56), we have

$$4l^2 - m^2 + 3(2l + m)^2 = 0$$

$$\text{or} \quad 8l^2 + 6lm + m^2 = 0$$

$$\text{or} \quad (2l + m)(4l + m) = 0$$

$$2l + m = 0 \text{ or } 4l + m = 0$$

When $2l + m = 0$, we have $m = -2l$. Therefore from (58),

$$n = 2l + m = 2l - 2l = 0$$

Hence

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{0}$$

The corresponding line of section is

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0} \quad \ldots..(58)$$

When $4l + m = 0$, we have $m = -4l$. Therefore from (58),

$$n = 2l + m = 2l - 4l = -2l$$

Hence

$$\frac{l}{1} = \frac{m}{-4} = \frac{n}{-2}$$

The corresponding line of section is

$$\frac{x}{1} = \frac{y}{-4} = \frac{z}{-2} \quad \ldots..(59)$$

If θ is the angle between the lines of section (58) and (59), then

$$\cos\theta = \frac{1.1 + (-2)(-4) + 0.(-2)}{\sqrt{\{1^2 + (-2)^2 + 0^2\}\{1^2 + (-4)^2 + (-2)^2\}}} = \sqrt{\frac{27}{35}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\sqrt{\frac{27}{35}}\right)$$

Example 6.10.2 Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Solution Let the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in lines given by

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \ldots\ldots..(60)$$

Since the lines lie on the plane and the cone, hence

$$al + bm + cn = 0 \quad \ldots.(61)$$

and $$mn + nl + lm = 0 \quad \ldots..(62)$$

From (61) we have

$$l = -\frac{(bm + cn)}{a}$$

Putting this value in (62) we get

$$mn - \frac{n(bm + cn)}{a} - \frac{m(bm + cn)}{a} = 0$$

or $$bm^2 + mn(b + c - a) + cn^2 = 0$$

or $$b\left(\frac{m}{n}\right)^2 + (b + c - a)\left(\frac{m}{n}\right) + c = 0 \quad \ldots..(63)$$

This is a quadratic equation in (m/n) with two roots, say (m_1/n_1) and (m_2/n_2). It shows that the plane cuts the cone in two lines. Now

$$\text{The product of roots} = \frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{c}{b}$$

$$\text{or} \qquad \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

By symmetry we can write

$$\frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

The lines of section are perpendicular if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Hence the result.

6.11 THE TANGENT LINE AND THE TANGENT PLANE

Now we shall find the condition that a given line through the point (α, β, γ)on the cone is a tangent line to the cone and we shall also obtain the equation of tangent plane at that point.

Let the equation of the cone be given as

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots\ldots (64)$$

Let $P(\alpha, \beta, \gamma)$ be any point on the cone (64). The equations of any line through $P(\alpha, \beta, \gamma)$ may be given as

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \text{ (say)} \qquad \ldots\ldots (65)$$

The coordinates of any point on this line will be $(lr + \alpha, mr + \beta, nr + \gamma)$. If the line (65) meets the cone (64) at this point, then

$$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 + 2f(mr + \beta)(nr + \gamma)$$
$$+2g(nr + \gamma)(lr + \alpha) + 2h(lr + \alpha)(mr + \beta) = 0$$

or $r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm)$

$$+2r\{(a\alpha + h\beta + g\gamma)l + (h\alpha + b\beta + f\gamma)m + (g\alpha + f\beta + c\gamma)n\}$$

$$+(a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta) = 0 \ldots\ldots (66)$$

Since the point $P(\alpha, \beta, \gamma)$ lies on the cone, hence

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0 \ldots. (67)$$

Hence (66) becomes

$r^2F(l,m,n) + 2r\{(a\alpha + h\beta + g\gamma)l + (h\alpha + b\beta + f\gamma)m + (g\alpha + f\beta + c\gamma)n\} = 0$

This is a quadratic equation in r with one root equal to zero. If the line (65) is a tangent to the cone, the other root must also vanish, i.e.

$$(a\alpha + h\beta + g\gamma)l + (h\alpha + b\beta + f\gamma)m + (g\alpha + f\beta + c\gamma)n = 0 \ldots (68)$$

This is the condition that the line (65) is a tangent line to the cone at $P(\alpha,\beta,\gamma)$.

Now the tangent plane to the cone (64) at point $P(\alpha,\beta,\gamma)$ is the locus of the tangent line (65) under the condition (68). Hence the equation of the tangent plane at $P(\alpha,\beta,\gamma)$ is obtained by substituting the proportionate values of l, m, n from (65) in equation (68) as

$(a\alpha + h\beta + g\gamma)(x-\alpha) + (h\alpha + b\beta + f\gamma)(y-\beta) + (g\alpha + f\beta + c\gamma)(z-\gamma) = 0$

or $(a\alpha + h\beta + g\gamma)x + (h\alpha + b\beta + f\gamma)y + (g\alpha + f\beta + c\gamma)z$

$$-(a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta) = 0$$

Using (67), we get the required equation of the tangent plane to the cone (64) at $P(\alpha,\beta,\gamma)$ as

$$(a\alpha + h\beta + g\gamma)x + (h\alpha + b\beta + f\gamma)y + (g\alpha + f\beta + c\gamma)z = 0$$

6.12 THE CONDITION OF TANGENCY

In this section we shall obtain the condition that the plane

$$ux + vy + wz = 0 \quad \ldots\ldots(69)$$

May touch the cone

$$F(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots\ldots(70)$$

If the plane (69) touches the cone (70) at the point $P(\alpha,\beta,\gamma)$, then it must be identical to the tangent plane at P

$$(a\alpha + h\beta + g\gamma)x + (h\alpha + b\beta + f\gamma)y + (g\alpha + f\beta + c\gamma)z = 0 \ldots.(71)$$

Since (69) and (71) represent the same plane, hence comparing the coefficients of x, y and z, we have

$$\frac{u}{a\alpha + h\beta + g\gamma} = \frac{v}{h\alpha + b\beta + f\gamma} = \frac{w}{g\alpha + f\beta + c\gamma} = -\frac{1}{k} \text{ (say)}$$

Therefore

$$a\alpha + h\beta + g\gamma + uk = 0 \ldots\ldots (72)$$

$$h\alpha + b\beta + f\gamma + vk = 0 \ldots\ldots (73)$$

$$g\alpha + f\beta + c\gamma + wk = 0 \ldots\ldots (74)$$

Also the point (α, β, γ) lies on the plane (69), hence

$$u\alpha + v\beta + w\gamma = 0 \ldots\ldots (75)$$

Eliminating α, β, γ and k from (72), (73), (74) and (75), we get the required condition as

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0 \quad \ldots\ldots (76)$$

$$\Rightarrow Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$$

Where $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af,$

$G = hf - bg, H = fg - ch$

6.13 RECIPROCAL CONE

Definition The reciprocal cone of a given cone is the locus of the lines through the vertex and right angles to the tangent planes of the given cone. In other words, the reciprocal cone of a given cone is the locus of the normals through the vertex to the tangent planes of the given cone.

Let the equation of the cone be given as

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \ldots\ldots (77)$$

Let the equation of a tangent plane to the cone (77) be

$$ux + vy + wz = 0 \ldots\ldots (78)$$

The dc's of the normal to the plane (78) are proportional to u, v, w. The vertex of the cone is the origin (0,0,0) Therefore the equations to the normal to the tangent plane (78) through the origin will be

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w} \quad \ldots\ldots (79)$$

The condition that the plane (78) touches the cone (77) is

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0 \ldots\ldots (80)$$

Where $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$, $F = gh - af$,

$G = hf - bg$, $H = fg - ch$

The locus of the normal (79) is obtained by eliminating u, v, w between (79) and (80), i.e.

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \dots \dots (81)$$

This is a homogeneous equation of second degree in x, y, z and therefore represents a cone with vertex at the origin. This is the reciprocal cone of the given cone (77).

6.14 ENVELOPING CONE

As the name suggests, it is a cone which envelopes a given surface. You will learn more about the enveloping cone of conicoids in unit-8. Let us begin with the formal definition:

Definition The locus of the tangent lines drawn from a given point to a given surface is called the enveloping cone or tangent cone to the surface. The point from which the tangent lines are drawn is called the vertex of the enveloping cone.

Let us derive the equation of the enveloping cone of a sphere with vertex at (α, β, γ). Suppose the equation of the sphere be given as

$$x^2 + y^2 + z^2 = a^2 \qquad \dots . (82)$$

The equations of a straight line passing through the point (α, β, γ) are given as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \ (= r) \dots \dots (83)$$

Then the coordinates of any point P on the straight line (83) are given by $(lr + \alpha, mr + \beta, nr + \gamma)$. If the line (83) meets the sphere (82) at point P, then

$$(lr + \alpha)^2 + (mr + \beta)^2 + (nr + \gamma)^2 = a^2$$

or $$r^2(l^2 + m^2 + n^2) + 2r(\alpha l + \beta m + \gamma n) + \alpha^2 + \beta^2 + \gamma^2 - a^2 = 0 \dots . (84)$$

This is a quadratic equation in r giving two roots, corresponding to which we have two points of intersection of the line (83) and the sphere (82). If the line (83) is a tangent line to the sphere, the points of intersection must coincide, i.e. the roots of the quadratic equation (84) must be equal, i.e.

$$\{2(\alpha l + \beta m + \gamma n)\}^2 = 4(l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

or $$(\alpha l + \beta m + \gamma n)^2 = (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2) \dots \dots (85)$$

The locus of the tangent line (83) gives the required enveloping cone of the sphere and can be obtained by eliminating l, m, n from (83) and (85), i.e.

$$\{\ (x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)\}^2$$

$$=\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}(\alpha^2+\beta^2+\gamma^2-a^2)$$

or $\{(\alpha x+\beta y+\gamma z-a^2)-(\alpha^2+\beta^2+\gamma^2-a^2)\}^2$

$$=\{(x^2+y^2+z^2-a^2)-2(\alpha x+\beta y+\gamma z-a^2)+(\alpha^2+\beta^2+\gamma^2-a^2)\}$$

$$\times(\alpha^2+\beta^2+\gamma^2-a^2)\ \ldots\ (86)$$

Let
$$\left.\begin{aligned} S &\equiv x^2+y^2+z^2-a^2\\ S' &\equiv \alpha^2+\beta^2+\gamma^2-a^2\\ T &\equiv \alpha x+\beta y+\gamma z-a^2 \end{aligned}\right]$$

Then (86) becomes $(T-S')^2=(S-2T+S')S'$

$$\text{or } T^2+S'^2-2TS'=SS'-2TS'+S'^2$$

$$\text{or }\ SS'=T^2$$

or $\quad (x^2+y^2+z^2-a^2)(\alpha^2+\beta^2+\gamma^2-a^2)=(\alpha x+\beta y+\gamma z-a^2)^2$

This is the required equation of the enveloping cone of the sphere.

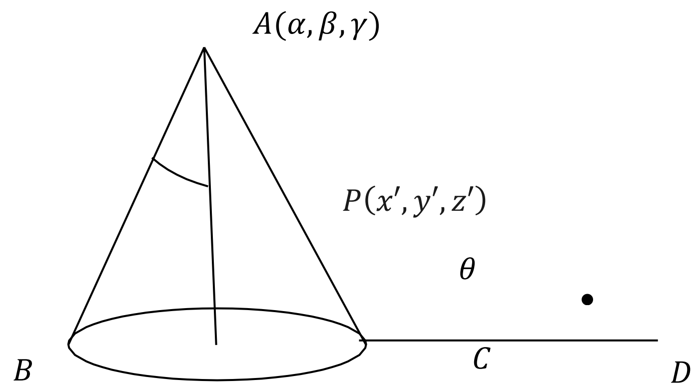
6.15 RIGHT CIRCULAR CONE

Definition A **right circular cone** is a surface generated by a moving line which passes through a fixed point (called **vertex**) and makes a constant angle θ with a fixed straight line through the vertex.

The constant angle θ is called the **semi-vertical angle** and the fixed straight line through the vertex is called the **axis** of the cone. The section of the right circular cone by a plane perpendicular to its axis is a circle.

Now we shall obtain the equation of a right circular cone with vertex at $A(\alpha,\beta,\gamma)$ and the axis AC with direction cosines proportional to l, m, n. The equations of the axis AC can be given as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ \ldots.\ (87)$$



Let $P(x', y', z')$ be any point on the surface of the cone. Then the direction cosines of the line AP are proportional to $x' - \alpha, y' - \beta, z' - \gamma$. If θ is the semi-vertical angle of the cone, then we have

$$\cos\theta = \frac{l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)}{\sqrt{l^2 + m^2 + n^2}\sqrt{(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2}}$$

or $\{l(x' - \alpha) + m(y' - \beta) + n(z' - \gamma)\}^2$

$$= (l^2 + m^2 + n^2)\{(x' - \alpha)^2 + (y' - \beta)^2 + (z' - \gamma)^2\} \cos^2\theta$$

Generalizing the coordinates (x', y', z'), we get the required equation of the right circular cone as

$\{l(x - \alpha) + m(y - \beta) + n(z - \gamma)\}^2$

$$= (l^2 + m^2 + n^2)\{(x - \alpha)^2 + (y - \beta)^2 + (z - \gamma)^2\} \cos^2\theta \text{ ... ... (88)}$$

6.16 ILLUSTRATIVE EXAMPLES

Example 6.16.1 Find the equation of the cone reciprocal to the cone

$$fyz + gzx + hxy = 0 \text{ .... (89)}$$

Solution Comparing the equation (89) with the general equation

$$a_0x^2 + b_0y^2 + c_0z^2 + 2f_0yz + 2g_0zx + 2h_0xy = 0 \quad \text{... (90)}$$

We have $a_0 = 0$, $b_0 = 0$, $c_0 = 0$, $f_0 = f/2$, $g_0 = f/2$, $h_0 = f/2$

Now the reciprocal cone of (90) is given by

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{.... (91)}$$

Where $= b_0c_0 - f_0^2 = -f^2/4$, $\quad B = c_0a_0 - g_0^2 = -g^2/4$,

$C = a_0b_0 - h_0^2 = -h^2/4$, $\quad F = g_0h_0 - a_0f_0 = gh/4$,

$$G = h_0 f_0 - b_0 g_0 = hf/4, \qquad H = f_0 g_0 - c_0 h_0 = fg/4$$

Putting these values in (91), we get

$$-\frac{f^2}{4}x^2 - \frac{g^2}{4}y^2 - \frac{h^2}{4}z^2 + 2.\frac{gh}{4}yz + 2.\frac{hf}{4}zx + 2.\frac{fg}{4}xy = 0$$

or $$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0$$

Example 6.16.2 Find the equation of the right circular cone with vertex at (1, −2, −1), semi-vertical angle 60^0 and the axis

$$\frac{x-1}{3} = \frac{y+2}{-4} = \frac{z+1}{5}$$

Solution The equation of right cicular cone is given by (88). Therefore we have

$$\{3(x-1) - 4(y+2) + 5(z+1)\}^2$$
$$= \{3^2 + (-4)^2 + 5^2\}\{(x-1)^2 + (y+2)^2 + (z+1)^2\} cos^2 60^0$$

Simplifying it we get

$$7x^2 - 7y^2 - 25z^2 + 80yz - 60zx + 48xy + 22x + 4y + 170z + 78 = 0$$

Example 6.16.3 Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the paraboloid $ax^2 + by^2 = 2cz$

Solution Let (α, β, γ) be a given point and

$$\left.\begin{array}{r} S \equiv ax^2 + by^2 - 2cz \\ S' \equiv a\alpha^2 + b\beta^2 - 2c\gamma \\ T \equiv a\alpha x + b\beta y - c(z+\gamma) \end{array}\right] \ldots\ldots(92)$$

Then the equation of the enveloping cone of the given paraboloid with vertex at

(α, β, γ) is

$$SS' = T^2$$

or $(ax^2 + by^2 - 2cz)(a\alpha^2 + b\beta^2 - 2c\gamma) = \{a\alpha x + b\beta y - c(z+\gamma)\}^2$ … (93)

Since three mutually perpendicular tangent lines are drawn to the given paraboloid from $P(\alpha, \beta, \gamma)$, the enveloping cone (93) will have three mutually perpendicular generators. Hence we must have

$$\text{coeff. of } x^2 + \text{coeff. of } y^2 + \text{coeff. of } z^2 = 0$$

or $$a(b\beta^2 - 2c\gamma) + b(a\alpha^2 - 2c\gamma) - c^2 = 0$$

or $$ab(\alpha^2 + \beta^2) - 2c(a+b)\gamma = c^2$$

The locus of the point (α, β, γ) is

$$(x^2 + y^2) - 2c(a+b)z = c^2$$

6.17 SUMMARY

In this unit, we have studied the following facts-

(1) A cone is a surface generated by a moving straight line passing through a fixed point and intersecting a given curve or touching a given surface.

(2) The condition that the general equation of second degree $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ represents a cone is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

(3) The equation of the cone with vertex $(\alpha,\ \beta,\ \gamma)$ and base conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ is given by $a(\alpha z - \gamma x)^2 + 2h(\alpha z - x\gamma)(\beta z - \gamma y) + b(\beta z - \gamma y\ )^2$ $+2g(\alpha z - x\gamma)(z - \gamma) + 2f(\beta z - y\gamma)(z - \gamma) + c(z - \gamma)^2 = 0$

(4) The angle between the lines in which the plane $ux + vy + wz = 0$ cuts the cone $F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is given by

$$\tan\theta = \frac{2P(u^2 + v^2 + w^2)^{1/2}}{(a + b + c)(u^2 + v^2 + w^2) - F(u, v, w)}$$

(5) The condition that the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has three mutually perpendicular generators is $a + b + c = 0$

(6) The equation of the tangent plane to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ at point $(\alpha,\ \beta,\ \gamma)$ is given by $(a\alpha + h\beta + g\gamma)x + (h\alpha + b\beta + f\gamma)y + (g\alpha + f\beta + c\gamma)z = 0$

(7) The condition that the plane

$$ux + vy + wz = 0$$

may touch the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is given by

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0$$

Where $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af,$

$G = hf - bg,\ H = fg - ch$

(8) The reciprocal cone of a given cone is the locus of the lines through the vertex and right angles to the tangent planes of the given cone. The reciprocal cone of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

is given by

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

Where $A = bc - f^2, B = ca - g^2, C = ab - h^2, F = gh - af,$

$G = hf - bg, H = fg - ch$

(9) The locus of the tangent lines drawn from a given point to a given surface is called the enveloping cone or tangent cone to the surface. The point from which the tangent lines are drawn is called the vertex of the enveloping cone.

(10) A right circular cone is a surface generated by a moving line which passes through a fixed point (called vertex) and makes a constant angle θ with a fixed straight line through the vertex. The constant angle θ is called the semi-vertical angle and the fixed straight line through the vertex is called the axis of the cone.

6.18 SELF ASSESSMENT QUESTIONS

(1) Prove that the equation of the cone whose vertex is the origin and base is the curve $f(x, y) = 0, z = k$ is $f(xk/z, yk/z) = 0$.

(2) Find the equation of the cone with the vertex at the origin and which passes through the curve $x^2 + y^2 + z^2 + x - 2y + z - 4 = 0$, $x^2 + y^2 + z^2 + 2x - 3y + 4z - 5 = 0$

[Ans: $2x^2 + y^2 - 5xy - 3yz + 4zx = 0$]

(3) Find the equation of the cone with vertex at the origin and direction cosines of its generators satisfy the relation $l^2 + 2m^2 - 3n^2 = 0$.

[Ans: $x^2 + 2y^2 - 3z^2 = 0$]

(4) Prove that a line which passes through (α, β, γ) and intersects the parabola $z^2 = 4ax$, $y = 0$ lies on the cone $(\beta z - \gamma y)^2 = 4a(\beta - y)(\beta x - \alpha y)$.

(5) Find the equation of the cone with vertex $(5,4,3)$ and with $3x^2 + 2y^2 = 6$, $y + z = 0$ as base.

[Ans: $147x^2 + 87y^2 + 101z^2 + 90yz - 210zx - 210xy + 84y + 84z - 294 = 0$]

(6) Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal to each other.

(7) Prove that the equation $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ represents a cone that touches the coordinate planes and find the equation of its reciprocal cone.

[Ans: $fyz + gzx + hxy = 0$]

(8) If the plane $2x - y + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines, find the value of c. [Ans: $c = 2$]

(9) Prove that the condition that the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

may have three mutually perpendicular tangent planes is

$$A + B + C = 0$$

Where $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$.

(10) Find the equation to the right circular cone whose vertex is $(-2, -3, 5)$, axis makes equal angles with the coordinate axes and semi-vertical angle is 30^0.

[Ans: $5x^2 + 5y^2 + 5z^2 - 8yz - 8zx - 8xy - 4x + 86y - 58z + 278 = 0$]

(11) Find the equation of the cone formed by rotating the line $2x + 3y = 6, z = 0$ about the y-axis. [Ans: $4x^2 - 9y^2 + 4z^2 + 36y - 36 = 0$]

(12) Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2y + 6z + 2 = 0$ with its vertex at $(1,1,1)$.

[Ans: $8x^2 + 9y^2 - 7z^2 - 8zx - 8x - 18y + 22z + 2 = 0$]

(13) Find the equation to the right circular cone whose vertex is the origin, axis is x- axis and semi-vertical angle is α. [Ans: $y^2 + z^2 = x^2 \tan^2 \alpha$]

6.19 FURTHER READINGS

(1) Shanti Narayan, P.K. Mittal (2007): Analytical Solid Geometry, S.Chand Publication, New Delhi.

(2) Abraham Adrian Albert (2016): Solid Analytic Geometry, Dover Publication.

(3) George Wentworth, D.E. Smith (2007): Plane and solid Geometry, Merchant books.

(4) D.M.Y. Sommerville (2016): Analytical Geometry of three dimensions, Cambridge university Press.

UNIT-7 CENTRAL CONICOIDS I

Structure

7.1 INTRODUCTION

In two dimensional geometry you studied conic sections (or conics) such as Circle, ellipse, hyperbola and parabola. A conic can be described as the intersection of a plane and a double-napped cone.

Do you know what happens when these conics are revolved about certain specific axes? It generates some interesting surfaces. For example, we obtain the surface of a sphere when a circle is revolved about its diameter. If an ellipse is revolved about its major or minor axis, a surface called **spheroid or ellipsoid of revolution** is obtained. Similarly we obtain a **paraboloid of revolution** by revolving a parabola about its axis of symmetry and a **hyperboloid of revolution** by revolving a hyperbola about conjugate axis or transverse axis. Let us see how this happens.

In general a surface is a locus of a variable point (x, y, z) represented by $f(x, y, z) = c$. We may assume it to be generated by a plane curve by revolving about an axis. If a plane curve $f(x, y) = 0, z = 0$ is revolved about x-axis, we obtain a surface of revolution given by $f\left(x, \sqrt{y^2 + z^2}\right) = 0$.

Suppose we revolve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ about its major axis (i.e. x-axis). The ellipsoid of revolution obtained is given by

$$\frac{x^2}{a^2}+\frac{\left(\sqrt{y^2+z^2}\right)^2}{b^2}=1$$

$$\text{or} \quad \frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

This surface is called a **prolate spheroid**. Similarly, if this ellipse is revolved about its minor axis (i.e. y-axis), we obtain an ellipsoid of revolution called **oblate spheroid**

$$\frac{x^2+z^2}{a^2}+\frac{y^2}{b^2}=1$$

Similarly, you can obtain a paraboloid of revolution $y^2+z^2=4ax$ by revolving the parabola $y^2=4ax, z=0$ about x-axis and hyperboloid of revolution

$$\frac{x^2+z^2}{a^2}-\frac{y^2}{b^2}=1$$

by revolving the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, z=0$ about conjugate axis (y-axis). **These surfaces are special cases of more general surfaces called conicoids.**

A **conicoid** is a surface whose sections by some specific planes (such as planes parallel to coordinate planes) are conics. For some conicoids , we can define a unique special point called **centre**. Such conicoids are called **central conicoids**. The examples are cone, ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets. The cylinders and paraboloids are examples of non-central conicoids.

In this unit, we shall study the central conicoids in details. We shall define centre and establish a standard equation for central conicoids. We shall study the equations and geometry of ellipsoids and hyperboloids. We shall discuss tangent lines, tangent planes, polar lines and polar planes for these surfaces.

7.2 OBJECTIVES

After reading this unit, you should be able to

- Define central conicoids
- Obtain standard equations of central conicoid
- Discuss some special central conicoids such as ellipsoid, hyperboloid of one sheet and hyperboloid of two sheet

- Define tangent lines and tangent planes at a point to a central conicoid
- Obtain the condition of tangency
- Discuss polar plane and polar lines for a central conicoid

7.3 STANDARD EQUATION OF A CENTRAL CONICOID

A **conicoid (or a quadric surface)** in the three dimensional rectangular Cartesian coordinate system is the set of points (x, y, z) in three dimensional space satisfying a general second degree equation in three variables.

i.e.

$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$
………. (1)

Let us shift the origin to a point $C\ (x_0, y_0, z_0)$ and consider new coordinate system of coordinate axes CX', CY', CZ' parallel to the given system with origin C. Then we have

$$x = x' + x_0, \qquad y = y' + y_0, \qquad z = z' + z_0$$

and equation (1) becomes

$$\sum a(x' + x_0)^2 + \sum 2f(y' + y_0)(z' + z_0) + \sum 2u(x' + x_0) + d = 0$$

Expanding above expression and simplifying, we get

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2u'x' + 2v'y' + 2w'z' + d' = 0$$

…….. (2)

Where

$$\left.\begin{aligned} u' &= ax_0 + hy_0 + gz_0 + u \\ v' &= hx_0 + by_0 + fz_0 + v \\ w' &= gx_0 + fy_0 + cz_0 + w \\ d' &= a{x_0}^2 + b{y_0}^2 + c{z_0}^2 + 2fy_0z_0 + 2gz_0x_0 + 2hx_0y_0 + 2ux_0 + 2vy_0 + 2wz_0 + d \end{aligned}\right]$$

……(3)

For a particular type of conicoids, the linear part of equation (2) vanishes. Let us choose the new origin $C\ (x_0, y_0, z_0)$ such that $u' = v' = w' = 0$, i.e.

$$\begin{aligned} ax_0 + hy_0 + gz_0 + u = 0 \\ hx_0 + by_0 + fz_0 + v = 0 \\ gx_0 + fy_0 + cz_0 + w = 0 \end{aligned}$$

or we can say that (x_0, y_0, z_0) is a solution of the system of equations

$$\left.\begin{aligned} ax + hy + gz + u = 0 \\ hx + by + fz + v = 0 \\ gx + fy + cz + w = 0 \end{aligned}\right] \quad \ldots\ldots\ldots (4)$$

If the system of equations (4) has a solution$(x_0, y_0, z_0) \in \mathbb{R}^3$, then the point $C\ (x_0, y_0, z_0)$ is called a **centre** of the given conicoid.

If $C\ (x_0, y_0, z_0)$ is a centre of the conicoid, then (2) becomes

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + d' = 0$$

Hence if a conicoid is represented by a general second degree equation $F(x, y, z) = 0$ and the conicoid has a centre $C\ (x_0, y_0, z_0)$, then by shifting the origin to the centre C , the equation assumes the following form with respect to the new coordinate system-

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0 \ \ldots\ldots.. (5)$$

Definition : A conicoid is called a **central conicoid** if it has a unique centre. If a conicoid has no centre or it has infinitely many centres, then it is called a **non-central conicoid.**

For example, consider a sphere

$$x^2 + y^2 + z^2 + 2x + 8y - 6z + 5 = 0$$

Comparing with equation (1), we get

$$a = b = c = 1, f = g = h = 0\ , u = 1, v = 4, w = -3, d = 5$$

The system of equations (4) becomes

$$\left.\begin{aligned} x + 1 = 0 \\ y + 4 = 0 \\ z - 3 = 0 \end{aligned}\right]$$

Which gives $x = -1, y = -4, z = 3$. Thus the system of equations (4) has a unique solution $(-1, -4,3)$. Hence the given sphere is a central conicoid with centre $(-1, -4,3)$. You can verify that every sphere is a central conicoid. What do you think about a cylinder? Take any equation of a cylinder and check it.

Note: The system of equations (4) has a unique solution if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

So you can use this fact to check whether a surface given by general second degree equation is a central conicoid or not.

Now suppose that a general second degree equation represents a central conicoid. We have seen that shifting the origin to the centre $C\ (x_0, y_0, z_0)$, the equation can be reduced to the following form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0$$
……….. (6)

Where
$$d' = ax_0{}^2 + by_0{}^2 + cz_0{}^2 + 2fy_0z_0 + 2gz_0x_0 + 2hx_0y_0 + 2ux_0 + 2vy_0 + 2wz_0 + d$$

Since $ax_0 + hy_0 + gz_0 = -u$, $hx_0 + by_0 + fz_0 = -v$, $gx_0 + fy_0 + cz_0 = -w$

Hence
$$d' = (ax_0 + hy_0 + gz_0)x_0 + (hx_0 + by_0 + fz_0)y_0 + (gx_0 + fy_0 + cz_0)z_0 + 2ux_0$$

$$+2vy_0 + 2wz_0 + d$$

$$= -ux_0 - vy_0 - wz_0 + 2ux_0 + 2vy_0 + 2wz_0 + d$$

$$= ux_0 + vy_0 + wz_0 + d$$

Suppose that the new axes are rotated with the following scheme

	x	y	z
X	l_1	m_1	n_1
Y	l_2	m_2	n_2
Z	l_3	m_3	n_3

Where $l_1, m_1, n_1;\ l_2, m_2, n_2; l_3, m_3, n_3$ are the direction cosines of the new coordinate axes CX, CY, CZ respectively. Then

$$X = l_1 x + m_1 y + n_1 z$$

$$Y = l_2 x + m_2 y + n_2 z$$

$$Z = l_3 x + m_3 y + n_3 z$$

By using these relations, equation (6) can be reduced to the form

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 + d' = 0$$

Where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

Therefore if S is a conicoid given by the general second degree equation $F(x, y, z) = 0$ which has a centre $C\ (x_0, y_0, z_0)$, then we can form a new coordinate system by shifting the origin to the centre C and then rotating the system about the new origin C, in which the equation reduced to the form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \ \ldots\ldots\ldots\ldots (7)$$

Equation (7) is called the **standard equation** of the central conicoid.

Since (7) has a unique centre, hence, we have

$$\Delta = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \neq 0$$

i.e. $$\lambda_1 \lambda_2 \lambda_3 \neq 0$$

Therefore $\lambda_1 \neq 0, \lambda_2 \neq 0, \lambda_3 \neq 0.$

Now the following five cases arise

Case 1 When $d' = 0$

In this case, equation (7) reduces to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$$

This represents a **cone**.

Case 2 When $d' \neq 0$ and $\lambda_1, \lambda_2, \lambda_3$ and d' have the same sign

In this case, the left hand side of (7) is not zero for any real values of x, y and z. This represents an **imaginary ellipsoid.**

Suppose $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0$ and $d' > 0$. Then equation (7) becomes

$$\frac{x^2}{\left(\frac{d'}{\lambda_1}\right)} + \frac{y^2}{\left(\frac{d'}{\lambda_2}\right)} + \frac{z^2}{\left(\frac{d'}{\lambda_3}\right)} = -1$$

or

$$\frac{x^2}{{a_1}^2} + \frac{y^2}{{b_1}^2} + \frac{z^2}{{c_1}^2} = -1$$

Where $a_1 = \sqrt{\frac{d'}{\lambda_1}}$, $b_1 = \sqrt{\frac{d'}{\lambda_2}}$ and $c_1 = \sqrt{\frac{d'}{\lambda_3}}$.

Case 3 When $d' \neq 0$ and the sign of λ_1, λ_2 and λ_3 are different from d'

In this case, the equation (7) becomes

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = -d'$$

or

$$\frac{x^2}{\left(\frac{-d'}{\lambda_1}\right)} + \frac{y^2}{\left(\frac{-d'}{\lambda_2}\right)} + \frac{z^2}{\left(\frac{-d'}{\lambda_3}\right)} = 1$$

Since $-d'/\lambda_1 > 0, -d'/\lambda_2 > 0$ and $-d'/\lambda_3 > 0$, hence we can write

$$\frac{x^2}{{a_1}^2} + \frac{y^2}{{b_1}^2} + \frac{z^2}{{c_1}^2} = 1$$

Where $a_1 = \sqrt{\frac{-d'}{\lambda_1}}$, $b_1 = \sqrt{\frac{-d'}{\lambda_2}}$ and $c_1 = \sqrt{\frac{-d'}{\lambda_3}}$. This central conicoid is called an **ellipsoid**.

Case 4 When $d' \neq 0$ and $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 < 0$ and $d' < 0$ (or any two of the four coefficients $\lambda_1, \lambda_2, \lambda_3$ and d' are of the same sign)

In this case, $-d'/\lambda_1 > 0, -d'/\lambda_2 > 0$ and $d'/\lambda_3 > 0$. Let $a_1 = \sqrt{\frac{-d'}{\lambda_1}}$, $b_1 = \sqrt{\frac{-d'}{\lambda_2}}$ and $c_1 = \sqrt{\frac{d'}{\lambda_3}}$. Then equation (7) becomes

$$\frac{x^2}{{a_1}^2} + \frac{y^2}{{b_1}^2} - \frac{z^2}{{c_1}^2} = 1$$

This central conicoid is called a **hyperboloid of one sheet**.

Case 5 When $d' \neq 0$ and $\lambda_1 > 0, \lambda_2 < 0, \lambda_3 < 0$ and $d' < 0$ (or any two of the coefficients $\lambda_1, \lambda_2, \lambda_3$ have the same sign as d')

In this case, $-d'/\lambda_1 > 0, d'/\lambda_2 > 0$ and $d'/\lambda_3 > 0$. Let $a_1 = \sqrt{\frac{-d'}{\lambda_1}}$, $b_1 = \sqrt{\frac{d'}{\lambda_2}}$ and $c_1 = \sqrt{\frac{d'}{\lambda_3}}$. Then equation (7) becomes

$$\frac{x^2}{{a_1}^2} - \frac{y^2}{{b_1}^2} - \frac{z^2}{{c_1}^2} = 1$$

This central conicoid is called a **hyperboloid of two sheets**.

Thus the standard equations for five types of central conicoids may be given as follows-

1. $ax^2 + by^2 + cz^2 = 0$ (Cone)

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ (Imaginary ellipsoid)

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Ellipsoid)

4. $\left.\begin{array}{r} \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \\ -\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{array}\right]$ (Hyperboloid of one sheet)

5. $\left.\begin{array}{r} \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \\ -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \\ -\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \end{array}\right]$ (Hyperboloid of two sheets)

The standard form representing ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets and imaginary ellipsoid may be given as

$$ax^2 + by^2 + cz^2 = 1$$

This equation represents an ellipsoid if a, b, c are all positive, a hyperboloid of one sheet if any one of them is negative and the remaining two are positive. It represents a hyperboloid of two sheets if any two of them are negative and the remaining one is positive. This represents an imaginary ellipsoid if all the a, b, c are negative.

Definition A conicoid S is called **symmetric** with respect to a point C if on shifting the origin to the point C, the new equation is symmetric with respect to the new origin C. Then the point C is called a **centre** of the conicoid S.

Consider the central conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Let $P(x_1, y_1, z_1)$ be any point on the central conicoid given by (8). Then you can verify that the point $Q(-x_1, -y_1, -z_1)$ also lies on (8). Hence the central conicoid given by standard form (8) is symmetric with respect to the origin $O(0,0,0)$. The origin is the centre of the conicoid (8).

Illustrative Example Reduce the second degree equation

$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0$ ….. (9)

to the standard form and identify the surface.

Solution comparing with the general second degree equation (1), we have

$$a = 11, b = 10, c = 6, f = -4, g = 2, h = -6, u = 36, v = -36, w = 18, d = 150$$

The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or}$$

$$\begin{vmatrix} 11-\lambda & -6 & 2 \\ -6 & 10-\lambda & -4 \\ 2 & -4 & 6-\lambda \end{vmatrix} = 0$$

or $$\lambda^3 - 27\lambda^2 + 180\lambda - 324 = 0$$

The factorization of above polynomial equation gives

$$(\lambda - 3)(\lambda - 6)(\lambda - 18) = 0$$

Therefore $\lambda = 3,6,18$.

The equations (4) becomes

$$\left.\begin{aligned} 11x - 6y + 2z + 36 &= 0 \\ -6x + 10y - 4z - 36 &= 0 \\ 2x - 4y + 6z + 18 &= 0 \end{aligned}\right] \quad \ldots\ldots\ldots (10)$$

These equations have unique solution $x = -2, y = 2, z = -1$. Hence the given equation (9) represents a central conicoid with centre $(-2,2,-1)$. The equation reduces to the standard form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

Where $d' = ux_0 + vy_0 + wz_0 + d$

$$= 36(-2) + (-36)(2) + 18(-1) + 150 = -12$$

Therefore the standard form of (9) may be given as

$$3x^2 + 6y^2 + 18z^2 - 12 = 0$$

Or

$$\frac{x^2}{4} + \frac{y^2}{2} + \frac{z^2}{(2/3)} = 1$$

Therefore the surface represented by (9) is an ellipsoid with centre $(-2,2,-1)$.

7.4 THE ELLIPSOID

The standard equation of an ellipsoid is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots\ldots(11)$$

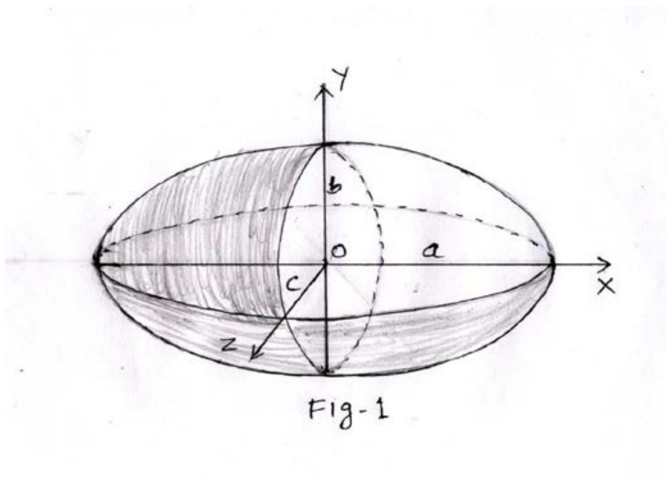


Fig-1

Now we have the following observations-

(i) If the point $P(x_1, y_1, z_1)$ lies on the ellipsoid given by (11), then the point $Q(-x_1, -y_1, -z_1)$ also lies on it. Hence the origin $O(0,0,0)$ is the centre of the ellipsoid.

(ii) The ellipsoid is symmetrical about the coordinate planes (i.e. xy-plane, yz-plane, zx-plane). These coordinate planes bisect all chords perpendicular to them. For instance the chord joining (x, y, z) and $(x, y, -z)$ drawn perpendicular to xy-plane is bisected by the xy-plane. These coordinate planes are called the **principal**

planes of the ellipsoid. The lines of intersection of these principal planes are the coordinate axes. These axes are called the **principal axes** of the ellipsoid.

(iii) From (11), we observe that the ellipsoid meets the x-axis at points where

$$y = 0, z = 0 \text{ and } \frac{x^2}{a^2} + \frac{0}{b^2} + \frac{0}{c^2} = 1$$

i.e. $x = \pm a, y = 0, z = 0$. Hence the ellipsoid (11) meets the x-axis at $A(a, 0,0)$ and $A'(-a, 0,0)$. Similarly, you can check that the points of intersection with y-axis are $B(0, b, 0)$ and $B'(0, -b, 0)$ and with z-axis are $C(0,0, c)$ and $C'(0,0, -c)$. The lengths AA', BB' and CC' are called the **principal diameters** of the ellipsoid and OA, OB and OC are called the **semi-axes** of the ellipsoid.

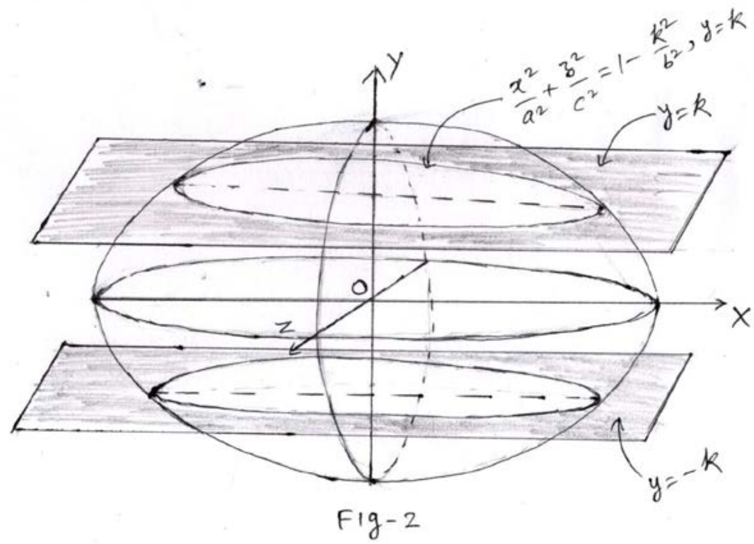


Fig-2

(iv) The ellipsoid is a closed surface (i.e. bounded surface)

Equation (11) can be written as

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}$$

You will observe that $\frac{y^2}{b^2} + \frac{z^2}{c^2}$ is negative for $|x| > a$, i.e. at least one of y and z has imaginary value. Therefore the surface does not exist when $|x| > a$, i.e, it is bounded by the planes $x = -a$ and $x = a$. Similarly you can check for yourself that the ellipsoid (11) is bounded by the planes $x = -b$ and $x = b$ and $x = -c$ and $x = c$.

(v) The section of the ellipsoid (11) by the plane $z = k$ is the ellipse given by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}, \qquad z = k$$

Since $-c < k < c$, i.e. $\frac{k^2}{c^2} < 1$, hence the section of the ellipsoid by the plane $z = k$ $(-c < k < c)$ is an ellipse. The centre of this ellipse lies on z-axis. Similarly, you can check that the sections of the ellipsoid by the planes parallel to yz-plane and zx-plane are ellipses.

(vi) When $b = c$, equation (11) becomes

$$\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$$

This ellipsoid of revolution is called a **prolate spheroid**. Similarly, if $c = a$ we obtain an ellipsoid of revolution called **oblate spheroid**

$$\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1$$

If $a = b = c$, then equation (11) becomes

$$x^2 + y^2 + z^2 = a^2$$

Which is a sphere of radius a with centre located at the origin.

7.5 THE HYPERBOLOID OF ONE SHEET

The standard equation of a hyperboloid of one sheet is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \dots\dots (12)$$

Now we have the following observations-

(i) The origin $O(0,0,0)$ is the centre of the hyperboloid as all chords passing through the origin are bisected at the origin.

(ii) The surface (12) is symmetrical about the coordinate planes (i.e. xy-plane, yz-plane, zx-plane). These coordinate planes bisect all chords perpendicular to them. Hence the coordinate planes are the **principal planes** of the surface. The lines of intersection of these principal planes are the coordinate axes, i.e. these coordinate axes are the **principal axes** of the surface (12).

(iii) The surface (12) meets the x-axis at points $A(a, 0,0)$ and $A'(-a, 0,0)$. Also the points of intersection with y-axis are $B(0, b, 0)$ and $B'(0, -b, 0)$ and the surface intersects the z-axis at imaginary points, i.e. the surface does not intersect z-axis.

(iv) The section of the surface (12) by the plane $z = k$ is given by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \qquad z = k$$

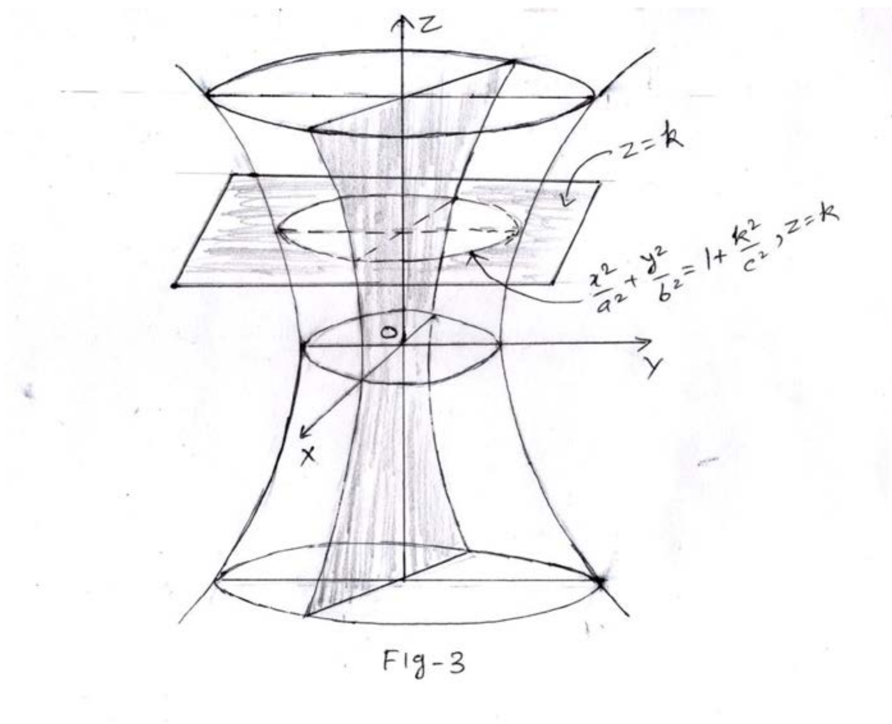


Fig 3

Here we have $-\infty < k < \infty$, i.e. the section of the surface (12) by the plane $z = k$ is always an ellipse for all real values of k. The centres of these ellipses lie on z-axis. The ellipse corresponding to $k = 0$ is called the principal ellipse. Similarly, you can check that the sections of this surface by the planes parallel to yz-plane and zx-plane are the following hyberbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{b^2}, \qquad y = k\,;(k < b)$$

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}, \qquad x = k\,;(k < a)$$

7.6 THE HYPERBOLOID OF TWO SHEETS

The standard equation of a hyperboloid of two sheets is given by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \ldots\ldots (13)$$

You will observe that

(i) The origin $O(0,0,0)$ is the centre of the surface as all chords passing through the origin are bisected at the origin.

(ii) The surface (13) is symmetrical about the coordinate planes (i.e. xy-plane, yz-plane, zx-plane). Hence the coordinate planes are the **principal planes** of the surface and the coordinate axes are the **principal axes** of the surface.

(iii) The surface (13) meets the x-axis at points $A(a, 0,0)$ and $A'(-a, 0,0)$.The surface does not intersect y-ais and z-axis.

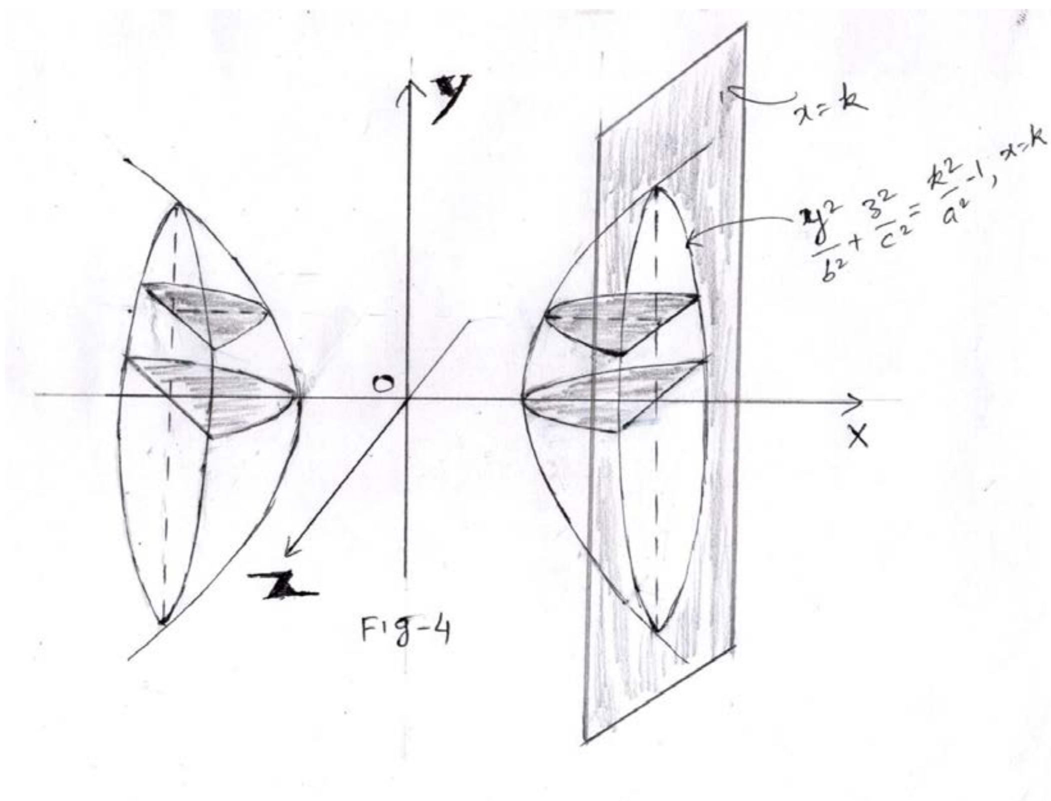


Fig-4

(iv) The section of the surface (13) by the plane $x = k$ is given by the equations

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \qquad x = k$$

Which represents an ellipse. For $-a < k < a$, this ellipse is imaginary. Hence no part of the surface lies in the region $-a < x < a$. you can check that the sections of this surface by the planes parallel to yz-plane and zx-plane are the following hyberbolas

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}, \qquad y = k$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \qquad z = k$$

You will notice that it is not a bounded surface.

7.7 TANGENT LINES AND TANGENT PLANES

First we shall discuss the intersection of a line with a central conicoid. Then we shall obtain conditions for a line to become a tangent to a given central conicoid. Finally we shall obtain the equation of a tangent plane to the central conicoid.

Let a central conicoid be given by equation (8). Let the equations of a straight line passing through a point $A(\alpha, \beta, \gamma)$ with direction cosines l, m, n be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots..(14)$$

Then the coordinates of any point P on the straight line (14) distant r from the point $A\ (\alpha, \beta, \gamma)$ are given by $(lr+\alpha, mr+\beta, nr+\gamma)$. If the line (14) meets the central conicoid (8) at point P, then

$$a(lr+\alpha)^2 + b(mr+\beta)^2 + c(nr+\gamma)^2 = 1$$

or $\quad r^2(al^2+bm^2+cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$

……………..(15)

This is a quadratic equation in r. Hence we get two values of r corresponding to which we have two points of intersection of the line (14) and the central conicoid (8). These two points may be real and distinct, coincident or imaginary depending upon the roots of equation (15).

If the line (14) is a tangent line to the conicoid (8), then the points of intersection must coincide, i.e. the roots of the quadratic equation (15) must be identical. It is possible if

$$\{2(a\alpha l + b\beta m + c\gamma n)\}^2 = 4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

or $\quad (a\alpha l + b\beta m + c\gamma n)^2 = (al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$ ……….. (16)

If the point $A\ (\alpha, \beta, \gamma)$ lies on the conicoid (8), then

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

Then (16) becomes

$$a\alpha l + b\beta m + c\gamma n = 0 \qquad \ldots\ldots\ldots\ldots (17)$$

This is the condition that the line (14) is a tangent to the central conicoid (8) at point $A\ (\alpha, \beta, \gamma)$.

There are infinitely many lines passing through (α, β, γ) satisfying the condition (17).

For example, consider the ellipsoid

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1 \quad \ldots\ldots (18)$$

The point $(2,0,0)$ lies on (18). Here $a = \frac{1}{4}, b = \frac{1}{9}, c = \frac{1}{16}$. The condition (17) gives

$$\frac{1}{4}(2)l + \frac{1}{9}(0)m + \frac{1}{16}(0)n = 0$$

$$\Rightarrow l = 0$$

Now $l^2 + m^2 + n^2 = 1$. Hence we have $l = 0, m^2 + n^2 = 1$. There are infinitely many sets of values (l, m, n) satisfying these conditions. For instance, $l = 0, m = 1, n = 0$ and $l = 0, m = 0, n = 1$ are two such sets of values. Therefore the lines lying in the plane parallel to yz-plane and passing through the point $(2,0,0)$ are tangent lines to the ellipsoid (18) at $(2,0,0)$. This plane containing all the tangent lines at a given point of any conicoid is called the **tangent plane** at that point. Hence the locus of the tangent lines to a conicoid at a point on it is called the tangent plane at that point.

In order to find the locus of these tangent lines, i.e. to obtain the equation of a tangent plane at $A\ (\alpha, \beta, \gamma)$ to the central conicoid (8), we have to eliminate l, m, n from (14) and (17). Hence we obtain

$$a\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma) = 0$$

or $$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

or $$a\alpha x + b\beta y + c\gamma z = 1 \qquad \ldots\ldots\ldots\ldots (19)$$

This equation represents the tangent plane at (α, β, γ) to the central conicoid (8).

7.8 CONDITION OF TANGENCY

Now you may ask a question. Is there a way to decide whether a given plane is a tangent plane to a given conicoid? The answer is yes. Let us see how we can obtain the condition of tangency.

Suppose equation (8) represents a given conicoid, i.e.

$$ax^2 + by^2 + cz^2 = 1$$

Assume that we are given a plane $lx + my + nz = p$ …………. (20)

The equation of the tangent plane at (α, β, γ) to the conicoid is given by equation (19), i.e. $a\alpha x + b\beta y + c\gamma z = 1$

If the plane (20) represents the tangent plane at (α, β, γ) to the conicoid, then equations (19) and (20) must be identical or the coefficients of these equations must be proportional, i.e.

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p}, \quad p \neq 0$$

i.e. $\alpha = \frac{l}{ap}, \beta = \frac{m}{bp}, \gamma = \frac{n}{cp}$ …………….. (21)

since the point (α, β, γ) lies on the conicoid (8), hence

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1$$

$$\Rightarrow a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1$$

$$\Rightarrow \quad \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2 \qquad \text{... ... (22)}$$

Which is the condition that the plane $lx + my + nz = p$ touches the conicoid $ax^2 + by^2 + cz^2 = 1$. The point at which the plane touches the conicoid is called the **point of contact** and is given by (21).

7.9 ILLUSTRATIVE EXAMPLES

Example 7.9.1 Find the equation of the tangent plane to the hyperboloid

$$\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{25} = 1$$

at (2,3,5).

Solution : Here $a = \frac{1}{4}, b = \frac{1}{9}, c = -\frac{1}{25}$ and $\alpha = 2, \beta = 3, \gamma = 5$.

The equation of tangent plane at (α, β, γ) to the conicoid $ax^2 + by^2 + cz^2 = 1$ is given by

$$a\alpha x + b\beta y + c\gamma z = 1$$

$$\Rightarrow \left(\frac{1}{4}\right)(2)x + \left(\frac{1}{9}\right)(3)y + \left(-\frac{1}{25}\right)(5)z = 1$$

or $$15x + 10y - 6z = 30$$

Example 7.9.2 Show that the plane $7x + 5y + 3z = 30$ touches the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$. Find the point of contact.

Solution : The equation of the ellipsoid may be written in the standard form as follows

$$\frac{7}{60}x^2 + \frac{1}{12}y^2 + \frac{1}{20}z^2 = 1$$

Here $a = \frac{7}{60}, b = \frac{1}{12}, c = \frac{1}{20}$.

The given plane is $7x + 5y + 3z = 30$. Therefore $l = 7, m = 5, n = 3, p = 30$

Now

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = \frac{60}{7}(7)^2 + 12(5)^2 + 20(3)^2$$

$$= 420 + 300 + 180$$

$$= 900 = 30^2 = p^2$$

Thus the condition of tangency (22) is satisfied. Hence the plane touches the ellipsoid. The point of contact is given by

i.e. $$\alpha = \frac{l}{ap} = \frac{60 \times 7}{7 \times 30} = 2, \beta = \frac{m}{bp} = \frac{12 \times 5}{1 \times 30} = 2, \gamma = \frac{n}{cp} = \frac{20 \times 3}{1 \times 30} = 2$$

i.e. (2,2,2) is the point of contact.

Example 7.9.3 Find the equation to the tangent planes to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0, 3x - 3y + 6z = 5$

Solution : The equation of any plane through the given line is

$$(x + 9y - 3z) + \lambda(3x - 3y + 6z - 5) = 0$$

or $$(1 + 3\lambda)x + (9 - 3\lambda)y + (6\lambda - 3)z = 5\lambda \quad \ldots\ldots. (23)$$

Suppose the plane (23) touches the given hyperboloid and let (α, β, γ) be the point of contact. The equation of the tangent plane at (α, β, γ) to the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$ is

$$2\alpha x - 6\beta y + 3\gamma z = 5 \quad \ldots\ldots\ldots (24)$$

Now (23) and (24) represent the same plane. Thus

$$\frac{2\alpha}{1+3\lambda} = \frac{-6\beta}{9-3\lambda} = \frac{3\gamma}{6\lambda - 3} = \frac{5}{5\lambda}$$

$$\Rightarrow \alpha = \frac{1+3\lambda}{2\lambda}, \qquad \beta = \frac{9-3\lambda}{-6\lambda}, \qquad \gamma = \frac{6\lambda-3}{3\lambda}$$

Since the point (α, β, γ) lies on the hyperboloid $2x^2 - 6y^2 + 3z^2 = 5$, hence

$$2\left(\frac{1+3\lambda}{2\lambda}\right)^2 - 6\left(\frac{9-3\lambda}{-6\lambda}\right)^2 + 3\left(\frac{6\lambda-3}{3\lambda}\right)^2 = 5$$

$$\Rightarrow \lambda^2 = 1 \text{ or } \lambda = \pm 1$$

From (23) the required equations of the tangent planes are

$$4x + 6y + 3z = 5 \text{ and } 2x - 12y + 9z = 5$$

Definition The director sphere of a central conicoid is the locus of the point of intersection of three mutually perpendicular tangent planes to that central conicoid.

In the following example, we shall obtain the equation of director sphere of a central conicoid.

Example 7.9.4 Find the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid $ax^2 + by^2 + cz^2 = 1$.

Given conicoid is

$$ax^2 + by^2 + cz^2 = 1 \qquad \ldots. (25)$$

Let the equations of any three mutually perpendicular tangent planes to the conicoid (25) be

$$l_1x + m_1y + n_1z = p_1, l_2x + m_2y + n_2z = p_2, l_3x + m_3y + n_3z = p_3 \quad \ldots. (26)$$

Then from the condition of tangency, we have

$$\frac{l_1{}^2}{a}+\frac{m_1{}^2}{b}+\frac{n_1{}^2}{c}=p_1{}^2,\quad \frac{l_2{}^2}{a}+\frac{m_2{}^2}{b}+\frac{n_2{}^2}{c}=p_2{}^2,\quad \frac{l_3{}^2}{a}+\frac{m_3{}^2}{b}+\frac{n_3{}^2}{c}=p_3{}^2$$

….(27)

Since the planes are mutually perpendicular, hence

$$\left.\begin{array}{c} l_1{}^2+l_2{}^2+l_3{}^2=1, m_1{}^2+m_2{}^2+m_3{}^2=1, n_1{}^2+n_2{}^2+n_3{}^2=1 \\ l_1m_1+l_2m_2+l_3m_3=0, m_1n_1+m_2n_2+m_3n_3=0, l_1n_1+l_2n_2+l_3n_3=0 \end{array}\right]$$

...(28)

Let (x_1, y_1, z_1) be the point of intersection of the tangent planes (26). Then

$$l_1x_1+m_1y_1+n_1z_1=p_1, l_2x_1+m_2y_1+n_2z_1=p_2, l_3x_1+m_3y_1+n_3z_1=p_3$$

Squaring and adding these equations we get

$$(l_1x_1+m_1y_1+n_1z_1)^2+(l_2x_1+m_2y_1+n_2z_1)^2+(l_3x_1+m_3y_1+n_3z_1)^2$$

$$=p_1{}^2+p_2{}^2+p_3{}^2$$

or $\quad x_1{}^2\sum l_1{}^2+y_1{}^2\sum m_1{}^2+z_1{}^2\sum n_1{}^2+2xy\sum l_1m_1+2yz\sum m_1n_1+2zx\sum n_1l_1$

$$=\frac{l_1{}^2}{a}+\frac{m_1{}^2}{b}+\frac{n_1{}^2}{c}+\frac{l_2{}^2}{a}+\frac{m_2{}^2}{b}+\frac{n_2{}^2}{c}+\frac{l_3{}^2}{a}+\frac{m_3{}^2}{b}+\frac{n_3{}^2}{c},$$

using (27)

or $x_1{}^2(1)+y_1{}^2(1)+z_1{}^2(1)+2xy(0)+2yz(0)+2zx(0)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, using (28)

or $\qquad x_1{}^2+y_1{}^2+z_1{}^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

The locus of the point (x_1, y_1, z_1) is

$$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

This is the required equation of the director sphere of the central conicoid (25).

7.10 POLAR PLANES AND POLAR LINES

Let a central conicoid be given as

$$ax^2+by^2+cz^2=1 \qquad ….. (29)$$

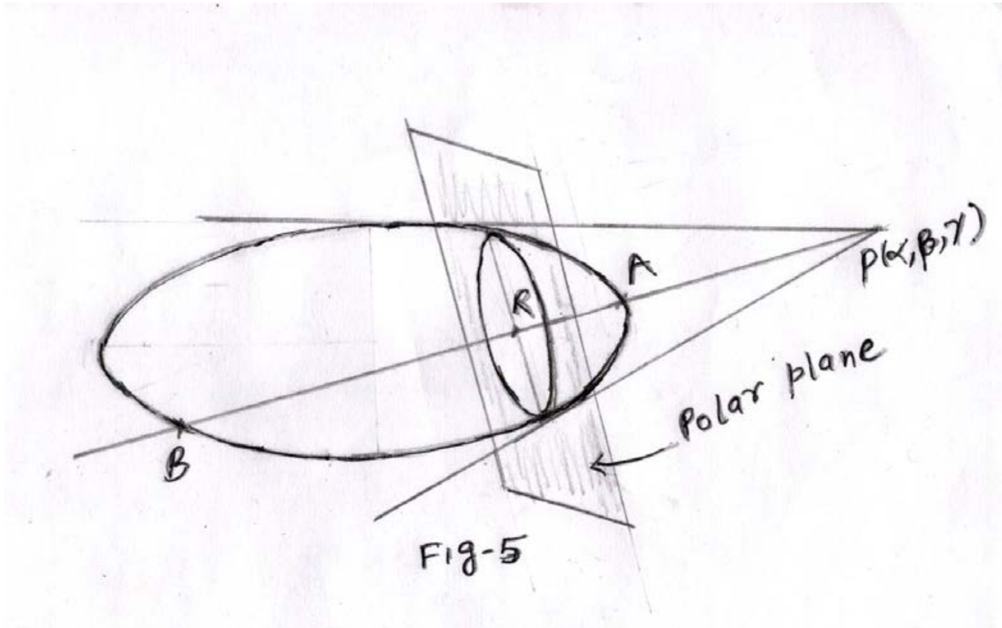


Fig 5

Let a straight line through a given point $P\ (\alpha, \beta, \gamma)$meet the conicoid (29) in two points A and B. Suppose R is a point on this straight line such that

$$\frac{1}{PA} + \frac{1}{PB} = \frac{2}{PR} \qquad \ldots\ldots (30)$$

Then the point P is called the **pole** and the locus of point R is called the **polar plane** of the point P with respect to the given conicoid.

Now we shall obtain the equation of the polar plane of $P\ (\alpha, \beta, \gamma)$ with respect to the conicoid (29).

The equations of any line through $P\ (\alpha, \beta, \gamma)$ may be given as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} (= r) \quad \ldots\ldots\ldots (31)$$

Any point on this line will be$(lr + \alpha, mr + \beta, nr + \gamma)$. If the line (31) meets the conicoid (29) in this point, then

$$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 = 1$$

or $\qquad r^2(al^2 + bm^2 + cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$

…………. (32)

Let r_1 and r_2 be the roots of above quadratic equation. Then $r_1 = PA$ and $r_2 = PB$.

From (32) we have

$$r_1 + r_2 = -2\frac{(a\alpha l + b\beta m + c\gamma n)}{(al^2 + bm^2 + cn^2)}$$

$$\text{and} \quad r_1 r_2 = \frac{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}{(al^2 + bm^2 + cn^2)}$$

Hence on dividing, we get

$$\frac{r_1 + r_2}{r_1 r_2} = -2\frac{(a\alpha l + b\beta m + c\gamma n)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}$$

$$\text{or} \quad \frac{1}{r_1} + \frac{1}{r_2} = -2\frac{(a\alpha l + b\beta m + c\gamma n)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)} \quad \ldots..(33)$$

Let $PR = r_3$. Then (30) becomes

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{2}{r_3}$$

Using (33), we have

$$\frac{2}{r_3} = -2\frac{(a\alpha l + b\beta m + c\gamma n)}{(a\alpha^2 + b\beta^2 + c\gamma^2 - 1)}$$

or $\quad a\alpha(lr_3) + b\beta(mr_3) + c\gamma(nr_3) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$ ….. (34)

If (x', y', z') are the coordinates of point R, then from (31) we have

$$x' - \alpha = lr_3, \qquad y' - \beta = mr_3, \qquad z' - \gamma = nr_3$$

Hence (34) becomes

$$a\alpha(x' - \alpha) + b\beta(y' - \beta) + c\gamma(z' - \gamma) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$$

or $\quad a\alpha x' + b\beta y' + c\gamma z' = 1$

The locus of point $R(x', y', z')$ is

$$a\alpha x + b\beta y + c\gamma z = 1 \quad ……. (35)$$

This represents the polar plane of the pole $P\ (\alpha, \beta, \gamma)$ with respect to the conicoid (29).

You will notice that the tangent plane at any point (x_1, y_1, z_1) to the conicoid (29) is given as

$$ax_1 x + by_1 y + cz_1 z = 1$$

If this tangent plane passes through $P\ (\alpha, \beta, \gamma)$, then

$$ax_1\alpha + bx_1\beta + cx_1\gamma = 1$$

Which shows that the point (x_1, y_1, z_1) lies on the polar plane (35) of the pole $P\ (\alpha, \beta, \gamma)$ with respect to the conicoid (29). That means the polar plane of $P\ (\alpha, \beta, \gamma)$ cuts the conicoid at points the tangent planes at which pass through the point $P\ (\alpha, \beta, \gamma)$. In other words, the polar palne (35) of $P\ (\alpha, \beta, \gamma)$ cuts the conicoid (29) in a conic and the line joining P to any point on this conic is a tangent line to the conicoid. The collection of all such tangent lines forms a cone called the tangent cone from $P\ (\alpha, \beta, \gamma)$ to the conicoid.

Note: If the point $P\ (\alpha, \beta, \gamma)$ lies on the conicoid (29), then the polar plane of P becomes the tangent plane at $P\ (\alpha, \beta, \gamma)$.

Polar lines:

Suppose we are given two points $A_1(\alpha_1, \beta_1, \gamma_1)$ and $A_2(\alpha_2, \beta_2, \gamma_2)$. The polar plane of $A_1(\alpha_1, \beta_1, \gamma_1)$ with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$a\alpha_1 x + b\beta_1 y + c\gamma_1 z = 1$$

If the point $A_2(\alpha_2, \beta_2, \gamma_2)$ lies on this plane, then

$$a\alpha_1\alpha_2 + b\beta_1\beta_2 + c\gamma_1\gamma_2 = 1$$

This equation shows that the point $A_1(\alpha_1, \beta_1, \gamma_1)$ lies on the polar plane of $A_2(\alpha_2, \beta_2, \gamma_2)$. Thus if the polar plane of any point A_1 with respect to a conicoid passes through a point A_2, then the polar plane of A_2 passes through the point A_1.

Suppose B_1 is any point on the line of intersection of the polar planes of A_1 and A_2. Then B_1 lies on the polar planes of A_1 and A_2. Hence the polar plane of B_1 must pass through A_1 and A_2, and therefore through the line A_1A_2. Similarly the polar plane of any other point B_2 lying on the line of intersection of the polar planes of A_1 and A_2 will pass through the line A_1A_2. Thus we can say that the lines A_1A_2 and B_1B_2 are such that the polar planes of all points on A_1A_2 pass through B_1B_2 and vice versa. The lines A_1A_2 and B_1B_2 are called **polar lines**.

Let us verify it analytically. Suppose the equations of the line A_1A_2 are given as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} (= r) \text{ ....(36)}$$

Any point on this line will be $P(lr + \alpha, mr + \beta, nr + \gamma)$. The polar plane of P with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$a(lr + \alpha)x + b(mr + \beta)y + c(nr + \gamma)z = 1$$

or $$(a\alpha x + b\beta y + c\gamma z - 1) + r(alx + bmy + cnz) = 0$$ …… (37)

This plane passes through the line of intersection of the planes

$$a\alpha x + b\beta y + c\gamma z - 1 = 0 \text{ and } alx + bmy + cnz = 0$$ …… (38)

We call this line of intersection as the line B_1B_2. You observe that for different values of r, the polar plane (37) always passes through the line (38), i.e. the line B_1B_2. Hence the polar plane of every point on A_1A_2 passes through the line B_1B_2 and vice versa. Thus the line A_1A_2 given by (36) and the line B_1B_2 given by (38) are polar lines.

7.11 ILLUSTRATIVE EXAMPLES

Example 7.11.1 Find the pole of the plane $4x + 6y + 8z = 18$ with respect to the conicoid $2x^2 + 3y^2 + 8z^2 = 1$.

Solution: Let the pole of the plane

$$4x + 6y + 8z = 18 \text{ ….. (39)}$$

with respect to the conicoid

$$2x^2 + 3y^2 + 8z^2 = 1 \text{ …… (40)}$$

be (α, β, γ). Now the polar plane of (α, β, γ) with respect to the conicoid (40) is

$$2\alpha x + 3\beta y + 8\gamma z = 1 \quad \text{…….. (41)}$$

Equations (39) and (41) represent the same plane. Hence

$$\frac{2\alpha}{4} = \frac{3\beta}{6} = \frac{8\gamma}{8} = \frac{1}{18}$$

i.e. $\alpha = \frac{1}{9}, \beta = \frac{1}{9}, \gamma = \frac{1}{18}$. Therefore the required pole is $\left(\frac{1}{9}, \frac{1}{9}, \frac{1}{18}\right)$.

Example 7.11.2 Find the conditions that the two given lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \dots\dots\dots (42)$$

$$\text{and} \quad \frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'} \dots\dots (43)$$

be polar lines with respect to the conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad \dots\dots\dots (44)$$

Solution: The polar line of the line (42) with respect to the conicoid (44) is the line of intersection of the planes

$$a\alpha x + b\beta y + c\gamma z - 1 = 0 \text{ and } alx + bmy + cnz = 0 \quad \dots\dots (45)$$

If the polar line of (42) is the line (43), then the line (43) must lie in the planes given by (45). Therefore

$$\left.\begin{aligned} a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' - 1 = 0 \\ a\alpha l' + b\beta m' + c\gamma n' = 0 \end{aligned}\right] \dots\dots\dots\dots (46)$$

$$\left.\begin{aligned} al\alpha' + bm\beta' + cn\gamma' = 0 \\ all' + bmm' + cnn' = 0 \end{aligned}\right] \dots\dots\dots\dots (47)$$

Equations (46) and (47) give the required conditions.

Example 7.11.3 Find the locus of the pole of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$.

Solution: Let a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ be

$$lx + my + nz = p \quad \dots\dots\dots (48)$$

Then from the condition of tangency, we have

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2 \quad \dots\dots (49)$$

Let (x_1, y_1, z_1) be the pole of the plane (48) with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$. Now the polar plane of (x_1, y_1, z_1) with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is

$$\alpha x_1 x + \beta y_1 y + \gamma z_1 z = 1 \quad \dots\dots\dots\dots (50)$$

The planes (49) and (50) are identical. Hence we have

$$\frac{l}{\alpha x_1} = \frac{m}{\beta y_1} = \frac{n}{\gamma z_1} = \frac{p}{1}$$

Which gives $= \alpha x_1 p,\ m = \beta y_1 p,\ n = \gamma z_1 p$. Therefore from (49) we have

$$\frac{(\alpha x_1 p)^2}{a} + \frac{(\beta y_1 p)^2}{b} + \frac{(\gamma z_1 p)^2}{c} = p^2$$

$$\text{or} \quad \frac{\alpha^2 {x_1}^2}{a} + \frac{\beta^2 {y_1}^2}{b} + \frac{\gamma^2 {z_1}^2}{c} = 1$$

The locus of the pole (x_1, y_1, z_1) is

$$\text{or} \qquad \frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1$$

7.12 SUMMARY

In this unit, we have studied the following facts-

(1) A conicoid (or a quadric surface) in the three dimensional rectangular Cartesian coordinate system is the set of points (x, y, z) in three dimensional space satisfying a general second degree equation in three variables.

$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$

(2) This general second degree equation represents a central conicoid if

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$$

(3) The centre (x_0, y_0, z_0) of the central conicoid is the unique solution of the following equations-

$$ax_0 + hy_0 + gz_0 + u = 0$$
$$hx_0 + by_0 + fz_0 + v = 0$$
$$gx_0 + fy_0 + cz_0 + w = 0$$

(4) For a central conicoid, the general equation of second degree can be reduced to the following standard form

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$$

Where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

and $d' = ux_0 + vy_0 + wz_0 + d$

(5) The standard equations for five types of central conicoids may be given as follows-

1. $ax^2 + by^2 + cz^2 = 0$ (Cone)

2. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1$ (Imaginary ellipsoid)

3. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Ellipsoid)

4. $\left.\begin{array}{l} \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \end{array}\right]$ (Hyperboloid of one sheet)

5. $\left.\begin{array}{l} \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \\ -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \\ -\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \end{array}\right]$ (Hyperboloid of two sheets)

(6) The standard form representing ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets and imaginary ellipsoid may be given as

$$ax^2 + by^2 + cz^2 = 1$$

This equation represents an ellipsoid if a, b, c are all positive, a hyperboloid of one sheet if any one of them is negative and the remaining two are positive. It represents a hyperboloid of two sheets if any two of them are negative and the remaining one is positive. This represents an imaginary ellipsoid if all the a, b, c are negative.

(7) We studied ellipsoid, hyperboloid of one sheet and hyperboloid of two sheets in details.

(8) The condition that the line given by

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

is a tangent to the central conicoid $ax^2 + by^2 + cz^2 = 1$ at (α, β, γ) is

$$a\alpha l + b\beta m + c\gamma n = 0$$

(9) The equation of the tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ at (α, β, γ) is $a\alpha x + b\beta y + c\gamma z = 1$.

(10) The condition that the plane $lx + my + nz = p$ is a tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

(11) The equation of the polar plane of the pole (α, β, γ) with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $a\alpha x + b\beta y + c\gamma z = 1$.

(12) The polar line of the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is the line of intersection of the planes $a\alpha x + b\beta y + c\gamma z - 1 = 0$ and $alx + bmy + cnz = 0$.

7.13 SELF ASSESSMENT QUESTIONS

(1) Prove that the equation

$$3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$$

represents a hyperboloid of one sheet. Also find its centre. [Ans: $(1,0,-1)$]

(2) Find the equation of the tangent plane to the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$ parallel to the plane $x + 4y - 2z = 0$. [Ans: $3x + 12y - 6z \pm 17 = 0$]

(3) Tangent planes are drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$ through (α, β, γ). Prove that perpendiculars to them from origin generate the cone

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = (\alpha x + \beta y + \gamma z)^2$$

(4) Find the polar plane of the point $(2, -3, 4)$ with respect to the conicoid $x^2 + 2y^2 + z^2 = 4$ [Ans: $-3y + 2z = 2$]

(5) Prove that the surface generated by the straight lines drawn through a fixed point (α, β, γ) at right angles to their polar with respect to the conicoid $ax^2 + by^2 + cz^2 = 1$ is $\sum \frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) = 0$

(6) Find the polar of the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ with respect to the conicoid $x^2 - 2y^2 + 3z^2 = 4$. [Ans : $\frac{x+6}{3} = \frac{y-2}{3} = \frac{z-3}{1}$]

7.14 FURTHER READINGS

(1) Shanti Narayan, P.K. Mittal (2007): Analytical Solid Geometry, S.Chand Publication, New Delhi.

(2) Abraham Adrian Albert (2016): Solid Analytic Geometry, Dover Publication.

(3) George Wentworth, D.E. Smith (2007): Plane and solid Geometry, Merchant books.

(4) D.M.Y. Sommerville (2016): Analytical Geometry of three dimensions, Cambridge university Press.

UNIT-8 CENTRAL CONICOIDS II

Structure

8.1 INTRODUCTION

In unit-7 you have studied different central conicoids such as ellipsoids, hyperboloids of one sheet and hyperboloids of two sheets. These conicoids may be enveloped by certain cones and cylinders. For example, if a fixed point is given, we can draw tangent lines to a central conicoid from this point. These tangent lines will lie on a cone with this given point as vertex. This cone is an enveloping cone of the conicoid. Similarly, if a line is given, then we can draw tangents to a conicoid parallel to this given line. The locus of all these tangent lines is an enveloping cylinder of the conicoid. In this unit, we shall obtain equations of enveloping cone and enveloping cylinder of a given central conicoid.

Also in unit-7 you studied condition of tangency and obtained equations of tangent planes to a central conicoid. A line through a point on a conicoid perpendicular to the tangent plane at this point is called the normal to the conicoid at that point. In this unit we shall obtain equations of the normal at a given point on a central conicoid. You will see that we can draw six normals to a central conicoid from a given point and the curve passing through the feet of these normals is a cubic curve. We shall obtain the equation of the cone on which these normals lie. In the next section, we shall discuss diametral planes and conjugate diameters of a conicoid which are helpful in exploring the geometry of ellipsoids and hyperboloids. Lastly, you will see how to define a plane intersecting a central conicoid in a conic with a given centre.

8.2 OBJECTIVES

After reading this unit, you should be able to

- Obtain equations of enveloping cone and enveloping cylinder of a central conicoid.
- Define normals to a central conicoid.
- Obtain equation of the cone through the six normals.
- Show that six normals can be drawn to a central conicoid from a given point and the curve through the feet of these normals is a cubic curve.
- Define and discuss the diametral planes and conjugate diameters of an ellipsoid and hyperboloid.
- Obtain the equation of the plane containing section with a given centre.

8.3 ENVELOPING CONE

The enveloping cone or tangent cone of a given surface is the locus of the tangent lines drawn from a given point to the given surface. The point from which the tangent lines are drawn is called the **vertex** of the enveloping cone. Suppose we want to obtain the equation of the enveloping cone of the central conicoid

$$ax^2 + by^2 + cz^2 = 1 \quad \text{.........} \ (1)$$

with the vertex at the point (α, β, γ).

The equations of a straight line passing through the point (α, β, γ) are given as

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \ (= r) \qquad \dots \dots \dots (2)$$

Then the coordinates of any point P on the straight line (2) are given by $(lr+\alpha, mr+\beta, nr+\gamma)$. If the line (2) meets the central conicoid (1) at point P, then

$$a(lr+\alpha)^2 + b(mr+\beta)^2 + c(nr+\gamma)^2 = 1$$

or $\quad r^2(al^2+bm^2+cn^2) + 2r(a\alpha l + b\beta m + c\gamma n) + a\alpha^2 + b\beta^2 + c\gamma^2 - 1 = 0$

……………(3)

This is a quadratic equation in r. Hence we get two values of r corresponding to which we have two points of intersection of the line (2) and the central conicoid (1). If the line (2) is a tangent line to the conicoid (1), then the points of intersection must coincide, i.e. the roots of the quadratic equation (3) must be equal. It is possible if

$$\{2(a\alpha l + b\beta m + c\gamma n)\}^2 = 4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

or $\quad (a\alpha l + b\beta m + c\gamma n)^2 = (al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1) \dots \dots (4)$

The locus of the tangent line (2) is the required enveloping cone of the conicoid (1). It is obtained by eliminating l, m, n from (2) and (4), i.e.

$$\{a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma)\}^2$$

$$= \{a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2\}(a\alpha^2+b\beta^2+c\gamma^2-1)$$

or $\{(a\alpha x + b\beta y + c\gamma z - 1) - (a\alpha^2+b\beta^2+c\gamma^2-1)\}^2$

$$= \{(ax^2+by^2+cz^2-1) - 2(a\alpha x + b\beta y + c\gamma z - 1) + (a\alpha^2+b\beta^2+c\gamma^2-1)\}(a\alpha^2+b\beta^2+c\gamma^2-1)$$

Let
$$\left.\begin{aligned} S &\equiv ax^2+by^2+cz^2-1 \\ S' &\equiv a\alpha^2+b\beta^2+c\gamma^2-1 \\ T &\equiv a\alpha x+b\beta y+c\gamma z-1 \end{aligned}\right] \dots \dots \dots (5)$$

Hence we have $(T-S')^2 = (S-2T+S')S'$

or $\quad T^2 + {S'}^2 - 2TS' = SS' - 2TS' + {S'}^2$

or $\quad SS' = T^2$

or $\quad (ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)=(a\alpha x+b\beta y+c\gamma z-1)^2$

This is the required equation of the enveloping cone of the conicoid $ax^2+by^2+cz^2=1$.

8.4 THE ENVELOPING CYLINDER

The enveloping cylinder of a given surface is the locus of the tangent lines to the surface drawn parallel to a given line. In other words, the enveloping cylinder is the cylinder whose generators touch a given surface and are directed in a given direction.

Let us obtain the equation of an enveloping cylinder of a central conicoid

$$ax^2+by^2+cz^2=1 \qquad \ldots\ldots\ldots\ldots\text{(6)}$$

Whose generators are parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \;\ldots\ldots\ldots\;\; (7)$$

Let $P(\alpha,\beta,\gamma)$ be any point on the enveloping cylinder of the conicoid (6). Since the generators of the cylinder are parallel to the line (7), hence the equations of the generator through $P(\alpha,\beta,\gamma)$ may be given as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\;(=r) \qquad \ldots\ldots\ldots.(8)$$

Then the coordinates of any point on the generator are given by $(lr+\alpha, mr+\beta, nr+\gamma)$. If the generator (8) meets the conicoid (6) at point this point, then

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1$$

or $\quad r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+a\alpha^2+b\beta^2+c\gamma^2-1=0$

…………..….(9)

If the generator (8) is a tangent to the conicoid (6), then the points of intersection must coincide, i.e. the roots of above quadratic equation (9) must be equal. It is possible if

$$\{2(a\alpha l+b\beta m+c\gamma n)\}^2=4(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

or $\quad (a\alpha l+b\beta m+c\gamma n)^2=(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1) \;\ldots\ldots\,(10)$

The locus of $P(\alpha, \beta, \gamma)$, i.e. the equation of the enveloping cylinder of the conicoid (6) is

$$(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)$$
……. (11)

8.5 ILLUSTRATIVE EXAMPLES

Example 8.5.1 Find the locus of the vertex of enveloping cone of the conicoid $ax^2 + by^2 + cz^2 = 1$ which has three mutually perpendicular generators.

Or

Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2 + by^2 + cz^2 = 1$

Solution: Let $P(\alpha, \beta, \gamma)$be the point whose locus is required. The equation of enveloping cone of the conicoid $ax^2 + by^2 + cz^2 = 1$ with vertex $P(\alpha, \beta, \gamma)$ is given by

$(ax^2 + by^2 + cz^2 - 1)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = (a\alpha x + b\beta y + c\gamma z - 1)^2$ ... (12)

If this enveloping cone has three mutually perpendicular generators, then the sum of the coefficients of $x^2, y^2$and z^2 in (12) should be equal to zero, i.e.

$$(a + b + c)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) - (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) = 0$$

or $$(b + c)\alpha^2 + (a + c)\beta^2 + (a + b)\gamma^2 = a + b + c$$

The required locus of point $P(\alpha, \beta, \gamma)$ is

$$(b + c)x^2 + (a + c)y^2 + (a + b)z^2 = a + b + c$$

Example 8.5.2 Find the locus of the luminous point which moves so that the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

casts a circular shadow on the plane $z = 0$.

Solution: Let $P(\alpha, \beta, \gamma)$be the luminous point. We have to find the locus of P such that the section of the enveloping cone of the given ellipsoid with vertex at P by the plane $z = 0$ is a circle.

Proceeding as in article 8.3, we can obtain the equation of the enveloping cone of the ellipsoid as

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)$$
$$=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2 \quad \dots (13)$$

The section of this enveloping cone by the plane $z=0$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2, z=0 \quad \dots (14)$$

This represents a conic in xy-plane. It is a circle if the coefficients of x^2 and y^2 are equal, and the coefficient of xy is zero, i.e.

$$\frac{1}{a^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{\alpha^2}{a^4}=\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{\beta^2}{b^4} \quad \dots\dots (15)$$

and $$\frac{-2\alpha\beta}{a^2b^2}=0 \quad \dots\dots (16)$$

Equation (16) gives $\alpha\beta=0$, i.e. $\alpha=0$ or $\beta=0$.

When $\alpha=0$, equation (15) becomes

$$\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{\beta^2}{b^4}$$

or $$\frac{\beta^2}{a^2b^2}+\frac{\gamma^2}{a^2c^2}-\frac{1}{a^2}=\frac{\gamma^2}{b^2c^2}-\frac{1}{b^2}$$

or $$c^2\beta^2+b^2\gamma^2-b^2c^2=a^2\gamma^2-a^2c^2$$

or $$c^2\beta^2+(b^2-a^2)\gamma^2=(b^2-a^2)c^2$$

The required locus of $P(\alpha,\beta,\gamma)$ is

or $$c^2y^2+(b^2-a^2)z^2=(b^2-a^2)c^2, \qquad x=0$$

When $\beta=0$, equation (15) becomes

$$\frac{1}{a^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{\alpha^2}{a^4}=\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)$$

or $$c^2\alpha^2+(a^2-b^2)\gamma^2=(a^2-b^2)c^2$$

The required locus of $P(\alpha,\beta,\gamma)$ is or $\quad c^2x^2 + (a^2 - b^2)z^2 = (a^2 - b^2)c^2, \qquad y = 0$

Example 8.5.3 Show that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to the z-axis meets the plane $z = 0$ in parabolas.

Solution: We know that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$, whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is given by equation (11), i.e.

$$(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)$$

The direction cosines of z-axis are 0,0,1. Hence if the generators are perpendicular to z-axis, we have

$$0l + 0m + 1n = 0$$

$$\Rightarrow n = 0$$

Thus the direction ratios of the generators can be taken as $l, m, 0$. The equation of the enveloping cylinder becomes

$$(alx + bmy)^2 = (al^2 + bm^2)(ax^2 + by^2 + cz^2 - 1)$$

The section of this cylinder by the plane $z = 0$ is

$$(alx + bmy)^2 = (al^2 + bm^2)(ax^2 + by^2 - 1), \qquad z = 0$$

or $\quad ab(m^2x^2 + ly - 2lmxy) - (al^2 + bm^2) = 0, \qquad z = 0$

or $\quad ab(mx - ly)^2 = (al^2 + bm^2), \qquad z = 0$

This equation represents a parabola in the plane $z = 0$.

8.6 NORMALS TO A CENTRAL CONICOID

You have studied tangent planes to a central conicoid in unit-7. A line through a point $P(\alpha, \beta, \gamma)$ on a conicoid perpendicular to the tangent plane at P is called the **normal** to the conicoid at P.

Let a given conicoid be $\quad ax^2 + by^2 + cz^2 = 1$ ……… (17)

The equation of the tangent plane at $P(x', y', z')$ to the conicoid (17) is given

$$ax'x + by'y + cz'z = 1 \qquad \text{………. (18)}$$

Hence the direction cosines of the normal to this plane through the point $P(x', y', z')$ are proportional to ax', by', cz'. Therefore the equations of the normal to the conicoid (17) at P may be given as

$$\frac{x - x'}{ax'} = \frac{y - y'}{by'} = \frac{z - z'}{cz'} \qquad \text{... ... ... (19)}$$

8.6.1 NORMALS FROM A GIVEN POINT

Suppose we are given a point $A(\alpha,\beta,\gamma)$. If the normal to the conicoid at $P(x',y',z')$ passes through $A(\alpha,\beta,\gamma)$, then from (19)

$$\frac{\alpha-x'}{ax'}=\frac{\beta-y'}{by'}=\frac{\gamma-z'}{cz'}=\lambda \text{ (say)} \qquad \ldots\ldots(20)$$

Therefore

$$x'=\frac{\alpha}{1+a\lambda},y'=\frac{\beta}{1+b\lambda},z'=\frac{\gamma}{1+c\lambda} \quad \ldots\ldots(21)$$

For a given point $A(\alpha,\beta,\gamma)$, equations (21) give points (x',y',z') on the conicoid the normal through which passes through $A(\alpha,\beta,\gamma)$. Since (x',y',z') lies on the conicoid (17), hence we have

$$ax'^2+by'^2+cz'^2=1$$

or $$a\left(\frac{\alpha}{1+a\lambda}\right)^2+b\left(\frac{\beta}{1+b\lambda}\right)^2+c\left(\frac{\gamma}{1+c\lambda}\right)^2=1$$

or $$\frac{a\alpha^2}{(1+a\lambda)^2}+\frac{b\beta^2}{(1+b\lambda)^2}+\frac{c\gamma^2}{(1+c\lambda)^2}=1 \quad \ldots\ldots(22)$$

This is a sixth degree equation in λ. It has six roots, i.e. six values of λ, corresponding to each of which there is a point on the conicoid determined by (21) such that the normals at these six points pass through the given point $A(\alpha,\beta,\gamma)$. **Hence there are six points on the conicoid (17), the normal at which pass through a given fixed point, i.e. six normals can be drawn to a central conicoid from a given point.**

8.6.2 CONE THROUGH THE SIX NORMALS

Now we shall show that all the six normals drawn from a fixed point (α,β,γ) to the conicoid (17) lie on a cone of second degree.

Let l,m,n be the direction cosines of the normal at $P(x',y',z')$ given by (19). Then

$$\frac{l}{ax'}=\frac{m}{by'}=\frac{n}{cz'}=p \text{ (say)}$$

or $$l=pax',m=pby',n=pcz'$$

or $$l=\frac{pa\alpha}{1+a\lambda},m=\frac{pb\beta}{1+b\lambda},n=\frac{pc\gamma}{1+c\lambda}$$

which gives

$$1 + a\lambda = \frac{pa\alpha}{l}\ , 1 + b\lambda = \frac{pb\beta}{m}\ , 1 + c\lambda = \frac{pc\gamma}{n}$$

Multiplying above equations by $(b-c), (c-a)$ and $(a-b)$, and then adding we have

$$\frac{pa\alpha}{l}(b-c) + \frac{pb\beta}{m}(c-a) + \frac{pc\gamma}{n}(a-b)$$

$$= (1 + a\lambda)(b-c) + (1 + b\lambda)(c-a) + (1 + c\lambda)(a-b)$$

or $$\frac{a\alpha}{l}(b-c) + \frac{b\beta}{m}(c-a) + \frac{c\gamma}{n}(a-b) = 0 \quad \ldots\ldots(23)$$

Now the equations of the normal through the point (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \ldots\ldots(24)$$

Eliminating l, m, n from (23) and (24), we have

or $$\frac{a\alpha}{x-\alpha}(b-c) + \frac{b\beta}{y-\beta}(c-a) + \frac{c\gamma}{z-\gamma}(a-b) = 0$$

This is the equation of the quadratic cone upon which the six normals from a given point (α, β, γ) to the conicoid (17) lie.

8.6.3 CUBIC CURVE THROUGH THE FEET OF THE NORMALS

So you have seen that six normals can be drawn to a central conicoid from a given point (α, β, γ). These normals intersect the conicoid in six points which are given by (21). We shall show that these six points lie on a cubic curve.

Equations (20) can be written as

$$(\alpha - x')by' = (\beta - y')ax'$$

$$(\beta - y')cz' = (\gamma - z')by'$$

$$(\gamma - z')ax' = (\alpha - x')cz'$$

Hence the six points lie on the cylinders given by

$$(\alpha - x)by = (\beta - y)ax$$

$$(\beta - y)cz = (\gamma - z')by$$

$$(\gamma - z)ax = (\alpha - x)cz$$

These cylinders intersect on a common curve on which the six points of intersection of the normals and the conicoid lie. Let the plane intersecting this curve be

$$Ax + By + Cz + D = 0$$

Since the points (x', y', z') lie on this plane, hence

$$Ax' + By' + Cz' + D = 0$$

$$\text{or } \frac{A\alpha}{1+a\lambda} + \frac{B\beta}{1+b\lambda} + \frac{C\gamma}{1+c\lambda} + D = 0 \quad \text{using (21)}$$

This is a cubic equation in λ having three roots. Hence the curve through the feet of the normals intersects the plane in three points, i.e. the curve is a cubic curve.

8.7 ILLUSTRATIVE EXAMPLES

Example 8.7.1 If the normal at a point P to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \dots\dots..(25)$$

meets the principal planes in G_1, G_2, G_3, show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$ and if

${PG_1}^2 + {PG_2}^2 + {PG_3}^2 = k^2$ then find the locus of P.

Solution: Let P be the point (x', y', z'). The equation of the tangent plane at $P(x', y', z')$ t the ellipsoid is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1$$

Hence the direction cosines of the normal to the ellipsoid at $P(x', y', z')$ are proportional to $x'/a^2, y'/b^2, z'/c^2$. Let the actual direction cosines of the normal at $P(x', y', z')$ be $p\,x'/a^2, p\,y'/b^2, p\,z'/c^2$. Then

$$\left(\frac{px'}{a^2}\right)^2 + \left(\frac{py'}{b^2}\right)^2 + \left(\frac{pz'}{c^2}\right)^2 = 1$$

$$\text{or} \quad \frac{1}{p^2} = \left(\frac{x'}{a^2}\right)^2 + \left(\frac{y'}{b^2}\right)^2 + \left(\frac{z'}{c^2}\right)^2 \quad \dots\dots(26)$$

Now the equations of the normal to the ellipsoid (25) at P may be given as

$$\frac{x-x'}{p\,x'/a^2}=\frac{y-y'}{p\,y'/b^2}=\frac{z-z'}{p\,z'/c^2}=r(\text{say}) \qquad \ldots\ldots(27)$$

If the normal (27) meets the plane $x=0$ in G_1, then

$$\frac{0-x'}{p\,x'/a^2}=\frac{y-y'}{p\,y'/b^2}=\frac{z-z'}{p\,z'/c^2}=PG_1$$

$$\text{or} \qquad PG_1=-\frac{a^2}{p}$$

Similarly if the normal (27) meets the plane $y=0$ and $z=0$ in points G_2 and G_3 respectively, then

$$PG_2=-\frac{b^2}{p}, PG_3=-\frac{c^2}{p}$$

Therefore $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$.

Now given that ${PG_1}^2+{PG_2}^2+{PG_3}^2=k^2$.

$$\Rightarrow \left(-\frac{a^2}{p}\right)^2+\left(-\frac{b^2}{p}\right)^2+\left(-\frac{c^2}{p}\right)^2=k^2$$

$$\Rightarrow \frac{1}{p^2}=\frac{k^2}{a^4+b^4+c^4}$$

Using (26), we have

$$\left(\frac{x'}{a^2}\right)^2+\left(\frac{y'}{b^2}\right)^2+\left(\frac{z'}{c^2}\right)^2=\frac{k^2}{a^4+b^4+c^4}$$

$$\text{or} \qquad \frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}=\frac{k^2}{a^4+b^4+c^4}$$

Hence the locus of point $P(x',y',z')$ is

$$\text{or} \qquad \frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^2}{a^4+b^4+c^4}$$

Example 8.7.2 Prove that the feet of the six normals drawn to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots(28)$$

from any point (α,β,γ) lie on the curve of intersection of the ellipsoid and the cone

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

Solution: Let (x', y', z') be any point on the ellipsoid (28). Then the equations of the normal at (x', y', z') to the ellipsoid are given by (27). Which gives

$$\frac{x-x'}{x'/a^2}=\frac{y-y'}{y'/b^2}=\frac{z-z'}{z'/c^2} \quad \ldots\ldots(29)$$

If this normal passes through the given point (α, β, γ), then

$$\frac{\alpha-x'}{x'/a^2}=\frac{\beta-y'}{y'/b^2}=\frac{\gamma-z'}{z'/c^2}=\lambda \text{ (say)}$$

Hence we have

$$x'=\frac{a^2\alpha}{a^2+\lambda}, y'=\frac{b^2\beta}{b^2+\lambda}, z'=\frac{c^2\gamma}{c^2+\lambda} \quad \ldots\ldots(30)$$

This gives the coordinates of the six feet of the normals drawn from the given point (α, β, γ). We can write these equations as

$$\lambda=\frac{a^2\alpha}{x'}-a^2, \qquad \lambda=\frac{b^2\beta}{y'}-b^2, \qquad \lambda=\frac{c^2\gamma}{z'}-c^2$$

Multiplying these equations by $(b^2-c^2), (c^2-a^2)$ and (a^2-b^2) and then adding we have

$$\lambda(b^2-c^2)+\lambda(c^2-a^2)+\lambda(a^2-b^2)$$

$$=\left(\frac{a^2\alpha}{x'}-a^2\right)(b^2-c^2)+\left(\frac{b^2\beta}{y'}-b^2\right)(c^2-a^2)+\left(\frac{c^2\gamma}{z'}-c^2\right)(a^2-b^2)$$

$$\text{Or} \quad \left(\frac{a^2\alpha}{x'}-a^2\right)(b^2-c^2)+\left(\frac{b^2\beta}{y'}-b^2\right)(c^2-a^2)+\left(\frac{c^2\gamma}{z'}-c^2\right)(a^2-b^2)=0$$

Simplifying the equation, we have

$$\frac{a^2(b^2-c^2)\alpha}{x'}+\frac{b^2(c^2-a^2)\beta}{y'}+\frac{c^2(a^2-b^2)\gamma}{z'}=0$$

Locus of (x', y', z') is

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

This equation is a homogeneoussecond degree equation and hence represents a cone. Therefore the six feet of normals drawn from (α,β,γ) lie on the curve of intersection of the ellipsoid and this cone .

Example 8.7.3 Prove that the lines drawn from the origin parallel to the normal to the conicoid $ax^2+by^2+cz^2=1$ ……… (31)

at points lying on its curve of intersection with the plane

$$lx+my+nz=p \qquad \text{………..} (32)$$

generates the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

Solution: Let $P(x',y',z')$ be any point on the conicoid (31). Then the equations of the normal to the conicoid at P may be given as

$$\frac{x-x'}{ax'}=\frac{y-y'}{by'}=\frac{z-z'}{cz'} \qquad \ldots\ldots\ldots (33)$$

Hence the equations of the line passing through the origin and parallel to the normal (33) may be given as

$$\frac{x}{ax'}=\frac{y}{by'}=\frac{z}{cz'} \qquad \ldots\ldots\ldots (34)$$

Since the point $P(x',y',z')$ lies on(31) and (32), hence

$$ax'^2+by'^2+cz'^2=1, \qquad lx'+my'+nz'=p$$

We can make the first equation homogeneous with the help of second equation, i.e.

$$ax'^2+by'^2+cz'^2=\left(\frac{lx'+my'+nz'}{p}\right)^2$$

or $$p^2\left\{\frac{(ax')^2}{a}+\frac{(by')^2}{b}+\frac{(cz')^2}{c}\right\}$$
$$=\left\{l\frac{(ax')}{a}+m\frac{(by')}{b}+n\frac{(cz')}{c}\right\}^2 \ldots\ldots (35)$$

The locus of line (34) is obtained by eliminating ax', by', cz' from (34) and (35), i.e.

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

Which is the equation of the required cone.

8.8 DIAMETRAL PLANES

Any chord through the centre of a conicoid is called a **diameter** of the conicoid. A plane which bisects a system of parallel chords is called a **diametral plane** of a conicoid. In other words, a diametral plane is the locus of middle points of a system of parallel chords drawn parallel to a given line or diameter. If a diametral plane bisects a system of chords parallel to a given line, we say that the diametral plane is **conjugate** to that line. Diametral planes which are perpendicular to the chords bisected by them are called **principal planes**. The lines of intersection of principal planes are called **principal axis**. Let us find the equation of a diametral plane of a central conicoid

$$ax^2+by^2+cz^2=1 \quad \ldots\ldots (36)$$

Let a fixed line be given as

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \ldots\ldots (37)$$

The equations of a chord drawn parallel to (37) with mid-point (α,β,γ) may be given as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ (=r) \quad \ldots\ldots\ldots (38)$$

Any point on this chord will be $(lr+\alpha, mr+\beta, nr+\gamma)$. If the chord (38) meets the conicoid (36) at point this point, then

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1$$

or $\quad r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+a\alpha^2+b\beta^2+c\gamma^2-1=0$

Since $(\alpha,\beta,\gamma)$is the mid-point of the chord (38), the two values of r given by above quadratic equation must be equal in magnitude but opposite in sign, i.e.

$$r_1=-r_2$$

or $$r_1+r_2=0$$

or $$a\alpha l + b\beta m + c\gamma n = 0 \quad …….. (39)$$

If l, m, n are given, then (38) represents system of parallel chords. The locus of middle point (α, β, γ) is

$$alx + bmy + cnz = 0 \; ………. (40)$$

This is the diametral plane conjugate to the line with dc's l, m, n.

For example, suppose we want to obtain the diametral plane of the conicoid $ax^2 + by^2 + cz^2 = 1$ conjugate to z-axis, i.e. diametral plane bisecting the chords parallel to z-axis.

The equations of z-axis are

$$\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$$

Here $l = 0, m = 0, n = 1$. Hence from (40) the equation of diametral plane is

$$z = 0, \qquad i.e.\ xy\text{-plane}$$

Similarly, it can be shown that the diametral planes bisecting chords parallel to x-axis and y-axis are yz-plane and xz-plane respectively. Also the coordinate planes are such that they bisect chords perpendicular to them. Hence the coordinate planes are principal planes and the coordinate axes are principal axes.

8.9 CONJUGATE DIAMETERS

You have seen that the coordinate planes are such diametral planes of the central conicoid $ax^2 + by^2 + cz^2 = 1$ that each bisects chords parallel to the line of intersection of the other two planes. Such planes are called conjugate diametral planes. Hence any three diametral planes which are such that each is the diametral plane of the line of intersection of the other two are called **conjugate diametral planes.**

Also you have seen that the coordinate axes are such that planes through any two bisect chords parallel to the third axis. Such lines are called conjugate diameters. Hence the three lines which are such that the plane containing any two is the diametral plane of the third are called **conjugate diameters**.

8.9.1 CONJUGATE DIAMETERS OF AN ELLIPSOID

Now we shall discuss the conjugate diameters of an ellipsoid given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots..(41)$$

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid (41). Then the direction cosines of the line OP are proportional to $x_1-0, y_1-0, z_1-0,$ i.e. x_1, y_1, z_1. Hence by (40), the diametral plane of OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \quad \ldots\ldots..(42)$$

If $Q(x_2, y_2, z_2)$ lies on the diametral plane (42) of OP, then

$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0 \quad \ldots\ldots..(43)$$

The symmetry of above equation indicates that if Q lies on the diametral plane of OP, then P lies on the diametral plane of OQ.

Let the diametral planes of OP and OQ intersect in diameter OR where $R(x_3, y_3, z_3)$ is one of the two points where the line of intersection of diametral planes of OP and OQ meets the ellipsoid.

Now the diametral plane of OR is

$$\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}=0 \quad \ldots\ldots..(44)$$

Since $R(x_3, y_3, z_3)$ lies on the diametral planes of OP and OQ, hence P and Q must lie on the diametral plane of OR, i.e.

$$\frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0 \quad \ldots\ldots..(45)$$

and $$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0 \quad \ldots\ldots..(46)$$

Thus the planes POQ, QOR, ROP are conjugate diametral planes and OR, OP, OQ are the corresponding conjugate semi-diameters.

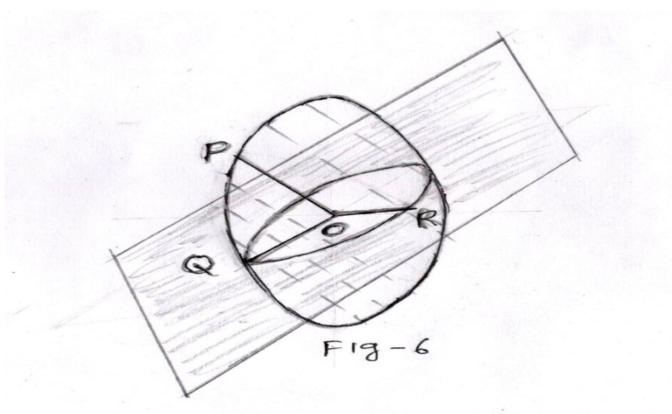


Fig-6

Now the points P, Q, R lie on the ellipsoid (41), hence

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1, \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1, \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \quad \ldots\ldots..(47)$$

Equations (44), (45), (46) and (47) indicate that the lines with direction cosines

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \ \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \ \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

are mutually perpendicular. Hence we have

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= a^2 \\ y_1^2 + y_2^2 + y_3^2 &= b^2 \\ z_1^2 + z_2^2 + z_3^2 &= c^2 \end{aligned}\right] \quad \ldots\ldots..(48)$$

$$\left.\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= 0 \\ y_1z_1 + y_2z_2 + y_3z_3 &= 0 \\ z_1x_1 + z_2x_2 + z_3x_3 &= 0 \end{aligned}\right] \quad \ldots\ldots..(49)$$

Now we shall discuss the properties of these conjugate diameters.

8.9.2 PROPERTIES OF CONJUGATE DIAMETERS

PROPERTY I The sum of squares of any three conjugate semidiameters of an ellipsoid is constant.

We have

$$OP^2 + OQ^2 + OR^2 = (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) + (x_3^2 + y_3^2 + z_3^2)$$

$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) + (z_1^2 + z_2^2 + z_3^2)$$

$$= a^2 + b^2 + c^2, \qquad \text{using (48)}$$

$$= \text{constant}$$

PROPERTY II The sum of squares of the projections of three conjugate semi-diameters on any line is constant

Suppose we are given a line with direction cosines l, m, n. Then the projection of the semi-diameter OP on this line

$$L_1 = l(x_1 - 0) + m(y_1 - 0) + n(z_1 - 0)$$

$$= lx_1 + my_1 + nz_1$$

Similarly, the projections of the semi-diameters OP and OQ on this line will be

$$L_2 = lx_2 + my_2 + nz_2, \qquad L_3 = lx_3 + my_3 + nz_3$$

Now $L_1{}^2 + L_2{}^2 + L_3{}^2 = (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2$

$+(lx_3 + my_3 + nz_3)^2$

$$= l^2(x_1{}^2 + x_2{}^2 + x_3{}^2) + m^2(y_1{}^2 + y_2{}^2 + y_3{}^2) + n^2(z_1{}^2 + z_2{}^2 + z_3{}^2)$$

$$+2lm(x_1y_1 + x_2y_2 + x_3y_3) + 2mn(y_1z_1 + y_2z_2 + y_3z_3) + 2ln(z_1x_1 + z_2x_2 + z_3x_3)$$

$= l^2a^2 + m^2b^2 + n^2c^2$ using (48)and (49)

$=$ constant

Similarly we can prove the following property-

PROPERTY III The sum of squares of the projections of three conjugate semi-diameters on any plane is constant.

PROPERTY IV The volume of the parallelopiped formed by three conjugate semi-diameters of an ellipsoid as coterminous edges is constant.

The volume of the parallelopiped having OP, OQ, OR as coterminous edges

$$V = 6 \times \text{volume of the tetrahedron } (O, PQR)$$

$$= 6 \times \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Hence $V^2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$= \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} \sum x_1^2 & \sum x_1 y_1 & \sum x_1 z_1 \\ \sum x_1 y_1 & \sum y_1^2 & \sum y_1 z_1 \\ \sum x_1 z_1 & \sum y_1 z_1 & \sum z_1^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix}$$

or $\quad V^2 = a^2 b^2 c^2 \quad$ or $V = abc =$ constant

PROPERTY V The sum of the squares of the areas of the faces POQ, QOR, ROP of the parallelopiped formed by three conjugate semi-diameters as coterminous edges is constant.

Let the areas of the faces QOR, ROP, POQ be A_1, A_2, A_3 respectively. Let $l_1, m_1, n_1;\ l_2, m_2, n_2$ and l_3, m_3, n_3 be the direction cosines of the normals to the planes QOR, ROP, POQ respectively. Then the projection of the area A_1 on the plane $x = 0$ is

$$A_1 l_1 = \frac{1}{2}(y_2 z_3 - y_3 z_2) \quad \dots\dots (50)$$

Now solving equations (43), (45) and (46), we have

$$\frac{x_1}{a} = \pm\frac{(y_2 z_3 - y_3 z_2)}{bc}, \frac{y_1}{b} = \pm\frac{(z_2 x_3 - z_3 x_2)}{ca}, \frac{z_1}{c} = \pm\frac{(x_2 y_3 - x_3 y_2)}{ab}$$

$$\frac{x_2}{a} = \pm\frac{(y_3 z_1 - y_1 z_3)}{bc} \quad \text{and so on.}$$

Hence (50) becomes

$$A_1 l_1 = \pm\frac{bcx_1}{2a}$$

Similarly

$$A_1 m_1 = \pm\frac{cay_1}{2b}, A_1 n_1 = \pm\frac{abz_1}{2c}$$

Squaring and adding, we have

$$A_1{}^2\left(l_1{}^2 + m_1{}^2 + n_1{}^2\right) = \frac{1}{4}\left(\frac{bcx_1}{a}\right)^2 + \frac{1}{4}\left(\frac{cay_1}{b}\right)^2 + \frac{1}{4}\left(\frac{abz_1}{c}\right)^2$$

or $$A_1{}^2 = \frac{1}{4}\left[\left(\frac{bcx_1}{2a}\right)^2 + \left(\frac{cay_1}{2b}\right)^2 + \left(\frac{abz_1}{2c}\right)^2\right] \quad \dots\dots\dots (51)$$

Similarly for the areas ROP and POQ, we have

$$A_2{}^2 = \frac{1}{4}\left[\left(\frac{bcx_2}{2a}\right)^2 + \left(\frac{cay_2}{2b}\right)^2 + \left(\frac{abz_2}{2c}\right)^2\right] \quad \text{........ (52)}$$

$$A_3{}^2 = \frac{1}{4}\left[\left(\frac{bcx_3}{2a}\right)^2 + \left(\frac{cay_3}{2b}\right)^2 + \left(\frac{abz_3}{2c}\right)^2\right] \quad \text{........ (53)}$$

Adding (51), (52) and (53)

$$A_1{}^2 + A_2{}^2 + A_3{}^2 = \frac{1}{4}\left[\frac{b^2c^2}{a^2}\sum x_1{}^2 + \frac{c^2a^2}{b^2}\sum y_1{}^2 + \frac{a^2b^2}{c^2}\sum z_1{}^2\right]$$

Using (48), we get

$$A_1{}^2 + A_2{}^2 + A_3{}^2 = \frac{1}{4}\left[\frac{b^2c^2}{a^2} \times a^2 + \frac{c^2a^2}{b^2} \times b^2 + \frac{a^2b^2}{c^2} \times c^2\right]$$

or $$A_1{}^2 + A_2{}^2 + A_3{}^2 = \frac{1}{4}[b^2c^2 + c^2a^2 + a^2b^2] = \text{constant}$$

8.9.3 CONJUGATE DIAMETERS OF THE HYPERBOLOIDS

Now we shall obtain relations for the conjugate semi-diameters of a hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \dots\dots..(54)$$

Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_3)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters of the hyperboloid (54). Then we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - \frac{z_1z_2}{c^2} = 0 \quad \dots\dots..(55)$$

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} - \frac{z_2z_3}{c^2} = 0 \quad \dots\dots..(56)$$

$$\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} - \frac{z_3z_1}{c^2} = 0 \quad \dots\dots..(57)$$

and

$$\left.\begin{aligned} x_1{}^2 + x_2{}^2 - x_3{}^2 &= a^2 \\ y_1{}^2 + y_2{}^2 - y_3{}^2 &= b^2 \\ z_1{}^2 + z_2{}^2 - z_3{}^2 &= c^2 \end{aligned}\right] \quad \dots\dots..(58)$$

Now the points P, Q, R lie on the hyperboloid (54), hence

$$\frac{{x_1}^2}{a^2}+\frac{{y_1}^2}{b^2}-\frac{{z_1}^2}{c^2}=1, \frac{{x_2}^2}{a^2}+\frac{{y_2}^2}{b^2}-\frac{{z_2}^2}{c^2}=1, \frac{{x_3}^2}{a^2}+\frac{{y_3}^2}{b^2}-\frac{{z_3}^2}{c^2}=1 \quad \dots\dots..(59)$$

Hence we have

$$\begin{aligned} OP^2+OQ^2-OR^2 &= ({x_1}^2+{y_1}^2+{z_1}^2)+({x_2}^2+{y_2}^2+{z_2}^2) - ({x_3}^2+{y_3}^2+{z_3}^2) \\ &= ({x_1}^2+{x_2}^2-{x_3}^2)+({y_1}^2+{y_2}^2-{y_3}^2)+({z_1}^2+{z_2}^2-{z_3}^2) \\ &= a^2+b^2+c^2, \qquad \text{using (58)} \\ &= \text{constant} \end{aligned}$$

Let the areas of the faces QOR, ROP, POQ be A_1, A_2, A_3 respectively. Then proceeding as in case of ellipsoid using relations (55), (56), (57), (58) and (59) we can show that

$$A_1{}^2+A_2{}^2-A_3{}^2=\frac{1}{4}[b^2c^2+c^2a^2-a^2b^2]$$

Similarly for the hyperboloid of two sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, we have

$$A_1{}^2-A_2{}^2-A_3{}^2=\frac{1}{4}[b^2c^2-c^2a^2-a^2b^2]$$

8.10 ILLUSTRATIVE EXAMPLES

Example 8.10.1 Find the equation to the plane through the extremities $P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$ and $R(x_3,y_3,z_3)$ of three conjugate semi-diameters of an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \dots\dots..(60)$$

Solution: Let the equation of the plane PQR be

$$lx+my+nz=p \quad \dots\dots\dots\dots (61)$$

Since $P(x_1,y_1,z_1), Q(x_2,y_2,z_2)$ and $R(x_3,y_3,z_3)$ lies on (61), hence

$$lx_1+my_1+nz_1=p \quad \dots\dots\dots (62)$$

$$lx_2+my_2+nz_2=p \quad \dots\dots\dots (63)$$

$$lx_3+my_3+nz_3=p \quad \dots\dots\dots (64)$$

Multiplying (62), (63) and (64) by x_1, x_2 and x_3 respectively and adding we have

$$l({x_1}^2 + {x_2}^2 + {x_3}^2) + m(x_1 y_1 + x_2 y_2 + x_3 y_3) + n(z_1 x_1 + z_2 x_2 + z_3 x_3)$$

$$= p(x_1 + x_2 + x_3)$$

Using (48) and (49), we have

$$la^2 = p(x_1 + x_2 + x_3)$$

$$\text{or} \quad l = \frac{p(x_1 + x_2 + x_3)}{a^2}$$

$$\text{Similarly} \qquad m = \frac{p(y_1 + y_2 + y_3)}{b^2} \quad \text{and} \quad l = \frac{p(z_1 + z_2 + z_3)}{c^2}$$

Putting these values in (61), we get the required equation of the plane PQR as

$$x\left(\frac{x_1 + x_2 + x_3}{a^2}\right) + y\left(\frac{y_1 + y_2 + y_3}{b^2}\right) + z\left(\frac{z_1 + z_2 + z_3}{c^2}\right) = 1$$

Example 8.10.2 Find the locus of the equal conjugate diameters of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_3)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters of the ellipsoid such that $OP = OQ = OR = r$

Now we know that $\qquad OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$

$$\Rightarrow 3r^2 = a^2 + b^2 + c^2$$

$$\Rightarrow r^2 = \frac{1}{3}(a^2 + b^2 + c^2) \quad \ldots\ldots(65)$$

Let the direction cosines of OP be l, m, n. Then the equations of the line OP are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \ldots\ldots(66)$$

and $x_1 = lr, y_1 = mr, z_1 = nr$.

Since P lies on the given ellipsoid, hence

$$\frac{{x_1}^2}{a^2} + \frac{{y_1}^2}{b^2} + \frac{{z_1}^2}{c^2} = 1$$

$$\text{or}\quad \frac{(lr)^2}{a^2}+\frac{(mr)^2}{b^2}+\frac{(nr)^2}{c^2}=l^2+m^2+n^2,\quad \text{as } l^2+m^2+n^2=1$$

$$\text{or}\quad r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=l^2+m^2+n^2$$

Using (65) we get

$$\frac{1}{3}(a^2+b^2+c^2)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=l^2+m^2+n^2$$

$$\text{or}\quad \left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=3\frac{(l^2+m^2+n^2)}{(a^2+b^2+c^2)}\quad \ldots..(67)$$

The locus of the line OP is obtained by eliminating l, m, n from (66) and (67), i.e.

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3\frac{(x^2+y^2+z^2)}{(a^2+b^2+c^2)}$$

$$\text{or}\quad \frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}(2c^2-a^2-b^2)=0$$

Which represents a cone.

8.11 SECTION WITH A GIVEN CENTRE

Suppose (α, β, γ) be given as middle point of a chord of the conicoid

$$ax^2+by^2+cz^2=1 \qquad \ldots\ldots (68)$$

We can find the equation of the plane which intersects the conicoid in a conic with centre as (α, β, γ). This plane will be the locus of all chords of (68) with (α, β, γ) as middle point.

The equations of a chord with (α, β, γ) as mid-point may be given as

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}\ (=r)\qquad \ldots\ldots\ldots.(69)$$

Any point on this chord will be $(lr+\alpha, mr+\beta, nr+\gamma)$. If the chord (69) meets the conicoid (68) at point this point, then

$$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1$$

$$\text{or}\quad r^2(al^2+bm^2+cn^2)+2r(a\alpha l+b\beta m+c\gamma n)+a\alpha^2+b\beta^2+c\gamma^2-1=0$$

Since $(\alpha, \beta, \gamma)$is the mid-point of the chord (69), the two values of r given by above quadratic equation must be equal in magnitude but opposite in sign, i.e.

$$r_1 = -r_2$$

or $$r_1 + r_2 = 0$$

or $$a\alpha l + b\beta m + c\gamma n = 0 \quad …….. (70)$$

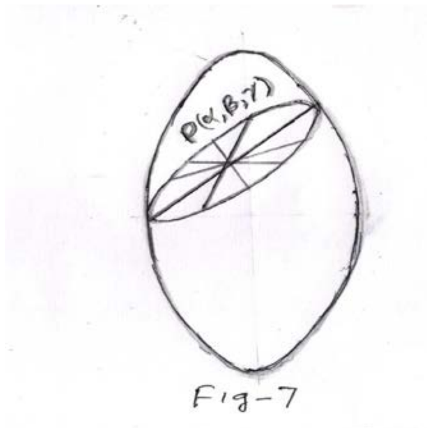


Fig-7

The locus of chord with a middle point (α, β, γ) is obtained by eliminating l, m, n from (69) and (70), i.e.

$$a\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma) = 0$$

or $$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 \; ….. (71)$$

Using the notations $S_1 \equiv a\alpha^2 + b\beta^2 + c\gamma^2 - 1, T \equiv a\alpha x + b\beta y + c\gamma z - 1$, the equation of the plane containing the section with centre (α, β, γ) may be given as

$$S_1 = T$$

8.12 ILLUSTRATIVE EXAMPLES

Example 8.12.1 Find the equation of the plane which cuts the conicoid $2x^2 - 3y^2 + 5z^2 = 1$ in a conic whose centre is at the point (2,1,3).

Solution The required equation of plane is given by

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2 \; ……. (72)$$

Here $\alpha = 2, \beta = 1, \gamma = 3$ and $a = 2, b = -3, c = 5$.

Hence (72) gives

$$2(2)x + (-3)(1)y + 5(3)z = 2(4) - 3(1) + 5(9)$$

or $$4x - 3y + 15z = 50$$

Example 8.12.2 Prove that the centres of sections of $ax^2 + by^2 + cz^2 = 1$ by the planes which are at a constant distance p from the origin lie on the surface

$$(ax^2 + by^2 + cz^2)^2 = p^2(a^2x^2 + b^2y^2 + c^2z^2)$$

Solution: The equation of the plane containing the section with centre (α, β, γ) is given as $$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

or $$-a\alpha x - b\beta y - c\gamma z + (a\alpha^2 + b\beta^2 + c\gamma^2) = 0$$

The length of the perpendicular drawn from the origin to this palne

$$p = \frac{a\alpha^2 + b\beta^2 + c\gamma^2}{\sqrt{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}}$$

or $$(a\alpha^2 + b\beta^2 + c\gamma^2)^2 = p^2(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)$$

Therfore the locus of the centre (α, β, γ) is

$$(ax^2 + by^2 + cz^2)^2 = p^2(a^2x^2 + b^2y^2 + c^2z^2)$$

Example 8.12.3 Find the locus of the centres of sections of $ax^2 + by^2 + cz^2 = 1$ which touch the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$

Solution Let the centre of one such section of the conicoid $ax^2 + by^2 + cz^2 = 1$ be (x_1, y_1, z_1). Then the equation of the plane containing the section is given as

$$axx_1 + byy_1 + czz_1 = ax_1{}^2 + by_1{}^2 + cz_1{}^2$$
…… (73)

If the plane (73) touches the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, then from the condition of tangency [see eqn 22, Unit-7] we have

$$\frac{(ax_1)^2}{\alpha} + \frac{(by_1)^2}{\beta} + \frac{(cz_1)^2}{\gamma} = (ax_1{}^2 + by_1{}^2 + cz_1{}^2)^2$$

The required locus of (x_1, y_1, z_1) is

$$\frac{a^2x^2}{\alpha} + \frac{b^2y^2}{\beta} + \frac{c^2z^2}{\gamma} = (ax^2 + by^2 + cz^2)^2$$

8.13 SUMMARY

In this unit, we have studied the following facts-

(1) The equation of the enveloping cone with vertex (α, β, γ) of the conicoid $ax^2 + by^2 + cz^2 = 1$ is $(ax^2 + by^2 + cz^2 - 1)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = (a\alpha x + b\beta y + c\gamma z - 1)^2$

(2) The equation of an enveloping cylinder of a central conicoid $ax^2 + by^2 + cz^2 = 1$ whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is

$(alx + bmy + cnz)^2 = (al^2 + bm^2 + cn^2)(ax^2 + by^2 + cz^2 - 1)$

(3) The equations of the normal to the conicoid $ax^2 + by^2 + cz^2 = 1$ at $P(x', y', z')$ may be given as

$$\frac{x - x'}{ax'} = \frac{y - y'}{by'} = \frac{z - z'}{cz'}$$

(4) Six normals can be drawn to a central conicoid from a given point. These six normals lie on a cone. These normals intersect the conicoid in six points which lie on a cubic curve.

(5) A plane which bisects a system of parallel chords is called a diametral plane of a conicoid. If a diametral plane bisects a system of chords parallel to a given line, we say that the diametral plane is conjugate to that line.

(6) The equation of a diametral plane of a central conicoid $ax^2 + by^2 + cz^2 = 1$ conjugate to the line with dc's l, m, n is given by

$$alx + bmy + cnz = 0$$

(7) The planes which are such that each bisects chords parallel to the line of intersection of the other two planes are called conjugate diametral planes. Any three diametral planes which are such that each is the diametral plane of the line of intersection of the other two are called conjugate diametral planes.

(8) The three lines which are such that the plane containing any two is the diametral plane of the third are called conjugate diameters.

(9) The sum of squares of any three conjugate semidiameters of an ellipsoid is constant.

$$OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2 = \text{constant}$$

(10) The sum of squares of the projections of three conjugate semi-diameters on any line is constant.

(11) The sum of squares of the projections of three conjugate semi-diameters on any plane is constant.

(12) The volume of the parallelopiped formed by three conjugate semi-diameters of an ellipsoid as coterminous edges is constant.

(13) The sum of the squares of the areas of the faces POQ, QOR, ROP of the parallelopiped formed by three conjugate semi-diameters as coterminous edges is constant.

$$A_1{}^2 + A_2{}^2 + A_3{}^2 = \frac{1}{4}[b^2c^2 + c^2a^2 + a^2b^2] = \text{constant}$$

(14) The locus of the chords with a given middle point (α, β, γ), i.e. The equation of the plane containing the section with centre (α, β, γ) of the conicoid $ax^2 + by^2 + cz^2 = 1$ is given as

$$a\alpha x + b\beta y + c\gamma z = a\alpha^2 + b\beta^2 + c\gamma^2$$

8.14 SELF ASSESSMENT QUESTIONS

(1) Find the locus of the vertices of enveloping cones of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if sections of cones by the plane $z = 0$ are circles.

(2) Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and which envelopes the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

[Ans. $\left(\frac{xl}{a^2} + \frac{ym}{b^2} + \frac{zn}{c^2}\right)^2 = \left(\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)^2$]

(3) Prove that the enveloping cylinder of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, whose generators are parallel to the line $\frac{x}{0} = \frac{y}{\sqrt{a^2 - b^2}} = \frac{z}{c}$ meets the plane $z = 0$ in a circle.

(4) Prove that the greatest value of the shortest distance between the x-axis and a normal to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $b - c$.

(5) Prove that the locus of the foot of the perpendicular from the centre to the plane through the extremities of three conjugate diameters of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$a^2x^2 + b^2y^2 + c^2z^2 = 3(x^2 + y^2 + z^2)^2$$

(6) Prove that the locus of middle points of chords of $x^2 + by^2 + cz^2 = 1$, which are parallel to $x = 0$ and touch $x^2 + y^2 + z^2 = r^2$ lie on the surface

$$by^2(bx^2 + by^2 + cz^2 - br^2) + cz^2(cx^2 + by^2 + cz^2 - br^2) = 0$$

(7) Prove that the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ whose centre is at the point $(a/3, b/3, c/3)$ passes through the extremities of the axes.

(8) Find the centre of the conic

$$\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{4} = 1, 2x + 2y - z = 3$$ [Ans. $\left(\frac{27}{52}, \frac{12}{13}, \frac{-3}{26}\right)$]

8.15 FURTHER READINGS

(1) Shanti Narayan, P.K. Mittal (2007): Analytical Solid Geometry, S.Chand Publication, New Delhi.

(2) Abraham Adrian Albert (2016): Solid Analytic Geometry, Dover Publication.

(3) George Wentworth, D.E. Smith (2007): Plane and solid Geometry, Merchant books.

(4) D.M.Y. Sommerville (2016): Analytical Geometry of three dimensions, Cambridge University Press.