Uttar Pradesh Rajarshi Tandon Open University

Bachelor of Science

UGPHS- 103

Electromagnetism

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Electromagnetism

Block

1 Electrostatics

UNIT - 1	**Electric Charge, Force and Fields**
UNIT - 2	**Electric Potential and Dipole**
UNIT - 3	**Dielectrics**

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Unit 1 Electric charge, force and fields

Structure

1.1 Introduction:

The process of supplying the electric charge (electrons) to an object or losing the electric charge (electrons) from an object is called charging. An uncharged object can be charged in different ways: charging by friction, charging by conduction, charging by induction.

Electric charge, basic property of matter carried by some elementary particles that governs how the particles are affected by an electric or magnetic field. Electric charge, which can be positive or negative, occurs in discrete natural units and is neither created nor destroyed.

Coulomb's law states that the electrical force between two charged objects is directly proportional to the product of the quantity of charge on the objects and inversely proportional to the square of the separation distance between the two objects.

Electric field, an electric property associated with each point in space when charge is present in any form. The magnitude and direction of the electric field are expressed by the value of E, called electric field strength or electric field intensity or simply the electric field.

Electric flux, property of an electric field that may be thought of as the number of electric lines of force (or electric field lines) that intersect a given area. Electric field lines are considered to originate on positive electric charges and to terminate on negative charges.

Gauss Law states that the total electric flux out of a closed surface is equal to the charge enclosed divided by the permittivity. The electric flux in an area is defined as the electric field multiplied by the area of the surface projected in a plane and perpendicular to the field.

The law relates the flux through any closed surface and the net charge enclosed within the surface. The law states that the total flux of the electric field E over any closed surface is equal to 1/?o times the net charge enclosed by the surface.

Differential form of Gauss law states that the divergence of electric field E at any point in space is equal to $1/\varepsilon_0$ times the volume charge density ρ, at that point. Del.E=ρ/ε_0. Where ρ is the volume charge density (charge per unit volume) and ε_0 the permittivity of free space. It is one of the Maxwell's equations.

The area integral of the electric field over any closed surface is equal to the net charge enclosed in the surface divided by the permittivity of space.

The first two of Maxwell's equations let us calculate the electric field (magnitude and direction) due to any static charge distribution. ... If a charge distribution has a high degree of symmetry, then Gauss' law alone can be used to determine the magnitude of the electric field.

Gauss's law can be used to derive Coulomb's law, and vice versa. Gauss's law states that: The net outward normal electric flux through any closed surface is proportional to the total electric charge enclosed within that closed surface.

The electric field of a ring of charge on the axis of the ring can be found by superposing the point charge fields of infinitesimal charge elements. The ring field can then be used as an element to calculate the electric field of a charged disc.

When you rub the plastic rod (polyethylene terephthalate, glycol modified, or PETG) with the wool cloth, the rod charges negative. When you rub the glass rod with the silk, the rod charges positive. If the charge present on the rod is positive, the electric field at P would point away from the rod. If the rod is negatively charged, the electric field at P would point towards the rod.

Coulomb's law states that: 'The magnitude of the electrostatic force of attraction or repulsion between two point charges is directly proportional to the product of the magnitudes of charges and inversely proportional to the square of the distance between them. The force is along the straight line joining them.

Electrostatics, the study of electromagnetic phenomena that occur when there are no moving charges—i.e., after a static equilibrium has been established. Charges reach their equilibrium positions rapidly, because the electric force is extremely strong.

Based on the same types of experiments like the one you performed, scientists were able to establish three laws of electrical charges: Opposite charges attract each other. Like charges repel each other. Charged objects attract neutral objects.

1.2 Objective:

After studying this unit you should be able to

- Explain and identify Concept of charge, Coulomb’s law, electric field, electric flux.
- Study and identify Gauss law (statement and derivation, integral and differential form).

- Explain and identify Application of Gauss law for charge distribution (linear, cylindrical and spherical).
- Study and identify Coulomb’s law from Gauss law.
- Explain Electric field due to charged ring, charged infinite rod and charged disc from Coulomb’s law.
- Study and identify Laws of electrostatics.

1.3 Concept of charge, Coulomb’s law, Electric field, Electric flux:

Concept of charge:

- There are two kinds of charge, positive and negative
- Like charges repel, unlike charges attract
- Positive charge comes from having more protons than electrons; negative charge comes from having more electrons than protons
- Charge is quantized, meaning that charge comes in integer multiples of the elementary charge e
- Charge is conserved

Probably everyone is familiar with the first three concepts, but what does it mean for charge to be quantized? Charge comes in multiples of an indivisible unit of charge, represented by the letter e. In other words, charge comes in multiples of the charge on the electron or the proton. These things have the same size charge, but the sign is different. A proton has a charge of +e, while an electron has a charge of -e.

Electrons and protons are not the only things that carry charge. Other particles (positrons, for example) also carry charge in multiples of the electronic charge. Those are not going to be discussed, for the most part, in this course, however.

Putting "charge is quantized" in terms of an equation, we say:

$$q = n\,e$$

q is the symbol used to represent charge, while n is a positive or negative integer, and e is the charge of electrons, 1.60×10^{-19} Coulombs.

The Law of Conservation of Charge:

The Law of conservation of charge states that the net charge of an isolated system remains constant.

If a system starts out with an equal number of positive and negative charges, there is nothing we can do to create an excess of one kind of charge in that system unless we bring in charge from outside the system (or remove some charge from the system). Likewise, if something starts out with a certain net charge, say +100 e, it will always have +100 e unless it is allowed to interact with something external to it.

Charge can be created and destroyed, but only in positive-negative pairs.

Table of elementary particle masses and charges:

particle	mass	charge	
electron	9.11×10^{-31} kg	-1.60×10^{-19} C	(-e)
proton	1.672×10^{-27} kg	$+1.60 \times 10^{-19}$ C	(+e)
neutron	1.674×10^{-27} kg	0	

Coulomb's law:

If two eclectically charge bodies are placed nearby each other there will be an attraction or a repulsion force acting on them depending upon the nature of the charge of the bodies. The formula for the force acting between two electrically charge bodies was first developed by Charles-Augustin de Coulomb and the formula he established for determining the value of force acting to nearby charge objects in known as Coulomb's law.

In his law, he stated that to similarly charged (either positive or negative) bodies will repeal each other and two dissimilarly charged bodies (one is positively charged and other is negatively charged) will attract each other. He had also stated that the force acting between the electrically charged bodies is proportional to the product of the charge of the charged bodies and inversely proportional to the square of the distance between the center of the charged bodies.

Coulomb's Law Formula:

Let us imagine, Q_1 and Q_2 are the electrical charges of two objects. d is the distance between the center of the objects.

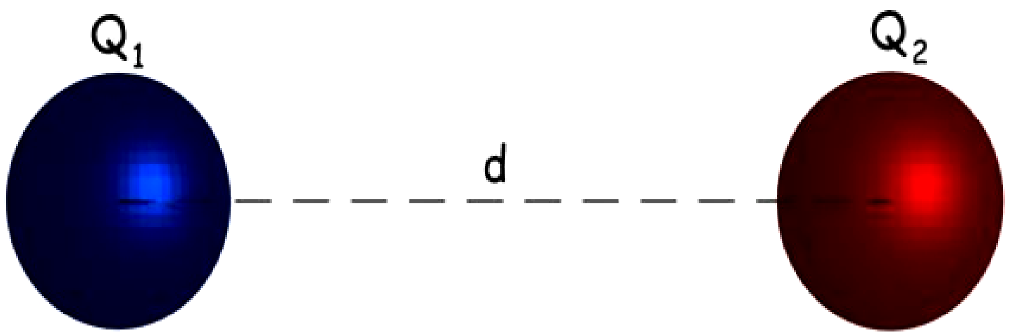


Fig 1.1 Coulomb's Law

The charged objects are placed in a medium of permittivity $\varepsilon = \varepsilon_o \varepsilon_r$

Then we can write the force 'F' as:

$$F = \frac{Q_1\, Q_2}{4\pi\epsilon_o\epsilon_r\, d^2}$$

First Law:

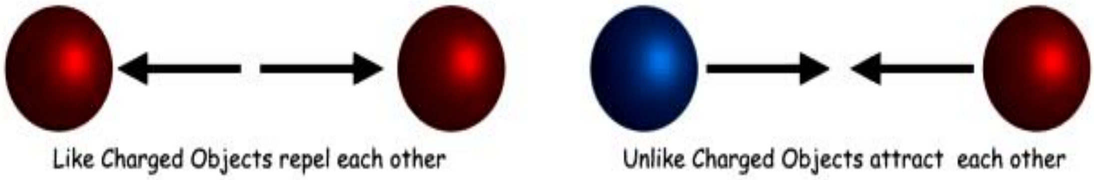


Fig 1.2 Coulomb's First Law

Like charged objects (bodies or particles) repel each other and unlike charged objects (bodies or particles) attract each other.

Second Law:

The force of attraction or repulsion between two electrically charged objects is directly proportional to the magnitude of their charge and inversely proportional to the square of the distance between them. Hence, according to the Coulomb's second law,

$$F \propto Q_1\, Q_2 \;\; \& \;\; F \propto \frac{1}{d^2}$$
$$\Rightarrow F \propto \frac{Q_1\, Q_2}{d^2} \Rightarrow F = k\frac{Q_1\, Q_2}{d^2}$$

Where,

1. 'F' is the repulsion or attraction force between two charged objects.

2. ‘Q_1’ and ‘Q_2’ are the electrical charged of the objects.

3. ‘d’ is distance between center of the two charged objects.

4. ‘k’ is a constant that depends on the medium in which charged objects are placed. In S.I. system, as well as in M.K.S. system $k=1/4\pi\varepsilon_o\varepsilon_r$. Hence, the above equation becomes.

$$F = \frac{Q_1 \, Q_2}{4\pi\epsilon_o\epsilon_r \, d^2}$$

The value of $\varepsilon_o = 8.854 \times 10^{-12}$ C^2/Nm^2.

$$Now, \; F = \frac{Q_1 \, Q_2}{4\pi \times \; 8.854 \; 10^{-12} \times \epsilon_r \; d^2}$$

$$= \; 8.9878 \; \times \; 10^9 \left[\frac{Q_1 \; Q_2}{\epsilon_r \; d^2}\right] \approx \; 9 \; \times \; 10^9 \left[\frac{Q_1 \; Q_2}{\epsilon_r \; d^2}\right]$$

Hence, Coulomb’s law can be written for medium as,

$$F_{medium} = \; 9 \; \times \; 10^9 \left[\frac{Q_1 \; Q_2}{\epsilon_r \; d^2}\right]$$

Then, in air or vacuum $\varepsilon_r = 1$. Hence, Coulomb’s law can be written for air medium as,

$$F_{air} = \; 9 \; \times \; 10^9 \left[\frac{Q_1 \; Q_2}{d^2}\right]$$

The value of ε_r would change depends on the medium. The expression for relative permittivity ε_r is as follows;

$$\epsilon_r = \frac{F_{air}}{F_{medium}} = \frac{Force\ developed\ between\ charged\ bodies\ in\ the\ air}{Force\ developed\ between\ same\ charged\ bodies\ in\ the\ medium}$$

Principle of Coulomb's Law:

Suppose if we have two charged bodies one is positively charged and one is negatively charged, then they will attract each other if they are kept at a certain distance from each other. Now if we increase the charge of one body keeping other unchanged, the attraction force is obviously increased. Similarly, if we increase the charge of the second body keeping the first one unchanged, the attraction force between them is again increased. Hence, the force between the charge bodies is proportional to the charge of either bodies or both.

$$F \propto Q_1 \ \ \& \ \ F \propto Q_2 \ \Rightarrow \ F \propto Q_1 Q_2$$

Now, by keeping their charge fixed at Q_1 and Q_2 if you bring them nearer to each other the force between them increases and if you take them away from each other the force acting between them decreases. If the distance between the two charge bodies is d, it can be proved that the force acting on them is inversely proportional to d^2.

$$F \propto \frac{1}{d^2}$$

This development of force between two same charged bodies is not the same in all mediums. As we discussed in the above formulas, ε_r would change for various medium. So, depends on the medium, creation of force can be varied.

$$F \propto \frac{1}{\epsilon_r}$$

Limitation of Coulomb's Law:

1. Coulomb's law is valid, if the average number of solvent molecules between the two interesting charge particles should be large.
2. Coulomb's law is valid, if the point charges are at rest.
3. It is difficult to apply the Coulomb's law when the charges are in arbitrary shape. Hence, we cannot determine the value of distance'd' between the charges when they are in arbitrary shape.

Electric Field:

An electric field is afield or space around an electrically charge object where any other electrically charged object will experience a force.

An electric field is measured by a term known as electric field intensity. If we place a positive unit charge near a positively charged object, the positive unit charge will experience a repulsive force. Due to this force, the positive unit charge will move away from the said charged object. The imaginary line through which the unit positive charge moves, is known as line of force.

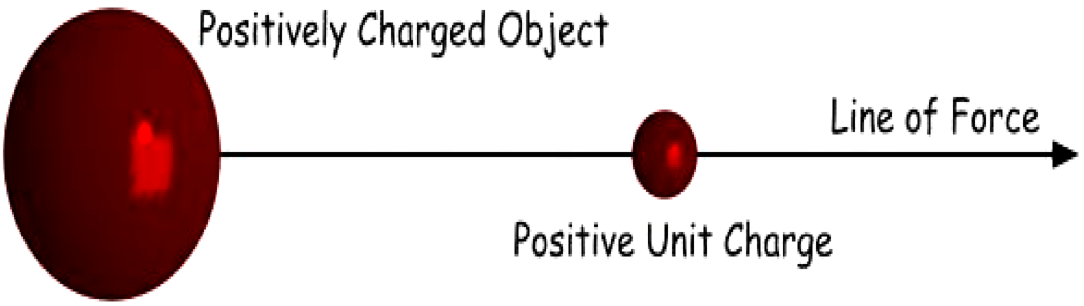


Fig 1.3 Electric Field due to positive charge

Similarly, if we place a positive unit in the field of a negatively charged object, the unit positive charge will experience an attractive force. Due to this force, the unit positive charge will come closer to the said negatively charged object. In that case, line through which the positive unit charge moves, is called line of force.

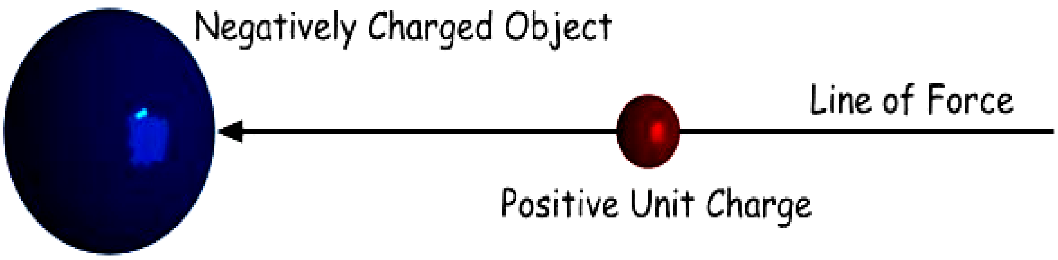


Fig 1.4 Electric Field due to negative charge

We can place a unit positive anywhere surround the positively charged object and each position where we place it, the unit positive charge follows a separate line force to move. Hence, we can say, the lines of force get radiated or come out from this charged object.

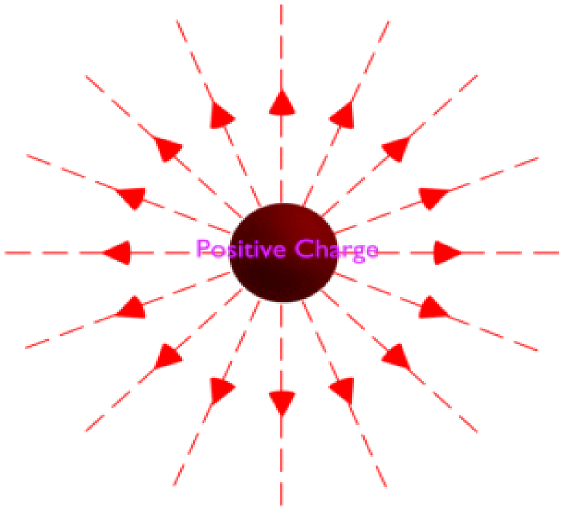


Fig 1.5 Positive Charges

But for a negatively charged object, these lines of force come into this negatively charged object.

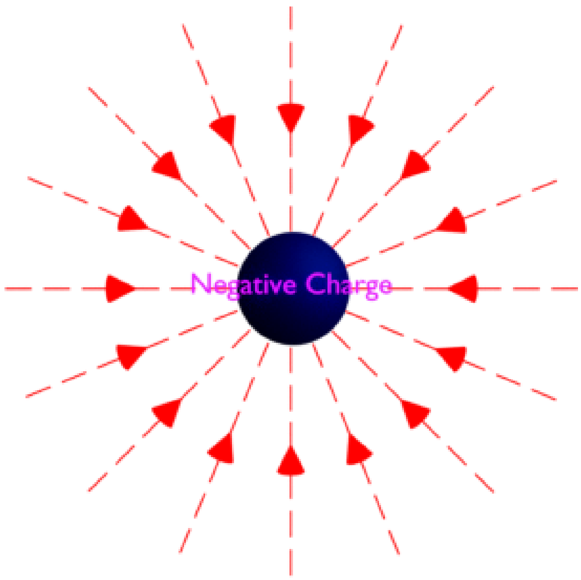


Fig 1.6 Negative Charges

Electric Flux:

Electric flux is the rate of flow of the electric field through a given area. Electric flux is proportional to the number of electric field lines going through a virtual surface.

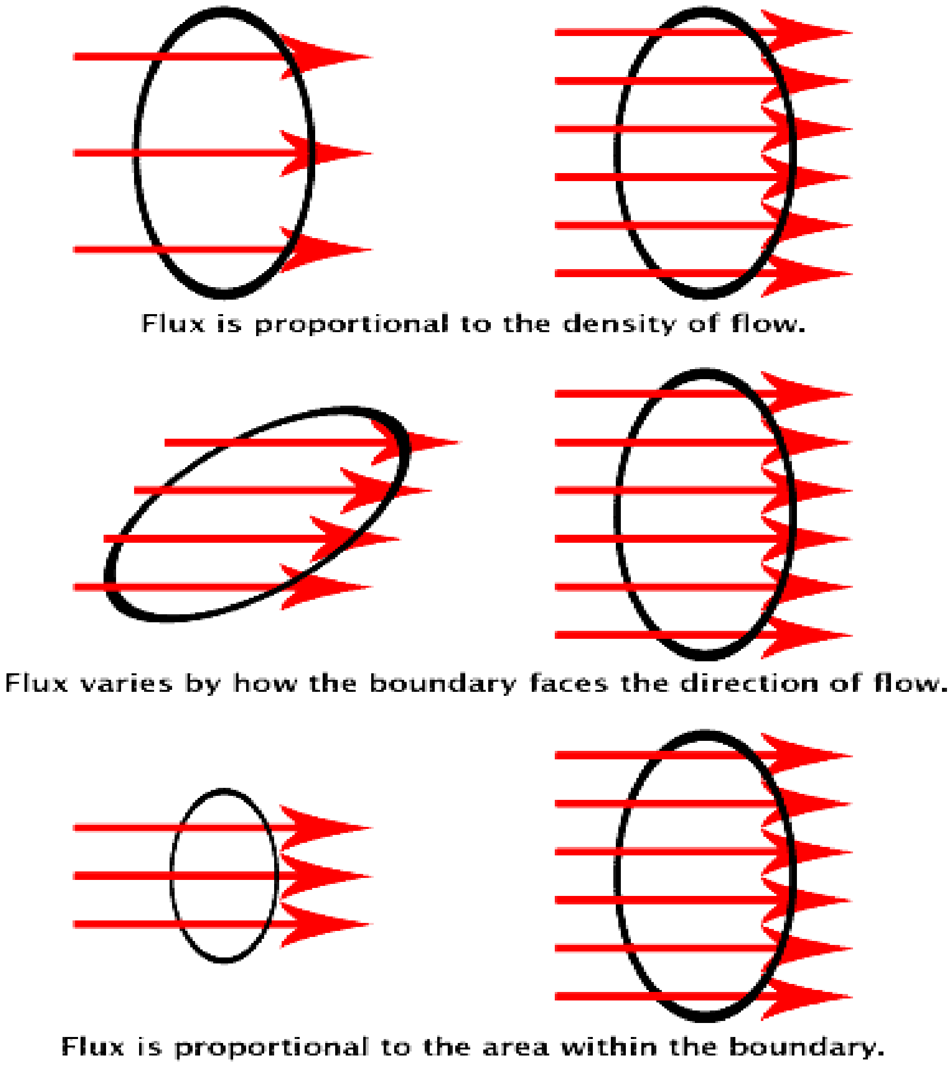


Fig 1.7 Electric Flux

Electric Flux: Electric flux visualized. The ring shows the surface boundaries. The red arrows for the electric field lines

If the electric field is uniform, the electric flux passing through a surface of vector area S

Is

$$\emptyset_E = E.S = EScos\theta$$

where E is the magnitude of the electric field (having units of V/m), S is the area of the surface, and θ is the angle between the electric field lines and the normal (perpendicular) to S.

For a non-uniform electric field, the electric flux dΦE through small surface area dS is given by

$$d\emptyset_E = E.dSd\emptyset E = E.dS$$

(the electric field, E, multiplied by the component of area perpendicular to the field). Gauss' Law describes the electric flux over a surface S as the surface integral:

$$\emptyset_E = \iint SE\cdot dS\Phi E = \iint SE\cdot dS$$

where E is the electric field and dS is a differential area on the closed surface S with an outward facing surface normal defining its direction.

It is important to note that while the electric flux is not affected by charges that are not within the closed surface, the net electric field, E, in the Gauss' Law equation, can be affected by charges that lie outside the closed surface. While Gauss' Law holds for all situations, it is only useful for "by hand" calculations when high degrees of symmetry exist in the electric field. Examples include spherical and cylindrical symmetry. Electric flux has SI units of volt meters (V m), or, equivalently, Newton meters squared per coulomb ($N\ m^2\ C^{-1}$). Thus, the SI base units of electric flux are $kg\cdot m^3\cdot s^{-3}\cdot A^{-1}$.

1.4 Gauss law (statement and derivation, integral and differential form):

Gauss Law:

"The total electric flux over a closed surface in an electric field is equal 1/ ε_o times the total charge enclosed by that surface"

Mathematically it may be expressed as

$$\phi = \oint_S \mathbf{E} \cdot \hat{n} dS = q/\varepsilon_o \quad \text{............(1)}$$

Where q is the net charge enclosed by the surface and εo is the permittivity (of free space) of the medium

Proof of Gauss law of electrostatics (Integral Form):

Consider a source producing the electric field E is a point charge +q situated at a point O inside a volume enclosed by an arbitrary closed surface S. let us consider a small area element dS around a point P on the surface where the electric field produced by the charge +q is E. if E is along OP and area vector dS is along the outward drawn normal to the area element dS.

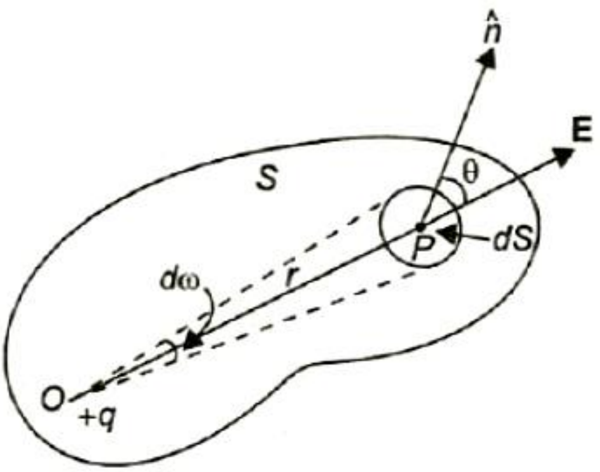


Fig 1.8 Gauss law of electrostatics (Integral Form)

The electric field strength E at the point P is given by

$$E = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2} \text{.............(2)}$$

Then the electric flux over the surface, therefore

$$\oint_S \mathbf{E}\cdot\hat{n}dS = \frac{q}{4\pi\epsilon_o}\oint_S \frac{\hat{n}dS}{r^2} \text{........(3)}$$

$$\frac{\hat{n}dS}{r^2} = d\omega \text{ (Solid angle)}$$

$$\oint_S d\omega = 4\pi$$

Equation (3) becomes

$$\oint_S \mathbf{E}\cdot\hat{n}dS = \frac{q}{4\pi\epsilon_o}4\pi$$

Or

$$\oint_S \mathbf{E} \cdot \hat{n} dS = \frac{q}{\epsilon_o} \text{..........(4)}$$

Equation (4) represents Gauss law (in integral form) for electrostatics for a single point charge (in integral form).

Gauss law in Differential Form:

Using divergence Theorem (Relates volume integral of divergence of a vector field to surface integral of the vector field)

$$\oint_S \mathbf{E} \cdot \hat{n} dS = \oint_V (\Delta \cdot \mathbf{E}) dV \text{.....(5)}$$

Using Equation (4)

$$\oint_V (\Delta \cdot \mathbf{E}) dV = \frac{q}{\epsilon_o} \text{........(6)}$$

and Let a charge q be distributed over a volume V of the closed surface S and p be the charge density;

then

$$\rho = \frac{q}{V}$$

or

$$q = \sum \rho dV$$

$$q = \int_V \rho dV$$

Substituting the value of net charge in terms of charge density, equation (6) becomes

$$\oint_V (\Delta \cdot \mathbf{E}) dV = \frac{1}{\epsilon_o} \int_V \rho dV$$

Or

$$\Delta \cdot \mathbf{E} = \frac{\rho}{\epsilon_o} \quad ....(7)$$

Or

$$Div \cdot \mathbf{E} = \frac{\rho}{\epsilon_o} \quad ....(8)$$

Equation (7) and (8) represent Gauss Law in differential form Differential form of Gauss law states that "the divergence of electric field E at any point in space is equal to 1/ε0 times the volume charge density,ρ, at that point".

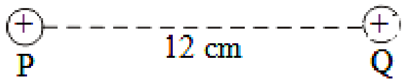
SAQ 1

a) Explain. i) Concept of charge ii) Electric field iii) Electric flux
b) Explain the Coulomb's law?

c) Write the equation of Gauss law for integral and differential form?

d) Two point charges, QA = +8 μC and QB = -5 μC, are separated by a distance r = 10 cm. What is the magnitude of the electric force. The constant k = 8.988 x 109 Nm2C

e) Two charged particles as shown in figure below. QP = +10 μC and Qq = +20 μC are separated by a distance r = 10 cm. What is the magnitude of the electrostatic force.



1.5 Application of Gauss law for charge distribution (linear, cylindrical, spherical):

Gaussian Surface of infinite wire (linear):

Consider an infinitely long line of charge with the charge per unit length being λ. We can take advantage of the cylindrical symmetry of this situation. By symmetry, The electric fields all point radially away from the line of charge, there is no component parallel to the line of charge We can use a cylinder (with an arbitrary radius (r) and length (l)) centred on the line of charge as our Gaussian surface.

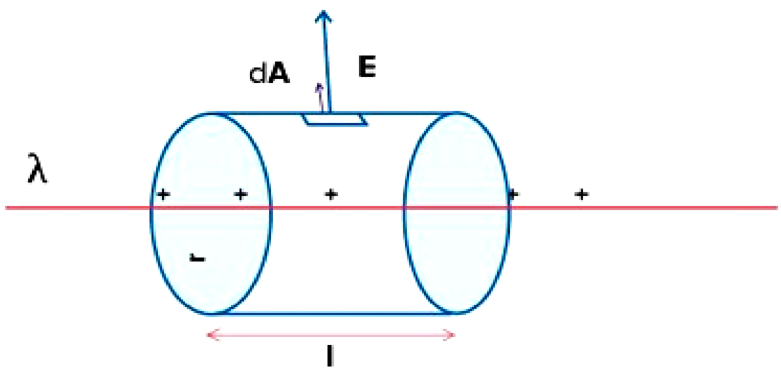


Fig 1.9 Gaussian Surface of infinite wire (linear)

As you can see in the above diagram, the electric field is perpendicular to the curved surface of the cylinder. Thus, the angle between the electric field and area vector is zero and $\cos\theta = 1$

The top and bottom surfaces of the cylinder lie parallel to the electric field. Thus the angle between area vector and the electric field is 90 degrees and $\cos\theta = 0$.

Thus, the electric flux is only due to the curved surface

According to Gauss Law,

$$\Phi = \rightarrow E.d \rightarrow A$$

$$\Phi = \Phi_{curved} + \Phi_{top} + \Phi_{bottom}$$

$$\Phi = \rightarrow E\ .\ d \rightarrow A = \int E\ .\ dA \cos 0 + \int E\ .\ dA \cos 90° + \int E\ .\ dA \cos 90°$$

$$\Phi = \int E\ .\ dA \times 1$$

Due to radial symmetry, the curved surface is equidistant from the line of charge and the electric field in the surface has a constant magnitude throughout.

$$\Phi = \int E . dA = E \int dA = E . 2\pi rl$$

The net charge enclosed by the surface is:

$$q_{net} = \lambda.l$$

Using Gauss theorem,

$$\Phi = E \times 2\pi rl = q_{net}/\varepsilon_0 = \lambda l/\varepsilon_0$$

$$E \times 2\pi rl = \lambda l/\varepsilon_0$$

$$E = \lambda/2\pi r\varepsilon_0$$

Gaussian Surface of Cylinder:

When a flux or electric field is produced on the surface of a cylindrical Gaussian surface due to any of the following:

- Uniform distribution of charge in an infinitely long line
- Uniform distribution of charge in an infinite plane
- Uniform distribution of charge on an infinitely long cylinder

Consider a point charge P at a distance r having charge density λ of an infinite line charge. The axis of rotation for the cylinder of length h is the line charge, following is the charge q enclosed in the cylinder:

$$q=\lambda h$$

Following is the flux out of the cylindrical surface with the differential vector area dA on surfaces a, b and c are given as:

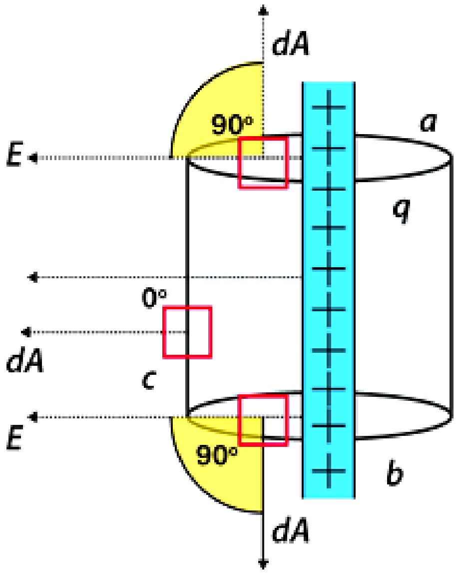


Fig 1.10 Gaussian Surface of Cylinder

$$\Phi_E = \oiint_A \mathbf{E} \cdot d\mathbf{A} = \iint_a \mathbf{E} \cdot d\mathbf{A} + \iint_b \mathbf{E} \cdot d\mathbf{A} + \iint_c \mathbf{E} \cdot d\mathbf{A}$$

$\emptyset_E = \iint aEdA\cos 90\circ + \iint bEdA\cos 90\circ + \iint cEdA\cos 0\circ = E\iint cdA \ \iint cdA = 2\pi rh$

(which is the surface area of the cylinder)

$\emptyset_E = E2\pi rh \ \emptyset_E == q\epsilon_0$ (by Gauss law)

$E2\pi rh = \lambda h\epsilon_0 \Rightarrow E = \lambda 2\pi\epsilon_0 r$

The above equation shows the cylindrical Gaussian surface with a uniform distribution of charges.

Electric field due to uniformly charged spherical shell:

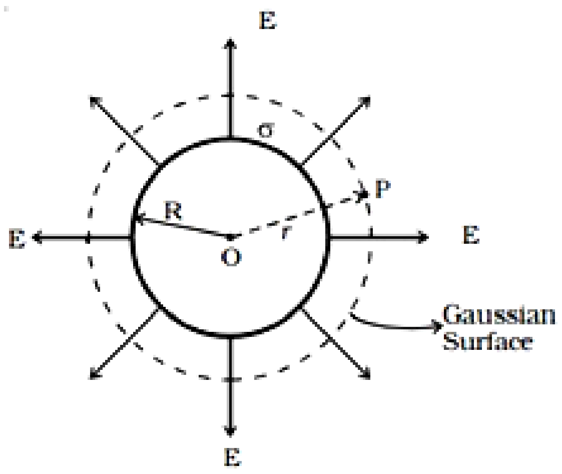


Fig 1.11 Field at a point outside the shell

Case (i) At a point outside the shell:

Consider a charged shell of radius R (Fig 1.20a). Let P be a point outside the shell, at a distance r from the centre O.

Let us construct a Gaussian surface with r as radius. The electric field E is normal to the surface.

The flux crossing the Gaussian sphere normally in an outward direction is,

$$\phi = \int_s \overline{E} \cdot \overline{ds} = \int_s E\, ds = E\,(4\pi r^2)$$

(since angle between E and ds is zero)

By Gauss's law, $E \cdot (4\pi r^2) = \frac{q}{\varepsilon_o}$

or $E = \frac{1}{4\pi\varepsilon_o}\frac{q}{r^2}$

It can be seen from the equation that, the electric field at a point outside the shell will be the same as if the total charge on the shell is concentrated at its centre.

Case (ii) At a point on the surface:

The electric field E for the points on the surface of charged spherical shell is,

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2} \quad (\because \; r = R)$$

Case (iii) At a point inside the shell:

Consider a point P' inside the shell at a distance r' from the centre of the shell. Let us construct a Gaussian surface with radius r'.

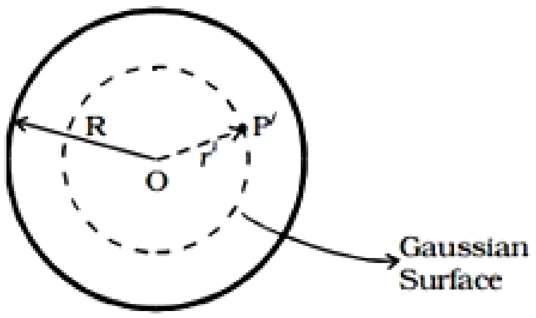


Fig 1.12 At a point inside the shell

The total flux crossing the Gaussian sphere normally in an outward direction is

$$\phi = \int_s \overline{E} \cdot \overline{ds} = \int_s E ds = E \times (4\pi r'^2)$$

since there is no charge enclosed by the gaussian surface, according to Gauss's Law

$$E \times 4\pi r'^2 = \frac{q}{\varepsilon_0} = 0 \qquad \therefore E = 0$$

(i.e) the field due to a uniformly charged thin shell is zero at all points inside the shell.

1.6 Coulomb's law from Gauss law:

To derive Coulomb's Law from gauss law or to find the intensity of electric field due to a point charge +q at any point in space using Gauss's law, draw a Gaussian sphere of radius r at the centre of which charge +q is located.

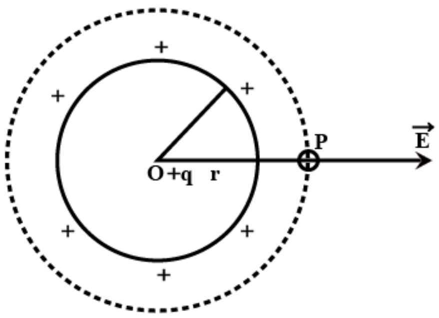


Fig 1.13 Coulomb's law from Gauss law

All the points on this surface are equivalent and according to the symmetric consideration the electric field E has the same magnitude at every point on the surface of the sphere and it is radially outward in direction. Therefore, for a area element dS around any point P on the Gaussian surface both E and dS are directed radially outward, that is ,the angle between E and dS is zero.

Therefore, the flux passing through the area element dS ,that is,

$$d\varphi = E.dS = EdS\cos 0^0 = EdS$$

Hence, the total flux through the entire Gaussian sphere is obtained as,

$$\Phi = \int EdS$$

Or

$$\varphi = E\int dS$$

But $\int dS$ is the total surface area of the sphere and is equal to $4\pi r^2$, that is,

$$\Phi = E(4\pi r^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

But according to <u>Gauss's law for electrostatics</u>

$$\Phi = q/\varepsilon_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Where q is the charge enclosed within the closed surface

By comparing equation (1) and (2) ,we get

$$E(4\pi r^2) = q/\varepsilon_0$$

Or

$$E = q/4\pi\varepsilon_0 r^2 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The equation (3) is the expression for the magnitude of the intensity of electric field E at a point, distant r from the point charge +q.

In vector form, $E=1/4\pi\varepsilon_0\, q/r^2 = 1/4\pi\varepsilon_0 qr/r^3$

In a second point charge q_0 be placed at the point at which the magnitude of E is computed, then the magnitude of the force acting on the second charge q_0 would be

$$F=q_0E$$

By substituting value of E from equation (3),we get

$$F=q_oq/4\pi\varepsilon_0 r^2 \text{ ……………..(4)}$$

The equation (4) represents the Coulomb's Law and it is derived from gauss law.

SAQ 2:

a) What is Gaussian Surface of Cylinder
b) What is Coulomb's law from Gauss law.
c) Write the equation Coulomb's law from Gauss law.

d) Determine the electric flux for a Gaussian surface that contains 100 million electrons.

e) A uniformly charged solid spherical insulator has a radius of 0.23 m. The total charge in the Volume is 3.2 pC. Find the E-field at a position of 0.14 m from the center of the sphere.

1.7 Electric field due to charged ring, charged infinite rod and charged disc from Coulomb's law:

Electric Field on the Axis of a Ring of Charge:

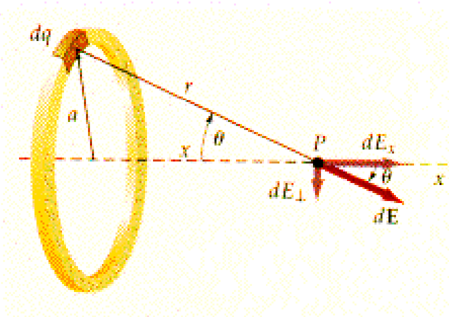


Fig.1.14. Electric Field on the Axis of a Ring of Charge

We determine the field at point P on the axis of the ring. It should be apparent from symmetry that the field is along the axis. The field dE due to a charge element dq is shown, and the total field is just the superposition of all such fields due to all charge elements around the ring. The perpendicular fields sum to zero, while the differential x-component of the field is

$$dE_x = \frac{kdq}{r^2}\cos\theta = \frac{kdq}{r^2}\frac{x}{r} = \frac{kxdq}{\left(x^2 + a^2\right)^{3/2}}$$

Where,

$$r^2 = x^2 + a^2 \quad and \quad \cos\theta = \frac{x}{r} = \frac{x}{\sqrt{x^2 + a^2}}$$

We now integrate, noting that r and x are constant for all points on the ring:

$$E_x = \int \frac{kxdq}{\left(x^2 + a^2\right)^{3/2}} = \frac{kx}{\left(x^2 + a^2\right)^{3/2}} \int dq = \frac{kxQ}{\left(x^2 + a^2\right)^{3/2}}$$

This gives the predicted result. Note that for x much larger than a (the radius of the ring), this reduces to a simple Coulomb field. This must happen since the ring looks like a point as we go far away from it. Also, as was the case for the gravitational field, this field has extreme at x=+/-a.

Electric field due to charged infinite rod:

We will calculate the electric field of a line of charge distribution by superposing the point charge field of several infinitesimal charge elements. Let us consider a straight charged wire of length L. The line charge density of this charged wire is . We have to calculate the electric field at point P which is Z distance apart from the line charge distribution. The situation is shown in the figure below

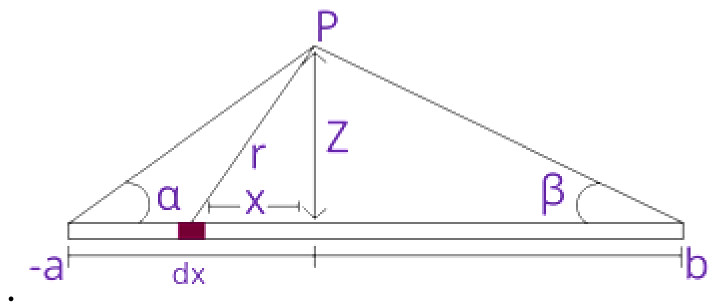


Fig.1.15. Electric field due to charged infinite rod

Let us consider an charge element of length dx at a distance x as shown in the figure. Charge contained within this element is λdx. The electric field at point P due to this charge element is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\hat{r}$$

and the radial part of the electric field from this charge element is,

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{r^2}\frac{z}{r}$$

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda z dx}{(z^2+x^2)^{\frac{3}{2}}}$$

We will integrate over the whole charged wire to get the total radial electric field at point P due to this line charge distribution.

$$E_z = \frac{\lambda z}{4\pi\epsilon_0}\int_{-a}^{b}\frac{dx}{(z^2+x^2)^{\frac{3}{2}}}$$

$$= \frac{\lambda}{4\pi\epsilon_0 z}\left[\frac{b}{(z^2+b^2)^{\frac{1}{2}}} + \frac{a}{(z^2+a^2)^{\frac{1}{2}}}\right]$$

Please be careful about the limit of integration. It runs from -a to b. Similarly, we can calculate the axial component of the electric field.

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-a}^{b}\frac{x dx}{(z^2+x^2)^{\frac{3}{2}}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{(z^2+b^2)^{\frac{1}{2}}} - \frac{1}{(z^2+a^2)^{\frac{1}{2}}}\right]$$

In a symmetric case of a=b, this axial component vanishes and we are left with the radial component only.

Electric field due to an infinitely long line charge distribution can be considered as a limiting case of the above solution. In this case a and b approach to the infinity. The axial component of the electric field vanishes again. Thus the electric field due to an infinitely long line charge distribution is

$$E_z = \frac{\lambda}{2\pi\epsilon_0 z}$$

and it does not have any axial component. This becomes obvious if we look at the axial symmetry of the problem. In the next section, we will exploit this symmetry to calculate the electric due to an infinitely long charged wire.

Electric Field due to Infinitely Long Line Charge (Gauss's Law Application):

We have to calculate the electric field at point P due to an infinitely long charged wire of charge density. The situation is shown in the diagram below. As already mentioned, the system has a cylindrical symmetry. This has significantly simplified the problem and we can use Gauss's law to calculate the electric field. Let us imagine a hypothetical cylindrical Gaussian surface as shown in the figure. Since the field is pointing radially outwards, the flux through the two ends of the cylinder is zero. Also, at every point on the cylindrical surface, the electric field is constant and is pointing normal to the surface. The surface area of the curved surface of length l is 2πrl . Thus total flux crossing through the cylindrical Gaussian surface is 2πrl.

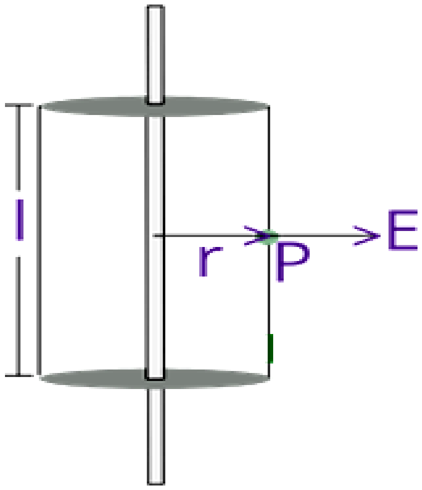


Fig.1.16. Electric Field due to Infinitely Long Line Charge

Total charge enclosed within this Gaussian surface is λl. Now according to Gauss's law

$$\vec{E} \cdot 2\pi r l \hat{r} = \frac{\lambda l}{\epsilon_0}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}$$

Points to Remember:

- *For an infinitely long charged wire electric field is proportional to line charge density and inversely proportional to the radial distance.*
- *In this case electric field is radial in nature and does not have any axial component.*
- *For a finitely long charged wire electric field is complicated and has both axial and radial component.*
- *However, in a symmetric case axial component vanishes and we are left with the radial component only.*

Charged Disk:

Consider an insulated circular disk of radius R with a positive surface charge density of σ (charge per unit area). We will now calculate E at a point P, which is at a distance z from the disk along its central axis.

Similar to the previous line of charge calculation, we will divide the disk into concentric rings and calculate the E by integrating. Let one of concentric rings have a radius of r and radial width of dr.

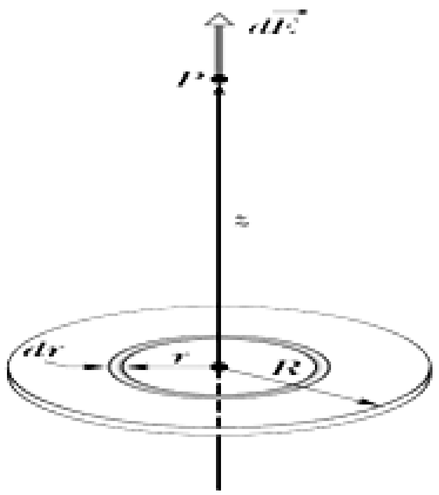


Fig 1.17 electric field due to Charged Disk

If σ is the charge per unit area and dA is the differential area of the ring, then the charge of the ring is

$$dq = \sigma\, dA = \sigma\,(2\pi r\, dr)$$

From the previous ring of charge calculation, dE due to flat ring is given by

$$dE = \frac{z\sigma 2\pi r\, dr}{4\pi\varepsilon_0 (z^2 + r^2)^{3/2}} = \frac{\sigma z}{4\varepsilon_0}\frac{2r\, dr}{(z^2 + r^2)^{3/2}}$$

Integrating this over the surface of the disk and rearranging, we get E of a charged disk as follows:

$$E = \int dE = \frac{\sigma z}{4\varepsilon_0} \int_0^R (z^2 + r^2)^{-3/2}(2r)\, dr \quad = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) \text{(after integration and rearranging)}$$

1.8 Laws of electrostatics:

In physics, electrostatics deals with the phenomena and properties of stationary or slow-moving electric charges. Electrostatic phenomena arise from the forces that electric charges exert on each other and are described by Coulomb's law. Even though electro statically induced forces seem to be rather weak.

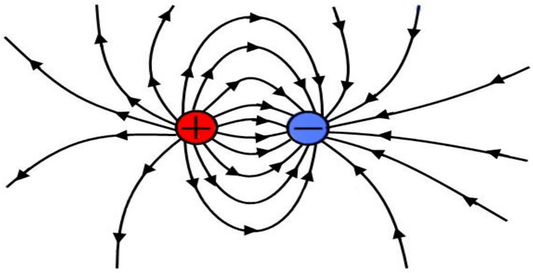


Fig 1.18 Electric field

Coulomb's Law of Electrostatics:

We begin with the magnitude of the electrostatic force between two point charges q and Q. It is convenient to label one of these charges, q, as a test charge, and call Q a source charge. As we develop the theory, more source charges will be added. If r is the distance between two charges, then the force of electrostatic formula is:

$$F = \frac{1}{4\pi\xi_0}\frac{qQ}{r^2} = k_e\frac{qQ}{r^2}$$

Or

$$F = k\frac{q_1 q_2}{d^2}$$

Electric field:

Electric field lines are useful for visualizing the electric field. Field lines begin on positive charge and terminate on negative charge. Electric field lines are parallel to the direction of the electric field, and the density of these field lines is a measure of the magnitude of the electric field at any given point.

We show charge with "q" or "Q" and smallest unit charge is 1.6021 x 10-19 Coulomb (C). One electron and a proton have same amount of charge.

Positively Charged Particles:

In this, numbers of positive ions are larger than the numbers of negative ions. Means, the numbers of protons are larger than the number of electrons. To neutralize positively charged particles, electrons from the surroundings come to this particle until the number of protons and electrons become equal.

Negatively Charged Particles:

Similarly numbers of electrons are larger than the number of protons. To neutralize negatively charged particles, since protons cannot move and cannot come to negatively charged particles, electrons moves to the ground or any other particle around

Neutral Particles:

Include equal numbers of protons and electrons. They have both protons, neutrons and electrons however, numbers of positive ions are equal to the numbers of negative ions.

Electrostatics Examples:

There are many examples of electrostatic phenomena:

- The attraction of the plastic wrap to your hand after you removes it from a package.
- The attraction of paper to a charged scale.
- The apparently spontaneous explosion of grain silos
- The damage of electronic components during manufacturing
- Photocopier & laser printer operation

SAQ 3:

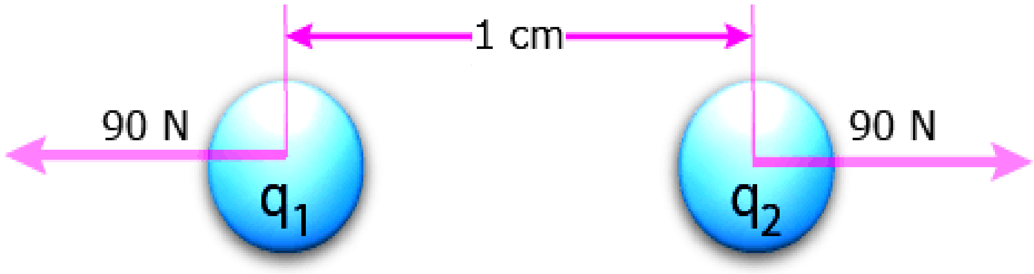
a) What is Electric field?
b) What is Coulomb’s Law of Electrostatics?
c) What is Positively Charged Particles?
d) What is Negatively Charged Particles?

Example:

Q.1. The force between two identical charges separated by 1 cm is equal to 90 N. What is the magnitude of the two charges?

Solution:

First, draw a force diagram of the problem



Define the variables:

F = 90 N

q = charge of first body

q = charge of second body

r = 1 cm

Use the Coulomb's Law equation

$$F = k\frac{q_1 q_2}{r^2}$$

The problem says the two charges are identical, so

q = q = q

Substitute this into the equation

$$F = k\frac{q^2}{r^2}$$

Since we want the charges, solve for q

$$q^2 = \frac{Fr^2}{k}$$

$$q = \sqrt{\frac{Fr^2}{k}}$$

Enter the values for the variables. Remember to convert 1 cm to 0.01 meters to keep the units consistent.

$$q = \sqrt{\frac{(90\ \text{N})(0.01\ \text{m})^2}{8.99\times 109\ \text{N}\cdot\text{m}^2/\text{C}^2}}$$

q = ±1.00×10 Coulombs

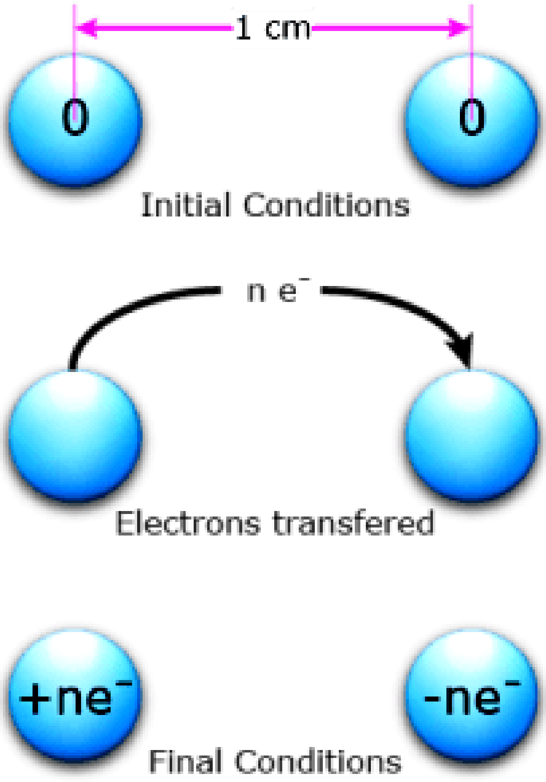
Since the charges are identical, they are either both positive or both negative. This force will be repulsive.

Two identical charges of ±1.00×10 Coulombs separated by 1 cm produce a repulsive force of 90 N.

Q.2. Two neutrally charged bodies are separated by 1 cm. Electrons are removed from one body and placed on the second body until a force of 1×10 N is generated between them. How many electrons were transferred between the bodies?

Solution:

First, draw a diagram of the problem



Define the variables:

F = coulomb force = 1×10^{-6} N

q = charge on first body

q = charge on second body

e = charge of a single electron = 1.60×10^{-19} C

k = 8.99×10 $N\bullet m^2/C^2$

r = distance between two bodies = 1 cm = 0.01 m

Start with the Coulomb's Law equation.

$$F = k\frac{q_1q_2}{r^2}$$

As an electron is transferred from body 1 to body 2, body 1 becomes positive and body two becomes negative by the charge of one electron. Once the final desired force is reached, n electrons have been transferred.

q = +ne

q = -ne

$$F = k\frac{(ne)(ne)}{r^2}$$

The signs of the charges give the direction of the force, we are more interested in the magnitude of the force. The magnitude of the charges are identical, so we can ignore the negative sign on q . This simplifies the above equation to:

$$F = k\frac{n^2e^2}{r^2}$$

We want the number of electrons, so solve the equation for n.

$$q = \sqrt{\frac{(10^{-6}\ N)(0.01\ m)^2}{(8.99\times10^9\ N\cdot m^2/C^2)(1.60\times10^{-19}\ C)^2}}$$

n = 6.59×10^8

6.59×10 electrons were transferred between the two bodies to produce an attractive force of 1×10^{-6} Newtons.

Q.3. A particle having surface charge density 4 x 10^{-6} c/m^2, is held at some distance from a very large uniformly charged plane. Calculate the electric field intensity at any point lying on uniformly charged plane. Here ε_0 = 8.85 x 10^{-12} $C^2N^{-1}m^{-2}$.

Solution:

$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2N^{-1}m^{-2}$

$\sigma = 4 \times 10^{-6}\ c/m^2$

$E = \sigma / (2\ \varepsilon_0)$

$E = 4 \times 10^{-6} / (2 \times 8.85 \times 10^{-12})$

$E = 2.26 \times 10^5\ NC^{-1}$

Q.4. A uniform electric field of magnitude E = 100 N/C exists in the space in X-direction. Using the Gauss theorem calculate the flux of this field through a plane square area of edge 10 cm placed in the Y-Z plane. Take the normal along the positive X-axis to be positive.

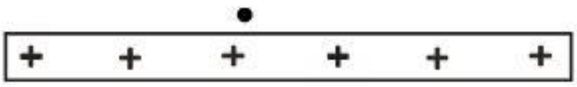
Solution:

The flux $\Phi = \int E.\cos\theta\ ds$.

As the normal to the area points along the electric field, $\theta = 0$.

Also, E is uniform so, $\Phi = E.\Delta S = (100\ N/C)\ (0.10m)^2 = 1\ N\text{-}m^2$.

Q.5. A particle of mass 5×10^{-6}g is kept over a large horizontal sheet of charge of density $4.0 \times 10^{-6}\ C/m^2$ (figure). What charge should be given to this particle so that if released, it does not fall down? How many electrons are to be removed to give this charge? How much mass is decreased due to the removal of these electrons?



Solution:

The electric field in front of the sheet is,

$E = \sigma/2\varepsilon_0 = (4.0 \times 10^{-6})/(2 \times 8.85 \times 10^{-12}) = 2.26 \times 10^5$ N/C

If a charge q is given to the particle, the electric force qE acts in the upward direction. It will balance the weight of the particle if

$q \times 2.26 \times 10^5$ N/C $= 5 \times 10^{-9}$ kg $\times 9.8$ m/s^2

or, $q = [4.9 \times 10^{-8}]/[2.26 \times 10^5]$C $= 2.21 \times 10^{-13}$ C

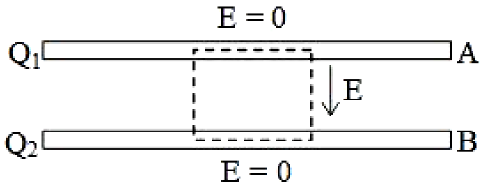
The charge on one electron is 1.6×10^{-19}C. The number of electrons to be removed;

$= [2.21 \times 10^{-13}]/[1.6 \times 10^{-19}] = 1.4 \times 10^6$

Mass decreased due to the removal of these electrons $= 1.4 \times 10^6 \times 9.1 \times 10^{-31}$ kg $= 1.3 \times 10^{-24}$ kg.

Q.6. Two conducting plates A and B are placed parallel to each other. A is given a charge Q1 and B a charge and Find the distribution of charges on the four surfaces.

Solution:

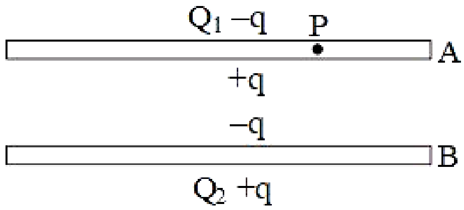


(a)

Consider a Gaussian surface as shown in figure (a). Two faces of this closed surface lie completely inside the conductor where the electric field is zero.

The flux through these faces is, therefore, zero. The other parts of the closed surface which are outside the conductor are parallel to the electric field and hence the flux on these parts is also zero.

The total flux of the electric field through the closed surface is, therefore, zero. From Gauss law, the total charge inside the closed surface should be zero. The charge on the inner surface of A should be equal and opposite to that on the inner surface of B.



(b)

The distribution should be like the one shown in figure (b). To find the value of q, consider the field at a point P inside the plate A. Suppose, the surface area of the plate (one side) is A.

Using the equation $E = \sigma/2\varepsilon_0$, the electric field at P;

- Due to the charge $Q1 - q = (Q1 - q)/2A\varepsilon_0$ (downward),
- Due to the charge $+q = q/2A\varepsilon_0$ (upward),
- Due to the charge $-q = q/2A\varepsilon_0$ (downward),
- Due to the charge $Q2 + q = (Q2 + q)/2A\varepsilon_0$ (upward).

The net electric field at P due to all the four charged surfaces is (in the downward direction)

$(Q1 - q)/2A\varepsilon_0 - q/2A\varepsilon_0 + q/2A\varepsilon_0 - (Q2 + q)/2A\varepsilon_0$

As the point P is inside the conductor, this field is should be zero.

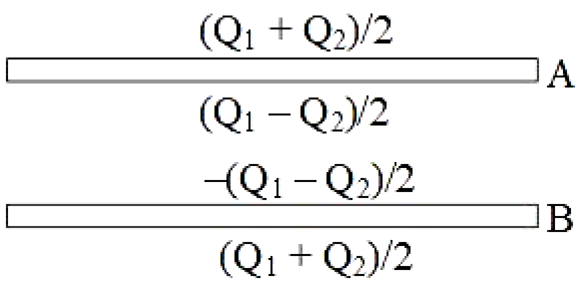
Hence, $Q1 - q - Q2 - q = 0$

or $q = (Q1 - Q2)/2$ . . . . . (i)

Thus, $Q1 - q = (Q1 + Q2)/2$ . . . . . . (ii)

and $Q2 + q = [Q1 + Q]2/2$

Using these equations, the distribution shown in the figure (a, b) can be redrawn as in the figure.

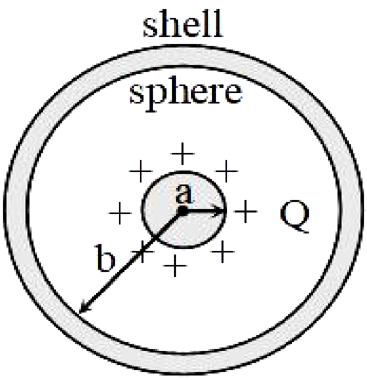


This result is a special case of the following result. When charged conducting plates are placed parallel to each other, the two outermost surfaces get equal charges and the facing surfaces get equal and opposite charges.

Q.7. A solid conducting sphere having a charge Q is surrounded by an uncharged concentric conducting hollow spherical shell. Let the potential difference between the surface of the solid sphere and that of the outer surface of hollow shell be V. What will be the new potential difference between the same two surfaces if the shell is given a charge -3Q?

Solution:

In case of a charged conducting sphere



$V_{in} = V_c = V_s = 1/4\pi\varepsilon_0$

and $V_{out} = 1/4\pi\varepsilon_0$

So if a and b are the radii of a sphere and spherical shell respectively,

the potential at their surfaces will be;

Vsphere = $1/4\pi\varepsilon_0$ [Q/a] and Vshell = $1/4\pi\varepsilon_0$ [Q/b] and so according to the given problem;

$V = V'sphere – V'shell = Q/4\pi\varepsilon_0 [1/a – 1/b] = V$ . . . . (1)

Now when the shell is given a charge (-3Q) the potential at its surface and also inside will change by;

$V_0 = 1/4\pi\varepsilon_0$ [-3Q/b]

So that now,

$V'sphere = 1/4\pi\varepsilon_0 [Q/a + V_0]$ and $V'shell = 1/4\pi\varepsilon_0 [Q/b + V_0]$

Hence, $V'sphere – V'shell = Q/4\pi\varepsilon_0 [1/a – 1/b] = V$ [from Eqn. (1)]

i.e., if any charge is given to external shell the potential difference between sphere and shell will not change.

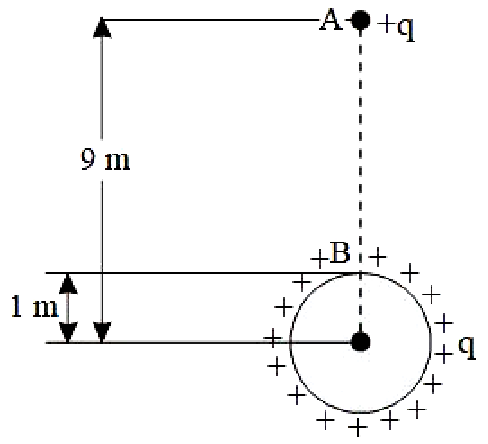
This is because by the presence of charge on the outer shell, potential everywhere inside and on the surface of the shell will change by the same amount and hence the potential difference between sphere and shell will remain unchanged.

Q.8. A very small sphere of mass 80 g having a charge q is held at height 9 m vertically above the centre of a fixed non conducting sphere of radius 1 m, carrying an equal charge q. When released it falls until it is repelled just before it comes in contact with the sphere. Calculate the charge q. [g = 9.8 m/s^2]

Solution:

Keeping in mind that here both electric and gravitational potential energy is changing and for an external point, a charged sphere behaves as the whole of its charge were concentrated at its centre.

Applying the law of conservation of energy between initial and final position, we have



$1/4\pi\varepsilon_0 \times (q.q/9) + mg \times 9 = 1/4\pi\varepsilon_0 \times (q^2/1) + mg \times 1$

or, $q2 = (80 \times 10^{-3} \times 9.8)/10^9 = 28\mu$

Q.9. A soccer goal, found is a city park, is made of tubing that supports an oddshaped hanging net behind the goal, but has a rectangular opening in front. The height of the opening is 2.5 m and the width is 3.2 m. If a uniform E-field, with a mangnitude of 0.1 N/C, passes through the goal from the front to the back, entering at 90º to the plane of the goal opening, what is the flux through the net? Also, find the flux through the net if the E-field enters the goal at a 60° angle to the plane of the front of the goal. In both cases, assume that there is no charge found inside the goal itself.

Solution:

No charge inside implies no total flux.

$\Phi total = 0 = \Phi net + \Phi front$

$0 = \Phi net + EAcos180$

$\Phi net = - 0.1(2.5)(3.2)cos180 = 0.8\ Nm2/C$

For part 2, the angle between the E-Field and the Area vector would be 30°.

$\Phi net = -EAcos150 = -0.1(2.5)(3.2)cos150 = 0.7\ Nm2/C$

Q.10. A cubic space (1.5 m on each side) contains positively charged particles. Imagine that the space is surrounded by a Gaussian surface of the exact same dimension as the cube and that the E-Field caused by the charges is normal to the faces of the Gaussian cube. If the E-field at each surface has a magnitude of 760 N/C, determine the number of charges per unit volume in the space described (i.e., find the charge density, ρ).

Solution:

$\Phi net = EAcos0 = q/\varepsilon o$

$760(6)(1.5)2 = q/8.85x10-12$

$q = 9.1x10-8C$

Now find the volume of the cube:

$V = (1.5)3 = 3.375\ m3$

Finally, determine the charge density:

$\rho = q/V = 9.1x10\text{-}8/3.375 = 2.7x10\text{-}8C/m3$

Q.11. A small charge (q = 6.0 mC) is found in a uniform E-field (E = 2.9 N/C). Determine the force on the charge.

Solution:

$F = qE$

$F = (6x10\text{-}3)(2.9) = 0.02$ N

Q.12.Charge q1 (positive) is located at position (0, 0.5m) and has a magnitude of 2.9x10-6 C. Charge q2 (same charge as q1) is located at the origin. Assume that these charges are unable to move. A third charge (q3 = +1.0x10-9 C and m = 4.0x10-25 kg) is located at (1.0 m, 0.25 m). Determine the force on and the acceleration of charge q3 at this position, and describe the trajectory the third charge would take when released in the field caused by the other two charges.

Solution:

The distance, r, from either q1 or q2 to q3:

$r2 = 12 + (0.25)2$

$r = 1.03$ m

The E-field from q1 and q2 can be calculated separately, then superpositioned:

E1 = kq1/r2 = k(2.9x10-6)/(1.03)2 = 2.46x104 N/C (pointing along the line that connects q3 and q1, away from q3, into the 4th quadrant, at 346^{0})

E2 = kq2/r2 = k(2.9x10-6)/(1.03)2 = 2.46x104 N/C (pointing along the line that connects q2 and q3, away from q3, into the 1st quadrant, at 14^{0})

The y-components of the E-fields cancel out.

The x-components add together to point in the +x direction.

Ex = (2.46x104cos346) + (2.46x104cos14) = 4.8x104 N/C

F = ma

qE = ma

1.0x10-9(4.8x104) = (4.0x10-25)a

a = 1.2x1020 m/s2

Once q3 begins to move it will get further from both q1 and q2, but it will stay equidistant from both,

ensuring that the net force is oriented at 0°. As it moves, the force (and the acceleration) will decrease.

Thus, it will continue to speed up, but at a lower rate at time goes on.

Summary:

1) Charge is quantized, meaning that charge comes in integer multiples of the elementary charge e

2) Putting "charge is quantized" in terms of an equation, we say:

$$q = n\,e$$

3) The Law of conservation of charge states that the net charge of an isolated system remains constant.

4) Charge can be created and destroyed, but only in positive-negative pairs

5) If two eclectically charge bodies are placed nearby each other there will be an attraction or a repulsion force acting on them depending upon the nature of the charge of the bodies

6) Like charged objects (bodies or particles) repel each other and unlike charged objects (bodies or particles) attract each other.

7) The force of attraction or repulsion between two electrically charged objects is directly proportional to the magnitude of their charge and inversely proportional to the square of the distance between them

8) An electric field is afield or space around an electrically charge object where any other electrically charged object will experience a force.

9) Electric flux is the rate of flow of the electric field through a area

10) “The total electric flux over a closed surface in an electric field is equal $1/\varepsilon_o$ times the total charge enclosed by that surface”.

11) *For an infinitely long charged wire electric field is proportional to line charge density and inversely proportional to the radial distance.*

12) Electric field lines are useful for visualizing the electric field

Terminal Question:

1. Explain the Concept of charge, Coulomb’s law, electric field,

2. What is electric flux

3. Explain the Gauss law in detail?

4. Explain the Concept of charge, Coulomb’s law, electric field, electric flux

5. Drive the equation of Coulomb’s law from Gauss law?

6. Explain the Application of Gauss law for charge distribution?

7. Explain the Laws of electrostatics

8. Explain the Electric field due to charged ring and charged infinite rod and charged disc from Coulomb’s law.

9. An infinitely long line of charge carries 0.4 C along each meter of length. Find the E-field 0.3m from the line of charge

10. Four charges are arranged in a square with sides of length 2.5 cm. The two charges in the top right and bottom left corners are +3.0 x 10^{-6} C. The charges in the other two corners are -3.0 x 10^{-6} C. What is the net force exerted on the charge in the top right corner by the other three charges?

11. A cubic space (1.5 m on each side) contains positively charged particles. Imagine that the space is surrounded by a Gaussian surface of the exact same dimension as the cube and that the EField caused by the charges is normal to the faces of the Gaussian cube. If the E-field at each surface has a magnitude of 760 N/C, determine the number of charges per unit volume in the space described(ie., find the charge density,ρ).

12. A point charge (q1) has a magnitude of 3x10-6C. A second charge (q2) has a magnitude of -1.5x10-6C and is located 0.12m from the first charge. Determine the electrostatic force each charge exerts on the other

13. Find the electric field acting on a 2.0 C charge if an electrostatic force of 10500 N acts on the particle.

Unit 02- Electric potential and dipole

Structure:

2.1 Introduction:

Electric potential energy is the energy that is needed to move a charge against an electric field. You need more energy to move a charge further in the electric field, but also more energy to move it through a stronger electric field.

Electrostatic potential energy is a potential energy (measured in joules) that results from conservative Coulomb forces and is associated with the configuration of a particular set of point charges within a defined system.

Electric field, an electric property associated with each point in space when charge is present in any form. The magnitude and direction of the electric field are expressed by the value of E, called electric field strength or electric field intensity or simply the electric field.

In physics, chemistry and biology, a potential gradient is the local rate of change of the potential with respect to displacement, i.e. spatial derivative, or gradient. This quantity frequently occurs in equations of physical processes because it leads to some form of flux.

The change of electric potential with respect to distance is called potential gradient. It is denoted by dv/dx. Hence, the negative of potential gradient is equal with electric field intensity.

In electrostatics, self energy of a particular charge distribution is the energy of required to assemble the charges from infinity to that particular configuration, without accelerating the charges. It is simply called the electrostatic potential energy stored in the system of charges.

Gauss' Law tells us that the electric field outside the sphere is the same as that from a point charge. This implies that outside the sphere the potential also looks like the potential from a point charge. What about inside the sphere? If the sphere is a conductor we know the field inside the sphere is zero.

The electric potential inside a charged spherical conductor of radius R is given by V = ke Q/R, and the potential outside is given by V = ke Q/r. Using Er = -dv/dr, derive the electric field inside and outside this charge distribution.

The electric field inside the hollow metallic sphere is zero, then work done by the charge is zero, suppose that Va is the potential on the inside and Vb is the potential on the surface then Vb-Va= zero or VB . Hence the potential is the same as on the surface. Due to the solid sphere, the gravitational potential is the same within the sphere.

The electric dipole moment is a measure of the separation of positive and negative electrical charges within a system, that is, a measure of the system's overall polarity. The dipole is represented by a vector from the negative charge towards the positive charge.

Dipole moments occur when there is a separation of charge. They can occur between two ions in an ionic bond or between atoms in a covalent bond; dipole moments arise from differences in electro negativity. The dipole moment is a measure of the polarity of the molecule.

When a dipole is placed in a uniform electric field and dipole vector direction is not parallel to field direction, each charge of dipole experiences a force. Once the dipole is aligned to electric field, the net force will be zero because they are in opposite direction.

If an electric dipole is placed in a non-uniform electric field, then the positive and the negative charges of the dipole will experience a net force. And as one end of the dipole is experiencing a force in one direction and the other end in the opposite direction, so the dipole will have a net torque also.

An electric dipole consists of two equal and opposite charges +q and -q separated by a small distance a. The Electric Dipole Moment P is defined as a vector of magnitude qa with a direction from the negative charge to the positive charge. In many molecules, though the net charge is zero, the nature of chemical bonds is such that the positive and negative charges do not cancel at every point.

Two perfect (infinitesimal) dipoles p_1 and p_2 are perpendicular and lie a distance r apart what is the torque on p_2 (about its center) due to p_1?

The forces the dipole exerts on each other are equal and opposite. Why isn't the torques? Because we calculated the torques about different centers. If we refer both torques to the coordinate origin (i.e. the position of dipole 1).

2.2 Objective:

After studying this unit you should be able to

a) Explain and identify Electric potential and electrostatic potential energy.
b) Study and identify Electric fields, potential gradient and their relationship.
c) Explain and identify Electrostatic self energy (conducting and dielectric sphere).
d) Explain and identify Electric potential due to spherical charge distribution (hollow and solid), graphical representation.
e) Study and identify Electric dipole and its behavior in uniform and non uniform electric field.
f) Explain and identify Electric field and potential due to electric dipole at a point in Cartesian and polar coordinates.
g) Study and identify Force between two electric dipoles.

2.3 Electric potential and electrostatic potential energy:

Electric potential:

Electric potential at a point in an electric field is define as the amount of work to be done to bring a unit positive electric charge from infinity to that point.

Similarly, the potential difference between two points is defined as the work required to be done for bringing a unit positive charge from one point to other point. When a body is charged, it can attract an oppositely charged body and can repulse a similar charged body. That means, the charged body has ability of doing work. That ability of doing work of a charged body is defined as **electrical potential** of that body.

If two electrically charged bodies are connected by a conductor, the electrons starts flowing from lower potential body to higher potential body, that means current starts flowing from higher potential body to lower potential body depending upon the potential difference of the bodies and resistance of the connecting conductor.

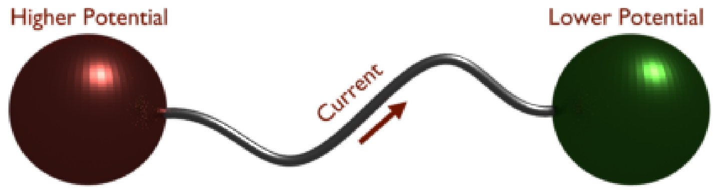


Fig.2.1 Current starts flowing from higher potential body to lower potential body

So, **electric potential** of a body in its charged condition determines whether it will take from or give up electric charge to other body.
Electric potential is graded as electrical level, and difference of two such levels, causes current to flow between them. This level must be measured from a reference zero level. The earth potential is taken as zero level. Electric potential above the earth potential is taken as positive potential and the electric potential below the earth potential is negative

The unit of electric potential is volt. To bring a unit charge from one point to another, if one joule work is done, then the potential difference between the points is said to be one volt. So, this can be represented as,

$$volt = \frac{joules}{coulomb}$$

If one point has electric potential 5 volt, then we can say to bring one coulomb charge from infinity to that point, 5 joule work has to be done.
If one point has potential 5 volt and another point has potential 8 volt, then 8 – 5 or 3 joules work to be done to move one coulomb from first point to second.

Potential at a Point due to Point Charge:

Let us take a positive charge + Q in the space. Let us imagine a point at a distance x from the said charge + Q. Now we place a unit positive charge at that point. As per Coulomb's law, the unit positive charge will experience a force,

$$F = \frac{Q}{4\pi\varepsilon_0\varepsilon_r x^2}$$

Now, let us move this unit positive charge, by a small distance dx towards charge Q.

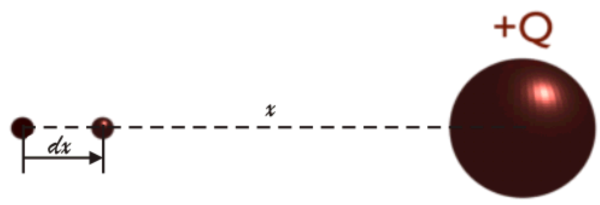


Fig.2.2 Potential at a Point due to Point Charge

During this movement the work done against the field is,

$$dw = -F.\,dx = -\frac{Q}{4\pi\epsilon_0\epsilon_r x^2}$$

So, total work to be done for bringing the positive unit charge from infinity to distance x, is given by,

$$-\int_{\infty}^{x} dw = -\int_{\infty}^{x} \frac{Q}{4\pi\epsilon_0\epsilon_r x^2}.\,dx = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left[\frac{1}{x}\right]_{\infty}^{x} = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left[\frac{1}{x} - \frac{1}{\infty}\right] = \frac{Q}{4\pi\epsilon_0\epsilon_r x}$$

As per definition, this is the electric potential of the point due to charge + Q. So, we can write,

$$V = \frac{Q}{4\pi\epsilon_0\epsilon_r x}$$

Potential Difference between Two Points:

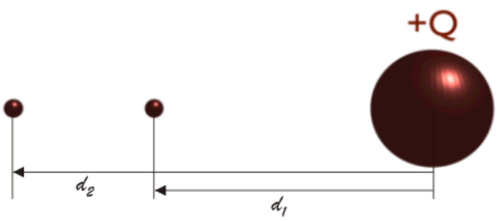


Fig.2.3 Potential Difference between Two Points

Let us consider two points at distance $d_1$meter and d_2 meter from a charge +Q.

We can express the electric potential at the point d_1 meter away from +Q, as,

$$V_{d_1} = \frac{Q}{4\pi\epsilon_0\epsilon_r d_1}$$

We can express the electric potential at the point d_2 meter away from +Q, as,

$$V_{d_2} = \frac{Q}{4\pi\epsilon_0\epsilon_r d_2}$$

Thus, the potential difference between these two points is

$$V_{d_1} - V_{d_2} = \frac{Q}{4\pi\epsilon_0\epsilon_r d_1} - \frac{Q}{4\pi\epsilon_0\epsilon_r d_2} = \frac{Q}{4\pi\epsilon_0\epsilon_r}\left[\frac{1}{d_1} - \frac{1}{d_2}\right]$$

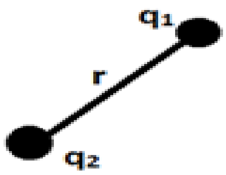
Electrostatic Potential Energy:

One way to measure the effects of these types of interactions between charges is to calculate the electrostatic potential energy of a system of charges. In general, potential energy is any kind of energy that is stored within a system. As this stored energy turns into kinetic energy, the object will start to move and will keep speeding up until all the potential energy has become kinetic energy.

A pair of charges will always have some potential energy because if they are released from rest, they will either start moving towards (if the charges are different) or away (if the charges are the same) from each other. Electrostatic potential energy is specifically the energy associated with a set of charges arranged in a certain configuration.

The potential energy (U_e) depends on the amount of charge that each object contains (*q*), how far apart the charges are (*r*), and Coulomb's constant (k):

$$\boxed{U_e = k\frac{q_1 q_2}{r}}$$



$$k = 9x10^9 \frac{N \cdot m^2}{C^2}$$

$$q_1, q_2 = charge\ of\ each\ object$$

$$r = distance\ between\ the\ charges$$

2.4 Electric fields, potential gradient and their relationship:

Electric Fields:

Definition: The region around the electric charge in which the stress or electric force exerted is called an electric field or electrostatic field. If the magnitude of charge is large, then it may create a huge stress around the region. The electric field is represented by the symbol E. The SI unit of the electric field is Newton per coulomb which is equal to volts per meter.

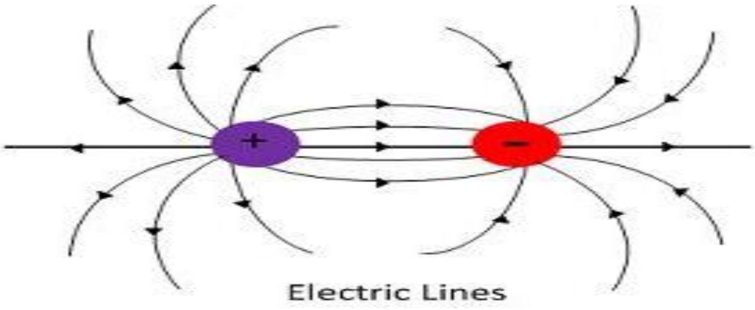


Fig.2.4 Electric Lines

The electric field is represented by the imaginary lines of force. For the positive charge, the line of force come out of the charge and for negative charge the line of force will move towards the charge. The electric field for positive and negative charges are shown below:

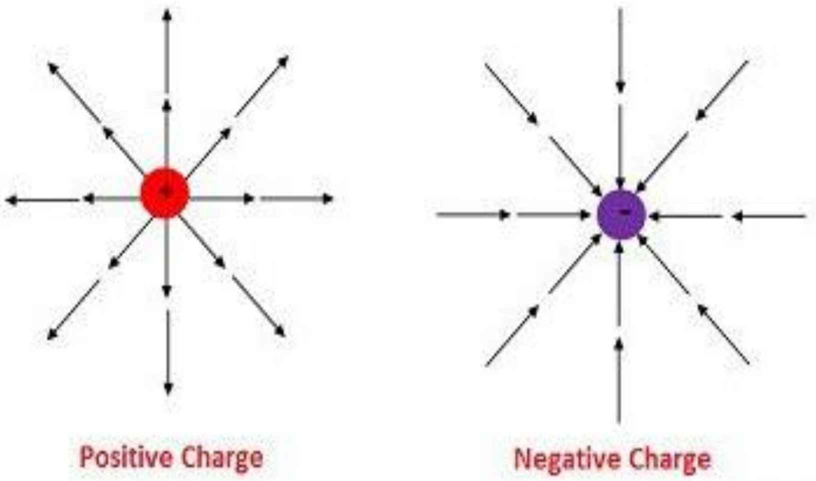


Fig.2.5 Electric field for positive and negative charges

Consider a unit charge Q placed in a vacuum. If another charge q is placed near the Q then according to Coulomb law, the charge Q applies a force on it. The charge Q produce an electric field around it, and when any other charge is placed near it, then the electric field of Q apply force on it.

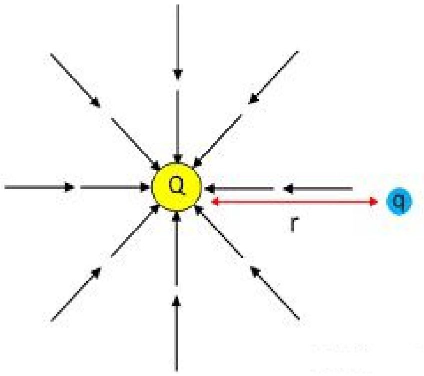


Fig.2.6 Charge Q produces an electric field

The electric field produced by the charge Q at a point r is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Where, Q – unit charge

r – Distance between the charges

A charge Q applies the force on a charge q is expressed by

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

The charge q also apply an equal and opposite force on the charge Q.

Types of an Electric Field

The electric field is mainly classified into two types. They are the uniform electric field and the non-uniform electric field.

1. Uniform Electric Field:

When the electric field is constant at every point, then the field is called the uniform electric field. The constant field is obtained by placing the two conductors parallel to each other, and the potential difference between them remains same at every point.

Fig.2.7 Uniform Electric Field

2. Non-Uniform Electric Field:

The field which is irregular at every point is called the non-uniform electric field. The non-uniform field has a different magnitude and directions.

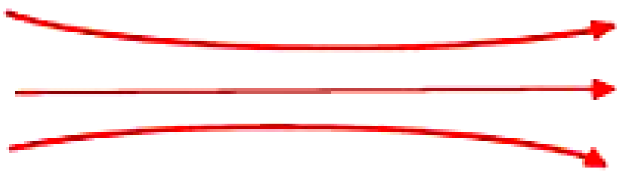


Fig.2.8 Non-Uniform Electric Field

Properties of an Electric Field:

The following are the properties of an electric field:

Field lines never intersect each other.

1. They are perpendicular to the surface charge.
2. The field is strong when the lines are close together, and it is weak when the field lines move apart from each other.
3. The number of field lines is directly proportional to the magnitude of the charge.
4. The electric field line starts from the positive charge and ends from negative charge.
5. If the charge is single, then they start or end at infinity.
6. The line curves are continuous in a charge-free region.

When the electric and magnetic field combines, they form the electromagnetic field.

Potential Gradient:

The potential gradient in a power system is defined as the rate of change of electric potential with respect to the distance from the base of the electrical structure. The resistance of the earth electrode is not concentrated at one point, but it is distributed over the soil around the electrode. When a fault current flows to ground, it results in a potential gradient around the electrode. This may be explained analytically as below

Consider that the base of the structure through which fault current is flowing to ground is a hemisphere of radius **b** as shown in fig below,

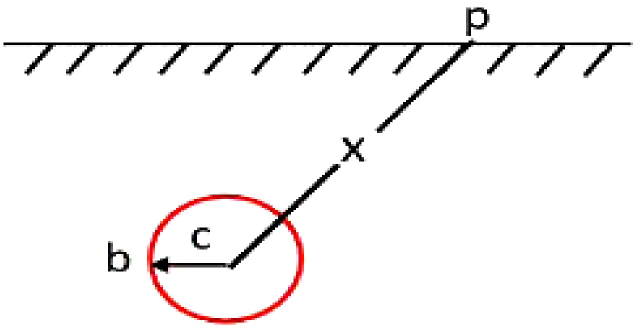


Fig 2.9 Fault current is flowing to ground is a hemisphere of radius **b**

$$e_x = \frac{\rho I_f}{2\pi \times b^2}$$

e_x – electrical field strength at a distance x in V/m

ρ – Resistivity in ohm-meters

I_f – fault current in amperes

x – Distance from the surface of the hemisphere in meters

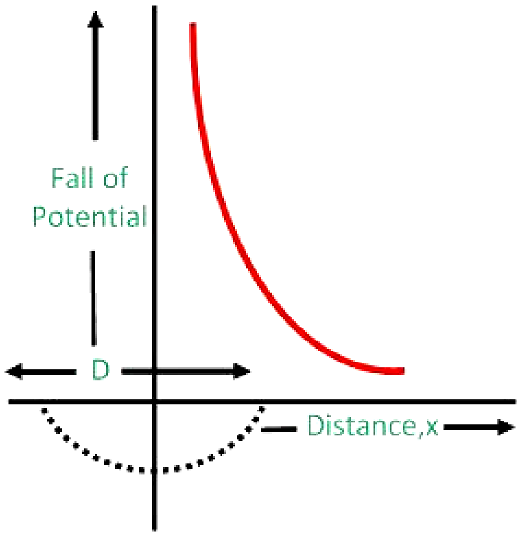


Fig.2.10 Curve between Distance and Potential

If a curve between the falls of potential with distance from the base of the structure is plotted, it is observed that the potential difference is quite definite near the electrode and the fault. The magnitude of potential gradient depends on the resistivity and fault current. If the magnitude of the potential gradient is high then it may affect the person by step and touch potential.

For safety purpose, the earthling system should be provided such that the potential difference due to the fault may not prove to be dangerous to the person approaching that electrode or while touching that structure

Establish the relation between electric field and potential gradient:

Let us consider two closely spaced equi-potential surfaces A and B as shown in figure.

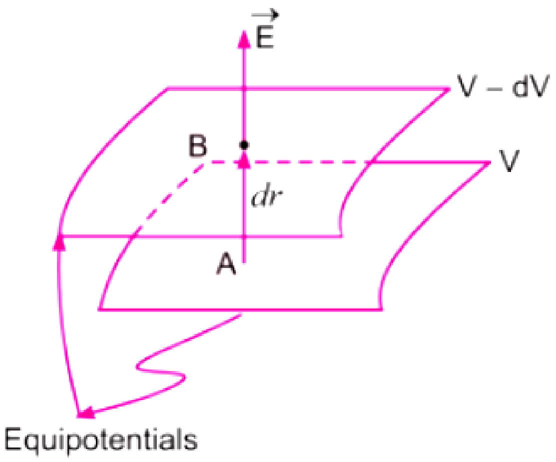


Fig.2.11 Two closely spaced equi-potential surfaces A and B

Let the potential of A be VA = V and potential of B be VB= V−dV is decrease in potential in the direction of electric field E normal to A and B.

Let dr be the perpendicular distance between the two equi-potential surfaces. When a unit positive charge is moved along this perpendicular from the surface B to surface A against the electric field, the work done in this process is

$$W_{BA} = -\vec{E}(dr)$$

This work done equals the potential difference VA−VB

∴ WBA=VA−VB

=V−(V−dV)

=dV

$$\therefore -\vec{E}.dr = dV$$

or,

$$\vec{E} = -\frac{dV}{dr} = \text{negative of potential gradient}$$

SAQ.1

a) What do you mean by Electric potential energy?
b) Define the electrostatic potential energy.
c) What do you mean by Electric fields and potential gradient?
d) An electron is accelerated from rest through a potential difference 24 V. What is the change in electric potential energy of the electron?
e) Two charges q_A = 5 μC and q_B = 8 μC are separated by a distance of 10 cm ($k = 9 \times 10^9$ N m^2 C^{-2}). What is the magnitude of the electric field at the center between q_A and q_B?

2.5 Electrostatic self energy (conducting and dielectric sphere):

Electrostatic self energy (conducting sphere):

We know what the electric field and potential from a point charge look like:

$$\mathbf{E} = \frac{kQ}{r^2}\hat{r} \quad \text{and} \quad V = \frac{kQ}{r}$$

Consider a charged sphere with a symmetrical distribution of charge. Gauss' Law tells us that the electric field outside the sphere is the same as that from a point charge. This implies that outside the sphere the potential also looks like the potential from a point charge.

What about inside the sphere? If the sphere is a conductor we know the field inside the sphere is zero. What about the potential?

Moving from a point on the surface of the sphere to a point inside, the potential changes by an amount:

$\Delta V = -\int \mathbf{E} \bullet \mathbf{ds}$

Because E = 0, we can only conclude that ΔV is also zero, so V is constant and equal to the value of the potential at the outer surface of the sphere.

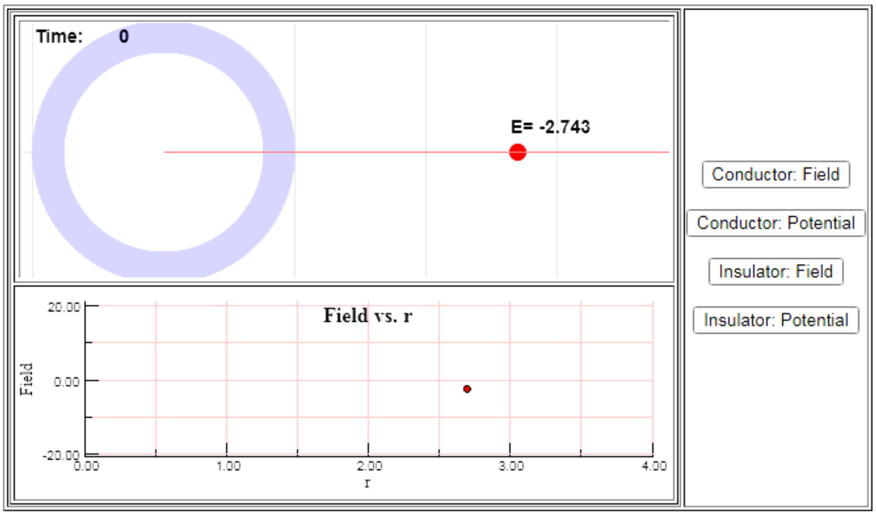


Fig.2.12 Potential near an Insulating Sphere

Now consider a solid insulating sphere of radius R with charge uniformly distributed throughout its volume. Once again, outside the sphere both the electric field and the electric potential are identical to the field and potential from a point charge.

What happens inside the sphere? Now the potential is not constant because there is a field inside the sphere. Using Gauss' Law we showed that the field inside a uniformly charged insulator is:

$$\mathbf{E} = \frac{k\,Q\,r}{R^3}\,\hat{\mathbf{r}}$$

Use this to calculate the potential inside the sphere. Starting from some point a distance r from the center and moving out to the edge of the sphere, the potential changes by an amount:

$$\Delta V = V(R) - V(r) = -\int_r^R \mathbf{E} \cdot \mathbf{ds} = -\int_r^R E\,dr = \frac{-kQ}{R^3}\int_r^R r\,dr$$

Integrating gives:

$$V(R) - V(r) = \frac{-kQ}{2R^3}\,(R^2 - r^2)$$

$$V(R) - V(r) = \frac{-kQ}{2R}\left(1 - \frac{r^2}{R^2}\right)$$

V(R) is simply kQ/R, which can be written as 2kQ/2R, so:

$$V(r) = \frac{2kQ}{2R} + \frac{kQ}{2R}\left(1 - \frac{r^2}{R^2}\right)$$

Therefore, for r < R,

$$V(r) = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Electrostatic self energy using dielectric sphere:

Self energy is equal to interaction energy of charges constituting the thick spherical shell.

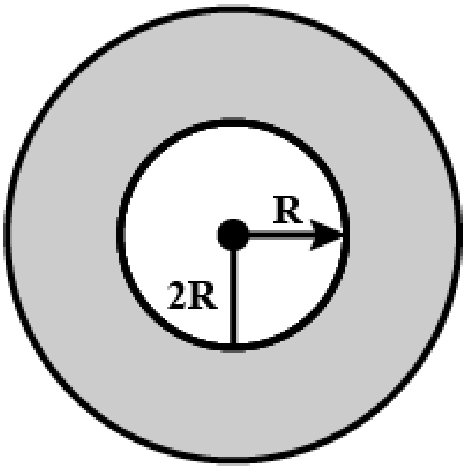


Fig.2.13 Dielectric Sphere

Consider core of outer radius r having charge q, layer of infinite small thickness dr with charge dq. Since core can be assumed to be a point charge at the centre with

$$q = \rho.\frac{4}{3}\pi(r^3 - R^3)$$

∴ Electrostatic interaction energy of point charge and the layer is given by

$$dU = \frac{1}{4\pi\varepsilon_0}\frac{q.dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{p\frac{4}{3}\pi(r^3 - R^3).\rho 4\pi r^2 dr}{r}$$

$$U = \int_{R}^{2R} \frac{\rho^2}{3\varepsilon_0}(r^4 - R^3 r).dr$$

$$U = \frac{\rho^2 4\pi}{3\varepsilon_0}\left[\frac{r^5}{5} - \frac{R^3 r^2}{2}\right]_R^{2R} = \frac{94}{15}\frac{\rho^2 \pi R^5}{\varepsilon_0}$$

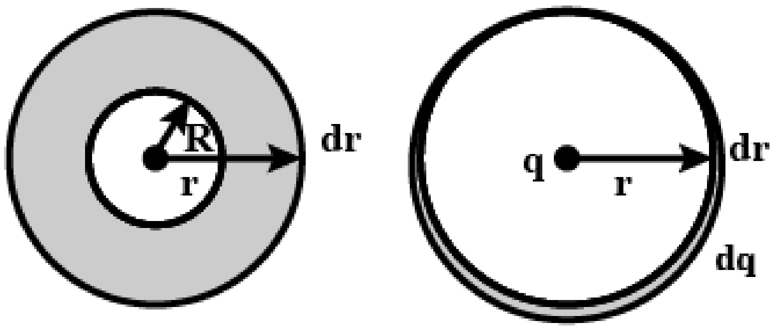


Fig 2.14

2.6 Electric potential due to spherical charge distribution (hollow and solid), graphical representation:

Electric Potential due to Solid spherical charge distribution and graphical representation:

Let us consider a uniform solid non-conducting sphere having total charge Q and radius a. The center of the sphere is located at point O. We have to calculate the electric potential due to this charged solid sphere at a test point P, either inside or outside of the sphere. This is shown in the diagram below. We denote the distance between O and P as . By definition, the potential is the work done to bring a unit charge from infinity to the test point P. The charge density is

$$\rho = \frac{Q}{\frac{4}{3}\pi a^3}$$

We will first draw two concentric spherical shells of radius +x and x+dx. The volume enclosed by this thick shell of interest is $4\pi x^2 dx$ .Charge enclosed within this thick spherical shell is volume multiplied by the charge density. Thus

$$dQ = \frac{Q}{\frac{4}{3}\pi a^3} \times 4\pi x^2 dx$$

$$= \frac{3Q}{a^3} x^2 dx$$

Now we can use the expression we have derived for the *Electric Potential due to a Charged Spherical* **Shell** to calculate the *Electric Potential due to a Charged Solid Sphere* at point P, either outside or inside of the sphere.

Electric Potential at a Point Outside of the Charged Sphere:

The most trivial situation is that the test point P is outside of the sphere. We use our ready-made equation from earlier exercise to calculate the potential at this point.

$$dV = \frac{dQ}{4\pi\epsilon_0 r}$$

Thus the potential due to the solid sphere will be

$$V = \frac{1}{4\pi\epsilon_0 r} \int dQ$$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}}$$

Again we note that the electric potential due to solid sphere at an external point is identical to the electric potential due to a point charge Q situated at the center of the sphere.

Electric Potential at a Point Inside of the Charged Sphere:

Here the situation is more complex than the earlier one, however, we can divide the sphere into two parts and use superposition principle to get the final potential. The first sphere has radius r and the second one is a thick spherical shell with an inner radius r and outer radius a. Let us assume that the charge of the inner sphere is Q', then

$$Q' = \frac{Q}{\frac{4}{3}\pi a^3} \times \frac{4}{3}\pi r^3$$

$$= \frac{Qr^3}{a^3}$$

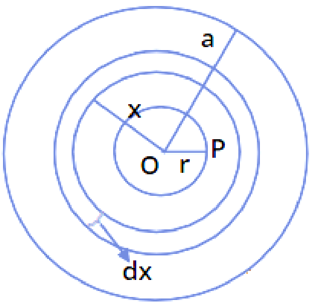


Fig.2.15 The spherical shell of the inner radius r and outer radius a

and the potential at a test point, P due to this sphere is

$$V' = \frac{Q'}{4\pi\epsilon_0 r}$$

$$= \frac{Qr^2}{4\pi\epsilon_0 a^3}$$

Now we have to calculate the potential at point P due to the spherical shell of the inner radius r and outer radius a. For that, we will divide the shell into several concentric shells. Let us consider one such shell with an inner radius of x and thickness dx. The charge contained in the shell is

$$dQ'' = \frac{Q}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3Q}{a^3} x^2 dx$$

The potential at an internal point P due to this thick shell is

$$dV'' = \frac{dQ''}{4\pi\epsilon_0 x} = \frac{3Q}{4\pi\epsilon_0 a^3} x dx$$

Thus the potential due to this part of the sphere is

$$V'' = \int_r^a \frac{3Q}{4\pi\epsilon_0 a^3} x dx$$

$$= \frac{3Q}{8\pi\epsilon_0 a^3}(a^2 - r^2)$$

and the total *electric potential due to the charged solid sphere* at an internal point P is

$$V = V' + V''$$

$$\boxed{V = \frac{Q}{8\pi\epsilon_0 a^3}(3a^2 - r^2)}$$

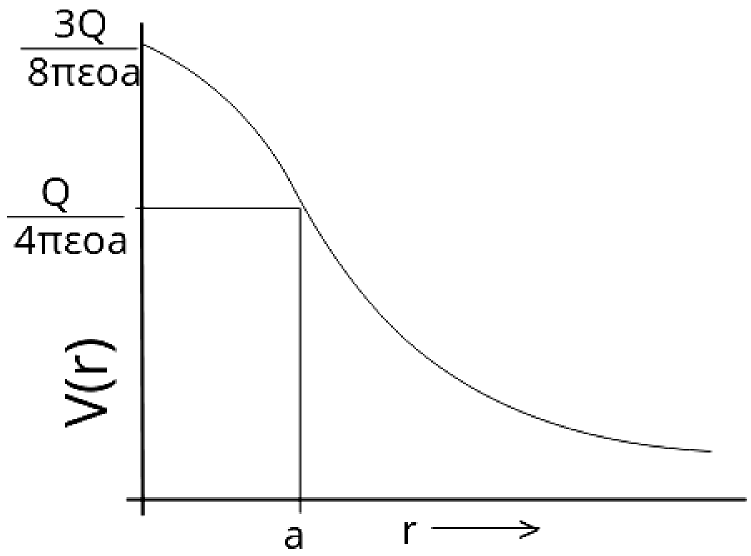


Fig.2.16 Plotted electrostatic potential due to a charged sphere

In the diagram above, we have plotted electrostatic potential due to a charged sphere as a function of distance from the center of the sphere.

Electric Potential due to Hallow spherical charge distribution and graphical representation:

We know what the electric field and potential from a point charge look like:

$$\mathbf{E} = \frac{kQ}{r^2}\,\hat{r} \quad \text{and} \quad V = \frac{kQ}{R}$$

Consider a charged sphere with a symmetrical distribution of charge. Gauss' Law tells us that the electric field outside the sphere is the same as that from a point charge. This implies that outside the sphere the potential also looks like the potential from a point charge.

What about inside the sphere? If the sphere is a conductor we know the field inside the sphere is zero. What about the potential?

Moving from a point on the surface of the sphere to a point inside, the potential changes by an amount:

$$\Delta V = -\int \mathbf{E} \bullet \mathbf{ds}$$

Because E = 0, we can only conclude that ΔV is also zero, so V is constant and equal to the value of the potential at the outer surface of the sphere.

Potential near an Insulating Sphere:

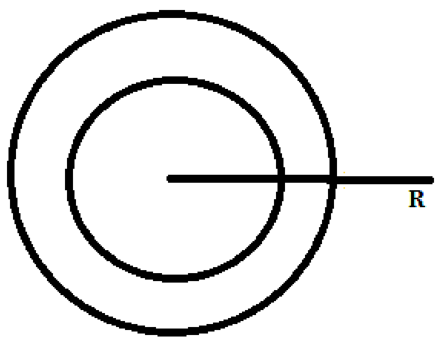


Fig.2.17 A solid insulating sphere of radius R with charge uniformly distributed

Now consider a solid insulating sphere of radius R with charge uniformly distributed throughout its volume. Once again, outside the sphere both the electric field and the electric potential are identical to the field and potential from a point charge.

What happens inside the sphere? Now the potential is not constant because there is a field inside the sphere. Using Gauss' Law we showed that the field inside a uniformly charged insulator is:

$$\mathbf{E} = \frac{k\,Q\,r}{R^3}\,\hat{\mathbf{r}}$$

Use this to calculate the potential inside the sphere. Starting from some point a distance r from the center and moving out to the edge of the sphere, the potential changes by an amount:

$$\Delta V = V(R) - V(r) = -\int_R^R \mathbf{E} \cdot \mathbf{ds} = -\int_r^R E\,dr = \frac{-kQ}{R^3}\int_r^R r\,dr$$

Integrating gives:

$$V(R) - V(r) = \frac{-kQ}{2R^3}(R^2 - r^2)$$

$$V(R) - V(r) = \frac{-kQ}{2R}\left(1 - \frac{r^2}{R^2}\right)$$

V(R) is simply kQ/R, which can be written as 2kQ/2R, so:

$$V(r) = \frac{2kQ}{2R} + \frac{kQ}{2R}\left(1 - \frac{r^2}{R^2}\right)$$

Therefore, for r < R,

$$V(r) = \frac{kQ}{2R}\left(3 - \frac{r^2}{R^2}\right)$$

Electric Potential due to Hallow spherical charge graphical representation:

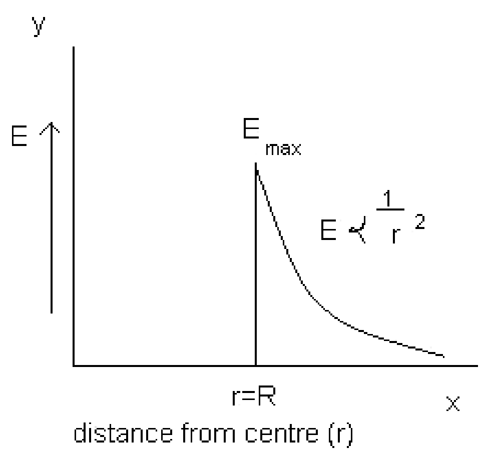


Fig.2.18 Hallow spherical charge graphical representation

SAQ.2

a) What do you mean by the Electrostatic self energy using conducting sphere?
b) Define the Electrostatic self energy using dielectric sphere.
c) What do you mean by Electric potential due to spherical charge distribution for hollow?
d) Define the Electric potential due to spherical charge distribution for solid.
e) A charge of 4×10^{-8} C is distributed uniformly on the surface of a sphere of radius 2 cm. It is covered by a concentric, hollow conducting sphere of radius 7 cm.
   (i) Find the electric field at a point 4 cm away from the centre.
   (ii) A charge of 6×10^{-8}C is placed on the hollow sphere. Find the surface charge density on the outer surface of the hollow sphere.

2.7 Electric dipole and its behavior in uniform and non uniform electric field:

Before we understand the properties of the torque acting on an electric dipole in a uniform electric field, let us brush up our understanding of electric dipole and torque clearly.

Electric dipole:

A pair of electric charges with an equal magnitude but opposite charges separated by a distance d is known as an electric dipole. The electric dipole moment for this is defined as the product of the magnitude of these charges and the distance between them. The electric dipole moment is a vector having a defined direction from the negative charge to the positive charge.

$$\vec{p} = q\,\vec{d}$$

Torque:

The measure of force that causes an object to rotate about an axis is known as torque. Torque is a vector quantity and its direction depends on the direction of the force on the axis. The magnitude of the torque vector is calculated as follows:

$$\tau = Fr\sin\Theta$$

Where r is the length of the moment arm, and θ is the angle between the moment arm and the force vector.

Introduction to Dipole in Uniform External Field

If a dipole is kept in an external electric field, it experiences a rotating effect. By external electric field, we mean electric field that is not induced by dipole itself. The rotating effect is also called torque on the dipole. How we can calculate the torque on a dipole and what are its applications? This can be done by calculating the net torque on opposite charges of the dipole.

Dipole in Uniform External Field

To find torque on a dipole from an external field, consider there is electric dipole $\vec{P}$ placed in an uniform external field $\vec{E}$. The uniform external electric field $\vec{E}$ is produced externally and is not induced by dipole.

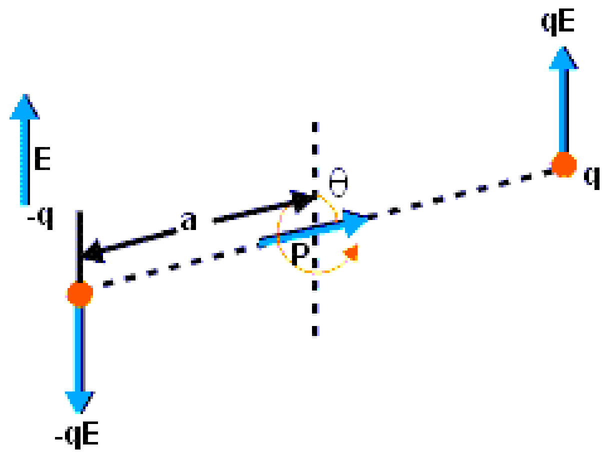


Fig.2.19 An electric dipole placed in non-uniform external electric field

The external electric field $\vec{E}$ will produce electric force of magnitude qE on positive charge in upward direction and on negative charge in downward direction. We can see that the dipole is in transitional equilibrium as net force on the dipole is zero. What about the rotational equilibrium? Is it also zero? If that was the case, then the dipole would have been stationary in position, but experimentally it is found that the dipole rotates with some angular velocity.

This is because, both the electrostatic force that is, qE acts a torque in a clockwise direction, thereby making the dipole to rotate in a uniform external electric field.

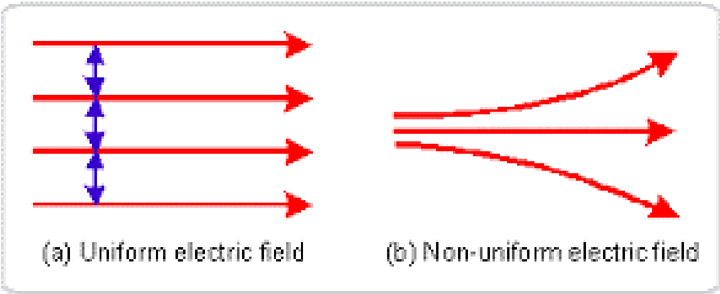


Fig.2.20 Uniform and Non-uniform electric field

Torque always acts in a couple, and its magnitude equals to the product of force and its arm. Arm is the distance between the point where the force acts and the point which rotates the dipole. In the dipole placed in the uniform external electric field, we take origin as the point. Torque is denoted by the symbol τ and as it has a direction, it is a vector quantity.

Mathematically,

Magnitude of torque = $q\ E \times 2a \sin\theta$

$\tau = 2\ q\ a\ E \sin\theta$

$\tau = p\ E \sin\theta$ (Since $p = 2\ q\ a$)

The vector form of torque is the cross product of dipole moment and electric field.

Observations in net force and torque:

Taking the nature of electric field and position of the dipole, following remarks will come out:

- If the dipole $\vec{P}$ and external electric field $\vec{E}$ are parallel, that is, angle between them is zero, then the dipole will feel zero torque
- If the external electric field $\vec{E}$ is non-uniform, then net force on the dipole will not be zero, and torque will still act on it
- If the dipole external electric field $\vec{E}$ are anti-parallel, that is, angle between them is non-zero, then the dipole will feel zero torque
- When the electric dipole $\vec{P}$ and electric field $\vec{E}$ are parallel, the direction of net force will be in direction of increasing electric field.

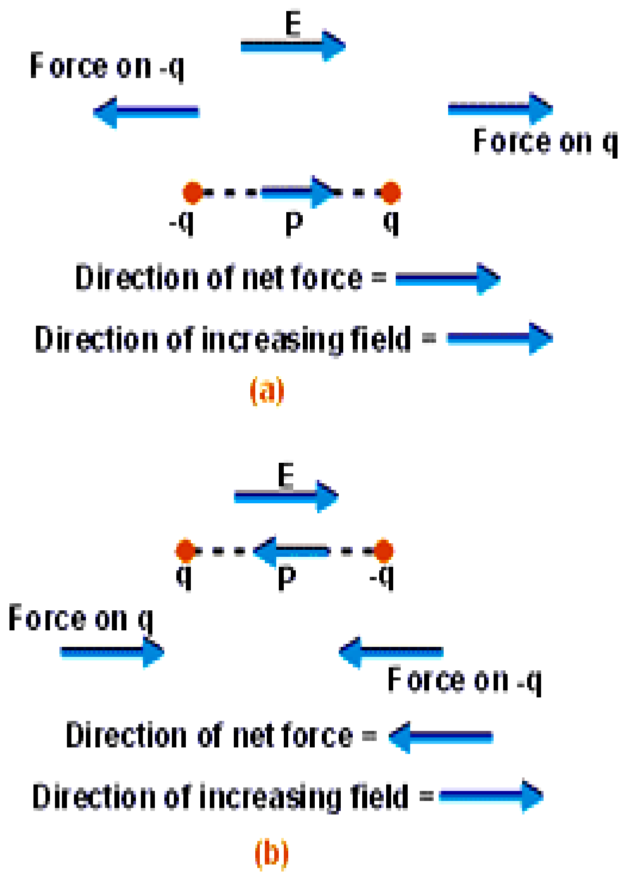


Fig.2.21 Direction of net force depends on orientation of electric dipole

- When the electric dipole $\vec{P}$ and electric field $\vec{E}$ are anti-parallel, then the direction of net force will be in direction of decreasing electric field

- Force and Torque on a dipole placed in a uniform external field $\vec{E}$ varies with the orientation of dipole in free space.

Physical Significance:

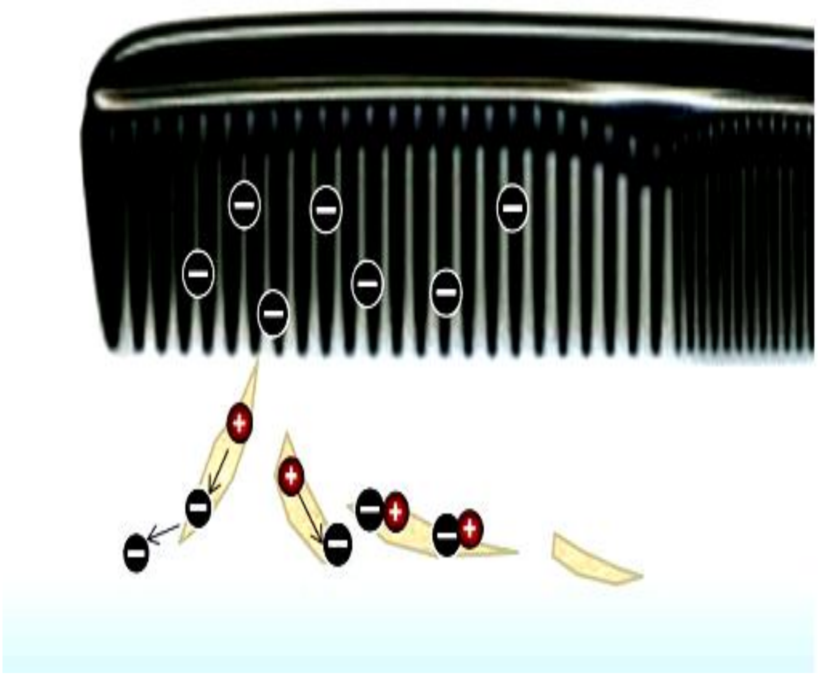


Fig.2.22 comb our dry hair and bring it near to some paper pieces, we find that the comb attracts the paper pieces

When we comb our dry hair and bring it near to some paper pieces, we find that the comb attracts the paper pieces. The comb gains charge, from our hair by the process of rubbing and induce a charge in the uncharged paper. In another way, the comb polarizes the pieces of paper that is, generate a net dipole moment in the direction of electric field. Also since the electric field is non-uniform, the paper pieces move in the direction of the comb.

1.8 Electric field and potential due to electric dipole at a point in Cartesian and Polar Coordinates:

An electric dipole consists of two equal and opposite charges +q and -q separated by a small distance a. The Electric Dipole Moment P is defined as a vector of magnitude qa with a direction from the negative charge to the positive charge. In many molecules, though the net charge is zero, the nature of chemical bonds is such that the positive and negative charges do not cancel at every point. There is a small separation between the positive charge centers and negative charge centers. Such molecules are said to be polar molecules as they have a non-zero dipole moment. The figure below shows an asymmetric molecule like water which has a dipole moment 6.2×10^{-30} C-m.

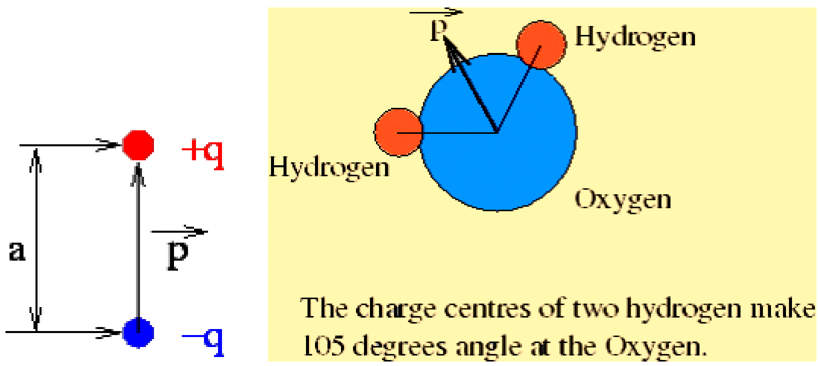


Fig.2.23 Polar r-θ coordinates

In the polar r-θ coordinates shown in the figure,

$$\vec{p} = p\cos\theta\hat{r} - p\sin\theta\hat{\theta}$$

Where $\hat{r}$ and $\hat{\theta}$ are unit vectors in the radial and tangential directions, taken respectively, in the direction of increasing r and increasing θ .

The electric potential at a point P with a position vector $\vec{r}$ is

$$\phi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{r_1} - \frac{q}{r_2}\right] = \frac{q}{4\pi\epsilon_0}\frac{r_2 - r_1}{r_1 r_2}$$

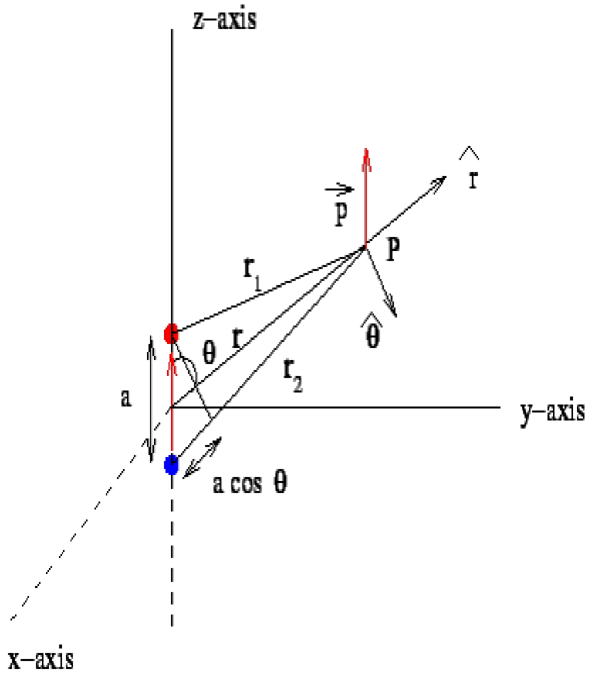


Fig.2.24 The electric potential at a point P with a position vector $\vec{r}$

If the distance a is small compared to r (i.e., if the point P is far away from the dipole), we may use

$$r_2 - r_1 \approx a\cos\theta \quad r_1 r_2 \approx r^2$$

Where θ is the angle between $\vec{r}$ and the dipole moment vector $\vec{p}$. This give

$$\phi(\vec{r}) \approx \frac{qa\cos\theta}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

Electric Field of a Dipole:

A. CARTESIAN COORDINATES:

It is convenient to define the Cartesian axes in the following way. Let the dipole moment vector be taken along the z-axis and position vector $\vec{r}$, of P in the y-z plane (We have denoted the point where the electric field is calculated by the letter P and the electric dipole moment vector as $\vec{p}$). We then have $\cos\theta = z/r$ with $r = \sqrt{y^2 + z^2}$.

Thus

$$\emptyset(x, y, z) = \frac{p cos\theta}{4\pi\epsilon_o r^2} = \frac{pz}{(y^2 + z^2)^{3/2}}$$

Since ø is independent of x, E_x =0. The y and z components are

$$E_y = -\frac{\partial}{\partial y}\left[\frac{pz}{4\pi\epsilon_o(y^2 + z^2)}\right]$$

$$= \frac{3pzy}{4\pi\epsilon_0(y^2 + z^2)^{5/2}} = \frac{3p}{4\pi\epsilon_0}\frac{yz}{r^5}$$

And

$$E_z = -\frac{\partial}{\partial z}\left[\frac{pz}{4\pi\epsilon_o(y^2 + z^2)}\right]$$

$$= -\frac{p}{4\pi\epsilon_0}\frac{1}{(y^2 + z^2)} + \frac{3pz^2}{4\pi\epsilon_0(y^2 + z^2)^{5/2}}$$

$$= \frac{p}{4\pi\epsilon_0}\frac{2z^2 - y^2}{r^5}$$

B. POLAR COORDINATES:

In polar (r-θ) coordinates, the radial and tangent components of the field are as follows:

$$E_r = -\frac{\partial \emptyset}{\partial r} = \frac{2pcos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1\partial \emptyset}{r\partial r} = \frac{psin\theta}{4\pi\epsilon_0 r^3}$$

GENERAL EXPRESSION:

A representation independent from for the dipole field can be obtain from the above,

We have

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta}$$

$$= \frac{p}{4\pi\epsilon_0 r^3}\left[2cos\theta\hat{r} + sin\theta\hat{\theta}\right]$$

Using

$$\vec{p} = pcos\theta\hat{r} - psin\theta\hat{\theta}$$

We get

$$\vec{E} = \frac{1}{4\pi\epsilon_0 r^3}[(3\vec{p}.\hat{r})\hat{r} - \vec{p}]$$

This from does not depend on any particular coordinate system. Note that, at large distances, the dipole field decrease as $1/r^3$ where as monopole field (i.e. field due to a point charge) decreases as $1/r^2$.

Dipole in a uniform Electric Field:

The net force on the dipole is zero. There is net torque acting on the dipole. If a is the length of the dipole, the torque is

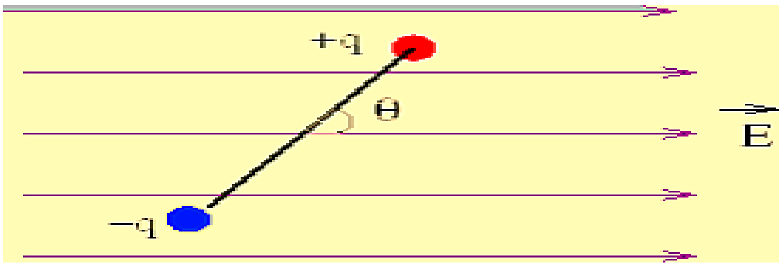


Fig.2.25 Net torque acting on the dipole

$$\tau = (qE) \times a \sin\theta = pE \sin\theta$$

Expressing in vector form,

$$\vec{\tau} = \vec{p} \times \vec{E}$$

If θ = 0° or 180°, (i.e. when the dipole is aligned parallel or anti-parallel to the field) the torque vanishes and the dipole is in equilibrium is stable if θ = 0° and unstable if θ =180°.

2.9 Force between two electric dipoles:

Two perfect (infinitesimal) dipoles p1 and p2 are perpendicular and lie a distance r apart what is the torque on p2 (about its center) due to p1? What are the forces on each? Why are the torque not equal and opposite?

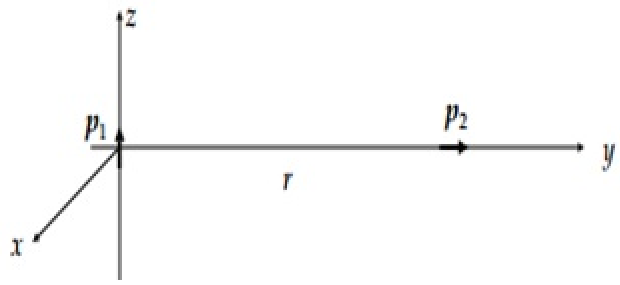


Fig.2.26 Dipoles p1 and p2 are perpendicular and lie a distance r

The field of each, at each other's position:

$$E_1(2) = \frac{2p_1 cos\theta}{r^3}\hat{r} + \frac{p_1 sin\theta}{r^3}\hat{\theta} = \frac{p_1}{r^3}\hat{\theta} = -\frac{p_1}{y^3}\hat{z} = -\frac{p_1}{a^3}\hat{z}$$

$$E_2(1) = \frac{2p_2 cos\theta'}{r^3}\widehat{r'} + \frac{p_2 sin\theta'}{r^3}\widehat{\theta'} = \frac{2p_2}{r^3}\widehat{r'} = -\frac{2p_2}{y^3}\hat{y} = -\frac{2p_2}{a^3}\hat{y}$$

The torque each:

$$N_1 = p_1 \times E_2(1) = p_1\hat{z} \times \frac{2p_2}{a^3}\hat{y} = -\frac{2p_1p_2}{a^3}\widehat{x}$$

$$N_2 = p_2 \times E_1(2) = p_2\hat{y} \times = \left(-\frac{p_1}{a^3}\hat{z}\right) = -\frac{p_1p_2}{a^3}\widehat{x}$$

Force on 2 due to field of 1:

$$F_2 = (p_2 \cdot \nabla)E_1\,(2) = p_2 \frac{\partial}{\partial y} E_1\,(2)\Big|_{y=a}$$

$$= p_2 \frac{\partial}{\partial y}\left(-\frac{p_1}{y^3}\hat{z}\right)|_{y=a} = \frac{3p_1p_2}{a^2}\hat{z}$$

By Newton's thirds law, the force on 2 by 1 is

$$F_1 = (p_1 \cdot \nabla)E_2\,(1) = -\frac{3p_1p_2}{a^4}\hat{z}$$

The forces the dipole exerts on each other are equal and opposite. Why isn't the torques? Because we calculated the torques about different centers. If we refer both torques to the coordinate origin (i.e. the position of dipole 1), then

$$N_1(0) = p_1 \times E_2(1) = -\frac{2p_1p_2}{a^3}\hat{x}$$

$$N_2(0) = p_2 \times E_1(2) + r \times F_2 = -\frac{p_1p_2}{a^3}\hat{x} + a\hat{y} \times \frac{3p_1p_2}{a^4}\hat{z}$$

$$= -\frac{p_1p_2}{a^3}\hat{x} + \frac{3p_1p_2}{a^3}\hat{x} = \frac{2p_1p_2}{a^3}\hat{x} = -N_1(0).$$

SAQ.3

a) What do you mean by Electric dipole and its behavior in uniform and non uniform electric field?

b) Define the Electric field and potential due to electric dipole at a point in polar coordinates.

c) Write the expression for Force between two electric dipoles.

d) What is the dipole moment for a dipole having equal charges -4C and 2C separated with a distance of 6cm?

Examples:

Q.1 An electron is accelerated from rest through a potential difference 12 V. What is the change in electric potential energy of the electron?

Solution:

$\Delta PE = q\ V =$

$(-1.60 \times 10^{-19}\ C)\ (12\ V)$

$= -19.2 \times 10^{-19}$ Joule

The minus sign indicates that the potential energy decreases.

Q.2 An electron is accelerated from rest through a potential difference 12 V. What is the change in electric potential energy of the electron?

Solution:

The charge on an electron (e) = -1.60×10^{-19} Coulomb

Electric potential = voltage (V) = 12 Volt

The change in electric potential energy of the electron (ΔPE) =?

$\Delta PE = q\ V = (-1.60 \times 10^{-19}\ C)(12\ V) = -19.2 \times 10^{-19}$ Joule

The minus sign indicates that the potential energy decreases.

Q.3 Two parallel plates are charged. The separation between the plates is 2 cm and the magnitude of the electric field between the plates is 500 Volt/meter. What is the change in potential energy of the proton when accelerated from the positively charged plate to the negatively charged plate?

Solution:

The magnitude of the electric field between the plates (E) = 500 Volt/meter

The distance between the plates (s) = 2 cm = 0.02 m

The charge on an proton = +1.60 x 10^{-19} Coulomb

The change in electric potential energy (ΔPE) =?

Electric potential:

V = E s

V = (500 Volt/m) (0.02 m)

V = 10 Volt

The change in electric potential energy:

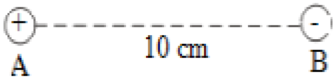
ΔPE = q V

ΔPE = (1.60 x 10^{-19} C) (10 V)

ΔPE = 16 x 10^{-19} Joule

ΔPE = 1.6 x 10^{-18} Joule

Q.4 Two point charges are separated by a distance of 10 cm. Charge on point A =+9 μC and charge on point B = -4 μC. k = 9 x 10^9 Nm^2C^{-2}, 1 μC = 10^{-6} C. What is the change in electric potential energy of charge on point B if accelerated to point A?



Solution:

Charge A (q_1) = +9 μC = +9 x 10^{-6} C

Charge B (q_1) = -4 μC = -4 x 10^{-6} C

k = 9 x 10^9 Nm^2C^{-2}

The distance between charge A and B (r) = 10 cm = 0.1 m = 10^{-1} m

The change in electric potential energy (ΔEP) =?

$$\Delta PE = \frac{kQq}{r}$$
$$\Delta PE = \frac{(9x10^9)(9x10^{-6})(4x10^{-6})}{(10^{-1})^2}$$
$$\Delta PE = \frac{(81x10^3)(4x10^{-6})}{10^{-2}}$$
$$\Delta PE = \frac{324x10^{-3}}{10^{-2}}$$
$$\Delta PE = 324x10^{-1}$$
$$\Delta PE = 32.4 Joule$$

Q.5 Two charges q_A = 1 μC and q_B = 4 μC are separated by a distance of 4 cm ($k=9x10^9$ Nm^2C^{-2}). What is the magnitude of the electric field at the center between q_A and q_B?

Solution:

Charge A (q_A) = 1 μC = 1 x 10^{-6} C

Charge B (q_B) = 4 μC = 4 x 10^{-6} C

k = 9 x 10^9 N m^2 C^{-2}

Distance between charge A and B (r_{AB}) = 4 cm = 0.04 meters

Distance between charge A and the center point (rA) = 0.02 meters

Distance between charge B and the center point (rB) = 0.02 meters

The magnitude of the electric field

The electric field produced by charge A at the center point:

$$E_A = k\frac{q_A}{r_A^2} = (9 x 10^9)\frac{1 x 10^{-6}}{0.02^2} = \frac{9 x 10^3}{0.0004} = \frac{9 x 10^3}{4 x 10^{-4}} = 2.25 \text{ x } 10^7 \text{ NC}^{-1}$$

Test charge is positive and charges A is positive so that the direction of the electric field points to charge B.

The electric field produced by charge B at the center point:

$$E_B = k\frac{q_B}{r_B^2} = (9 x 10^9)\frac{4 x 10^{-6}}{0.02^2} = \frac{36 x 10^3}{0.0004} = \frac{36 x 10^3}{4 x 10^{-4}} = 9 \text{ x } 10^7 \text{ NC}^{-1}$$

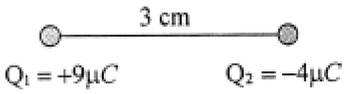
Test charge is positive and charge B is positive so that the direction of the electric field points to charge A.

The resultant of the electric field at the center point:

E_A and E_B have the opposite direction.

$E = E_B - E_A = 9 \times 10^7 - 2.25 \times 10^7 = 6.75 \times 10^7\ NC^{-1}$

Q.6 According to figure below, where the point P is located so that the magnitude of the electric field at point P = 0 ? ($k = 9 \times 10^9\ Nm^2C^{-2}$, $1\ \mu C = 10^{-6}\ C$)



Solution:

If point P located at the left of Q1; the electric field produced by Q1 on point P points to leftward (away from Q1) and the electric field produced by Q2 on point P points to rightward (point to Q1). The direction of the electric field is opposite so that the electric field at point P = 0.

Known:

$Q1 = +9\ \mu C = +9 \times 10^{-6}\ C$

$Q2 = -4\ \mu C = -4 \times 10^{-6}\ C$

$k = 9 \times 10^9\ Nm^2C^{-2}$

Distance between charge 1 and charge 2 = 3 cm

Distance between Q1 and point P (r1P) = a

Distance between Q2 and point P (r2P) = 3 + a

Position of point P=?

Point P located at leftward of Q1.

The electric field produced by Q1 at point P:

$$E_1 = k\frac{Q_1}{r_{1P}{}^2} = (9x10^9)\frac{9x10^{-6}}{a^2} \quad = \quad \frac{81x10^3}{a^2}$$

Test charge is positive and Q_1 is positive so that the direction of the electric field to leftward.

The electric field produced by Q2 at point P:

$$E_2 = k\frac{Q_2}{r_{2P}{}^2} = (9x10^9)\frac{4x10^{-6}}{(3+a)^2} = \frac{36x10^3}{9+6a+a^2}$$

Test charge is positive and Q_2 is negative so that the direction of the electric field to rightward.

Resultant of the electric field at point A:

E_1 and E_2 have opposite direction.

$E_1 - E_2 = 0$

$E_1 = E_2$

$$\frac{81x10^3}{a^2} = \frac{36x10^3}{9+6a+a^2}$$

$$\frac{2.25}{a^2} = \frac{1}{9+6a+a^2}$$

$(2.25)(9 + 6a + a^2) = a^2$

$20.25 + 13.5a + 2.25 a^2 = a^2$

$a^2 - 2.25 a^2 - 13.5a - 20.25 = 0$

$-1.25 a^2 - 13.5a - 20.25 = 0$

Use quadratic formula to find a :

a = -1.25, b = -13.5, c = -20.25

$$a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = \frac{-(-13.5) \pm \sqrt{(-13.5)^2 - 4(-1.25)(-20.25)}}{2(-1.25)}$$

$$a = \frac{13.5 \pm \sqrt{182.25 - 101.25}}{-2.5}$$

$$a = \frac{13.5 \pm \sqrt{81}}{-2.5} = \frac{13.5 \pm 9}{-2.5}$$

$$a_1 = \frac{13.5 + 9}{-2.5} = \frac{22.5}{-2.5} = -9$$

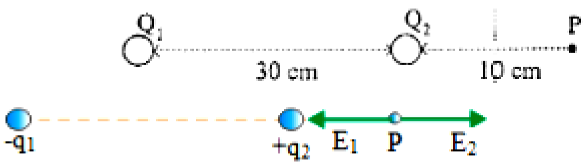
$$a_2 = \frac{13.5 - 9}{-2.5} = \frac{4.5}{-2.5} = -1.8$$

Distance between Q2 and point P (r_{2P}) = 3 + a = 3 – 1.8 = 1.2 cm or 3 + a = 3 – 9 = -6 cm.

Distance between Q1 and point P (r_{1P}) = a = -9 cm or -1.8 cm.

Point P located at 1.2 cm rightward of Q_2.

Q.7 Two charges Q_1 = -40 μC and Q_2 = +5 μC as shown in figure below (k = 9 x 10^9 Nm^2C^2 and 1 μC = 10^{-6} C). What is the magnitude of the electric field at point P?



Solution:

Known:

Charge q_1 = -40 μC = -40 x 10^{-6} C

Charge q_2 = +5 μC = +5 x 10^{-6} C

Distance between q_1 and point P (r_1) = 40 cm = 0.4 m = 4 x 10^{-1} m

Distance between q_2 and point P (r_2) = 10 cm = 0.1 = 1 x 10^{-1} m

k = 9 x 10^9 N m^2 C^{-2}

the magnitude of the electric field at point P =?

The electric field 1:

$E_1 = k\, q_1 / r_1^2$

$E_1 = (9 \times 10^9)(40 \times 10^{-6}) / (4 \times 10^{-1})^2$

$E_1 = (360 \times 10^3) / (16 \times 10^{-2})$

$E_1 = 22.5 \times 10^5$ N/C

The electric field 2:

$E_2 = k\, q_2 / r_2^2$

$E_2 = (9 \times 10^9)(5 \times 10^{-6}) / (1 \times 10^{-1})^2$

$E_2 = (45 \times 10^3) / 1 \times 10^{-2}$

$E_2 = 45 \times 10^5$ N/C

Resultant of the electric field:

The resultant of the electric field at point P :

$E = E_2 - E1 = (45 - 22.5) \times 10^5 = 22.5 \times 10^5$ N/C

$E = 2.25 \times 10^6$ N/C

The direction of the electric field points to rightward (same direction as E_2).

Q.8. A potentiometer wire has a length of 2m and a resistance of 10 ohm. It is connected in series with a cell of e.m.f. 4V and internal resistance 6 ohm. Find the potential gradient on the wire. Find also where a cell of e.m.f. 1V will balance on the wire.

Solution:

Given: Length of potentiometer wire = l_{AB} = 2 m, Resistance of potentiometer wire = R_{AB} = 10 ohm, e.m.f. of cell = E = 4 V, Internal resistance of cell = r = 6 ohm, e.m.f. of cell = e = 1 V

To Find: Potential gradient =? Balancing length = l =?

$$I = E/(R_{AB} + r) = 4/(10 + 6) = 4/16 = 0.25 \text{ A}$$

$$V_{AB} = I\, R_{AB} = 0.25 \times 10 = 2.5 \text{ V}$$

$$\text{Potential gradient} = V_{AB}/l_{AB} = 2.5/2 = 1.25 \text{ V m}^{-1}$$

$$\text{E.m.f. of cell} = \text{Potential drop x Balancing length}$$

$$1 = 1.25 \times l_{AB}$$

$$l_{AB} = 1/1.25 = 0.8 \text{ m}$$

Potential gradient = 1.25 V m^{-1} and balancing length = 0.8 m

Q.9 A potentiometer wire has a length of 4m and a resistance of 10 ohm. It is connected in series with a cell of e.m.f. 4V and internal resistance 2 ohm. Find the potential gradient on the wire. Find also where a cell of e.m.f. 1.5 V will balance on the wire.

Solution:

Given: Length of potentiometer wire = l_{AB} = 4m, Resistance of potentiometer wire = R_{AB} = 10 ohm, e.m.f. of cell = E = 4 V, Internal resistance of cell = r = 2 ohm, e.m.f. of cell = e = 1.5 V

To Find: Potential gradient =? Balancing length = l =?

$I = E/(R_{AB} + r) = 4/(10 + 2) = 4/12 = (1/3)$ A

$V_{AB} = I\, R_{AB} = (1/3) \times 10 = (10/3)$ V

Potential gradient = $V_{AB}/l_{AB} = (10/3)/4 = 10/12 = 5/6$

$= 0.8333 \text{ V m}^{-1}$

E.m.f. of cell = Potential drop x Balancing length

$1.5 = (5/6) \times l_{AB}$

$l_{AB} = (1.5 \times 6)/5 = 1.8$ m

Potential gradient = 0.8333 V m^{-1} and balancing length = 1.8 m

Q.10. A potentiometer wire has a length of 2m and a resistance of 10 ohm. It is connected in series with a cell of e.m.f. 2V and a resistance 990 ohm. Find the potential gradient on the wire.

Solution:

Given: Length of potentiometer wire = l_{AB} = 2m, Resistance of potentiometer wire = R_{AB} = 10 ohm, e.m.f. of cell = E = 2 V, Resistance in series = R = 990 ohm.

To Find: Potential gradient =?

$I = E/(R_{AB} + R) = 2/(10 + 990) = 2/1000 = 0.002$ A

$V_{AB} = I\ R_{AB} = 0.002 \times 10 = 0.02$ V

Potential gradient = $V_{AB}/l_{AB} = 0.02/2 = 0.01\ V\ m^{-1}$

Potential gradient = $0.01\ V\ m^{-1}$

Q.11 A uniform electric field of magnitude E = 100 N/C exists in the space in X-direction. Using the Gauss theorem calculate the flux of this field through a plane square area of edge 10 cm placed in the Y-Z plane. Take the normal along the positive X-axis to be positive.

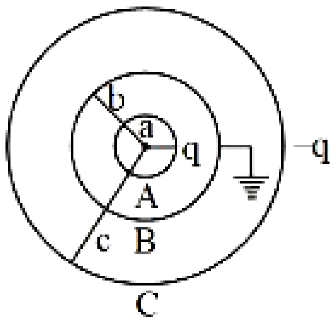
Solution:

The flux $\Phi = \int E.\cos\theta\ ds$.

As the normal to the area points along the electric field, $\theta = 0$.

Also, E is uniform so, $\Phi = E.\Delta S = (100\ N/C)\ (0.10m)^2 = 1\ N\text{-}m^2$.

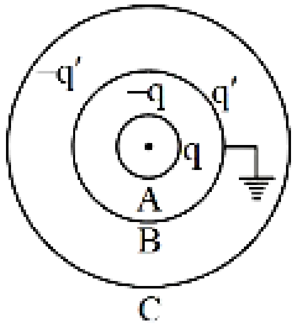
Q.12 The figure shows three concentric thin spherical shells A, B and C of radii a, b, and c respectively. The shells A and C are given charges q and -q respectively and the shell B is earthed. Find the charges appearing on the surfaces of B and C.



Solution:

As shown in the previous worked out example, the inner surface of B must have a charge -q from the Gauss law. Suppose, the outer surface of B has a charge q.

The inner surface of C must have a charge -q' from Gauss law. As the net charge on C must be -q, its outer surface should have a charge q' – q. The charge distribution is shown in the figure.



The potential at B,

- Due to the charge q on:

$$A = q/4\pi\varepsilon_0 b,$$

- Due to the charge –q on the inner surface of:

$$B = -q/4\pi\varepsilon_0 b,$$

- Due to the charge q’ on the outer surface of:

$$B = q'/4\pi\varepsilon_0 b,$$

- Due to the charge –q’, on the inner surface of:

$$C = -q'/4\pi\varepsilon_0 c,$$

- Due to the charge q’ – q on the outer surface of:

$$C = (q' - q)/4\pi\varepsilon_0 c.$$

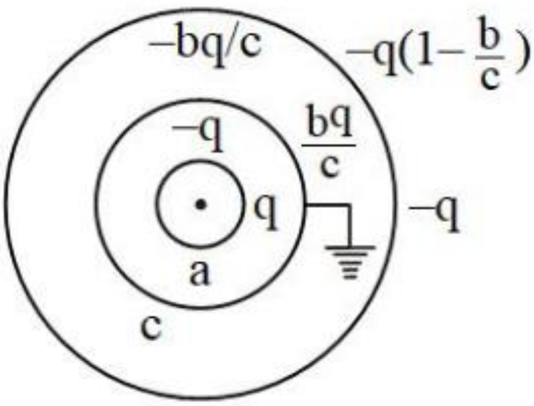
The net potential is,

$$VB = q'/4\pi\varepsilon_0 b - q/4\pi\varepsilon_0 c$$

This should be zero as the shell B is earthed. Thus

$$q' = q \times b/c$$

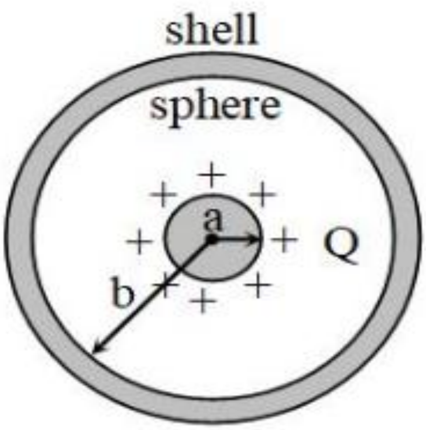
The charge on various surface are as shown in the figure:



Q.13 A solid conducting sphere having a charge Q is surrounded by an uncharged concentric conducting hollow spherical shell. Let the potential difference between the surface of the solid sphere and that of the outer surface of hollow shell be V. What will be the new potential difference between the same two surfaces if the shell is given a charge -3Q?

Solution:

In case of a charged conducting sphere



$$V_{in} = V_c = V_s = 1/4\pi\varepsilon_0$$

And

$$V_{out} = 1/4\pi\varepsilon_0$$

So if a and b are the radii of a sphere and spherical shell respectively, the potential at their surfaces will be;

$$Vsphere = 1/4\pi\varepsilon_0 \,[Q/a]$$

And

$$Vshell = 1/4\pi\varepsilon_0 \,[Q/b]$$

And so according to the given problem;

$$V = V'sphere - V'shell = Q/4\pi\varepsilon_0 \,[1/a - 1/b] = V \,.\,.\,.\,.\,.\,.\,.\,(1)$$

Now when the shell is given a charge (-3Q) the potential at its surface and also inside will change by;

$$V_0 = 1/4\pi\varepsilon_0\,[-3Q/b]$$

So that now,

$$V'sphere = 1/4\pi\varepsilon_0\,[Q/a + V_0] \text{ and } V'shell = 1/4\pi\varepsilon_0\,[Q/b + V_0]$$

Hence

$$V'sphere - V'shell = Q/4\pi\varepsilon_0\,[1/a - 1/b] = V$$ ……… [From eqn. (1)]

i.e., if any change is given to external shell the potential difference between sphere and shell will not change.

This is because by the presence of charge on the outer shell, potential everywhere inside and on the surface of the shell will change by the same amount and hence the potential difference between sphere and shell will remain unchanged.

Q.14 What is the dipole moment for a dipole having equal charges -2C and 2C separated with a distance of 2cm?

Solution: The calculated dipole moment for this condition is, p = q x d.

Thus, p = 2 x 0.02 = 0.04 C-m.

Q.15 A sample of HCl gas is placed in a uniform electric field of magnitude 3×10^4 N C^{-1}. The dipole moment of each HCl molecule is 3.4×10^{-30} Cm. Calculate the maximum torque experienced by each HCl molecule.

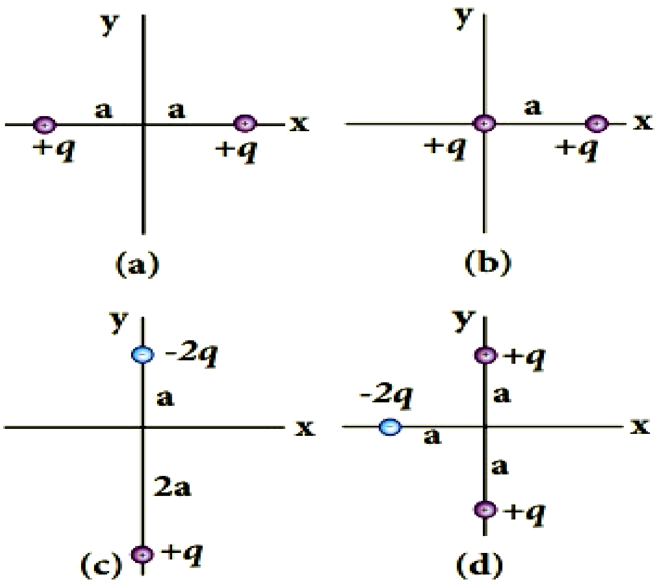
Solution:

The maximum torque experienced by the dipole is when it is aligned perpendicular to the applied field.

τ_{max} = pE sin 90 = 3 .4×10^{-30} ×3 ×10^{4} N m

τ_{max} = 10.2 ×10^{-26} N m

Q.16 Calculate the electric dipole moment for the following charge configurations.



Solution:

Case (a) The position vector for the +q on the positive x-axis is ai and position vector for the +q charge the negative x axis is -a i ^ . So the dipole moment is,

$$\vec{p} = (+q)(a\hat{i}) + (+q)(-a\hat{i}) = 0$$

Case (b) In this case one charge is placed at the origin, so its position vector is zero. Hence only the second charge +q with position vector ai contributes to the dipole moment, which is $\vec{p}$ = qa $\hat{i}$.

From both cases (a) and (b), we can infer that in general the electric dipole moment depends on the choice of the origin and charge configuration. But for one special case, the electric dipole moment is independent of the origin. If the total charge is zero, then the electric dipole moment will be the same irrespective of the choice of the origin. It is because of this reason that the electric dipole moment of an electric dipole (total charge is zero) is always directed from –q to +q, independent of the choice of the origin.

Case (c)

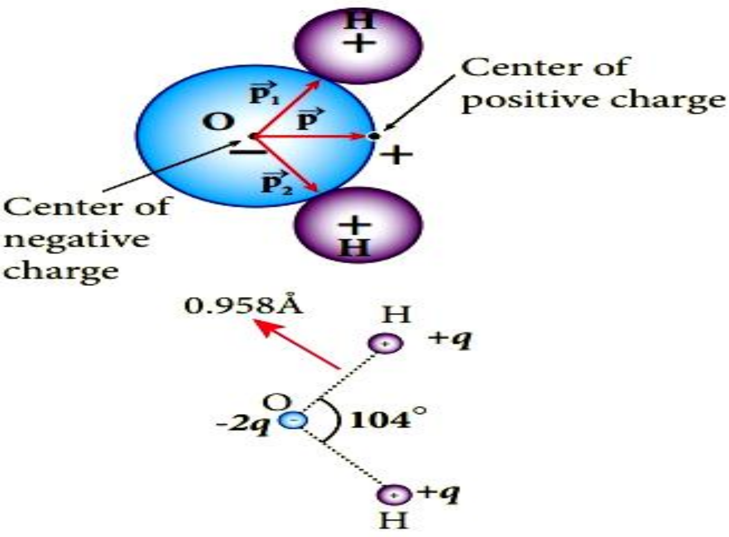
$$\vec{p} = (-2q)a\hat{j} + q(2a)(-\hat{j}) = -4qa\hat{j}.$$

Note that in this case p is directed from -2q to +q.

Case (d)

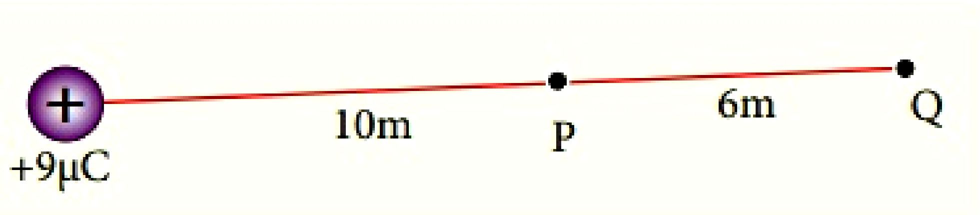
$$\vec{p} = -2qa(-\hat{i}) + qa\hat{j} + qa(-\hat{j}) = 2qa\hat{i}$$

The water molecule (H2O) has this charge configuration. The water molecule has three atoms (two H atom and one O atom). The centers of positive (H) and negative (O) charges of a water molecule lie at different points, hence it possess permanent dipole moment. The O-H bond length is 0.958×10^{-10} m due to which the electric dipole moment of water molecule has the magnitude p = 6.1 x 10^{-30} Cm. The electric dipole moment $\vec{p}$ is directed from center of negative charge to the center of positive charge, as shown in the figure.



Q.17 (a) Calculate the electric potential at points P and Q as shown in the figure below.

(b) Suppose the charge +9µC is replaced by -9µC find the electrostatic potentials at points P and Q.



(c) Calculate the work done to bring a test charge +2µC from infinity to the point P. Assume the charge +9µC is held fixed at origin and +2µC is brought from infinity to P.

Solution:

(a) Electric potential at point P is given by

$$V_P = \frac{1}{4\pi\varepsilon_\circ}\frac{q}{r_P} = \frac{9\times10^9\times9\times10^{-6}}{10} = 8.1\times10^3 V$$

Electric potential at point Q is given by

$$V_Q = \frac{1}{4\pi\varepsilon_\circ}\frac{q}{r_Q} = \frac{9\times10^9\times9\times10^{-6}}{16} = 5.06\times10^3 V$$

Note that the electric potential at point Q is less than the electric potential at point P. If we put a positive charge at P, it moves from P to Q. However if we place a negative charge at P it will move towards the charge +9μC.

The potential difference between the points P and Q is given by

$\Delta V = V_P - V_Q = +3 .04 \times 10^3 V$

(b) Suppose we replace the charge +9 μC by -9 μC, then the corresponding potentials at the points P and Q are,

$V_P = -8 . 1\times10^3$ V,

$V_Q = - 5.06 \times 10^3 V$

Note that in this case electric potential at the point Q is higher than at point P.

The potential difference or voltage between the points P and Q is given by

$\Delta V = V_P - VQ = -3 .04 \times 10^3 V$

(c) The electric potential V at a point P due to some charge is defined as the work done by an external force to bring a unit positive charge from infinity to P. So to bring the q amount of charge from infinity to the point P, work done is given as follows.

$W = qV$

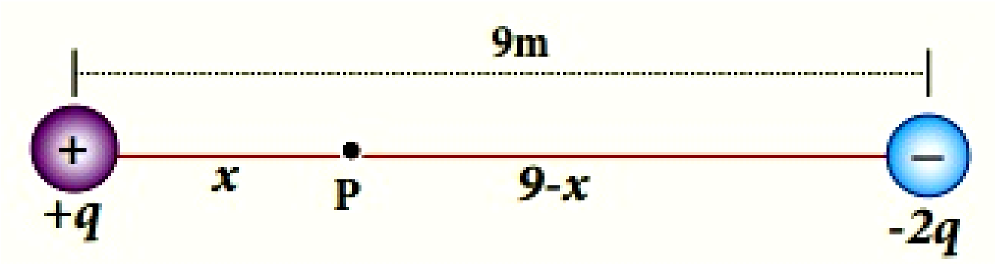
$W_Q = 2 \times 10^{-6} \times 5.06 \times 10^{3}$ J $= 10.12 \times 10^{-3}$ J.

Q.18 Consider a point charge +q placed at the origin and another point charge -2q placed at a distance of 9 m from the charge +q. Determine the point between the two charges at which electric potential is zero.

Solution:

According to the superposition principle, the total electric potential at a point is equal to the sum of the potentials due to each charge at that point.

Consider the point at which the total potential zero is located at a distance x from the charge +q as shown in the figure.



The total electric potential at P is zero.

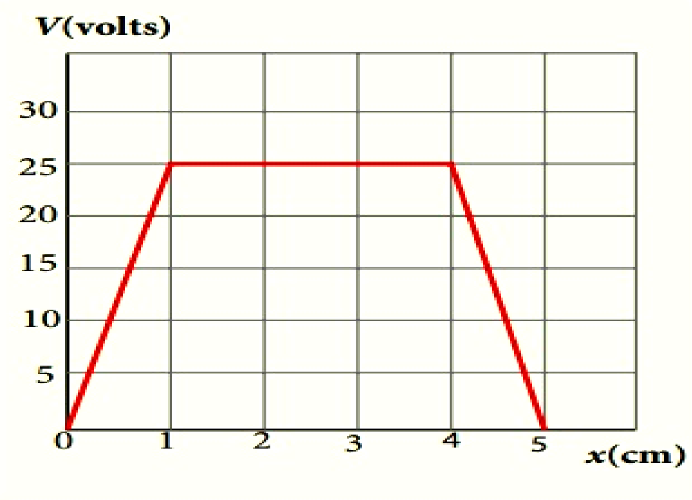
$$V_{tot} = \frac{1}{4\pi\varepsilon_\circ}\left(\frac{q}{x} - \frac{2q}{(9-x)}\right) = 0$$

which gives $\quad \dfrac{q}{x} = \dfrac{2q}{(9-x)}$

or $\quad \dfrac{1}{x} = \dfrac{2}{(9-x)}$

Hence, $\quad x = 3 \text{ m}$

Q.19 The following figure represents the electric potential as a function of x – coordinate. Plot the corresponding electric field as a function of x.



Solution:

In the given problem, since the potential depends only on x, we can use

$$\vec{E} = -\frac{dV}{dx}\hat{i}$$

(the other two terms ∂V/∂y and ∂V/∂z are zero)

From 0 to 1 cm, the slope is constant and so dV/dx = 25V cm^{-1}.

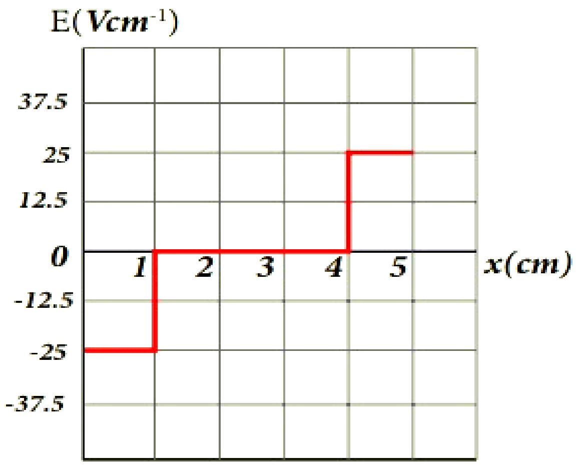
So $\vec{E} = -25\text{V cm}^{-1}\ \hat{i}$

From 1 to 4 cm, the potential is constant, V = 25 V. It implies that dV/dx = 0. So $\vec{E}$ = 0

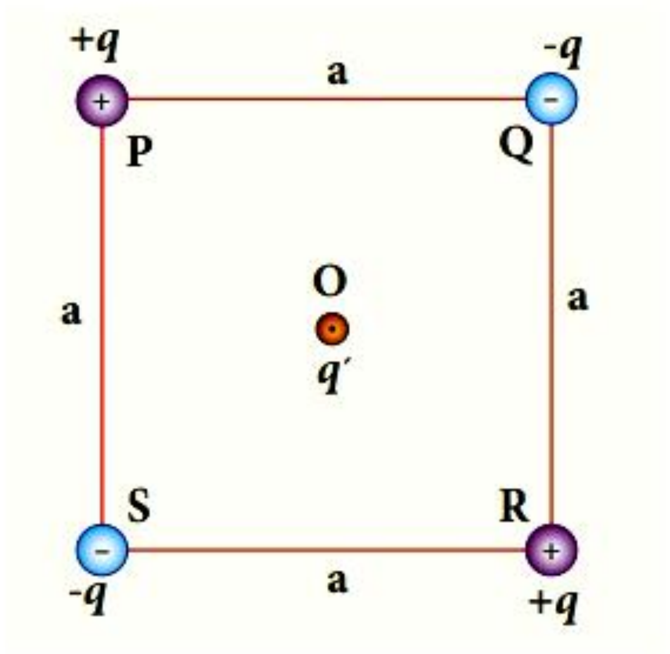
From 4 to 5 cm, the slope $dV/dx = -25V\ cm^{-1}$.

So $\vec{E}$ = $+25V\ cm^{-1}\ \hat{i}$

The plot of electric field for the various points along the x axis is given below.



Q.20 Four charges are arranged at the corners of the square PQRS of side a as shown in the figure.(a) Find the work required to assemble these charges in the given configuration. (b) Suppose a charge q′ is brought to the center of the square, by keeping the four charges fixed at the corners, how much extra work is required for this?



Solution:

(a) The work done to arrange the charges in the corners of the square is independent of the way they are arranged. We can follow any order.

(i) First, the charge +q is brought to the corner P. This requires no work since no charge is already present, $W_P = 0$

(ii) Work required to bring the charge –q to the corner Q = (-q) x potential at a point Q due to +q located at a point P.

$W_Q = -q \times (1/4\pi\varepsilon).\ q/a$

$= -(1/4\pi\varepsilon).\ q^2/a$

$$W_Q = -q \times \frac{1}{4\pi\varepsilon_\circ}\frac{q}{a} = -\frac{1}{4\pi\varepsilon_\circ}\frac{q^2}{a}$$

(iii) Work required to bring the charge +q to the corner R= q × potential at the point R due to charges at the point P and Q.

$$W_R = q \times \frac{1}{4\pi\varepsilon_\circ}\left(-\frac{q}{a}+\frac{q}{\sqrt{2}a}\right)$$
$$= \frac{1}{4\pi\varepsilon_\circ}\frac{q^2}{a}\left(-1+\frac{1}{\sqrt{2}}\right)$$

(iv) Work required to bring the fourth charge –q at the position S = q × potential at the point S due the all the three charges at the point P, Q and R

$$W_S = -q \times \frac{1}{4\pi\varepsilon_\circ}\left(\frac{q}{a}+\frac{q}{a}-\frac{q}{\sqrt{2}a}\right)$$
$$W_S = -\frac{1}{4\pi\varepsilon_\circ}\frac{q}{a}\left(2-\frac{1}{\sqrt{2}}\right)$$

(b) Work required to bring the charge q′ to the center of the square = q′ × potential at the center point O due to all the four charges in the four corners

The potential created by the two +q charges are canceled by the potential created by the -q charges which are located in the opposite corners. Therefore the net electric potential at the center O due to all the charges in the corners is zero.

Hence no work is required to bring any charge to the point O. Physically this implies that if any charge q′ when brought close to O, then it moves to the point O without any external force.

Q.21 A water molecule has an electric dipole moment of 6.3×10^{-30} Cm. A sample contains 10^{22} water molecules, with all the dipole moments aligned parallel to the external electric field of magnitude 3×10^{5} N C^{-1}. How much work is required to rotate all the water molecules from $\theta = 0^o$ to 90^o?

Solution:

When the water molecules are aligned in the direction of the electric field, it has minimum potential energy. The work done to rotate the dipole from $\theta = 0^o$ to 90^o is equal to the potential energy difference between these two configurations.

$W = \Delta U = U(90^o) - U(0^o)$

From the equation (1.51), we write $U = -pE\cos\theta$, Next we calculate the work done to rotate one water molecule from $\theta = 0^o$ to 90^o.

For one water molecule

$W = -pE\cos 90^o + pE\cos 0^o = pE$

$W = 6.3 \times 10^{-30} \times 3 \times 10^{5} = 18.9 \times 10^{-25}$ J

For 10^{22} water molecules, the total work done is

$W_{tot} = 18.9 \times 10^{-25} \times 10^{22} = 18.9 \times 10^{-3}$ J

Summary:

1) Electric potential energy is the energy that is needed to move a charge against an electric field. You need more energy to move a charge further in the electric field, but also more energy to move it through a stronger electric field.

2) Electric potential energy Ue is the potential energy stored when charges are out of equilibrium (like gravitational potential energy). Electric potential is the same, but per charge, Ueq.

3) Electrostatic potential energy is a potential energy (measured in joules) that results from conservative Coulomb forces and is associated with the configuration of a particular set of point charges within a defined system.

4) Electric field, an electric property associated with each point in space when charge is present in any form. The magnitude and direction of the electric field are expressed by the value of E, called electric field strength or electric field intensity or simply the electric field.

5) The SI unit of the electric field is volts per meter (V/m). This unit is equivalent to Newton's per coulomb. These are derived units where Newton is a unit of force and Coulomb is the unit of charge.

6) The potential gradient is the potential difference per unit length. The SI unit of the potential gradient can be determined by substituting the unit of potential difference or voltage and length. Therefore, the unit of potential difference is volt/meter.

7) The relationship between potential and field (E) is a differential: electric field is the gradient of potential (V) in the x direction. This can be represented as: Ex=−dVdx. Thus, as the test charge is moved in the x direction, the rate of the change in potential is the value of the electric field.

8) Gauss' Law tells us that the electric field outside the sphere is the same as that from a point charge. This implies that outside the sphere the potential also looks like the potential from a point charge. What about inside the sphere? If the sphere is a conductor we know the field inside the sphere is zero.

9) The electric potential inside a charged spherical conductor of radius R is given by V = ke Q/R, and the potential outside is given by V = ke Q/r. Using Er = -dv/dr, derive the electric field inside and outside this charge distribution.

10) The electric field inside a hollow metallic sphere is zero. So the work done is also zero. Suppose Va is the potential on the inside and Vb is the potential on the surface, then Vb - Va = 0 or Vb = Va. Hence the potential is the same inside as on the surface. Due to the solid sphere, the gravitational potential is the same within the sphere.

11) The electric dipole moment is a measure of the separation of positive and negative electrical charges within a system, that is, a measure of the system's overall polarity. The dipole is represented by a vector from the negative charge towards the positive charge.

12) Electric dipole moment is defined as the product of charge and the distance between the charges, and is directed from negative to positive charge. The SI unit of electric dipole moment is coulomb meter (Cm).

13) When a dipole is placed in a uniform electric field and dipole vector direction is not parallel to field direction, each charges of dipole experiences a force. ... Once the dipole is aligned to electric field, the net force will be zero because they are in opposite direction.

14) If an electric dipole is placed in a non-uniform electric field, then the positive and the negative charges of the dipole will experience a net force. And as one end of the dipole is experiencing a force in one direction and the other end in the opposite direction, so the dipole will have a net torque also.

15) An electric dipole consists of two equal and opposite charges +q and -q separated by a small distance a. The Electric Dipole Moment P is defined as a vector of magnitude qa with a direction from the negative charge to the positive charge. In many molecules, though the net charge is zero, the nature of chemical bonds is such that the positive and negative charges do not cancel at every point.

16) The forces the dipole exerts on each other are equal and opposite. Why isn't the torques? Because we calculated the torques about different centers. If we refer both torques to the coordinate origin (i.e. the position of dipole 1).

Terminal Question:

1) Explain the Electric potential and electrostatic potential energy in detail.
2) What do you mean by Electric fields and potential gradient?
3) Derive the relationship between Electric fields and potential gradient.
4) Explain the concept the Electrostatic self energy using conducting and dielectric sphere.
5) Explain and derive the expression for Electric potential due to spherical charge distribution using hollow and also its graphical representation.
6) Explain and derive the expression for Electric potential due to spherical charge distribution using solid and also its graphical representation.
7) What do you mean by Electric dipole and also its behavior in uniform and non uniform electric field?
8) Derive the expression for Electric field and potential due to electric dipole at a point in Cartesian and polar coordinates.
9) Derive the expression in detail for Force between two electric dipoles.
10) An electron is accelerated from rest through a potential difference 18 V. What is the change in electric potential energy of the electron?
11) Two point charges are separated by a distance of 10 cm. Charge on point A =+14 μC and charge on point B = -6 μC. k = 9 x

10^9 Nm^2C^{-2}, 1 $\mu C = 10^{-6}$ C. What is the change in electric potential energy of charge on point B if accelerated to point A?

12) A potentiometer wire has a length of 4m and a resistance of 12 ohm. It is connected in series with a cell of e.m.f. 8V and internal resistance 10 ohm. Find the potential gradient on the wire. Find also where a cell of e.m.f. 2V will balance on the wire.

13) Consider a point charge +q placed at the origin and another point charge -4q placed at a distance of 12 m from the charge +q. Determine the point between the two charges at which electric potential is zero.

Unit 03- Dielectrics

Structure:

3.1 Introduction

3.2 Objective

3.3 Capacitor and its capacity, principle of capacitor, energy stored in field of capacitor.

3.4 Capacity of partially filled parallel plate capacitor, expression for induced charge.

3.5 Effect of dielectrics slab introduced inside plates of charged capacitor when its remains connected with battery and when it is disconnected from battery.

3.6 Spherical plate's capacitor and cylindrical plates capacitor.

3.7 Change in electrical properties when N small charged drops coalesce to form a large drop.

3.8 Three electric vectors (**D, E, P**), dielectric constant, dielectric strength, electrical susceptibility.

3.9 Polarization, surface and volume charge density, Gauss law in dielectrics.

3.1 Introduction:

The capacitor is a component which has the ability or "capacity" to store energy in the form of an electrical charge producing a potential difference (*Static Voltage*) across its plates, much like a small rechargeable battery.

$$c \propto \frac{\epsilon A}{d}$$

A capacitor is a device that is used to store charges in an electrical circuit. A capacitor works on the principle that the capacitance of a conductor increases appreciably when an earthed conductor is brought near it. Hence, a capacitor has two plates separated by a distance having equal and opposite charges.

Energy stored in a capacitor is electrical potential energy, and it is thus related to the charge Q and voltage V on the capacitor. Thus the energy stored in a capacitor, E_{cap}, is Ecap=QV2 E cap = Q V 2 , where Q is the charge on a capacitor with a voltage V applied.

A capacitor is charged by connecting it to a power supply. Then the connections to the power supply are removed, and a piece of dielectric is inserted between the plates. ... The charge on the plates stays the same, while the potential difference decreases.

If the wires to the battery are disconnected, the charge remains on the plates -- and the voltage across the plates remains the same. If the wires are connected to each other, current will flow and the capacitor will discharge. Then there will be no voltage across the capacitor nor any charge on the plates.

A spherical capacitor consists of a hollow or a solid spherical conductor surrounded by another concentric hollow spherical conductor. The capacitance of a spherical capacitor is derived as : By Gauss law charge enclosed by gaussian sphere of radius r be. $q=\epsilon_o EA = \epsilon_o E(4\pi r^2)$.

The capacitor is used to store large amounts of electric current in a small space. It is often used to store the electric charge. ... The Cylindrical capacitor is a type of capacitor that possess the shape of a cylinder having an inner radius as a and outer radius as b.

The relationship among the three vectors D, E, P in the metre-kilogram-second (mks) or SI system is: $D = \varepsilon_0 E + P$ (ε_0 is a constant, the permittivity of a vacuum).

The *dielectric constant* is a measure of the amount of electric potential energy, in the form of induced polarization that is stored in a given volume of material under the action of an electric field. It is expressed as the ratio of the *dielectric permittivity* of the material to that of a vacuum or dry air.

Dielectric constant, also called relative permittivity or specific inductive capacity, property of an electrical insulating material (a dielectric) equal to the ratio of the capacitance of a capacitor filled with the given material to the capacitance of an identical capacitor in a vacuum without the dielectric material. The insertion of a dielectric between the plates of, say, a parallel-plate capacitor always increases its capacitance, or ability to store opposite charges on each plate, compared with this ability when the plates are separated by a vacuum.

Electric susceptibility, quantitative measure of the extent to which an electric field applied to a dielectric material causes polarization, the slight displacement of positive and negative charge within the material.

Polarization occurs when an electric field distorts the negative cloud of electrons around positive atomic nuclei in a direction opposite the field. This slight separation of charge makes one side of the atom somewhat positive and the opposite side somewhat negative.

According to electromagnetism, charge density is defined as a measure of electric charge per unit volume of the space in one, two, or three dimensions. To be specific, the linear surface or volume charge density is the amount of electric charge per surface area or volume, respectively.

Integral form ("big picture") of Gauss's law: The flux of electric field out of a closed surface is proportional to the charge it encloses. The above is Gauss's law in free space (vacuum). For a dielectric, just replace ε_0. With ε = ε.

we will describe microscopic picture of a dieiecnic in which We will define the local field (El,), arid the average macmscopip field inside the dielectric (Ei). Further, we will derive the relationship between the local field and the macroscopic field.

The Clausius–Mossotti relation expresses the dielectric constant (relative permittivity, ε_r) of a material in terms of the atomic polarizability, α, of the material's constituent atoms and/or molecules, or a homogeneous mixture thereof. It is named after Ottaviano-Fabrizio Mossotti and Rudolf Clausius. It is equivalent to the Lorentz–Lorenz equation.

3.2 Objective:

After studying this unit you should be able to

- Study and identify Capacitor and its capacity, principle of capacitor, energy stored in field of capacitor.
- Explain and identify Capacity of partially filled parallel plate capacitor, expression for induced charge.
- Study and identify Effect of dielectrics slab introduced inside plates of charged capacitor when its remains connected with battery and when it is disconnected from battery.

- Study and identify Spherical plates capacitor and cylindrical plates capacitor.
- Explain and identify Change in electrical properties when N small charged drops coalesce to form a large drop.
- Study and identify three electric vectors (**D, E, P**), dielectric constant, dielectric strength, electrical susceptibility.
- Study and identify Polarization, surface and volume charge density, Gauss law in dielectrics.
- Explain and identify Macroscopic and microscopic properties of dielectrics. Clausius – Mossotte formula.

3.3 Capacitor and its capacity, principle of capacitor, energy stored in field of capacitor.

Capacitor and its capacity:

The capacitor is a component which has the ability or "capacity" to store energy in the form of an electrical charge producing a potential difference (*Static Voltage*) across its plates, much like a small rechargeable battery.

There are many different kinds of capacitors available from very small capacitor beads used in resonance circuits to large power factor correction capacitors, but they all do the same thing, they store charge.

In its basic form, a capacitor consists of two or more parallel conductive (metal) plates which are not connected or touching each other, but are electrically separated either by air or by some form of a good insulating material such as waxed paper, mica, ceramic, plastic or some form of a liquid gel as used in electrolytic capacitors. The insulating layer between a capacitors plates is commonly called the **Dielectric**.

Fig.3.1 Capacitor

Due to this insulating layer, DC current can not flow through the capacitor as it blocks it allowing instead a voltage to be present across the plates in the form of an electrical charge.

The conductive metal plates of a capacitor can be either square, circular or rectangular, or they can be of a cylindrical or spherical shape with the general shape, size and construction of a parallel plate capacitor depending on its application and voltage rating.

When used in a direct current or DC circuit, a capacitor charges up to its supply voltage but blocks the flow of current through it because the dielectric of a capacitor is non-conductive and basically an insulator. However, when a capacitor is connected to an alternating current or AC circuit, the flow of the current appears to pass straight through the capacitor with little or no resistance.

There are two types of electrical charge, a positive charge in the form of Protons and a negative charge in the form of Electrons. When a DC voltage is placed across a capacitor, the positive (+ve) charge quickly accumulates on one plate while a corresponding and opposite negative (-ve) charge accumulates on the other plate. For every particle of +ve charge that arrives at one plate a charge of the same sign will depart from the -ve plate.

Then the plates remain charge neutral and a potential difference due to this charge is established between the two plates. Once the capacitor reaches its steady state condition an electrical current is unable to flow through the capacitor itself and around the circuit due to the insulating properties of the dielectric used to separate the plates.

The flow of electrons onto the plates is known as the capacitors **Charging Current** which continues to flow until the voltage across both plates (and hence the capacitor) is equal to the applied voltage Vc. At this point the capacitor is said to be “fully charged” with electrons.

The strength or rate of this charging current is at its maximum value when the plates are fully discharged (initial condition) and slowly reduces in value to zero as the plates charge up to a potential difference across the capacitors plates equal to the source voltage.

The amount of potential difference present across the capacitor depends upon how much charge was deposited onto the plates by the work being done by the source voltage and also by how much capacitance the capacitor has and this is illustrated below.

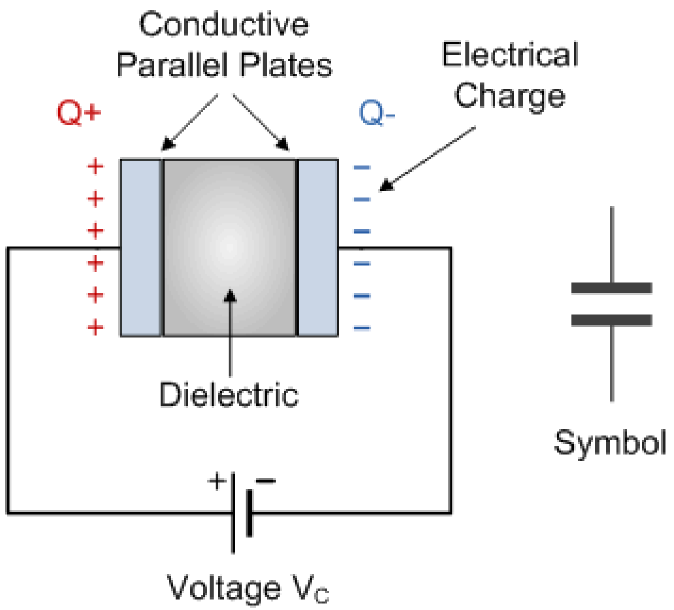


Fig.3.2 Parallel Plates Capacitor and symbol

The parallel plate capacitor is the simplest form of capacitor. It can be constructed using two metal or metalized foil plates at a distance parallel to each other, with its capacitance value in Farads, being fixed by the surface area of the conductive plates and the distance of separation between them. Altering any two of these values alters the the value of its capacitance and this forms the basis of operation of the variable capacitors.

Also, because capacitors store the energy of the electrons in the form of an electrical charge on the plates the larger the plates and/or smaller their separation the greater will be the charge that the capacitor holds for any given voltage across its plates. In other words, larger plates, smaller distance, more capacitance.

By applying a voltage to a capacitor and measuring the charge on the plates, the ratio of the charge Q to the voltage V will give the capacitance value of the capacitor and is therefore given as:

$$C = Q/V$$

this equation can also be re-arranged to give the familiar formula for the quantity of charge on the plates as:

$$Q = C \times V$$

Although we have said that the charge is stored on the plates of a capacitor, it is more exact to say that the energy within the charge is stored in an “electrostatic field” between the two plates. When an electric current flows into the capacitor, it charges up, so the electrostatic field becomes much stronger as it stores more energy between the plates.

Likewise, as the current flowing out of the capacitor, discharging it, the potential difference between the two plates decreases and the electrostatic field decreases as the energy moves out of the plates.

The property of a capacitor to store charge on its plates in the form of an electrostatic field is called the Capacitance of the capacitor. Not only that, but capacitance is also the property of a capacitor which resists the change of voltage across it.

The Capacitance of a Capacitor:

Capacitance is the electrical property of a capacitor and is the measure of a capacitors ability to store an electrical charge onto its two plates with the unit of capacitance being the **Farad** (abbreviated to F) named after the British physicist Michael Faraday.

Capacitance is defined as being that a capacitor has the capacitance of **One Farad** when a charge of **One Coulomb** is stored on the plates by a voltage of **One volt**. Note that capacitance, C is always positive in value and has no negative units. However, the Farad is a very large unit of measurement to use on its own so sub-multiples of the Farad are generally used such as micro-farads, nano-farads and pico-farads, for example.

Standard Units of Capacitance:

- Microfarad (μF) 1μF = 1/1,000,000 = 0.000001 = 10^{-6} F
- Nanofarad (nF) 1nF = 1/1,000,000,000 = 0.000000001 = 10^{-9} F
- Picofarad (pF) 1pF = 1/1,000,000,000,000 = 0.000000000001 = 10^{-12} F

Then using the information above we can construct a simple table to help us convert between pico-Farad (pF), to nano-Farad (nF), to micro-Farad (μF) and to Farads (F) as shown.

Pico-Farad (pF)	Nano-Farad (nF)	Micro-Farad (μF)	Farads (F)
1,000	1.0	0.001	
10,000	10.0	0.01	
1,000,000	1,000	1.0	
	10,000	10.0	
	100,000	100	
	1,000,000	1,000	0.001
		10,000	0.01

		100,000	0.1
		1,000,000	1.0

Working Principle of a Capacitor:

To demonstrate how a capacitor works, let us consider a most basic structure of a capacitor. It is made of two parallel conducting plates separated by a dielectric that is parallel plate capacitor. When we connect a battery (DC Voltage Source) across the capacitor, one plate (plate-I) gets attached to the positive end, and another plate (plate-II) to the negative end of the battery. Now, the potential of that battery is applied across that capacitor. At that situation, plate-I is in positive potency with respect to the plate-II. At steady state condition, the current from the battery tries to flow through this capacitor from its positive plate (plate-I) to negative plate (plate-II) but cannot flow due to the separation of these plates with an insulating material.

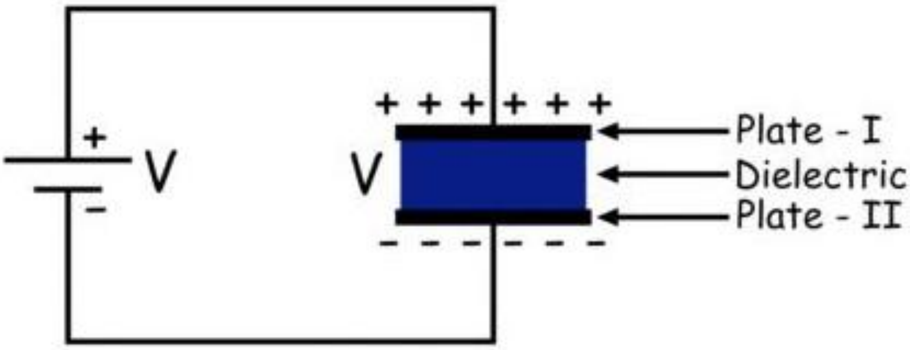


Fig.3.3 Parallel Plate Capacitor with dielectric

An electric field appears across the capacitor. As time goes on, positive plate (plate I) will accumulate positive charge from the battery, and negative plate (plate II) will accumulate negative charge from the battery. After a certain time, the capacitor holds maximum amount of charge as per its capacitance with respect to this voltage. This time span is called charging time of this capacitor.

After removing this battery from this capacitor, these two plates hold positive and negative charge for a certain time. Thus this capacitor acts as a source electrical energy.

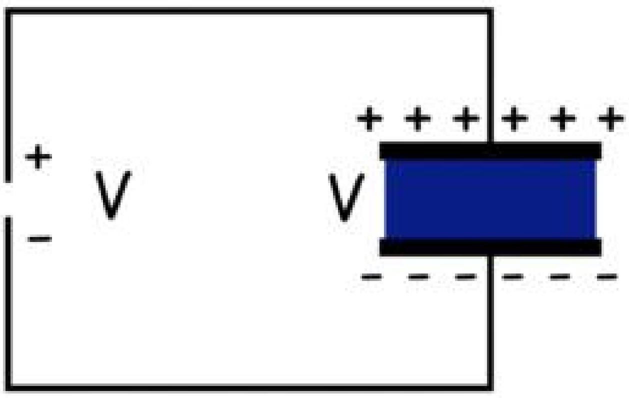


Fig.3.4 Capacitor acts as a source electrical energy

If two ends (plate I and plate II) are connected to a load, a current will flow through this load from plate-I to plate-II until all charges get vanished from both plates. This time span is known as discharging time of the capacitor.

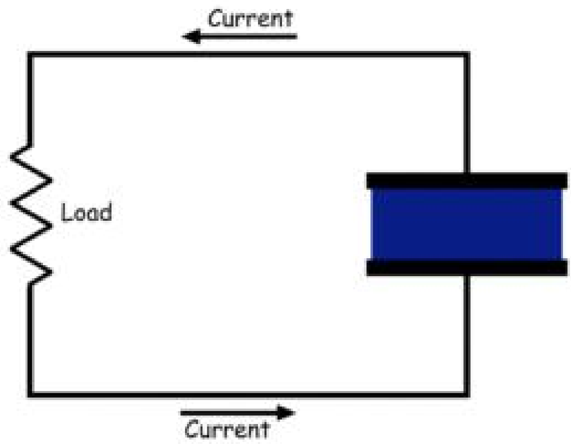


Fig.3.5 Discharging time of the capacitor.

Capacitor in a DC Circuit:

Suppose a capacitor is connected across a battery through a switch.

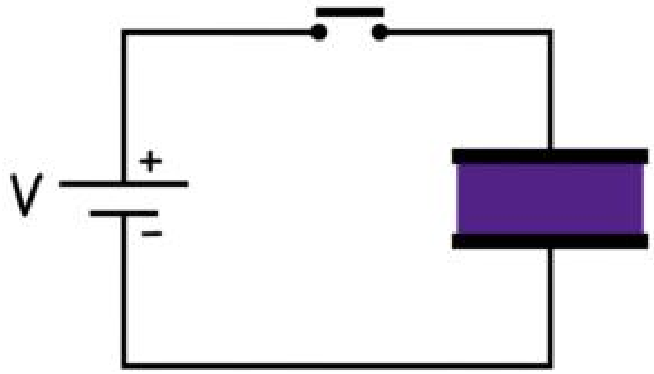


Fig.3.6 Capacitor in a DC Circuit

When the switch is ON, i.e., at t = $^{+}0$, a current will start flowing through this capacitor. After a certain time (i.e. charging time) capacitor never allow current to flow through it further. It is because of the maximum charges is accumulated on both plates and capacitor acts as a source which has a positive end connected to the positive end of the battery and has a negative end connected to the negative end of the battery with the same potency.

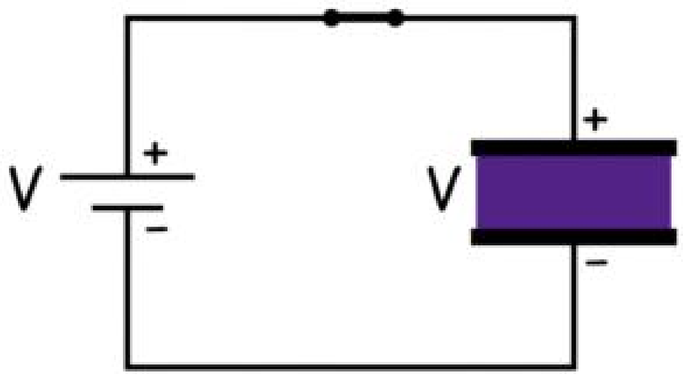
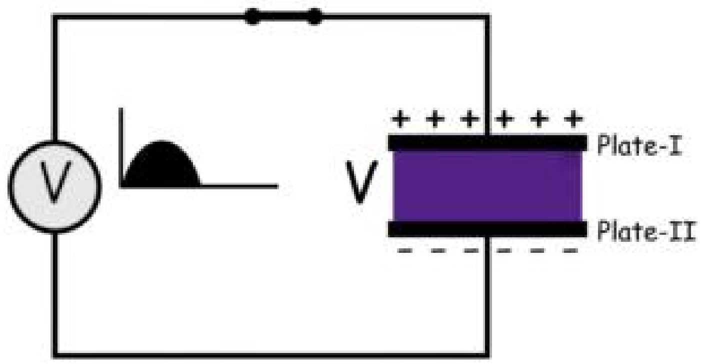


Fig.3.7 Capacitor connected battery with the same potency

Due to zero potential difference between battery and capacitor, no current will flow through it. So, it can be said that initially a capacitor is short-circuited and finally open circuited when it gets connected across a battery or DC source.

Capacitor in an AC Circuit:

Suppose a capacitor is connected across an AC source. Consider, at a certain moment of positive half of this alternating voltage, plate-I gets positive polarity and plate-II negative polarity. Just at that moment, plate-I accumulates positive charge and plate-II accumulates negative charge.



Fig,3.8 Capacitor in an AC Circuit

But at the negative half of this applied AC voltage, plate-I gets a negative charge and plate-II positive charge. There is no flow of electrons between these two plates due to dielectric placed between the plates but they change their polarity with the change of source polarity. The capacitor plates get charged and discharged alternatively by the AC.

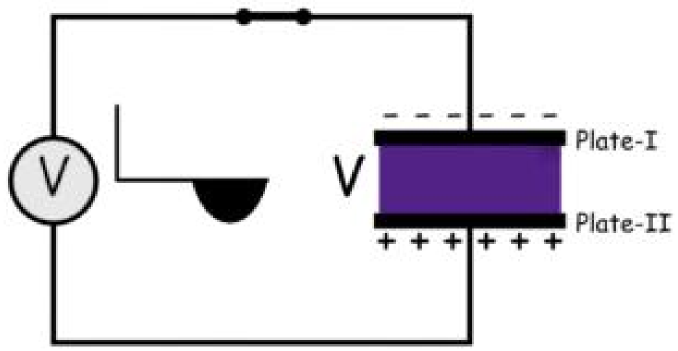


Fig.3.9 Capacitor plates get charged and discharged alternatively by the AC

Energy Stored in Capacitor:

While capacitor is connected across a battery, charges come from the battery and get stored in the capacitor plates. But this process of energy storing is step by step only. At the very beginning, capacitor does not have any charge or potential. i.e. V = 0 volts and q = 0 C.

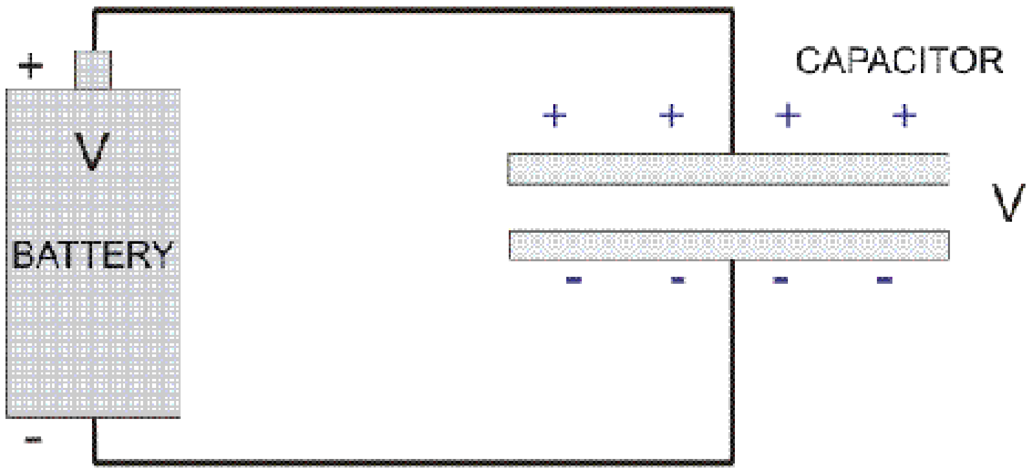


Fig.3.10 Energy Stored in Capacitor

Now at the time of switching, full battery voltage will fall across the capacitor. A positive charge (q) will come to the positive plate of the capacitor, but there is no work done for this first charge (q) to come to the positive plate of the capacitor from the battery. It is because of the capacitor does not have own voltage across its plates, rather the initial voltage is due to the battery. First charge grows little amount of voltage across the capacitor plates, and then second positive charge will come to the positive plate of the capacitor, but gets repealed by the first charge. As the battery voltage is more than the capacitor voltage then this second charge will be stored in the positive plate.

At that condition a little amount of work is to be done to store second charge in the capacitor. Again for the third charge, same phenomenon will appear. Gradually charges will come to be stored in the capacitor against pre-stored charges and their little amount of work done grows up.

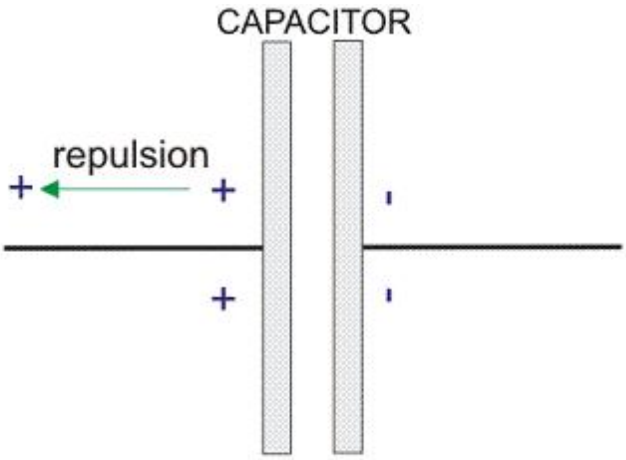


Fig.3.11 Capacitor against pre-stored charges

It can't be said that the capacitor voltage is fixed. It is because of the capacitor voltage is not fixed from the very beginning. It will be at its maximum limit when potency of capacitor will be equal to that of the battery.

As storage of charges increases, the voltage of the capacitor increases and also energy of the capacitor increases. So at that point of discussion the energy equation for the capacitor can't be written as energy

$$(E) = V.q$$

As the voltage increases the electric field (E) inside the capacitor dielectric increases gradually but in opposite direction i.e. from positive plate to negative plate.

$$E = -\frac{dV}{dx}$$

Here dx is the distance between two plates of the capacitor.

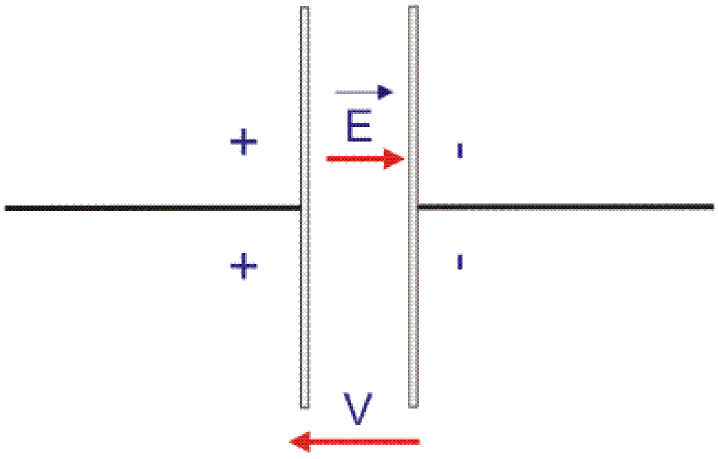


Fig.3.12 Two Plate capacitor

Charge will flow from battery to the capacitor plate until the capacitor gains as same potency as the battery. So, we have to calculate the energy of the capacitor from the very begging to the last moment of charge getting full. Suppose, a small charge q is stored in the positive plate of the capacitor with respect to the battery voltage V and a small work done is dW. Then considering the total charging time, we can write that,

$$\int_0^W dW = \int_0^Q V.dQ$$

$$W = \int_0^Q \frac{q}{C}.dq, \quad [as\ C = \frac{q}{V}] \quad Or,\ W = \frac{1}{2}.\frac{Q^2}{C} \quad Or,\ W = \frac{1}{2}.CV^2$$

Now we go for the energy loss during the charging time of a capacitor by a battery.

As the battery is in the fixed voltage the energy loss by the battery always follows the equation, W = V.q, this equation is not applicable for the capacitor as it does not have the fixed voltage from the very beginning of charging by the battery.

Now, the charge collected by the capacitor from the battery is

$$W_{cap} = \frac{1}{2}.CV^2 = \frac{1}{2}.Q.V.$$

Now charge lost by the battery is

$$W_{loss} = V.Q - \frac{1}{2}.Q.V = \frac{1}{2}.Q.V$$

This half energy from total amount of energy goes to the capacitor and rest half of energy automatically gets lost from the battery and it should be kept in mind always.

3.4 Capacity of partially filled parallel plate capacitor, expression for induced charge:

Capacity of partially filled parallel plate capacitor:

Capacity of a parallel plate condenser which is partially filled with a dielectric medium- By the distance between the plates is d and in between the plates there is a dielectric medium of thick-ness t and dielectric constant K. If each plate is given a charge +q then surface charge.

σ=A/q

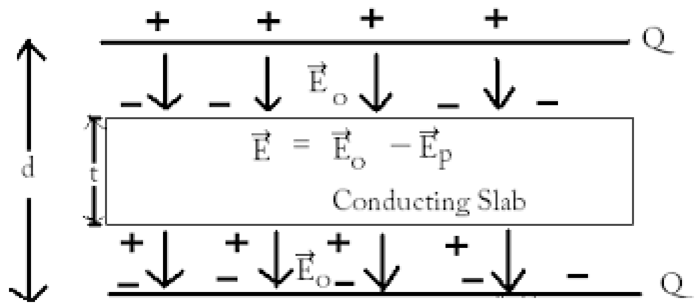


Fig.3.13 Parallel plate capacitor

Where A is the surface area of the plates. If the distance between the plates in negligible in Comparison to their area then the intensity of electric field in the area filled with air between the plates.

$$E_o = \frac{\sigma}{\varepsilon_o}$$

Electric field intensity within the dielectric medium is

$$E = \frac{\sigma}{K\varepsilon_o}$$

By the definition of potential difference, the potential difference between the plates.

V= Work done in moving a unit charge from one plate (negative) to another (positive) plate.

= Work done in moving a unit charge a distance (d−t) in air and distance t in dielectric medium

$$V = E_o \times (d-t) + Et$$

On substituting the values of Eo and E,

$$V = \frac{\sigma}{\varepsilon_o}(d-t) + \frac{\sigma}{K\varepsilon_o}t$$

$$\text{or } V = \frac{\sigma}{\varepsilon_o}\left[(d-t) + \frac{t}{K}\right]$$

$$\text{or } V = \frac{q}{A\varepsilon_o}\left[(d-t) + \frac{t}{K}\right]$$

Hence capacity of the condenser

$$C = \frac{q}{V}$$

$$C = \frac{\varepsilon_o A}{\left[(d-t) + \frac{t}{K}\right]}$$

Expression for Induced Charge:

$$E = \frac{E0}{K} \quad \text{(Q constant)}$$

Where:

E = field with the dielectric between plates

E_0 = field with vacuum between the plates

E is smaller when the dielectric is present → surface charge density smaller. The surface charge on conducting plates does not

change, but an induced charge of opposite sign appears on each surface of the dielectric neutral(only charge distribution)

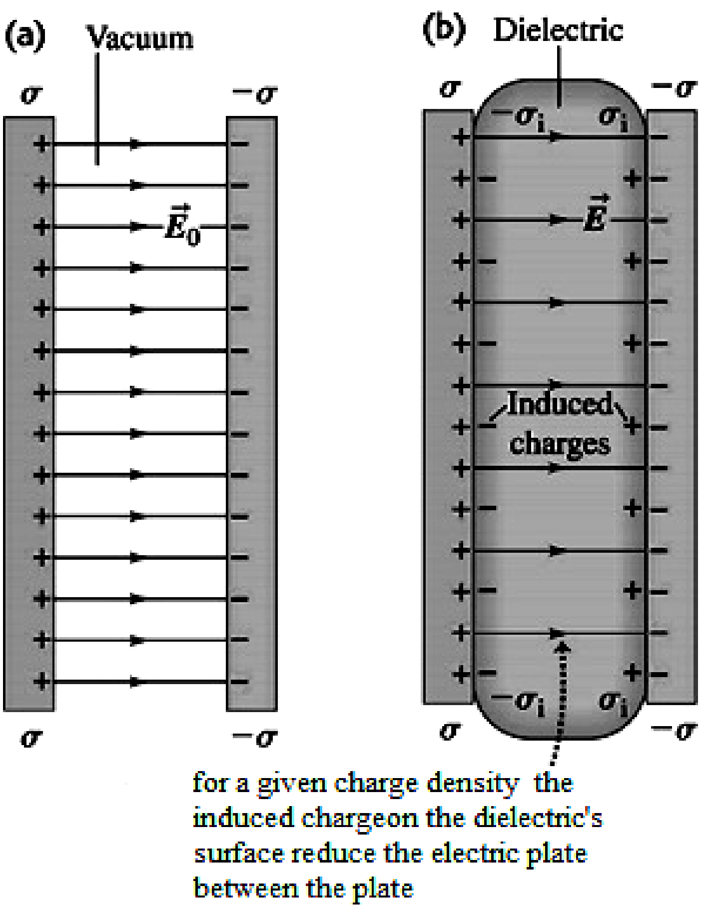


Fig.3.14 Induced charges in plates

$$E = E_0 - E_i$$

Where E_0 is electric field due to plate charge E_i is electric field due to induced charge on dielectric

$$\frac{E_0}{K} = E_0 - E_i$$

$$\frac{E_0}{K} = E_0 - E_i$$

$$E_i = E_0 \left(1 - \frac{1}{K}\right)$$

$$\frac{Q_i}{A\epsilon_0} = \frac{Q}{A\epsilon_0} \left(1 - \frac{1}{K}\right)$$

$$Q_i = Q \left(1 - \frac{1}{K}\right)$$

Where Q_i is the induced charge and Q is charge of capacitor before insertion of dielectric

3.5 Effect of dielectrics slab introduced inside plates of charged capacitor when its remains connected with battery and when it is disconnected from battery:

Effect of dielectrics slab introduced inside plates of charged capacitor when its remains connected with battery:

While a capacitor remains connected to a battery and dielectric slab is slipped between the plates, the potential difference between the plates remains uncharged. The introduction of dielectric slab increases the charge of capacitor which flows from the battery.

Dielectrics with Battery:

Consider a second case where a battery supplying a potential difference $|\Delta V_0|$ remains connected as the dielectric is inserted. Experimentally, it is found (first by Faraday) that the charge on the plates is increased by a factor K_e :

$$Q = \kappa_e Q_0$$

where Q_0 is the charge on the plates in the absence of any dielectric.

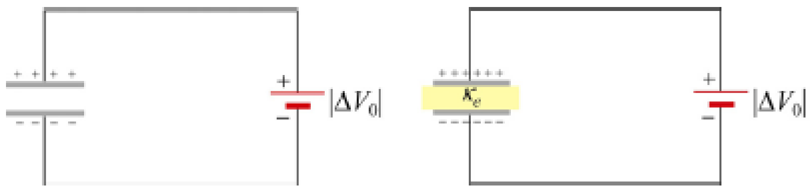


Fig.3.15. Inserting a dielectric material between the capacitor plates

While maintaining a constant potential difference $0|\Delta V_0|$

The capacitance becomes

$$\boxed{C = \frac{Q}{|\Delta V_0|} = \frac{\kappa_e Q_0}{|\Delta V_0|} = \kappa_e C_0}$$

which is the same as the first case where the charge Q_0 is kept constant, but now the charge has increased.

Effect of dielectrics slab introduced inside plates of charged capacitor when it is disconnected from battery:

If a dielectric slab of dielectric constant K is filled in between the plates of a capacitor after charging the capacitor (i.e., after removing the connection of battery with the plates of capacitor) the potential difference between the plates reduces to 1/Ktimes and the potential energy of capacitor reduces to 1/K. times but there is no change in the charge on the plates.

Dielectrics without Battery:

As shown in Figure, a battery with a potential difference $|\Delta V_0|$ across its terminals is first connected to a capacitor C_0, which holds a charge $Q_0 = C_0 |\Delta V_0|$. We then disconnect the battery, leaving Q0=const.

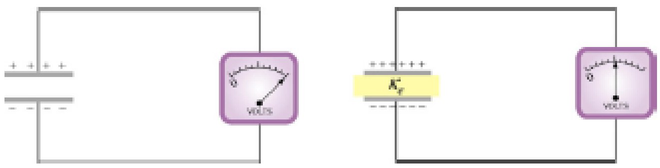


Fig.3.16 Dielectrics without Battery

Inserting a dielectric material between the capacitor plates while keeping the charge Q_0 constant

If we then insert a dielectric between the plates, while keeping the charge constant, experimentally it is found that the potential difference decreases by a factor of K_e :

$$|\Delta V| = \frac{|\Delta V_0|}{\kappa_e}$$

This implies that the capacitance is changed to

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{Q_0}{|\Delta V_0|/\kappa_e} = \kappa_e \frac{Q_0}{|\Delta V_0|} = \kappa_e C_0}$$

Thus, we see that the capacitance has increased by a factor of K_e .The electric field within the dielectric is now

$$E = \frac{|\Delta V|}{d} = \frac{|\Delta V_0|/\kappa_e}{d} = \frac{1}{\kappa_e}\left(\frac{|\Delta V_0|}{d}\right) = \frac{E_0}{\kappa_e}$$

We see that in the presence of a dielectric, the electric field decreases by a factor of K_e.

SAQ 1:

a) What do you mean by Capacitor and its capacity?
b) Discuss the principle of capacitor.
c) What do you mean energy stored in field of capacitor?
d) What do you mean by induced charge?
e) What is the charge stored when the voltage across a 30 μF capacitor is 6 V?
f) What is the capacitance of a capacitor that stores 18 μC of charge when connected to a 9V battery?

3.6 Spherical plate's capacitor and cylindrical plate's capacitor:

Spherical plate's capacitor:

A spherical capacitor consists of a solid or hollow spherical conductor of radius a, surrounded by another hollow concentric spherical of radius b shown below in figure

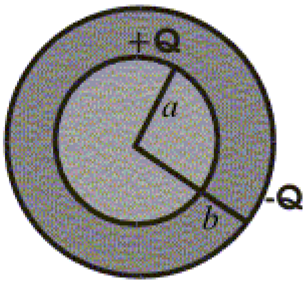


Fig.3.17 Spherical plates capacitor

- Let +Q be the charge given to the inner sphere and -Q be the charge given to the outer sphere.

- The field at any point between conductors is same as that of point charge Q at the origin and charge on outer shell does not contribute to the field inside it.

- Thus electric field between conductors is E=Q2πϵ0r2

$$E = \frac{Q}{2\pi\epsilon_0 r^2}$$

Potential difference between two conductors is

$$V = V_a - V_b$$
$$= -\int E.dr$$

Where limits of integration goes from a to b.

On integrating we get potential difference between to conductors as

$$V = \frac{Q(b-a)}{4\pi\epsilon_0 ba}$$

Now , capacitance of spherical conductor is

$$C = \frac{Q}{V}$$

or,

$$C = \frac{4\pi\epsilon_0 ba}{(b-a)} \cdot$$

again if radius of outer conductor approaches to infinity then from equation 6 we have

$$C = 4\pi\epsilon_0 a \cdot$$

- Above equation 2 gives the capacitance of single isolated sphere of radius a.
- Thus capacitance of isolated spherical conductor is proportional to its radius.

Spherical capacitor when inner sphere is earthed:

- If a positive charge of Q coulombs is given to the outer sphere B, it will distribute itself over both its inner and outer surfaces.
- Let the charges of Q1 and Q2 coulombs be at the inner and outer surfaces respectively of sphere B where Q=Q1+Q2,
- The charge + Q1 on the inner surface of outer sphere B will induce a charge of -Q1 coulombs on the outer surface of inner sphere A and

+Q1 coulombs on the inner surface of sphere A, which will go to earth.

- Now there are two capacitors connected in parallel.
    i. One capacitor consists outer surface of sphere B and earth having capacitance C1=4πϵ0b farads
    ii. Second capacitor consisting of inner surface of outer sphere B and the outer surface of inner sphere A having capacitance

$$C_2 = \frac{4\pi\epsilon_0 ba}{(b-a)}$$

Final Capacitance:

$$C = C_1 + C_2 = 4\pi\epsilon_0 b + \frac{4\pi\epsilon_0 ba}{(b-a)} = \frac{4\pi\epsilon_0 b^2}{(b-a)}$$

Cylindrical plate's capacitor:

- A cylindrical capacitor is made up of a conducting cylinder or wire of radius a surrounded by another concentric cylindrical shell of radius b (b>a).

- Let L be the length of both the cylinders and charge on inner cylinder is +Q and charge on outer cylinder is -Q.

- For calculate electric field between the conductors using Gauss's law consider a Gaussian surface of radius r and length L^1 as shown in below figure .

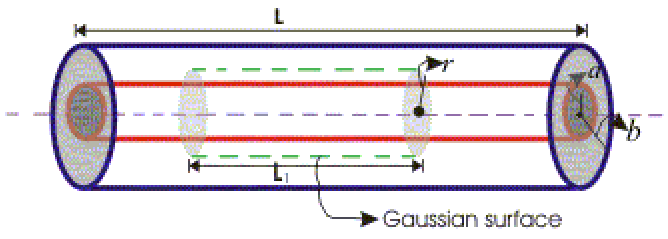


Fig.3.18 Cylindrical plates capacitor

- According to Gauss's law flux through this surface is q/ϵ0 where q is net charge inside this surface.
- We know that electric flux is given by

$$\begin{aligned}\phi &= E.A \\ &= EAcos\theta \\ &= EA\end{aligned}$$

Since electric field is constant in magnitude on the Gaussian surface and is perpendicular to this surface. Thus,

$$\phi=E(2\pi rL)$$

Since $\phi=q/\epsilon_0$

$$\Rightarrow E(2\pi rL) = \frac{\lambda L}{\epsilon_0}$$

Where $\lambda=Q/L$= charge per unit length

So,

$$E=\lambda/2\pi\epsilon_0 r$$

If potential at inner cylinder is V_a and V_b is potential of outer cylinder then potential difference between both the cylinders is

$$V=V_a \text{ and } V_b=\int Edr$$

Where limits of integration goes from a to b.

Potential of inner conductor is greater than that of outer conductor because inner cylinder carries positive charge. Thus potential difference is

$$V = \frac{Qln(b/a)}{2\pi\epsilon_0 L}$$

Thus capacitance of cylindrical capacitor is

$$C = \frac{Q}{V}$$
$$C = \frac{4\pi\varepsilon_o r_1 r_2}{r_1 - r_2}$$

- From the above equation it can easily be concluded that capacitance of a cylindrical capacitor depends on length of cylinders.
- More is the length of cylinders , more charge could be stored on the capacitor for a given potential difference

3.7 Change in electrical properties when N small charged drops coalesce to form a large drop:

Let r, q and v be the radius, charge the potential of a small drop.

The total charge on the bigger drop is the sum of all charge on small drop s

Q = Nq

The volume of N small drops = Nx 4/3 πr cube

And for the bigger drop = 4/3 πR cube

Hence,

4/3 πR cube = N x 4/3 π r cube

R = N raise to power 1/3 r

So the potential on bigger drop

V = 1/ 4π Ɛo Q/R

= 1/4πƐ0 Nq/N raise to power 1/3r

= N raise to power 2/3 x 1/ 4πƐo q/r

V = N raise to power 2/3 x v

And the capacitance C = 4πƐoR

C = 4πƐoN raise to power 1/3 r

= N raise to power 1/3 (4πƐor)

C = N raise to power 1/3c

3.8 Three electric vectors (D, E, P), dielectric constant, dielectric strength, electrical

Susceptibility:

Three electric vectors (D, E, P) (Or) Relation between polarization vector (P), displacement (D) and electric field (E):

Let us derive the relation between polarization vector (P), displacement (D) and electric field (E):

In the last article of polarization, we have discussed about the effect on dielectric placed in an external electric field E_0 and there will be electric field due to polarized charges, this field is called electric field due to polarization (E_p). (You can see the figure in that article).

Rewrite equation (1) of that article, that is:

$$E = E_0 – E_p \quad (1)$$

Polarization vector, P = P is equal to the bound charge per unit area or equal to the surface density of bound charges (because surface charge density is charge per unit area),

$$\text{Thus } P = q_b/A = \sigma_p \quad (2)$$

Where q_b is bound charge and σ_p is surface density of bound charges. P is also defined as the electric dipole moment of material per unit volume.

$$P = np$$

Where n is number of molecules per unit volume.

Displacement vector, D= D is equal to the free charge per unit area or equal to the surface density of free charges,

$$\text{Thus } D = q/A = \sigma \quad (3)$$

Where q is free charge and σ is surface density of free charges.

As for parallel plate capacitor (already derived in earlier articles):

$$E = \sigma / \varepsilon_0 \quad (4)$$

$$E_p = \sigma_p / \varepsilon_0 \quad (5)$$

By substituting equations 4 and 5 in equation 1, we get

$$E = \sigma / \varepsilon_0 - \sigma_p / \varepsilon_0$$

$$\text{Or } \varepsilon_0 E = \sigma - \sigma_0$$

By putting equations 2 and 3 in above equation, we get

$$\varepsilon_0 E = D - P$$

$$\text{or } D = \varepsilon_0 E + P$$

This is the relation between D, E and P.

Dielectric Constant:

The dielectric constant (Dk) of a plastic or dielectric or insulating material can be defined as the ratio of the charge stored in an insulating material placed between two metallic plates to the charge that can be stored when the insulating material is replaced by vacuum or air. It is also called as electric permittivity or simply permittivity.

And, at times referred as relative permittivity, because it is measured relatively from the permittivity of free space (ε_0).

Dielectric constant characterizes the ability of plastics to store electrical energy. Typical values of ε for dielectrics are:

Material	**Dielectric Constant (ε)**
Vacuum	**1.000**
Dry Air	1.0059
Foam Polyethylene	1.6
Fluoropolymers	2.0
Polypropylene	2.1
Butyl Rubber	2.3
SBR	2.9
Silicone Rubber	3.2
Plexiglass	3.4
PVC	4.0
Glass	3.8-14.5
Distilled Water	~80

A dielectric constant of 2 means an insulator will absorb twice more electrical charge than vacuum.

Applications include:

- Use of materials in the production of capacitors used in radios and other electrical equipment. Commonly used by circuit designers to compare different printed-circuit-board (PCB) materials.

- Development of materials for energy storage applications.

For example, polymer-based dielectric composites are highly desirable for applications ranging from electronic packaging, embedded capacitors, to energy storage. These composites are highly flexible with a low process temperature and they exhibit a relatively high dielectric constant, low dielectric loss, high dielectric strength.

Calculate Dielectric Constant:

In other words, dielectric constant can also be defined as the ratio of the capacitance induced by two metallic plates with an insulator between them, to the capacitance of the same plates with air or a vacuum between them.

An insulating material with higher dielectric constant is needed when it is to be used in E&E applications where high capacitance is needed.

If a material were to be used for strictly insulating purposes, it would be better to have a lower dielectric constant.

The dielectric constant formula is:

$$\varepsilon = \frac{C}{C_\circ}\ ,\quad C_\circ = \frac{\varepsilon_\circ A}{t}$$

Where:

- C = capacitance using the material as the dielectric capacitor
- C_0 = capacitance using vacuum as the dielectric
- ε_0 = Permittivity of free space (8.85×10^{-12} F/m i.e. Farad per metre)
- A = Area of the plate/sample cross section area
- T = Thickness of the sample

Dielectric Constant Units:

This electrical property is a dimension less measure.

The most generally used standard tests to calculate dielectric constant for plastics are ASTM D2520, ASTM D150 or IEC 60250 (of course there exist several other methods as well, but they are not discussed here).

The method includes:

A sample is placed between two metallic plates and capacitance is measured. A second run is measured without the specimen between the two electrodes. The ratio of these two values is the dielectric constant.

- The test can be conducted at different frequencies, often between the 10Hz and 2MHz range
- The sample must be flat and larger than the 50mm (2 in) circular electrodes used for the measurement.

Polar Plastics Vs Non-polar Plastics:

- Dielectric properties of a polymer largely depend upon their structure. The structure determines whether a polymer is polar or non-polar and this in turn decided the electrical properties of the polymer.
- In polar polymers (PMMA, PVC, Nylon, PC etc.), dipoles are created due to imbalance in the distribution of electrons. These dipoles tend to align in the presence of electric field. Hence, this creates dipole polarization of the material making these materials only moderately good as insulators.

- While non-polar polymers (PTFE, PP, PE, PS) have symmetrical molecules and are truly covalent. There are no polar dipoles present in them and hence in presence of electric field does not align the dipoles. However, slight electron polarization occurs due to the movement of electrons in the direction of electric field, which is effectively instantaneous. These polymers have high resistivities and low dielectric constant.

- Polar plastics have a tendency to absorb moisture from the atmosphere. Presence of moisture raises the dielectric constant and lowers the resistivity. With rise in temperature, there is faster movement of polymer chains and fast alignment of dipoles. This invariably raises the dielectric constant values for polar plastics.
- Non-polar plastics are not affected by moisture and rise in temperature.

Factors Influencing Dielectric Constant:

- Frequency - Dielectric constant decreases abruptly as frequency increases
- Moisture &Temperature
- Voltage
- Structure & morphology (see polar plastics vs non-polar plastics)
- Presence of other materials in the plastic
- Weathering and Deterioration

Dielectric Strength:

Dielectric Strength reflects the electric strength of insulating materials at various power frequencies. Or it can be defined as the measure of dielectric breakdown resistance of a material under an applied voltage and is expressed as Volts per unit thickness. It is an indicator of how good an insulator a material is.

In other words, it is the voltage per unit thickness at which a material will conduct electricity. The higher the value, the more electrically insulating a material is.

It is an important property sought for materials used in applications where electrical field is present and is a vital parameter for electrical industry applications.

Applications of Dielectric Strength:

- Development of materials for energy storage applications
- Dielectric materials for capacitors
- Thin films in high speed digital circuitry

The dielectric strength depends on:

- The type of the plastic and electrodes
- The shape of the plastic and electrodes
- The rate with which the field is increased, and
- The medium that surrounds the insulator

Unit for Dielectric Strength is kV by mm of thickness (customary units sometimes refer to in V/mil)

Measure Dielectric Strength:

The most generally used standard tests to calculate dielectric strength are ASTM D149 or IEC 60243-1 (ofcourse there exist several other methods as well, but they are not discussed here). The measurement of dielectric strength is usually carried out either by the:

- Short-time method
- Slow rate-of-rise method
- Step-by-Step method

Short-time method

In this method, the voltage is applied across the two electrodes and increased continuously at a uniform rate (500 V/sec) until the breakdown occurs. Breakdown is defined as when an electrical burn- through punctures the sample or decomposition occurs in the sample.

Slow rate-of-rise method:

In this test method, the voltage is applied to the test electrodes from the starting voltage 50% of the breakdown voltage until breakdown occurs.

Step-by-Step method:

The voltage is applied to the test electrodes at the preferred starting voltage in steps and duration until breakdown occurs.

Specimen Size - The recommended specimen type is a 4 inch plaque or larger. Any specimen thickness can be used.

Dielectric strength is calculated by dividing the breakdown voltage by the thickness of the sample.
Most plastics have good dielectric strengths (in the order of 100 to 300 kV/cm)

Factors Affecting Dielectric Strength:

- **The dielectric strength of an insulation material usually decreases with increase in temperature:**
  It is approximately inversely proportional to the absolute temperature. At the same time, it is equally important to note that below room temperature, dielectric strength is substantially independent of temperature change.

- **Mechanical loading has a pronounced effect on dielectric strength:**
  Since, a mechanical stress may introduce internal flaws which serve as leakage paths, mechanical loaded insulators may show substantially reduced values of dielectric strength.
- **Dielectric Strength of an insulating material is influenced by the fabrication details:**
  For example, flow lines in a compression molding or weld lines in an injection molding may serve as paths of least resistance of leakage currents, this reducing the dielectric strength. Even nearly invisible minute flaws in a plastics insulator may reduce the dielectric strength to one-third this normal value.

Electric susceptibility:

Is the quantitative measure of the extent to which an electric field applied to a dielectric material causes polarization, the slight displacement of positive and negative charge within the material. For most linear dielectric materials, the polarization P is directly proportional to the average electric field strength E so that the ratio of the two, P/E, is a constant that expresses an intrinsic property of the material. The electric susceptibility, χ_e, in the centimetre-gram-second (cgs) system, is defined by this ratio; that is, $\chi_e = P/E$. In the metre-kilogram-second (mks) system, electric susceptibility is defined slightly differently by including the constant permittivity of a vacuum, ε_0, in the expression; that is, $\chi_e = P/(\varepsilon_0 E)$. In both systems the electric susceptibility is always a dimensionless positive number. Because of the slight difference in definition, the value of the electric susceptibility of a given material in the mks system is 4π times its value in the cgs system.

SAQ 2:

a) What do you mean by spherical plate's capacitor?

b) Define the cylindrical plate's capacitor.

c) Write are the applications of Dielectric Constant?

d) What do you mean by Dielectric Strength?

e) What is the unit of Dielectric Strength?

f) Define the Electric susceptibility.

g) A Cylindrical capacitor having a length of 10 cm is made of two concentric rings with an inner radius as 4 cm and outer radius as 8 cm. find the capacitance of the capacitor.

3.9 Polarization, surface and volume charge density, Gauss law in dielectrics:

Polarization:

We have shown that dielectric materials consist of many permanent or induced electric dipoles. One of the concepts crucial to the understanding of dielectric materials is the average electric field produced by many little electric dipoles which are all aligned. Suppose we have a piece of material in the form of a cylinder with area A and height h, as shown in Figure 5.5.3, and that it consists of N electric dipoles, each with electric dipole moment $\vec{p}$ spread uniformly throughout the volume of the cylinder.

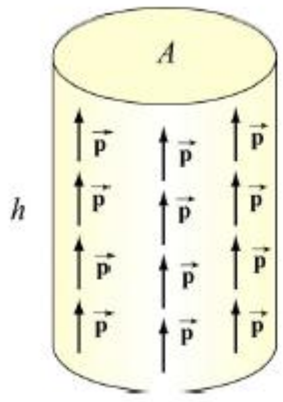


Fig.3.19 A cylinder with uniform dipole distribution

We furthermore assume for the moment that all of the electric dipole moments $\vec{p}$ are aligned with the axis of the cylinder. Since each electric dipole has its own electric field associated with it, in the absence of any external electric field, if we average over all the individual fields produced by the dipole, what is the average electric field just due to the presence of the aligned dipoles.

To answer this question, let us define the polarization vector $\vec{p}$ to be the net electric dipole moment vector per unit volume:

$$\vec{\mathbf{P}} = \frac{1}{\text{Volume}} \sum_{i=1}^{N} \vec{\mathbf{p}}_i$$

In the case of our cylinder, where all the dipoles are perfectly aligned, the magnitude of $\vec{p}$ is equal to

$$P = \frac{Np}{Ah}$$

and the direction of $\vec{p}$ is parallel to the aligned dipoles

Now, what is the average electric field these dipoles produce? The key to figuring this out is realizing that the situation shown in Figure 5.5.4(a) is equivalent that shown in Figure (b), where all the little ± charges associated with the electric dipoles in the interior of the cylinder are replaced with two equivalent charges, $\pm Q_p$, on the top and bottom of the cylinder, respectively.

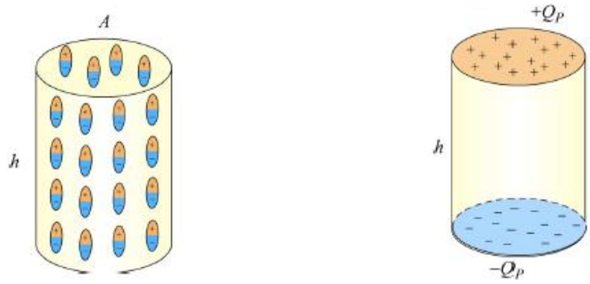


Fig.3.20 (a) A cylinder with uniform dipole distribution.

(b) Equivalent charge distribution.

The equivalence can be seen by noting that in the interior of the cylinder, positive charge at the top of any one of the electric dipoles is canceled on average by the negative charge of the dipole just above it. The only place where cancellation does not take place is for electric dipoles at the top of the cylinder, since there are no adjacent dipoles further up. Thus the interior of the cylinder appears uncharged in an average sense (averaging over many dipoles), whereas the top surface of the cylinder appears to carry a net positive charge. Similarly, the bottom surface of the cylinder will appear to carry a net negative charge

How do we find an expression for the equivalent charge QP in terms of quantities we know? The simplest way is to require that the electric dipole moment QP produces, QPh, is equal to the total electric dipole moment of all the little electric dipoles. This gives QPh= Np , or

$$Q_P = \frac{Np}{h}$$

To compute the electric field produced by QP, we note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_p that is equal to the magnitude of the polarization:

$$\sigma_P = \frac{Q_P}{A} = \frac{Np}{Ah} = P$$

Note that the SI units of P (C.m)/m^3 , are or C/m^2 , which is the same as the surface charge density. In general if the polarization vector makes an angle θ with ˆ**n** , the outward normal vector of the surface, the surface charge density would be

$$\sigma_P = \vec{\mathbf{P}} \cdot \hat{\mathbf{n}} = P\cos\theta$$

Thus, our equivalent charge system will produce an average electric field of magnitude EP = P/ε_0. Since the direction of this electric field is opposite to the direction of ˆ**P**, in vector notation, we have

$$\vec{\mathbf{E}}_P = -\vec{\mathbf{P}} / \varepsilon_0$$

Thus, the average electric field of all these dipoles is opposite to the direction of the dipoles themselves. It is important to realize that this is just the average field due to all the dipoles. If we go close to any individual dipole, we will see a very different field.

We have assumed here that all our electric dipoles are aligned. In general, if these dipoles are randomly oriented, then the polarization $\vec{P}$ given in Eq. will be zero, and there will be no average field due to their presence. If the dipoles have some tendency toward a preferred orientation, then $\vec{P} \neq \vec{0}$, leading to a non-vanishing average field $\vec{E}_p$

Let us now examine the effects of introducing dielectric material into a system. We shall first assume that the atoms or molecules comprising the dielectric material have a permanent electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field , $\vec{P} = \vec{0}$. due to the random alignment of dipoles, and the average electric field $\vec{E}_p$ is zero as well. However, when we place the dielectric material in an external field $\vec{E}_0$ the dipoles will experience a torque $\vec{\tau} = \vec{p} \times \vec{E}_0$ that tends to align the dipole vectors $\vec{p}$ with $\vec{E}_0$. The effect is a net polarization $\vec{P}$ parallel $\vec{E}_0$ to , and therefore an average electric field of the dipoles $\vec{E}_p$□anti-parallel to $\vec{E}_0$, i.e., that will tend to reduce the total electric field strength below $\vec{E}_0$. The total electric field $\vec{E}$ is the sum of these two fields:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0 - \vec{\mathbf{P}} / \varepsilon_0$$

In most cases, the polarization $\vec{P}$ is not only in the same direction as $\vec{E}_0$, but also linearly proportional to $\vec{E}_0$ (and hence $\vec{E}$.) This is reasonable because without the external field there $\vec{E}_0$ would be no alignment of dipoles and no polarization $\vec{P}$. We write the linear relation between $\vec{P}$ and as $\vec{E}$

$$\vec{\mathbf{P}} = \varepsilon_0 \chi_e \vec{\mathbf{E}}$$

where χe is called the electric susceptibility. Materials they obey this relation are linear dielectrics. Combing Equations gives

$$\vec{\mathbf{E}}_0 = (1+\chi_e)\vec{\mathbf{E}} = \kappa_e \vec{\mathbf{E}}$$

Where

$$\kappa_e = (1+\chi_e)$$

is the dielectric constant. The dielectric constant K_e is always greater than one since χ_e >. This implies

$$E = \frac{E_0}{\kappa_e} < E_0$$

Thus, we see that the effect of dielectric materials is always to decrease the electric field below what it would otherwise be.

In the case of dielectric material where there are no permanent electric dipoles, a similar effect is observed because the presence of an external field $\vec{E}_0$ induces electric dipole moments in the atoms or molecules. These induced electric dipoles are parallel to $\vec{E}_0$, again leading to a polarization $\vec{P}$ parallel to $\vec{E}_0$, and a reduction of the total electric field strength.

Surface charge density:

According to electromagnetism, charge density is defined as a measure of electric charge per unit volume of the space in one, two, or three dimensions. To be specific, the linear surface or volume charge density is the amount of electric charge per surface area or volume, respectively.

Surface charge describes the electric potential difference between the inner and outer surface of different states like solid and liquid, liquid and gas, or gas and liquid. The surface charge density is present only in conducting surfaces and describes the whole amount of charge q per unit area A.

Formula of Surface Charge Density:

The surface charge density formula is given by,

$\sigma = q / A$

Where,

- σ is surface charge density ($C\cdot m^{-2}$)
- q is charge {Coulomb(C)}
- A is surface area (m^2)

Volume Charge Density:

In electromagnetism, the charge density tells how much charge is present in a given length, area or volume. The Greek symbol Pho (ρ) denotes electric charge, and the subscript V indicates the volume charge density.

Formula of Volume Charge Density:

The charge in terms of volume charge density is expressed as,

$\rho=q/v$

Where,

ρ is the charge density,

q is the charge(C),

v is the total volume in m^3.

Gauss's Law for Dielectrics:

Consider again a parallel-plate capacitor shown in Figure

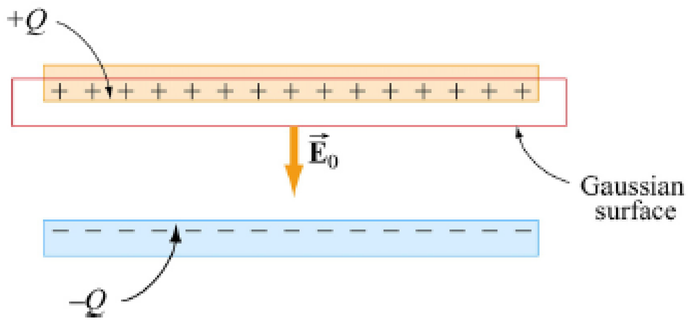


Fig.3.21 Gaussian surface in the absence of a dielectric

When no dielectric is present, the electric field $\vec{E}_0$ in the region between the plates can be found by using Gauss's law:

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_0 A = \frac{Q}{\varepsilon_0}, \quad \Rightarrow \quad E_0 = \frac{\sigma}{\varepsilon_0}$$

We have see that when a dielectric is inserted (Figure), there is an induced charge Q_p of opposite sign on the surface, and the net charge enclosed by the Gaussian surface is $Q- Q_p$

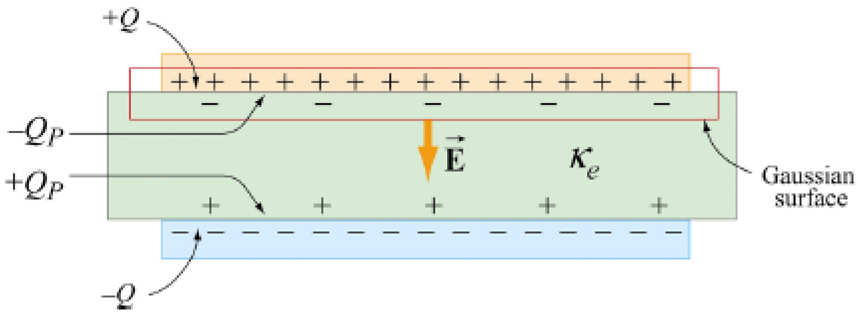


Fig3.22 Gaussian surface in the presence of a dielectric

Gauss's law becomes

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q - Q_P}{\varepsilon_0}$$

Or

$$\boxed{E = \frac{Q - Q_P}{\varepsilon_0 A}}$$

However, we have just seen that the effect of the dielectric is to weaken the original field E_0 by a factor K_e . Therefore

$$E = \frac{E_0}{\kappa_e} = \frac{Q}{\kappa_e \varepsilon_0 A} = \frac{Q - Q_P}{\varepsilon_0 A}$$

from which the induced charge Q_p can be obtained as

$$Q_P = Q\left(1 - \frac{1}{\kappa_e}\right)$$

In terms of the surface charge density, we have

$$\sigma_P = \sigma\left(1 - \frac{1}{K_e}\right)$$

Note that in the limit $K_e = 1$, $Q_p = 0$ which corresponds to the case of no dielectric material. Substituting the Equations we see that Gauss's law with dielectric can be rewritten as

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{K_e \varepsilon_0} = \frac{Q}{\varepsilon}$$

Where $\varepsilon = K_e\ \varepsilon_0$ is called the dielectric permittivity. Alternatively, we may also write

$$\oiint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{A}} = Q$$

Where $\vec{D} = \varepsilon_0 K \vec{E}$ is called the electric displacement vector E.

3.10 Macroscopic and microscopic properties of dielectrics, Clausius – Mossotte formula:

Macroscopic and microscopic properties of dielectrics:

The average (macroscopic) behavior of dielectrics. In this section, we will study the microscopic picture of a dielectric in a uniform electric field. Let us consider a dielectric in a uniform electric field as shown in Fig

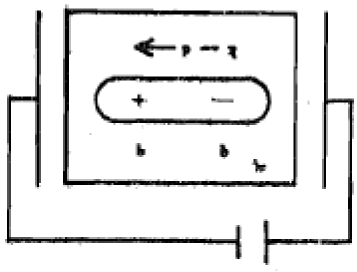


Fig.3.23 Dielectric in a uniform electric field

In an electric field, the electrons and atomic nuclei of the dielectric material experience forces in opposite directions. We know that the electrons in a dielectric cannot move freely as in a conductor. Hence each atom becomes a tiny dipole with the positive and' negative charge centers slightly separated. Taking the charge separation as a**,** the charge as q the dipole moment p in the direction of field associated with the atom or molecule

p = qa

Above EQ gives the dipole moment induced in the atom/molecule by the field. Hence we call it as induced dipole moment. If there are n such dipoles in an element of volume V of the material, we can define the polarization vector P as the (dielectric) dipole moment per unit volume as

$$p = \frac{npV}{V}$$

Within the dielectric the charges neutralized each other, the negative charge of one Atom/molecule is neutralized by the positive charge of its neighbor, Thus within the bulk of the material, the electric field produces on charge density but only a dipole moment density. However, at the surface this charge cancellation is not complete, and a polarization charge densities of apposite signs app at the two surfaces, perpendicular to the field. Now what is the consequence of the appearance of polarization charges?

The consequence of this is that the electric field inside the dielectric is less than the! Electric field causing the polarization, The polarization charges give rise to an electric field in the opposite direction. This field opposes the electric field causing polarization, It is shown' in below Fig.

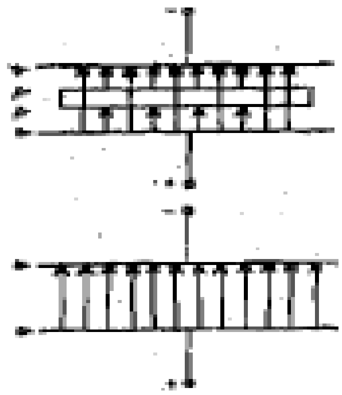


Fig.3.24 Field inside dielectric

Hence we conclude that inside the dielectric, the average electric field is less than the electric field causing polarization. However, the macroscopic or average field is not a satisfactory measure of the focal field responsible for the polarization of each atom. , Let us denote the field at the site or location of the atom or molecule as the local field. In next section, we will calculate the local field inside a dielectric

Definition of Local field:

In this section we will define the local field in a dielectric material. This is the field on a unit positive charge kept at a location or site from which an atom or molecule has been removed provided the other charges remain unaffected. Fig shows a site in a uniformly polarized medium from which a molecule/atom is removed when all other charges are kept intact at their positions.

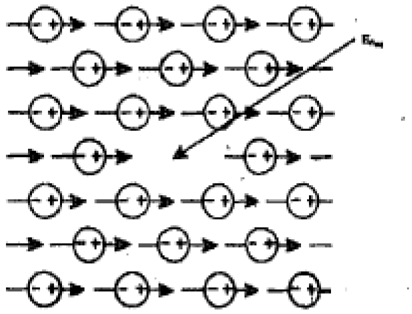


Fig.3.25 A site in a uniformly polarized medium

The extent of the charge separation depends on the magnitude of the local field. Hence we conclude that the induced dipole moment, p, is directly proportional to the local field, $\mathbf{E}_{\mathbf{Ioc}}$**,** Thus we have,

$$P = a\ E_{Ioc}$$

Where a is the constant of proportionality and is known **as** atomic/molecular polarizability and $\mathbf{E}_{\mathbf{Ioc}}$**,** the local field.

Clausius – Mossotte formula:

In a liquid we would expect an individual atom to be polarized by a field obtained in a spherical cavity rather than by the average (macroscopic) field. Thus using Equations we have

$$p = n$$

$$\mathbf{P} = n\alpha\ \mathbf{E}_{loc}$$

$$\mathbf{P} = n\alpha\mathbf{E} + \frac{\mathbf{P}}{3\varepsilon_0}$$

This can be return as

$$\mathbf{P} = \frac{n\alpha}{1 - \frac{n\alpha}{3\varepsilon_0}} \mathbf{E}$$

The susceptibility **x** was defined by the equation

$$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$$

Hence

$$= \frac{n\alpha / \varepsilon_0}{1 - n\alpha / 3\varepsilon_0}$$

Above Eq. gives the relation between susceptibility and atomic/molecular polarizability. This is one form of Clausius-Mossotti Equation

SAQ 3:

a) What do you mean by polarization?

b) Define the Surface charge density.

c) What is the Gauss's Law for Dielectrics?

d) Define the Clausius–Mossotte formula**.**

e) Calculate the polarization produced in dielectric medium of dielectric constant 9 when it is subjected to an electric field of 200 Vm–1. (e_0 = 8.854 × 10–12 Fm–1)

f) Calculate the surface charge density of a conductor whose charge is 4 C in an area of 8 m^2.

Example:

Q.1. What is the charge stored when the voltage across a 50 μF capacitor is 9 V?

Solution: $Q = VC = 9\ V \times 50\ \mu F = 450\ \mu F$

Q.2. What is the capacitance of a capacitor that stores 12 μC of charge when connected to a 6 V battery?

Solution:

$C=Q/V =12\ \mu C/ 6V = 2\ \mu C$

Q.3. Work out the voltage across the plates of a 10 μF capacitor when it has a charge of 50 μC.

Solution:

$V=Q/C = 50\ \mu C/ 10\ F = 5V$

Q.4. Calculate the energy stored in a capacitor with a charge of 200 μC and 9 V across its plates.

Solution:

$W = ½\ QV = (200\ \mu C \times 9\ V) / 2$

$= 0.9\ mJ$

Q.5. Calculate the energy stored in a 1 μF capacitor charged to 50 V.

Solution:

$W = \frac{1}{2} V^2 C = (2500\ V \times 1\ mF) / 2$

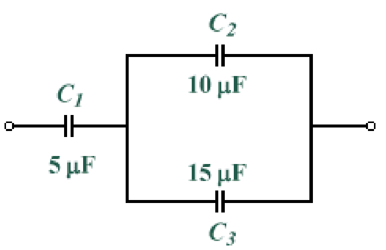
$= 1.25\ mJ$

Q.6. What is the combined capacitance of: a) a 2.2 μF capacitor and a 4.7 μF capacitor in parallel? b) Two 100 μF capacitors in series?

Solution : (a) $C_{TOTAL} = C_1 + C_2 = 2.2\ \mu F + 4.7\ \mu F = 6.9\ \mu F$

$$\text{(b) } C_{TOTAL} = \frac{C_1 \times C_2}{C_1 + C_2} = \frac{10{,}000\ \mu F}{200\ \mu F} = 50\ \mu F$$

Q.7. What is the total combined capacitance of the network shown below?



Solution:

$$C_{TOTAL} = \underline{C_1 \times (C_2 + C_3)} = \underline{5\ \mu F \times 25\ \mu F} = 4.167\ \mu F$$

$C_1 + (C_2 + C_3)$ 30 μF

Q.8. A parallel plate capacitor has square plates of side 5 cm and separated by a distance of 1 mm. (a) Calculate the capacitance of this capacitor. (b) If a 10 V battery is connected to the capacitor, what is the charge stored in any one of the plates? (The value of εo = 8.85 x 10^{-12} Nm2 C^{-2})

Solution:

(a) The capacitance of the capacitor is

$$C = \frac{\varepsilon_0 A}{d} = \frac{8.85 \times 10^{-12} \times 25 \times 10^{-4}}{1 \times 10^{-3}}$$

$= 221.2 \times 10^{-13}$ F

$C = 22.12 \times 10^{-12}$ F $= 22.12$ pF

(b) The charge stored in any one of the plates is Q = CV, Then

$= 22.12 \times 10^{-12} \times 10 = 221.2 \times 10^{-12}$ C $= 221.2$ pC

Q.9. A parallel plate capacitor filled with mica having εr = 5 is connected to a 10 V battery. The area of the parallel plate is 6 m^2 and separation distance is 6 mm.

(a) Find the capacitance and stored charge.

(b) After the capacitor is fully charged, the battery is disconnected and the dielectric is removed carefully. Calculate the new values of capacitance, stored energy and charge.

Solution:

(a) The capacitance of the capacitor in the presence of dielectric is

$$C = \frac{\varepsilon_r \varepsilon_0 A}{d} = \frac{5 \times 8.85 \times 10^{-12} \times 6}{6 \times 10^{-3}}$$
$$= 44.25 \times 10^{-9} F = 44.25 \text{nF}$$

The stored charge is

$$Q = CV = 44.25 \times 10^{-9} \times 10$$
$$= 442.5 \times 10^{-9} C = 442.5 nC$$

The stored energy is

$$U = \frac{1}{2} CV^2 = \frac{1}{2} \times 44.25 C \times 10^{-9} \times 100$$
$$= 2.21 \times 10^{-6} J = 2.21 \, \mu J$$

(b) After the removal of the dielectric, since the battery is already disconnected the total charge will not change. But the potential difference between the plates increases. As a result, the capacitance is decreased.

New capacitance is

$$C_0 = \frac{C}{\varepsilon_r} = \frac{44.25 \times 10^{-9}}{5}$$
$$= 8.85 \times 10^{-9} F = 8.85 nF$$

The stored charge remains same and 442.5 nC. Hence newly stored energy is

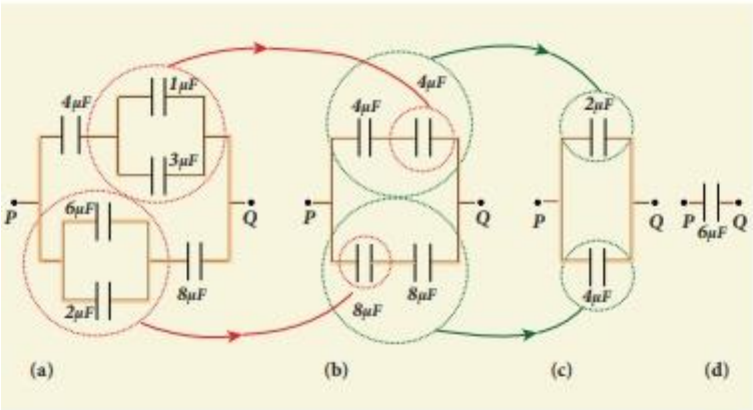
$$U_0 = \frac{Q^2}{2C_0} = \frac{Q^2\varepsilon_r}{2C} = \varepsilon_r U$$
$$= 5\times 2.21\ \mu J = 11.05\ \mu J$$

The increased energy is

$\Delta U = 11.05\mu\ J - 2.21\ \mu\ J = 8.84\ \mu\ J$

When the dielectric is removed, it experiences an inward pulling force due to the plates. To remove the dielectric, an external agency has to do work on the dielectric which is stored as additional energy. This is the source for the extra energy 8.84 μJ.

Q.10. Find the equivalent capacitance between P and Q for the configuration shown below in the figure (a).



Solution:

The capacitors 1 μF and 3μF are connected in parallel and 6μF and 2 μF are also separately connected in parallel. So these parallel combinations reduced to equivalent single capacitances in their respective positions, as shown in the figure (b).

Ceq = 1μF + 3μF = 4μF

Ceq = 6μF + 2μF = 8μF

From the figure (b), we infer that the two 4 μF capacitors are connected in series and the two 8 μF capacitors are connected in series. By using formula for the series, we can reduce to their equivalent capacitances as shown in figure (c).

$$\frac{1}{C_{eq}} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \qquad \Rightarrow C_{eq} = 2\ \mu F$$

and

$$\frac{1}{C_{eq}} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \qquad \Rightarrow C_{eq} = 4\ \mu F$$

From the figure (c), we infer that 2μF and 4μF are connected in parallel. So the equivalent capacitance is given in the figure (d).

Ceq = 2μF + 4μF = 6μF

Thus the combination of capacitances in figure (a) can be replaced by a single capacitance 6 μF.

Q.11. A parallel plate capacitor is kept in the air has an area of $0.50m^2$ and separated from each other by distance 0.04m. Calculate the parallel plate capacitor.

Solution:

Given:

Area A = 0.50 m^2,

Distance d = 0.04 m,

relative permittivity k = 1,

ϵo = 8.854 × 10^{-12} F/m

The parallel plate capacitor formula is expressed by,

$$C = k\frac{\epsilon_0 A}{d}$$

= 8.854×10^{-12} × 0.50 / 0.04

= 4.427 x 10^{-12} / 0.04

Therefore, C = 110.67 x 10^{-12} F

Q.12. Determine the area of parallel plate capacitor in the air if the capacitance is 25 nF and separation between the plates is 0.04m.

Solution:

Given:

Capacitance = 25 nF,

Distance d = 0.04 m,

Relative permittivity k = 1,

ϵ_o = 8.854 × 10^{-12} F/m

The parallel plate capacitor formula is expressed by,

$$C = k\frac{\epsilon_0 A}{d}$$

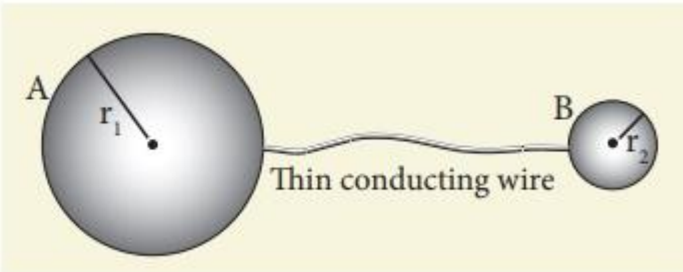
$$A = \frac{dC}{k\epsilon_0}$$

$= 0.04 \times 25\times10^{-9} / 1\times8.854\times10^{-12}$

$A = 1 \text{ x}10^{-9} / 8.854 \times10^{-12}$

Therefore, area of parallel plate capacitor is 112.94 m^2.

Q.13. Two conducting spheres of radius $r_1 = 8$ cm and $r_2 = 2$ cm are separated by a distance much larger than 8 cm and are connected by a thin conducting wire as shown in the figure. A total charge of Q = +100 nC is placed on one of the spheres. After a fraction of a second, the charge Q is redistributed and both the spheres attain electrostatic equilibrium.



(a) Calculate the charge and surface charge density on each sphere.

(b) Calculate the potential at the surface of each sphere.

Solution

(a) The electrostatic potential on the surface of the sphere A is

$$V_A = \frac{1}{4\pi\varepsilon_\circ}\frac{q_1}{r_1}$$

The electrostatic potential on the surface of the sphere A is

$$V_B = \frac{1}{4\pi\varepsilon_\circ}\frac{q_2}{r_2}$$

Since VA = VB. We have

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} \Rightarrow q_1 = \left(\frac{r_1}{r_2}\right)q_2$$

But from the conservation of total charge, $Q = q_1 + q_2$, we get $q_1 = Q - q_2$. By substituting this in the above equation,

$$Q - q_2 = \left(\frac{r_1}{r_2}\right)q_2$$

so that $\quad q_2 = Q\left(\frac{r_2}{r_1 + r_2}\right)$

Therefore,

$$q_2 = 100 \times 10^{-9} \times \left(\frac{2}{10}\right) = 20nC$$

and $\quad q_1 = Q - q_2 = 80nC$

The electric charge density for sphere A is

$$\sigma_1 = \frac{q_1}{4\pi r_1^2}$$

The electric charge density for sphere B is

$$\sigma_2 = \frac{q_2}{4\pi r_2^2}$$

Therefore,

$$\sigma_1 = \frac{80\times10^{-9}}{4\times64\times10^{-4}} = 0.99\times10^{-6}Cm^{-2}$$

and

$$\sigma_2 = \frac{20\times10^{-9}}{4\pi\times4\times10^{-4}} = 3.9\times10^{-6}Cm^{-2}$$

Note that the surface charge density is greater on the smaller sphere compared to the larger sphere ($\sigma_2 \approx 4\sigma_1$) which confirms the result $\sigma_1 / \sigma_1 = r_2 / r_2$.

The potential on both spheres is the same. So we can calculate the potential on any one of the spheres.

$$V_A = \frac{1}{4\pi\varepsilon_\circ}\frac{q_1}{r_1} = \frac{9\times10^9\times80\times10^{-9}}{8\times10^{-2}} = 9kV$$

Q.14. Dielectric strength of air is 3×10^6 V m^{-1}. Suppose the radius of a hollow sphere in the Van de Graff generator is R = 0.5 m, calculate the maximum potential difference created by this Van de Graaff generator.

Solution:

The electric field on the surface of the sphere (by Gauss law) is given by

$$E = \frac{1}{4\pi\varepsilon_\circ}\frac{Q}{R^2}$$

The potential on the surface of the hollow metallic sphere is given by

$$V = \frac{1}{4\pi\varepsilon_\circ}\frac{Q}{R} = ER$$

with Vmax = EmaxR

Here, Emax = 3 ×10^6 V/m.

So the maximum potential difference created is given by

Vmax = 3 × 10^6 × 0.5

= 1.5 × 10^6 V (or) 1.5 M volt.

Q.15. Find the capacitance of a conducting sphere of radius R.

Solution: Let charge Q is given to sphere. The field outside the sphere at distance r is

$$E = = \frac{kQ}{r^2}$$

$$\therefore -\frac{dV}{dr} = E$$

$$\therefore \int_0^v dV = -\int_\infty^R E dr$$

$$\Rightarrow V = kQ\left[-\frac{1}{r}\right]_\infty^R$$

$$\Rightarrow V = \frac{kQ}{R}$$

$$\therefore C = \frac{Q}{V} = \frac{R}{1/4\pi\varepsilon_0} = 4\pi\varepsilon_0 R$$

Q.16. A parallel plate air capacitor is made using two plates 0.2m square, spaced 1cm apart. It is connected to a 50V battery. (a) What is the capacitance? (b) What is the charge on each plate? (c) What is the electric field between two plates? (d)If the battery is disconnected and then the plates are pulled apart to a separation of 2cm, what are the answers to the above parts?

Solution:

$$C_a = \frac{\varepsilon_0 A}{d_0} = \frac{8.85 \times 10^{-12} \times 0.2 \times 0.2}{0.01}$$

$$Q = C_0 V_0 = (3.54 \times 10^{-5} \times 50)\mu C = 1.77 \times 10^{-5} \mu C$$

$$E_0 = \frac{V_0}{d_0} = \frac{50}{0.01} = 5000 V/m$$

If the battery is disconnected, the charge on the capacitor plates remains constant while the potential difference between plates can change.

$$\Rightarrow C = \frac{A\varepsilon_0}{2d} = 1.77 \times 10^{-5} \mu f$$

$$\Rightarrow Q = Q_0 = 1.77 \times 10^{-3} \mu F$$

$$\therefore V = \frac{Q}{C} = \frac{Q_0}{C_0/2} = 2V_0 = 100 volts$$

$$\therefore E = \frac{V}{C} = \frac{2V_0}{2d_0} = E_0 = 5000 V/m$$

Q.17. A parallel plate conductor connected in the battery with a plate area of 3.0 cm^2 and plate separation is of 3mm if the charge stored on the plate is 4.0pc. Calculate the voltage of the battery?

Solution:

Area A = 3.0 cm^2 = 3.0×10^{-4} m^2

$$C_a = \frac{\varepsilon_0 A}{d_0}$$

$$C_a = \frac{\varepsilon_0 A}{d_0} = \frac{8.85 \times 10^{-12}\left(3 \times 10^{-4}\right)}{3 \times 10^{-3}}$$

$C_a = 8.85 \times 10^{-13}$

$$C = \frac{Q}{V}$$

$$V = \frac{Q}{C}$$

$$V = \frac{4 \times 10^{-12}}{8.85 \times \times 10^{-13}}$$

V = 4.52 V

Q.18. A Cylindrical capacitor having a length of 8 cm is made of two concentric rings with an inner radius as 3 cm and outer radius as 6 cm. Find the capacitance of the capacitor.

Solution:

Given:

Length L = 8 cm

inner radius a = 3 cm

outer radius b = 6 cm

Formula for cylindrical capacitor is

$$C = 2\pi\epsilon_0 L/ln(\frac{b}{a})$$

$$C = (2)(3.14)(8.85 * 10^{-12})(8 * 10^{-2})/ln(\frac{6}{3})$$

$$C = 444.62X10^{-14}/0.301$$

$$C = 1.477X10^{-11}F.$$

Q.19. Determine the value of polarizing light angles when the refractive index of green color glass is 1.515 and that of violet color is 1.521.

Solution:

Given:

n_1 = 1.515

n_2 = 1.521

The in_1 = $\tan^{-1}$ (1.515)

= 56^o 57’

The $in_2 = \tan^{-1}(1.521)$

$= 56^o 68'$

Q.20. Consider a parallel plate capacitor which is maintained at potential of 200 V. If the separation distance between the plates of the capacitor and area of the plates are 1 and 20 cm^2. Calculate the displacement current for the time in μs.

Solution

Potential difference between the plates of the capacitor, V = 200 V

The distance between the plates,

$d = 1 \text{ mm} = 1 \times 10^{-3}$ m

Area of the plates of the capacitor,

$A = 20$ cm2 $= 20 \times 10^{-4}\ m^2$

Time is given in micro-second, $\mu s = 10^{-6}$ s

Displacement current

$$I_d = \varepsilon_\circ \frac{d\Phi_B}{dt} \Rightarrow I_d = \varepsilon_\circ \frac{EA}{t}$$

But electric field, E = V/d

Therefore,

$$I = \frac{V}{d} I_d = \varepsilon_o \frac{VA}{td} = 8.85 \times 10^{-12} \times \frac{200 \times 20 \times 10^{-4}}{10^{-6} \times 1 \times 10^{-3}}$$

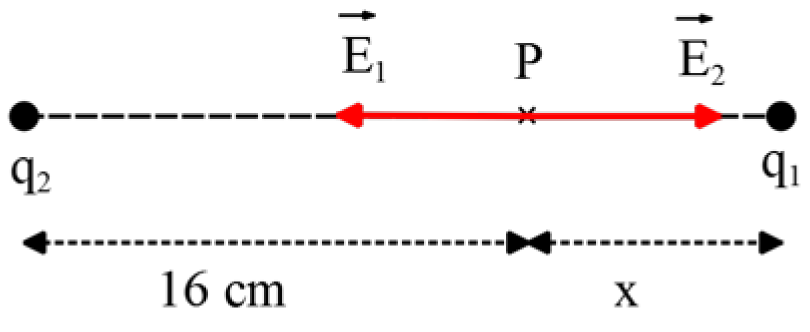
$= 35400 \times 10^{-7} = 3.5$ mA

Q.21. The electric field due to charges $q_1 = 2\mu C$ and $q_2 = 32\mu C$ at distance 16cm from charge q_2 is zero. What is the distance between the two charges?

Solution:

Since the two charges q_1 and q_2 are positive, somewhere between them the net electric force must be zero that is at the given point the magnitude of the fields are equal (remember that the electric field of a positive charge at field point is outward). Therefore, we get

$$\vec{E}_{net} = 0 \quad \Longrightarrow \quad |\vec{E}_1| = |\vec{E}_2|$$



Now use the definition of electric field to evaluate the relation above:

$$k\frac{|q_1|}{r_1^2} = k\frac{|q_2|}{r^2} \quad \Longrightarrow \quad \frac{2}{x^2} = \frac{32}{16^2}$$

Taking square root of the both sides, we obtain

$$\frac{1}{x} = \frac{4}{16} \quad \Longrightarrow \quad x = 4\text{cm}$$

As shown in the figure, the distance of the two charges is d=x+16=4+16=20cm.

Q.22. *A solid elemental dielectric with 3 × 10^{28}* atoms/m^3 *shows an electronic polarizability of 10^{-40}* F-m^2. *Assuming the internal electric field to be a Lorentz field, calculate the dielectric constant of the material.*

Solution: Number density of dielectric atoms, N = 3 × 10^{28}/m^3

Electronic polarizability, $\alpha_e = 10^{-40}$ F-m^2

Calculate the dielectric constant, ϵ_r = ?

$$\alpha_e = \frac{\epsilon_0 (\epsilon_r - 1)}{N}$$

$$\text{(or)} \qquad \epsilon_r = \frac{\alpha_e N}{\epsilon_0} + 1 = \frac{10^{-40} \times 3 \times 10^{28}}{8.85 \times 10^{-12}} + 1 = \frac{3}{8.85} + 1 = 1.339$$

Q.23. Calculate the polarization produced in dielectric medium of dielectric constant 6 when it is subjected to an electric field of 100 Vm–1. (e_0 = 8.854 × 10–12 Fm–1)

Solution:

Given data:

$$\text{Dielectic constant } \varepsilon_r = 6$$

$$\text{Permitivity of free space } \varepsilon_0 = 8.854 \times 10^{-12} \text{ Fm}^{-1}$$

$$\text{Electric field } E = 100 \text{ Vm}^{-1}$$

$$\text{Polarization } P = \varepsilon_0 (\varepsilon_r - 1) E$$

$$= 8.854 \times 10^{-12} (6-1)\ 100$$

$$\boxed{P = 4.425 \times 10^{-9} \text{ Vm}^{-2} \text{ F}}$$

$P = 4.425 \times 10^{-9}$ Vm^{-2} F

Q.24. Calculate the electronic polarizability of neon. The radius of neon atom is 0.158 nm. ($e_0 = 8.854 \times 10^{-12}$ Fm^{-1})

Solution:

Given data:

$$\text{Radius of neon atom } R = 0.158 \text{ nm} \quad \text{(or)}$$

$$= 0.158 \times 10^{-9} \text{ m}$$

$$\text{Permitivity of free space } \varepsilon_0 = 8.854 \times 10^{-12} \text{ Fm}^{-1}$$

$$\text{Electronic polarizability} \quad \alpha_e = 4\pi\varepsilon_0 R^3$$

$$\alpha_e = 4(3.14)(8.854 \times 10^{-12})(0.158 \times 10^{-9})^3$$

$$= 4.386 \times 10^{-40} \text{ Fm}^2$$

$$\text{Electronic polarizability} \quad \boxed{\alpha_e = 4.386 \times 10^{-40} \text{ Fm}^2}$$

Q.25.The dielectric constant of a helium gas at NTP is 1.0000684. Calculate the electron polarizability of helium atoms if the gas contains 2.7

$\times 10^{26}$ atoms/m^3 and hence calculate the radius of helium atom (e_0= 8.854 $\times 10^{-12}$ Fm^{-1})

Solution:

Given data:

$$\text{Relative permitivity } \varepsilon_r = 1.0000684$$
$$\text{No. of atoms in the gas } N = 2.7 \times 10^{26} \text{ atoms/m}^3$$
$$\text{Permitivity of free space } \varepsilon_0 = 8.854 \times 10^{-12} \text{ Fm}^{-1}$$

i) We know polarization

$$P = \varepsilon_0 (\varepsilon_r - 1) E$$
$$\text{and } P = N\alpha_e E$$

From the above two equations, we can write

$$N\alpha_e = \varepsilon_0 (\varepsilon_r - 1)$$

$$\alpha_e = \frac{\varepsilon_0 (\varepsilon_r - 1)}{N}$$

$$= \frac{8.854 \times 10^{-12} (1.0000684 - 1)}{2.7 \times 10^{26}}$$

$$\boxed{\alpha_e = 2.242 \times 10^{-42} \text{ Fm}^2}$$

ii) Electronic polarizability

$$\alpha_e = 4\pi\varepsilon_0 R^3$$

$$R = \frac{\alpha_e}{4\pi\varepsilon_0}$$

$$= \frac{2.242 \times 10^{-42}}{4 \times 3.14 \times 8.854 \times 10^{-12}}$$

Radius of helium atom $\boxed{R = 0.0201 \times 10^{-30} \text{ m}}$

Q.26. Calculate the surface charge density of a conductor whose charge is 5 C in an area of 10 m^2.

Solution:

Given:

Charge $q = 5$ C,

Area $A = 10\ m^2$

Surface charge density formula is given by,

$\sigma = q / A$

$= 5 / 10$

Therefore, $\sigma = 0.5\ C/m^2$

Q.27. Calculate the surface charge density of the sphere whose charge is 12 C and radius is 9 cm.

Solution:

Given:

Charge $q = 12$ C,

Radius $r = 9$ cm.

The surface charge density formula is given by,

$\sigma = q / A$

For a sphere, area $A = 4\ \pi\ r^2$

$A = 4\ \pi\ (0.09)^2$

$A = 0.1017\ m^2$

Surface charge density, $\sigma = q / A$

$\sigma = 12 / 0.1017$

$= 117.994$

Therefore, $\sigma = 117.994\ C \cdot m^{-2}$

Q.28. Find the volume charge density if the charge of 10 C is applied across the area of $2m^3$.

Solution:

Given:

Charge q = 10 C

Volume $v = 2m^3$.

The volume charge density formula is

$$\rho = \frac{q}{v}$$
$$\rho = \frac{10C}{2m^3}$$
$$\rho = 5\,C/m^3$$

Q.29 n charged drops, each of radius r and charge q, coalesce to from a big drop of radius R and charge Q. If V is the electric potential and E is the electric field at the surface of a drop, then.

Solution: For each small drop

$$V_1 = \frac{q}{4\pi \in_0 r}$$

When n small drops coalesce to from one big drop of radius R, then as

$$\frac{4}{3}\pi R^3 = n \times \frac{4}{3}\pi r^3, \ \therefore R = n^{1/3} r$$

Total charge Q = nq

∴ Potential of big drop,

$$V_b = \frac{Q}{4\pi \in_0 R} = \frac{nq}{4\pi \in_0 m^{1/3} r}$$

$$V_b = n^{2/3} V,$$

Summary:

1) *The capacitor* is a component which has *the ability* or "*capacity*" to store energy in *the* form of an electrical charge producing a *potential* difference (Static Voltage) across *its* plates, much like a small rechargeable battery.
2) A capacitor is an electronic component that stores and releases electricity in a circuit. It also passes alternating current without passing direct current. A capacitor is an indispensible part of

electronic equipment and is thus almost invariably used in an electronic circuit.

3) Capacitance or electric capacity indicates the ability of a system to store charge. It is defined as the ratio of charge stored in the conductors to the potential difference across the conductors. Its SI unit is Farad (F).
4) Energy stored in a capacitor is electrical potential energy, and it is thus related to the charge Q and voltage V on the capacitor.
5) As the dielectric slab is introduced there is some charge distribution in the slab and because of this the electric field between the two plates is decreased, due to which the capacitor can hold more charge. Thus, the capacity to hold charge of the capacitor is increased.
6) If the wires to the battery are disconnected, the charge remains on the plates -- and the voltage across the plates remains the same. If the wires are connected to each other, current will flow and the capacitor will discharge. Then there will be no voltage across the capacitor nor any charge on the plates.
7) A spherical capacitor consists of a hollow or a solid spherical conductor surrounded by another concentric hollow spherical conductor.
8) The capacitor is used to store large amounts of electric current in a small space. The cylindrical capacitor includes a hollow or a solid cylindrical conductor surrounded by the concentric hollow spherical cylinder.

9) The relationship among the three vectors D, E, P in the metre-kilogram-second (mks) or SI system is: $D = \varepsilon_0 E + P$ (ε_0 is a constant, the permittivity of a vacuum).

10) Dielectric constant (ϵ_r) is defined as the ratio of the electric permeability of the material to the electric permeability of free space (i.e., vacuum) and its value can derived from a simplified capacitor model.

11) Dielectric strength is defined as the electrical strength of an insulting material. In a sufficiently strong electric field the insulating properties of an insulator breaks down allowing flow of charge. Dielectric strength is measured as the maximum voltage required to produce a dielectric breakdown through a material.

12) Electrical susceptibility (χ) of a dielectric material is defined as $\chi = \in r - 1$where $\in r$ is its relative permittivity. An isolated parallel plate capacitor carries some charge and the field in the dielectric present between its plates is E. Express the electric field due to induced charge on dielectric surface in terms of χ and E.

13) Electric polarization refers to the separation of center of positive charge and the center of negative charge in a material. The separation can be caused by a sufficiently high-electric field.

14) Surface charge density (σ) is the quantity of charge per unit area, measured in coulombs per square meter ($C \cdot m^{-2}$), at any point on a surface charge distribution on a two dimensional surface.

15) In electromagnetism, charge density is the amount of electric charge per unit length, surface area, or volume. Volume charge density (symbolized by the Greek letter ρ) is the quantity

of charge per unit volume, measured in the SI system in coulombs per cubic meter ($C{\cdot}m^{-3}$), at any point in a volume.

16) Integral form ("big picture") of Gauss's law: The flux of electric field out of a closed surface is proportional to the charge it encloses. The above is Gauss's law in free space (vacuum). For a dielectric, just replace ε_0 with $\varepsilon = \varepsilon$.

17) A polarizable quantum mechanics and molecular mechanics model has been extended to account for the difference between the macroscopic electric field and the actual electric field felt by the solute molecule. This enables the calculation of effective microscopic properties which can be related to macroscopic susceptibilities directly comparable with experimental results.

18) The Clausius-Mossotti equation relates the dielectric constant of a material to the polarisability of its atoms. It finds natural explanation in terms of the (often omitted) delta function in the electric field of an ideal dipole. This avoids the subtleties of the rather tricky conventional derivation.

Terminal Question:

1) What is the capacitor and explain in detail?
2) Explain the Working Principle of a Capacitor.
3) Explain the effect when Capacitor in a DC and Circuit Capacitor in an AC Circuit?
4) Explain the Dielectrics with Battery and without Battery in detail.
5) Explain and define Spherical capacitor in details.

6) Explain the cylinder capacitor in details.

7) Explain how to calculate Dielectric Constant?

8) Write short notes on: (i) Dielectric constant, (ii) Dielectric strength, (iii) Electrical susceptibility.

9) What is the Relation between polarization vector (P), displacement (D) and electric field (E)?

10) What is the difference between Polar Plastics Vs Non-polar Plastics

11) Explain the Dielectric Strength and also write the applications.

12) Explain the method of measurement Dielectric Strength.

13) Explain the surface and volume charge density.

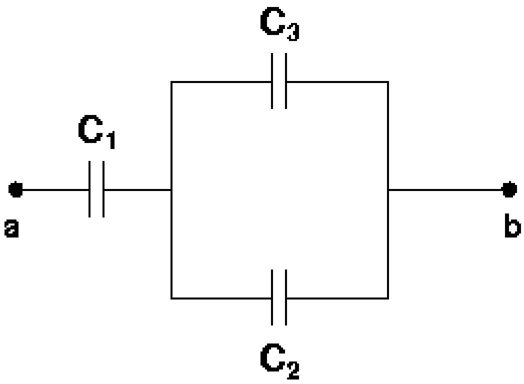
14) What do you mean by the Gauss law in dielectrics?

15) Derive the expression of Clausius – Mossotte formula.

16) Consider two plates separated by d=1.5 cm , where the electric field between them is 100 V/m, and the charge on the plates is 30.0 μC. What is the capacitance?

17) Consider a capacitor made of two 0.05 m^2 plates separated by 0.5 mm. If the capacitance is 3.0 nf, what is the relative permeability, k, of the material between the plates?

18) What is the capacitance of the following segment of a circuit? When C_1=C_2=C_3=3.0 μf.



19) A capacitor has a charge of 3.0 *nC* when the voltage across the capacitor is 12 *V*. What is the energy stored in the capacitor?

20) A cylindrical capacitor is constructed using two coaxial cylinders of the same length 10 cm of radii 5 mm and 10 mm. (a) calculate the capacitance, (b) another capacitor of the same length is constructed with cylinders of radii 8 mm and 16 mm. Calculate the capacitance .

21) *A parallel plate capacitor has an area of 100* cm^2, *a plate separation of 1* cm *and is charged to a potential of 100* V. *Calculate the capacitance of the capacitor and the charge on the plates.*

22) What is the magnitude of the electric force acting on an electron located in an electric field with an intensity of 5.0×10^3 N per coulomb (electrons have a charge of -1.6×10^{-19}C)?

23) Glass has a dielectric constant of 4.1. If capacitor A originally has a capacitance of 1.5 F, what is its capacitance after glass, a dielectric material, is inserted to completely fill the space between the plates of capacitor A?

24) Wood has a dielectric constant of 2.80. If capacitor A originally has a capacitance of 0.5 F, what is its capacitance after wood, a dielectric material, is inserted to completely fill the space between the plates of capacitor A?

25) The electronic polarisability is 0.18x 10 ^{40} fm^2. Find the relative dielectric constant at 0C and 1 atmospheric pressure.

26) A capacitor has capacitance of 0.019 F when uses wax paper $_r$ 1.85 between the electrodes of Aluminum foil. The wax

paper is to be replaced by plastic film $_r$ 2.15 of same dimensions. Taking other factors being equal, obtain the change (increase/decrease) in capacitance.

27) The radium of the Helium atom is about 0.55 AU. Calculate the polarisability of Helium and its relative permeability. The number of Helium atoms in a volume of 1 m is 2.7×10^{25} atoms.

28) A capacitor uses Aluminum oxide as the dielectric with relative permeability = 8. An effective surface area of 360 cm^2 gives a capacitance of 6 F. Calculate the field strength and the total dipole moment induced in oxide layer if a potential difference of 15 volts exists across the capacitor.

29) A long thin rod circular of length 50 cm and radius 7 cm has a total charge of 5 mC, which is uniformly distributed over it. Find the Surface charge density.

Uttar Pradesh Rajarshi Tandon

Open University

Bachelor of Science

UGPHS-103

Electromagnetism

Block

2 Magnetostatics

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Unit 04 - Electric current and magnetic fields

Structure

4.1 Introduction:

Electric current, is a movement of electric charge carriers, such as subatomic charged particles (e.g., electrons having negative charge, protons having positive charge), ions (atoms that have lost or gained one or more electrons), or holes (electron deficiencies that may be thought of as positive particles).

In electromagnetism, current density is the amount of charge per unit time that flows through a unit area of a chosen cross section. ... In SI base units, the electric current density is measured in amperes per square meter.

The magnitude of current density is also equivalent to the ratio of current (I) to area (A). In equation form, current density can be written as… The SI unit of current density is the ampere per square meter [A/m^2].

Ohm's law states that the voltage or potential difference between two points is directly proportional to the current or electricity passing through the resistance, and directly proportional to the resistance of the circuit. The formula for Ohm's law is V=IR.

Joule's law, when an electric current passes through a conductor, heat H is produced, which is directly proportional to the resistance R of the conductor, the time t for which the current flows, and to the square of the magnitude of current I.

In physics a drift velocity is the average velocity attained by charged particles, such as electrons, in a material due to an electric field. In general, an electron in a conductor will propagate randomly at the Fermi velocity, resulting in an average velocity of zero.

A stationary charge will produce only an electric field in the surrounding space. If the charge is moving, a magnetic field is also produced.

Interaction of a magnetic field with a charge. How does the magnetic field interact with a charged object? If the charge is at rest, there is no interaction. If the charge moves, however, it is subjected to a force, the size of which increases in direct proportion with the velocity of the charge.

Current is generally defined as the rate of flow of charge. Magnetic field due to a current-carrying conductor depends on the current in the conductor and distance of the point from the conductor. The direction of the magnetic field is perpendicular to the conductor.

The Biot Savart Law states that it is a mathematical expression which illustrates the magnetic field produced by a stable electric current in the particular electromagnetism of physics. It tells the magnetic field toward the magnitude, length, direction, as well as closeness of the electric current.

When an electric current flow through a solenoid magnetic field is set up around solenoid similar to that of a bar magnet. One end of a solenoid act as a north pole and other as south pole. Magnetic field is represented by straight magnetic field lines parallel and very close to each other.

A Toroid is shaped like a solenoid bent into a circular shape such as to close itself into a loop-like structure. The magnetic field inside the Toroid, along with the circular turn, is constant in magnitude and its direction inside the Toroid is clockwise as per the right-hand thumb rule for circular loops.

Magnetic force is always perpendicular to velocity, so that it does not work on the charged particle. The particle's kinetic energy and speed thus remain constant. The direction of motion is affected, but not the speed.

Lorentz force, the force exerted on a charged particle q moving with velocity v through an electric field E and magnetic field B. The entire electromagnetic force F on the charged particle is called the Lorentz force (after the Dutch physicist Hendrik A. Lorentz) and is given by $F = qE + qv \times B$.

You might expect that there are significant forces between current-carrying wires, since ordinary currents produce significant magnetic fields and these fields exert significant forces on ordinary currents. But you might not expect that the force between wires is used to define the ampere. It might also surprise you to learn that this force has something to do with why large circuit breakers burn up when they attempt to interrupt large currents.

What is the mechanism by which one magnet exerts a force on another? The answer is related to the fact that all magnetism is caused by current, the flow of charge. Magnetic fields exert forces on moving charges, and so they exert forces on other magnets, all of which have moving charges.

A Cyclotron is a machine that accelerates charged particles or ions to high energies. In a Cyclotron, charged particles accelerate outwards from the centre along a spiral path. These particles are held to a spiral trajectory by a static magnetic field and accelerated by a rapidly varying electric field.

In a Betatron, the changing magnetic field from the primary coil accelerates electrons injected into the vacuum torus, causing them to circle around the torus in the same manner as current is induced in the secondary coil of a transformer (Faraday's Law).

4.2 Objective:

After studying this unit you should be able to

- Study and identify Electric current and current density. Ohm's law and Joule's law, drift velocity.
- Explain and identify Magnetic field around stationary charge, moving charge and current carrying conductor.
- Explain Biot-Savart law and its application to straight conductor, circular loop, solenoid and toriod carrying current.
- Study and identify Magnetic field due to moving charge, Lorentz force
- Explain and identify Force between two current carrying conductor and two moving charges.
- Explain Cyclotron (principle, construction, working, limitations and modification), Betatron.

4.3 Electric current and current density:

Electric Current Definition:

Today at the flick of a switch or turn of a knob we are having instant power. This is possible due to the electric current. It is one of the important discoveries that helped us to revolutionize our way of living. From the time we wake up till night, our life is fully dependent on electricity. Electricity represents the follow of electric current. Electric current is known as the rate of flow of negative charges of the conductor. It means the continuous flow of electrons in an electric circuit is called an electric current.

What is the Electric Current?

The conducting material consists of a large number of free electrons which is moving from one atom to the other at random. When the potential difference is applied across a wire, then loosely attached free electrons will start moving towards the positive terminal of the cell.

This continuous flow of electrons makes the existence of the electrical current. Therefore the flow of currents in the wire is from the negative terminal to the positive terminal through the external circuit.

This traditional flow of current is so firmly established that it is still in use. Thus, the conventional direction of the flow of the electric current is from the positive terminal of the cell to the negative terminal of the cell through the external circuit.

On the basis of the flow of electric charge, the current can be classified into two types, which are alternating current and direct current. In direct current, the charges flow in one direction but in alternating current, the charges flow in both the direction.

The Formula for Electric Current:

The magnitude of the flow of current at any section of the conductor is defined as the rate of flow of electrons.

Mathematically, this can be represented as:

$I = Q/t$

Where,

I = Electric current,

Q = Electric Charge

T = Time

Electric current is the rate of change of electric charge through a circuit. This electric current is related to the voltage and resistance of the circuit. Using Ohm's law, we can represent as the formula:

$I= V/R$

Where,

V = Electric Voltage,

R = The resistance of the metallic wire,

I = Electric current

Since we measure the charge in coulombs and time in seconds, therefore the unit of electric current is coulomb/Sec or amperes. The ampere is the SI unit of the electric current. The symbol for electric current is I. Thus, an electric wire is said to carry a current of 1 ampere when charge flows through it with the rate of one coulomb per second.

Current Density Definition:

We can define current as the flow of electrically charged particles, mostly in those atoms which are electron-deficient. The standard symbol of current is capital I. The standard unit of current is ampere and it is denoted by A. Conversely, a current of one ampere is one coulomb of charge(6.24 x 10^{18} charge carriers) going past a given point per second. According to Physicists, Current is considered to move from relatively positive to negative points, and this is known as conventional current. Electrons are known to be the common negatively charged carriers and circulate from relatively negative to positive points. In this article, we will learn about the concepts of Current Density in a detailed manner.

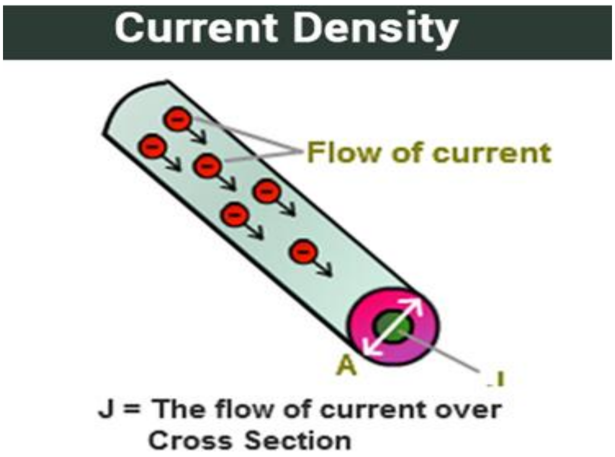


Fig.4.1 Current density

Types of Current:

Current can be divided into two types.

Direct Current:

- Direct current travels towards the same direction at all points, although the instantaneous magnitude can differ.
- An example of DC is the current generated by an electrochemical cell.

Alternating Current:

- The flow of charge carriers is towards opposite direction periodically in an alternating current.
- The number of AC cycles per second is known as frequency and calculated in Hertz.

What is Current Density?

The amount of electric current travelling per unit cross-section area is called as current density and expressed in amperes per square meter. More the current in a conductor, higher will be the current density. However, the current density alters in different parts of an electrical conductor and the effect takes place with alternating currents at higher frequencies.

Electric current always creates a magnetic field. Stronger the current, more intense is the magnetic field. Varying AC or DC creates an electromagnetic field and this is the principle based on which signal propagation takes place.

Current density is a vector quantity having both a direction and a scalar magnitude. The electric current flowing through a solid having units of charge per unit time is calculated towards the direction perpendicular to the flow of direction.

It is all about the amount of current flowing across the given region.

Current Density Formula:

The formula for Current Density is given as,

J = I / A

Where,

I = current flowing through the conductor in Amperes

A = cross sectional area in m^2.

Current density is expressed in A/m^2.

Ohm's law and Joule's law:

Ohm's law:

Ohm's law states that the voltage or potential difference between two points is directly proportional to the current or electricity passing through the resistance, and directly proportional to the resistance of the circuit. The formula for Ohm's law is V=IR. This relationship between current, voltage and relationship was discovered by German scientist Georg Simon Ohm.

Ohm's Law Definition:

Most basic components of electricity are voltage, current, and resistance. Ohm's law shows a simple relation between these three quantities. Ohm's law states that the current through a conductor between two points is directly proportional to the voltage across the two points.

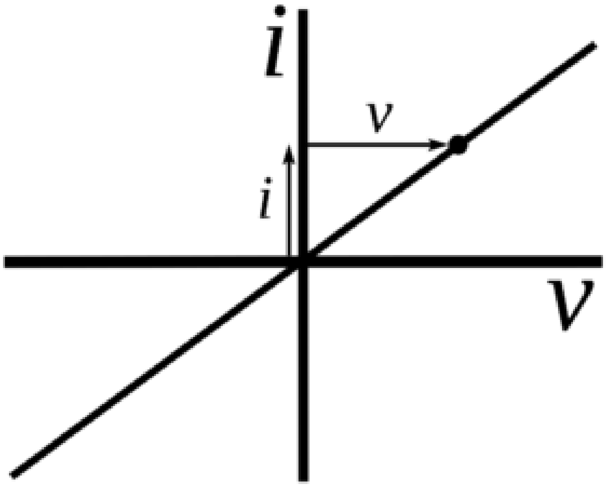


Fig.4.2. V-I Characteristics of Ohm's Law

Ohm's Law Formula:

Voltage= Current× Resistance

V= I×R

V= voltage, I= current and, R= resistance

The SI unit of resistance is ohms and is denoted by Ω

This law is one of the most basic laws of electricity. It helps to calculate the power, efficiency, current, voltage, and resistance of an element of an electrical circuit.

Applications of Ohm's Law:

Ohm's law helps us in determining voltage, current or impedance or resistance of a linear electric circuit when the other two quantities are known to us. It also makes power calculation simpler.

How do we establish the current-voltage relationship?

In order to establish the current-voltage relationship, the ratio V / I remains constant for a given resistance, therefore a graph between the potential difference (V) and the current (I) must be a straight line.

How do we find the unknown values of resistance?

It is the constant ratio that gives the unknown values of resistance,

$$\frac{V}{I} = R$$

For a wire of uniform cross-section, the resistance depends on the length l and the area of cross-section A. It also depends on the temperature of the conductor. At a given temperature the resistance,

$$\mathrm{R} = \rho \frac{\mathrm{l}}{\mathrm{A}}$$

where ρ is the specific resistance or resistivity and is characteristic of the material of wire. The specific resistance or resistivity of the material of the wire is,

$$\rho = \frac{\mathrm{RA}}{\mathrm{l}}$$

If 'r' is the radius of the wire, then the cross-sectional area, $A = \pi r^2$. Then the specific resistance or resistivity of the material of the wire is,

$$\rho = \frac{\pi r^2 R}{l}$$

Limitations of ohms law:

1. Ohm's law is not applicable to unilateral networks. Unilateral networks allow the current to flow in one direction. Such types of network consist elements like a diode, transistor, etc.

2. Ohm's law is also not applicable to non – linear elements. Non-linear elements are those which do not have current exactly proportional to the applied voltage that means the resistance value of those elements changes for different values of voltage and current. Examples of non – linear elements are the thyristor.

Joule's law:

When an electric current passes through wire heat energy is produced. It is due to the collision

of electrons with the atoms. In order to continue steady current, work has to be done on electric

charges.

Statement: Amount of work done on electric charge on steady current is directly proportional to amount of heat.

Work a Heat

Proof:

Consider a conductor through which electric current q is passing in time t let the potential

difference between two ends of wire is V.

We know that

$v = W/q$

or

$W = q \times V$_(i)

According to Ohm's law $V = IR$

putting the value of V in equation

$W = q \times IR$

But

$I = q/t$

Or

$Q = It$

putting the value of q in equation

$W = It \cdot IR$

$W = I^2Rt$

Drift Velocity:

Subatomic particles like electrons move in random directions all the time. When electrons are subjected to an electric field they do move randomly, but they slowly drift in one direction, in the direction of the electric field applied. The net velocity at which these electrons drift is known as drift velocity.

Drift velocity can be defined as:

The average velocity attained by charged particles, (eg. electrons) in a material due to an electric field.

The SI unit of drift velocity is m/s. It is also measured in $m^2/(V.s)$.

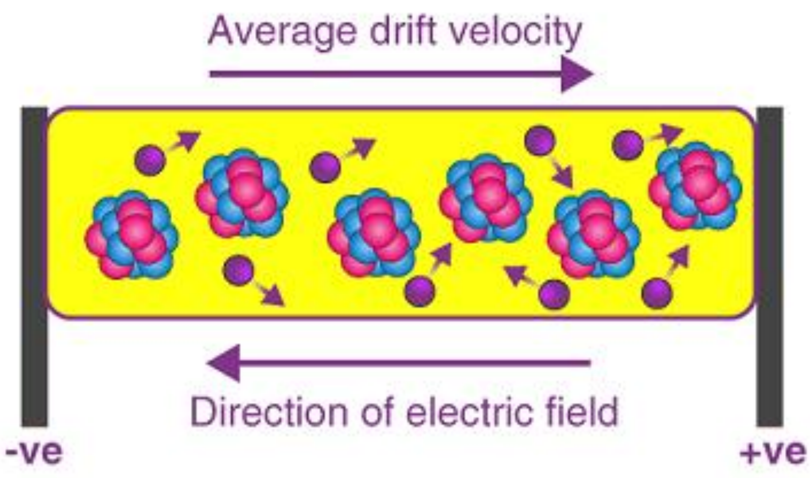


Fig.4.3. Average drift velocity and the direction of the electric field

Net velocity of the electrons:

Every material above absolute zero temperature which can conduct like metals will have some free electrons moving at random velocity. When a potential is applied around a conductor the electrons will tend to move towards the positive potential, but as they move, they will collide with atoms and will bounce back or lose some of their kinetic energy. However, due to the electric field, the electrons will accelerate back again, and these random collisions will keep happening but as the acceleration is always in the same direction due to the electric field the net velocity of the electrons will also be in the same direction.

Formula To Calculate Drift Velocity:

We can use the following formula in order to calculate drift velocity:

I = nAvQ $\quad v = \frac{I}{nAQ}$

Where,

- I is the current flowing through the conductor which is measured in amperes
- n is the number of electrons
- A is the area of the cross-section of the conductor which is measured in m^2
- v is the drift velocity of the electrons
- Q is the charge of an electron which is measured in Coulombs

Example:

Let’s consider a current of 3A that is flowing in a copper conductor with a cross-section of $1mm^2$ ($1\times10^{-6}m^2$)

We know that for copper, n = 8.5×10^{28} per m^3

So according to the formula we have,

$3 = 8.5\times10^{28}\times1\times10^{-6}\times v\times1.6\times10^{-19}$

Where, $Q=1.6\times10^{-19}$ C

Therefore,

$v=2.205882\times10^{-4}\ ms^{-1}$

If the intensity of the electric field is increased then the electrons are accelerated more rapidly towards the positive direction, opposite to the direction of the electric field applied.

Mobility of an electron:

The drift velocity of an electron for a unit electric field is known as mobility of the electron.

Mobility of an electron can be calculated by:

$\mu= V_d / E$ or $\mu = v/E$ where $V_d = v$=drift vilocity

Relation between Drift Velocity and Electric Current:

Mobility is always a positive quantity and depends on the nature of the charge carrier, the drift velocity of an electron is very small usually in terms of $10^{-3}ms^{-1}$. Hence, at this velocity it will take approx. 17 mins for electrons to pass through a conductor of 1 meter, but it's surprising that we can turn on electronic appliances in our home at lightning speeds with a flick of a switch this is because an electric current is not established with the drift velocity but with the speed of light.

As soon as the electric field is established the current starts flowing inside the conductor at the speed of light and not at the speed at which the electrons are drifting, hence there is a negligible small delay between an input and an output in turning on of an electric bulb.

Relation between Drift Velocity and Current Density:

We can define current density as the total amount of current passing through a unit cross-sectional conductor in unit time. From drift velocity, we know the formula for drift velocity as:

I = nAvQ

J = I/A = nVQ

Where,

- J is the current density measured in Amperes per square meter
- v is the drift velocity of the electrons

Then, we can say that drift velocity of the electrons and its current density is directly proportional to each other. Also, when the electric field intensity increases, the drift velocity increases, and the current flowing through the conductor also increases.

4.4 Magnetic field around stationary charge:

A stationary charged particle does not interact with a static magnetic field. A charge placed in a magnetic field experiences a magnetic force. The charge must be moving, for no magnetic force acts on a stationary charge.

A stationary charge will produce only an electric field in the surrounding space. If the charge is moving, a magnetic field is also produced.

It is because magnetic force acts on moving charges . Since the charge is stationary , no magnetic force will act on it. F = q v B , Where F is magnetic force on charge q travelling with velocity v in a magnetic field of intensity B.

There is no force on a stationary charge, or on a charge moving parallel to the field. The direction of the force experienced by a positive charge is opposite to that experienced by a negative charge if the charges are moving in the same direction.

Since current is defined as the rate of flow of charge, what can you conclude about the magentic field produced by a stationary charge? What about moving charges? The magnetic field produced by a stationary charge is zero. A moving charge is a current so it will produce a magnetic field.

Magnetic field around moving charge:

What is the mechanism by which one magnet exerts a force on another? The answer is related to the fact that all magnetism is caused by current, the flow of charge. Magnetic fields exert forces on moving charges, and so they exert forces on other magnets, all of which have moving charges.

Right Hand Rule 1

The magnetic force on a moving charge is one of the most fundamental known. Magnetic force is as important as the electrostatic or Coulomb force. Yet the magnetic force is more complex, in both the number of factors that affects it and in its direction, than the relatively simple Coulomb force. The magnitude of the magnetic force F on a charge q moving at a speed v in a magnetic field of strength B is given by

$$F = qvB \sin \theta,$$

where θ is the angle between the directions of v and B. This force is often called the Lorentz force. In fact, this is how we define the magnetic field strength B—in terms of the force on a charged particle moving in a magnetic field. The SI unit for magnetic field strength B is called the tesla (T) after the eccentric but brilliant inventor Nikola Tesla (1856–1943). To determine how the tesla relates to other SI units, we solve F = qvB sin θ for B.

B=Fqvsinθ

Because sin θ is unitless, the tesla is

1 T=1 N C· m/s=1 NA· m1 T=1 N C· m/s=1 NA· m

(note that C/s = A). Another smaller unit, called the gauss (G), where 1 G = 10−4 T, is sometimes used. The strongest permanent magnets have fields near 2 T; superconducting electromagnets may attain 10 T or more. The Earth's magnetic field on its surface is only about 5×10^{-5} T, or 0.5 G.

The direction of the magnetic force F is perpendicular to the plane formed by v and B, as determined by the right hand rule 1 (or RHR-1), which is illustrated in Figure 1. RHR-1 states that, to determine the direction of the magnetic force on a positive moving charge, you point the thumb of the right hand in the direction of v, the fingers in the direction of B, and a perpendicular to the palm points in the direction of F. One way to remember this is that there is one velocity, and so the thumb represents it. There are many field lines, and so the fingers represent them. The force is in the direction you would push with your palm. The force on a negative charge is in exactly the opposite direction to that on a positive charge.

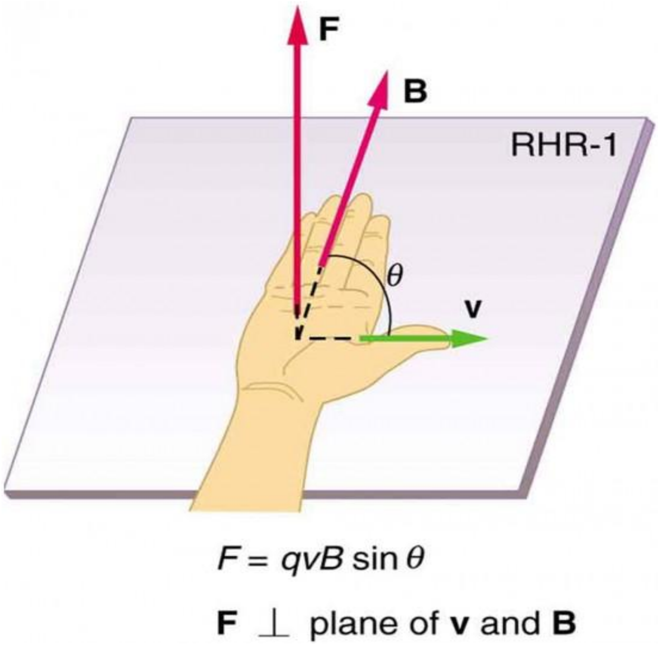


Fig.4.4. Magnetic fields exert forces on moving charges. This force is one of the most basic known. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by v and B and follows right hand rule–1 (RHR-1) as shown. The magnitude of the force is proportional to q, v, B, and the sine of the angle between v and B.

Magnetic field around current carrying conductor:

Because charges ordinarily cannot escape a conductor, the magnetic force on charges moving in a conductor is transmitted to the conductor itself.

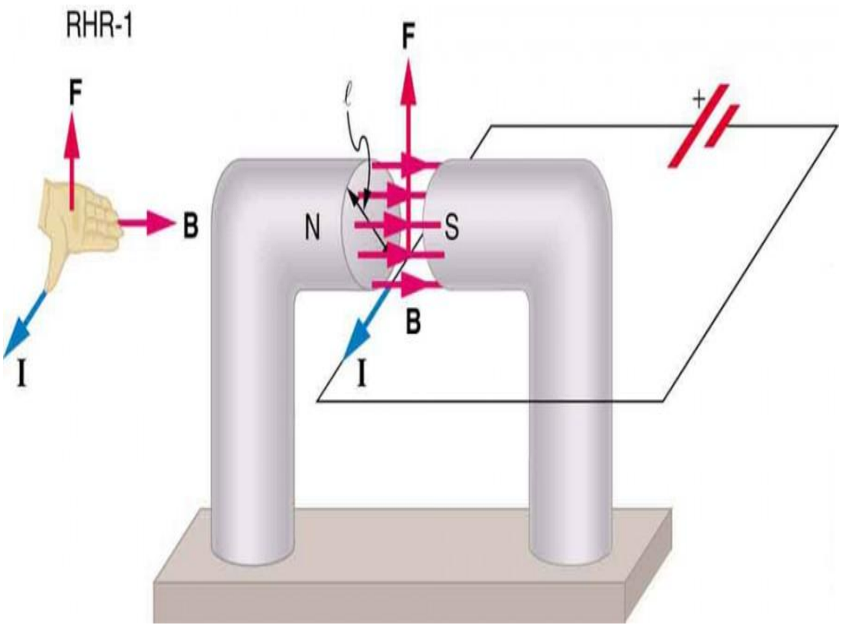


Fig.4.5. The magnetic field exerts a force on a current-carrying wire in a direction given by the right hand rule 1 (the same direction as that on the individual moving charges). This force can easily be large enough to move the wire, since typical currents consist of very large numbers of moving charges.

We can derive an expression for the magnetic force on a current by taking a sum of the magnetic forces on individual charges. (The forces add because they are in the same direction.) The force on an individual charge moving at the drift velocity v_d is given by $F = qv_dB \sin \theta$.

Taking B to be uniform over a length of wire l and zero elsewhere, the total magnetic force on the wire is then

$F = (qv_dB \sin \theta)(N)$,

Where N is the number of charge carriers in the section of wire of length l. Now, $N = nV$, where n is the number of charge carriers per unit volume and V is the volume of wire in the field.

Noting that $V = Al$, where A is the cross-sectional area of the wire, then the force on the wire is $F = (qv_dB \sin \theta)(nAl)$. Gathering terms,

$$F=(nqAv_d)lB\sin\theta$$

$$F=(nqAv_d)lB\sin\theta.$$

Because $nqAv_d = I$ (see Current),

$$F=IlB\sin\theta$$

$$F=IlB\sin\theta$$

is the equation for magnetic force on a length l of wire carrying a current I in a uniform magnetic field B, as shown in Figure 2. If we divide both sides of this expression by l, we find that the magnetic force per unit length of wire in a uniform field is $\frac{F}{l}=IB\sin\theta$. The direction of this force is given by RHR-1, with the thumb in the direction of the current I. Then, with the fingers in the direction of B, a perpendicular to the palm points in the direction of F, as in Figure 2.

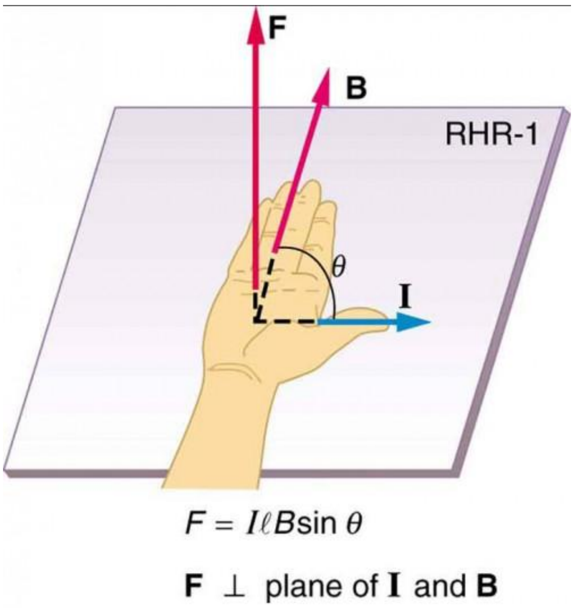


Fig.4.6. The force on a current-carrying wire in a magnetic field is $F = IlB \sin \theta$. Its direction is given by RHR-1.

Magnetic force on current-carrying conductors is used to convert electric energy to work. (Motors are a prime example—they employ loops of wire and are considered in the next section.) Magneto-hydrodynamics (MHD) is the technical name given to a clever application where magnetic force pumps fluids without moving mechanical parts. (See Figure 4.7)

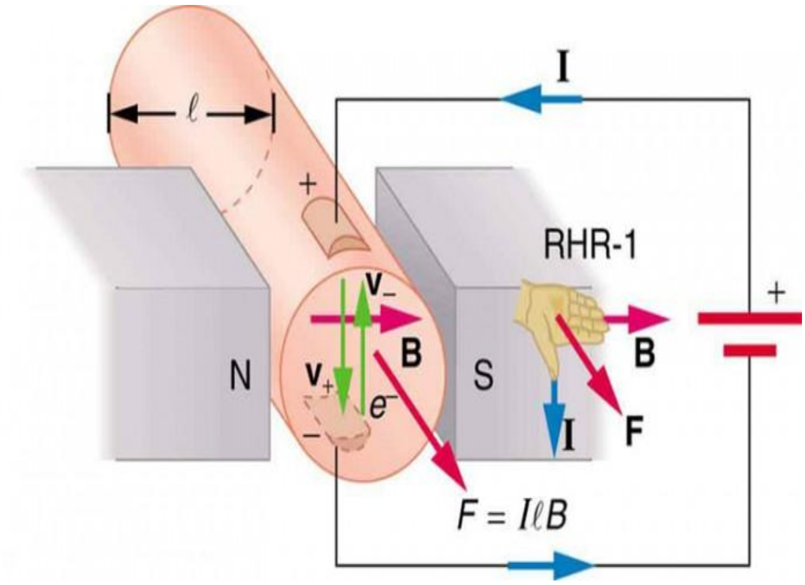


Fig.4.7. Magneto-hydrodynamics. The magnetic force on the current passed through this fluid can be used as a non-mechanical pump.

A strong magnetic field is applied across a tube and a current is passed through the fluid at right angles to the field, resulting in a force on the fluid parallel to the tube axis as shown. The absence of moving parts makes this attractive for moving a hot, chemically active substance, such as the liquid sodium employed in some nuclear reactors. Experimental artificial hearts are testing with this technique for pumping blood, perhaps circumventing the adverse effects of mechanical pumps. (Cell membranes, however, are affected by the large fields needed in MHD, delaying its practical application in humans.) MHD propulsion for nuclear submarines has been proposed, because it could be considerably quieter than conventional propeller drives. The deterrent value of nuclear submarines is based on their ability to hide and survive a first or second nuclear strike. As we slowly disassemble our nuclear weapons arsenals, the submarine branch will be the last to be decommissioned because of this ability Existing MHD drives are heavy and inefficient—much development work is needed.

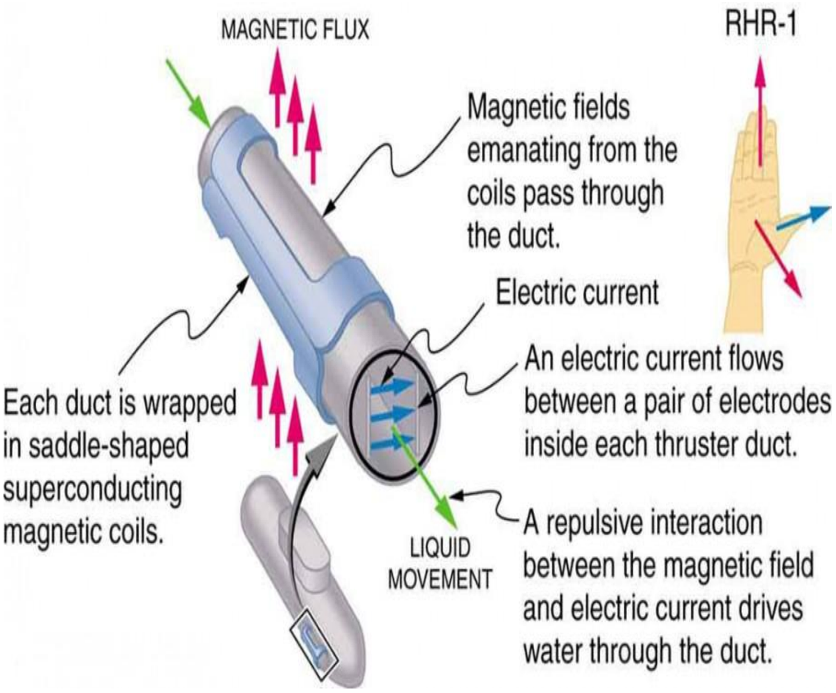


Fig.4.8. An MHD propulsion system in a nuclear submarine could produce significantly less turbulence than propellers and allow it to run more silently. The development of a silent drive submarine was dramatized.

SAQ.1

a) Define the Electric current and current density.

b) What do you mean by Ohm's law and Joule's law?

c) Define the drift velocity.

d) What do you mean by Magnetic field around stationary charge?

e) Calculate the electric current passing through the circuit in which the voltage and resistance be 15V and 10 Ω respectively?

4.5 Biot-Savart law and its application to straight conductor:

What is Biot-Savart Law?

Biot-Savart's law is an equation that gives the magnetic field produced due to a current carrying segment. This segment is taken as a vector quantity known as the current element.

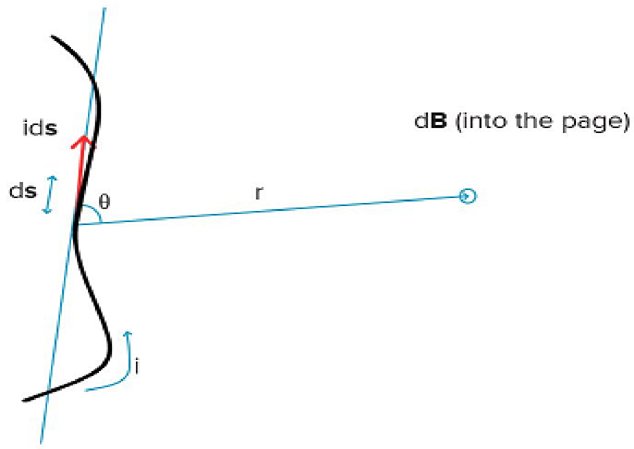


Fig.4.9 Magnetic field produced due to a current carrying segment

What is the Formula of Biot-Savart's Law?

Consider a current carrying wire 'i' in a specific direction as shown in the above figure. Take a small element of the wire of length ds. The direction of this element is along that of the current so that it forms a vector i ds.

To know the magnetic field produced at a point due to this small element, one can apply Biot-Savart's Law. Let the position vector of the point in question drawn from the current element be r and the angle between the two be θ. Then,

$$|dB| = \left(\frac{\mu_0}{4\pi}\right)\left(\frac{Idl\sin\Theta}{r^2}\right)$$

Where

- μ_0 is the permeability of free space and is equal to $4\pi \times 10^{-7}$ TmA^{-1}.

The direction of the magnetic field is always in a plane perpendicular to the line of element and position vector. It is given by the right-hand thumb rule where the thumb points to the direction of conventional current and the other fingers show the magnetic field's direction.

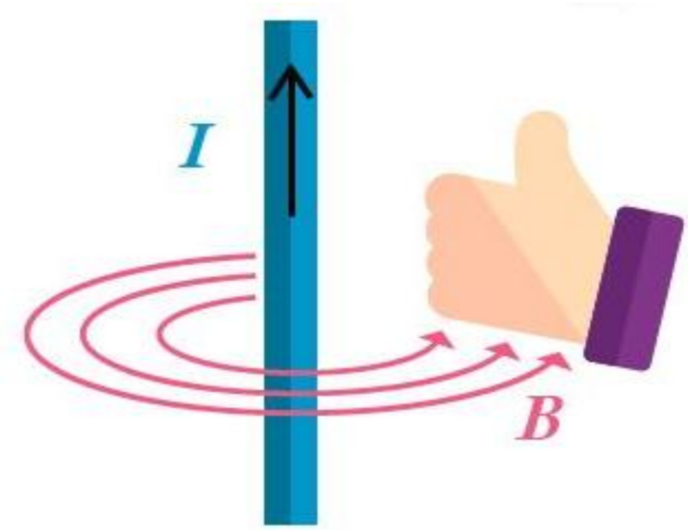


Fig.4.10. A plane perpendicular to the line of element and position vector

In the figure shown above, the direction of the magnetic field is pointing into the page.
This can be expressed in terms of vectors as:

$d\vec{B} = \mu_0 4\pi$ i $\vec{ds} \times \hat{}rr2$

Let us use this law in an example to calculate the Magnetic field due to a wire carrying current in a loop.

Example of Biot-Savart's Law:

The magnetic field of Current Loop:

Consider a current loop of radius R with a current 'i' flowing in it. If we wish to find the electric field at a distance l from the center of the loop due to small element ds, we can use the Biot-Savart Law as:

$d\vec{B} = \mu_0 4\pi$ i $d\vec{s} \times \hat{}rr2$

Consider the current element ids at M which is coming out of a plane in the figure. Since r is in the plane of the page, the two of them are perpendicular to each other. Furthermore, the magnetic field produced db is also in the plane of the page.

$dB = \mu_0 4\pi$ i ds . 1. sin $90^0 r2 = \mu_0 4\pi$ i dsr2

But from the figure,

R2 + l2 = r2

$dB = \mu_0 4\pi$ i dsR2 + l2

Now, if we consider the diametrically opposite element at N, it produces a field such that it's component perpendicular to the axis of the loop is opposite to that of the field produced at M. Thus, only the axial components remain. We can divide the loop into diametrically opposite pairs and apply the same logic.

Also, note that from the figure that

$\alpha = \theta$

$\therefore \cos\theta = R\sqrt{R2 + l2}$

Thus,

$dB \cos\theta = \mu_0 4\pi \text{ i } dsR2 + l2 \times R\sqrt{R2 + l2}$

The total field will be thus,

$B = \int \mu_0 4\pi \text{ i } ds\ R(R2 + l2)32 = \mu_0 4\pi \text{ i } R(R2 + l2)32 \int ds$

$B = \mu_0 4\pi \text{ i } R(R2 + l2)32 \times 2\pi R$

$B = \mu_0 \text{ i } R22(R2 + l2)32$

The right-hand thumb rule can be used to find the direction of magnetic field.

Applications of Biot-Savart's Law:

Some of Biot-Savart's Law applications are given below.

- We can use Biot–Savart law to calculate magnetic responses even at the atomic or molecular level.
- It is also used in aerodynamic theory to calculate the velocity induced by vortex lines.

Importance of Biot-Savart Law

Following are the importance of Biot-Savart law:

- Biot-Savart law is similar to the Coulomb's law in electrostatics.

- The law is applicable for very small conductors too which carry current.
- The law is applicable for symmetrical current distribution.

Genetic Field Due to a Straight Current Carrying wire:

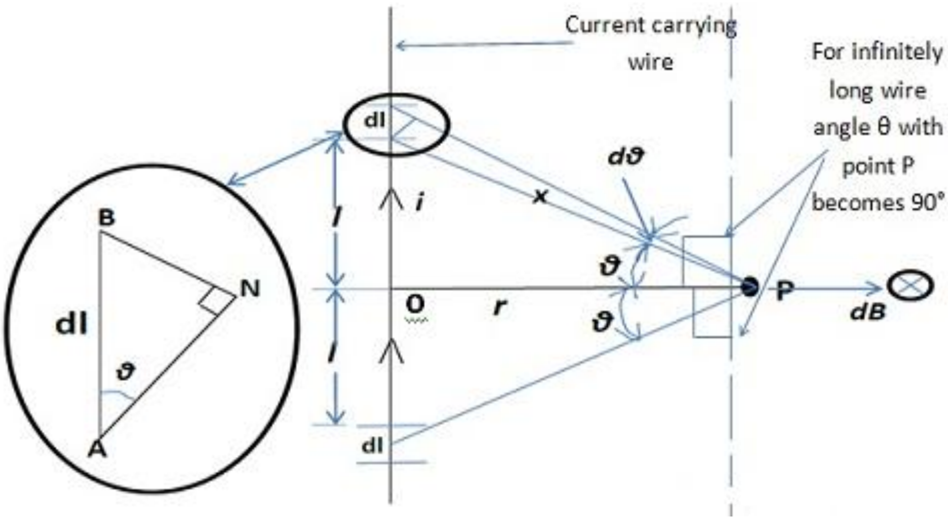


Fig.4.11. Field Due to a Straight Current Carrying wire

According to the Biot-Savart law, magnetic field dB at point P due to current element idl in the above diagram is given by:

$\therefore B= \int(\mu_o/4\pi)idl\cos\theta/x^2$

$=(\mu_o/4\pi)\int idl\cos\theta/x^2$……………….(i)

$dB = (\mu_o/4\pi)idl \sin(90°-\theta) /x^2$

$= (\mu_o/4\pi)idl\cos\theta /x^2$

- Considering triangle ABN: $\cos\theta = AN/dl$

$AN = dl \cos\theta$

- Considering triangle ANP: $\sin(d\theta)\tilde{}d\theta = AN/x$

$AN = x(d\theta)$

- Using the value of AN from the above 2 equations:

$dl\cos\theta = xd\theta$……………………..(ii)

- Considering triangle AOP

$\cos\theta = r/x$

$\therefore x = r/\cos\theta$…………………(iii)

- Using the values of $dl\cos\theta$ from eq.(ii) and x from eq.(iii) in eq.(i):

$B = \int(\mu_o/4\pi)ixd\theta/x^2$

$= \int(\mu_o/4\pi)id\theta/x$

$= \int(\mu_o/4\pi)i(\cos\theta)dx/r$

$\therefore B = (\mu_o/4\pi)(\sin\theta_2 + \sin\theta_1)$

- For infinitely long wire ($\theta_1 = 90°$ $\theta_2 = 90°$):The above equation becomes

$\therefore B = \mu_o i/(2\pi r)$

Magnetic Field on the Axis of a Circular Current Loop:

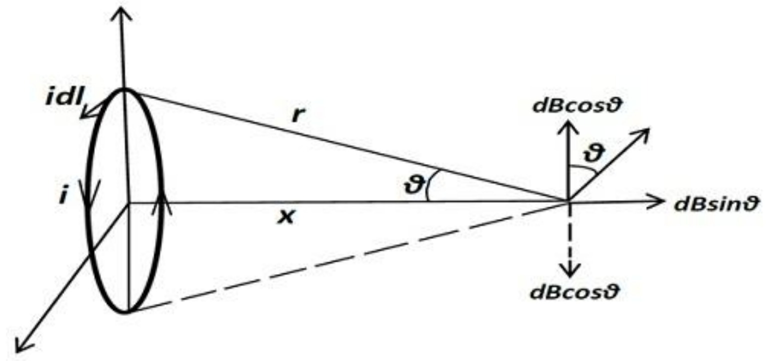


Fig.4.12. Magnetic Field on the Axis of a Circular Current Loop

Magnetic field *dB* at point *P* due to current element *idl*, making right angle to the line joining point P and current element, will be given by Biot-Savart law as:

$dB = (\mu_o/4\pi)idl\ \sin(90°)/r^2 = (\mu_o/4\pi)idl/r^2$

- As we can see in the diagram, the magnetic field dB will have 2 component, i) the vertical component dBcosθ, and ii) the horizontal component dBsinθ
- It is also evident from the diagram that the vertical component dBcosθ will be cancelled by the equal and opposite component due to current element at the opposite of the above current element (due to symmetry).
- So, the total magnetic field will only be due to the horizontal component (dBsinθ) along the positive x-axis

$dB\sin\theta = (\mu_o/4\pi)idl(\sin\theta)/r^2$

$\sin\theta = R/r = R/\sqrt{(x^2 + R^2)}$

$\therefore dB\sin\theta = (\mu_o/4\pi)iRdl/(x^2 + R^2)^{3/2}$

- So, the total magnetic field will be:

$$dB\sin\theta = \frac{\mu_o}{4\pi}\frac{iRdl}{(x^2+R^2)^{\frac{3}{2}}}$$

$$B_{total} = \int dB\sin\theta = \frac{\mu_o}{4\pi}\frac{iR}{(x^2+R^2)^{\frac{3}{2}}}\int_0^{2\pi R} dl$$

$$\therefore B_{total} = \frac{\mu_o}{4\pi}\frac{iR}{(x^2+R^2)^{\frac{3}{2}}}2\pi R = \frac{\mu_o iR^2}{2(x^2+R^2)^{\frac{3}{2}}}$$

- For magnetic field at the center of current loop($x = 0$):

$$B_{total} = \frac{\mu_o iR^2}{2(0^2+R^2)^{\frac{3}{2}}} = \frac{\mu_o iR^2}{2R^3} = \boxed{\frac{\mu_o i}{2R}}$$

Ampere's Circuital Law:

- Ampere's circuital law states that line integral of magnetic field forming a closed loop around the current(i) carrying wire, in the plane normal to the current, is equal to the μ_o times the net current passing through the close loop.

$$\oint \vec{B}.\vec{dl} = \mu_o i$$

Here μ_o = permeability of free space = $4\pi\times10^{-15}N/A^2$, I is net current.

- This law is based on the assumption that the closed loop consists of small elemental parts of length dl, and the total magnetic field of the closed loop will be the integral of magnetic field and the length of these elements $\oint \vec{B}.\vec{dl}.$ This closed loop is called Amperian loop
- Further, this integral will be equal to the multiplication of net current passing through this closed loop and the permeability of free space($\mu_o i$)

Proof-1(Regular coil):

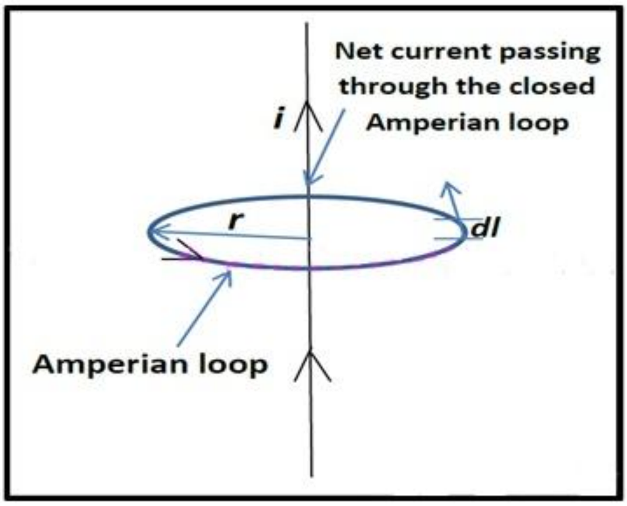


Fig.4.13. Ampere's Circuital Law using Regular coil

To prove: $\int B.dl = \mu_o i$

Starting from the left hand side, we can see in the diagram that angle between the element *dl* and magnetic field *B* is *0°*

$$\int \vec{B}.\vec{dl} = \int Bdl\ cos0 = B\int\ dl$$

We know that magnetic field due to a long current carrying wire is:

$B = \mu_o i/(2\pi r)$

Also, the integral of element will form the whole circle of circumference ($2\pi r$):

$\int dl = 2\pi r$

Now putting the value of B and $\int dl$ in the equation, we get:

$B\int dl = \mu_o i/(2\pi r) \times 2\pi r = \mu_o i$

$\therefore \int B.dl = \mu_o i$

Proof-2(Irregular coil):

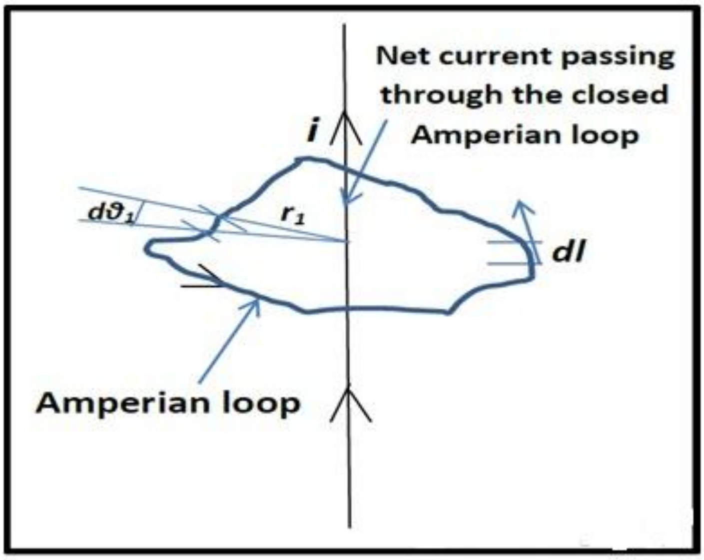


Fig.4.14. Ampere's Circuital Law using Irregular coil

To prove: $\int B.dl = \mu_o i$

Starting from the left hand side:

$\int B.dl_1 = \int \mu_o i/(2\pi r_1) \times dl_1$

We know that: $d\theta_1 = dl_1/r_1$

$\therefore \int \mu_o i/(2\pi r_1) \times dl_1 = \mu_o i/(2\pi)\int d\theta_1 = \mu_o i$

$\int B.dl = \mu_o i$

The Solenoid:

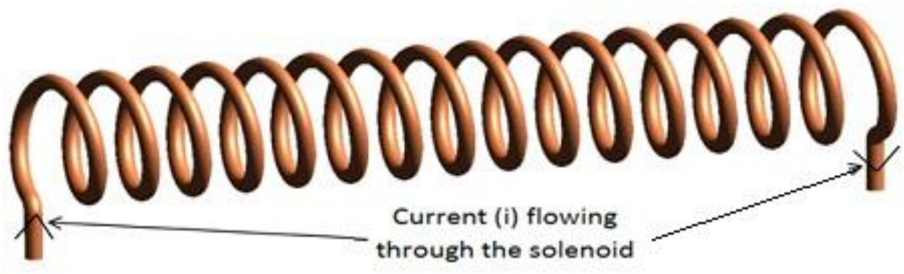


Fig.4.15. Current flowing through the solenoid

The figure above shows a solenoid, which is actually a wire, twisted in many close circular turns, and when the length of solenoid is large compared to the radius of circular turns, then, that solenoid is known as long solenoid. We are going to discuss long solenoid in this section.

Taking a small element dx from the solenoid of *n* number of turns per unit length, at a distance *x* from the point *P* inside the solenoid where magnetic field due to current *i* is to be calculated

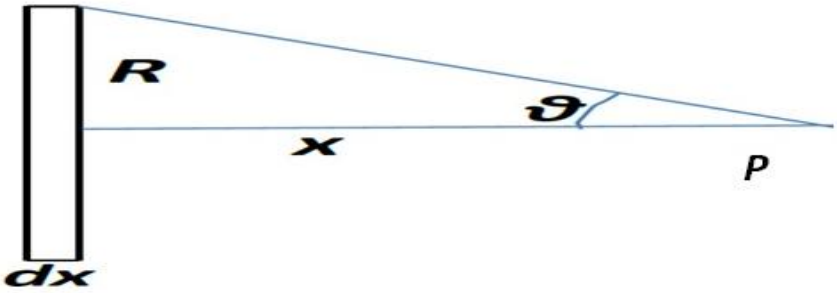


Fig.4.16. Vector diagram

Number of turns inside the element *dx* will be *n×dx*

We already know that magnetic field on the axis of circular loop is given by:

$dB = (\mu_o/2)\ in(dx)R^2/(x^2 + R^2)^{3/2}$

From the above triangle, we can write:

$\tan\theta = R/x$

$x = R \cot\theta$

On differentiating: $dx = -R\ (cosec^2\theta)\ d\theta$

Putting the value of *x* and *dx* in the equation of *dB*, we get:

$$dB = \frac{\mu_o}{2} inR^2 \times \frac{-R cosec^2\theta d\theta}{(R^2 + R^2 cot^2\theta)^{\frac{3}{2}}} = -\frac{\mu_o}{2}\frac{inR^3 cosec^2\theta}{R^3 cosec^3\theta} d\theta$$

$$\therefore B = -\frac{\mu_o}{2} in \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = -\frac{\mu_o}{2} in(-\cos\theta)_{\theta_1}^{\theta_2}$$

$$\therefore B = \frac{\mu_o}{2} in(\cos\theta_2 - \cos\theta_1)$$

Putting the values of $\theta_1 = 180°$, and $\theta_2 = 0°$, we get:

$B = (\mu_o/2)\ in(\cos 0° - \cos 180°) = (\mu_o/2)\ 2in = \mu_o ni$

$\therefore B = \mu_o ni$

For magnetic field at the end (corner) of the solenoid ($\theta_1 = 90°$, $\theta_2 = 0°$)

$\therefore B = \mu_o ni/2$

Note: In the above equations n = number of turns per unit length.

For any point outside the solenoid, the magnetic field is 0(for ideal solenoid).

Solenoids are used in electromagnets, transformers etc.

The Toroid:

A toroid is simply a solenoid bent into a closed circular loop**.** As toroid has no end points, magnetic flux leakage (loss) is minimized, and hence flux linkage is maximized as compared to a solenoid.

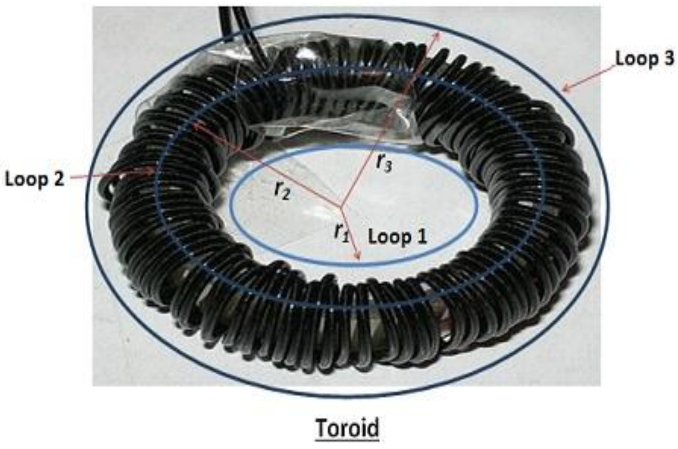


Fig.4.17. Toroid in a closed circular loop

Case-1: Magnetic field at a point in the empty space inside the toroid. We will take an Amperian loop (loop 1). By the Ampere's circuital law:

$$\oint \vec{B_1}.\vec{dl} = \mu_o i$$

We can see in the diagram above that current passing through the inside of the loop 1 is 0

$\int B_1.dl = \mu_o \times 0 = 0$

$\therefore B_1 = 0$

Case-2: Magnetic field at a point inside the toroid (between the turns). We will take another Amperian loop (loop2) of radius r_2. By the Ampere's circuital law:

$$\oint \vec{B_2}.\vec{dl} = \mu_o i$$

We can see in the diagram above that net current passing through the inside of the loop 2 is Ni, where N is the total number of turns in the toroid

$B_2 \int dl = \mu_o \times Ni$

$B_2 2\pi r_2 = \mu_o Ni$

$\therefore B_2 = \mu_o Ni/(2\pi r_2) = \mu_o ni$

Here n = number of turns per unit length of toroid = $N/(2\pi r_2)$

Note: The equation of magnetic field due to toroid is same as that of magnetic field due to solenoid.

Case-3: Magnetic field at a point outside the toroid. We will take another Amperian loop (loop3) of radius r_3. By the Ampere's circuital law:

$$\oint \vec{B_3}.\vec{dl} = \mu_o i$$

We can see in the diagram above that net current passing through the inside of the loop 2 is 0 (Ni current going out of the loop, and Ni current entering the loop, so net current is o)

$\int B_3.dl = \mu_o \times 0 = 0$

$\therefore B_3 = 0$

- Toroid are used in toroidal transformers, toroidal inductors etc.

4.6 Magnetic field due to moving charge:

Experiments show that the magnetic field of moving charge can be expressed as:

$$\mathbf{B} = \frac{\mu_o}{4\pi}\frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

$\mu_o \equiv 4\pi \times 10^{-7}$ $N \cdot s^2/C^2$ is called the permeability of free space. The constant ε_o that is used in electric field calculations is called the permittivity of free space. Note that $\varepsilon_o\mu_o = 1/c^2$.

Example:

Two protons with a vertical displacement of *r* between them move in the x-y plane parallel to the x-axis at the same speed *v* (small compared to *c*). When they are both at $x = 0$, what is the ratio of the electric/magnetic forces between them?

$F_E = kq^2/r^2$

To get F_B acting on top charge, first find B caused by bottom charge:

$$B = \frac{\mu_o}{4\pi}\frac{qv}{r^2}$$

So, the force this field exerts on the top charge is:

$$F = q(-v)\times B = \frac{\mu_o}{4\pi}\frac{q^2v^2}{r^2}$$
$$F_B = q(-v\hat{i})\times\frac{\mu_o}{4\pi}\frac{qv}{r^2}\hat{k}$$

Comparing the ratio of F_B to F_E:

$$\frac{F_B}{F_E} = \frac{v^2}{c^2}$$

Lorentz Force:

Hendrik Antoon Lorentz was a Dutch physicist who explained the theories related to electromagnetic radiation. He mainly concentrated on the relationship between magnetism, light, and electricity.

What is Lorentz Force?

Lorentz force is defined as the combination of the magnetic and electric force on a point charge due to electromagnetic fields. It is used in electromagnetism and is also known as the electromagnetic force. In the year 1895, Hendrik Lorentz derived the modern formula of Lorentz force.

What is Lorentz Force Formula?

Lorentz force formula for the charged particle is as follows:

$F=q(E+v*B)$

Where,

- F is the force acting on the particle
- q is the electric charge of the particle
- v is the velocity
- E is the external electric field
- B is the magnetic field

Lorentz force formula for **continuous charge distribution** is as follows:

$dF=dq(E+v*B)$

Where,

- dF is a force on a small piece of the charge
- dq is the charge of a small piece

When a small piece of charge distribution is divided by the volume dV, the following is the formula:

$f = \rho (E+v*B)$

Where,

- f is the force per unit volume
- ρ is the charge density

With the help of the right-hand rule, it becomes easy to find the direction of the magnetic part of the force.

What is the importance of Lorentz force?

Lorentz force explains the mathematical equations along with the physical importance of forces acting on the charged particles that are traveling through the space containing electric as well as the magnetic field. This is the importance of the Lorentz force.

Right-Hand Rule:

The right-hand rule is useful to find the magnetic force as it becomes easy to visualize the direction as given in Lorentz force law.

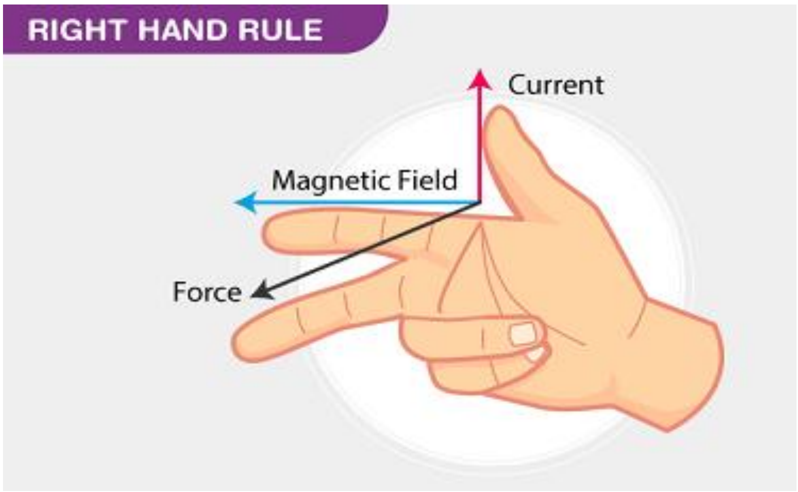


Fig.4.18. The right-hand rule

From the above figure, it is understood that the magnetic force is perpendicular to both the magnetic field and charge velocity.

Applications of Lorentz Force

The following are the applications of Lorentz force:

- Cyclotrons and other particle accelerators use Lorentz force.
- A bubble chamber uses Lorentz force to produce the graph for getting the trajectories of charged particles.
- Cathode ray tube televisions use the concept of Lorentz force to deviate the electrons in a straight line so land on specific spots on the screen.

SAQ.2

a) What do you mean by Biot-Savart law?
b) Define the solenoid and toriod carrying current.
c) What do you mean by Magnetic field due to moving charge?
d) Define the Lorentz force.
e) A circular coil of radius 5.68×10^{-2} m and with 50 turns is carrying a current of 0.10 A. Determine the magnetic field of the circular coil at the center.

4.7 Force between two current-carrying conductors:

AB and CD are two straight very long parallel conductors placed in air at a distance a. They carry currents I_1 and I_2 respectively.(Fig) The magnetic induction due to current I_1 in AB at a distance a is

$$B_1 = \frac{\mu_o I_1}{2\pi a}$$

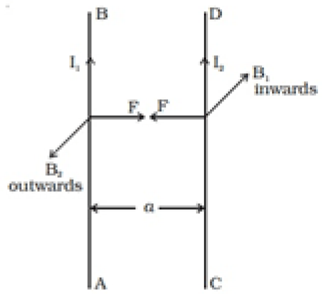


Fig4.19. Force between two long parallel current-carrying conductor

This magnetic field acts perpendicular to the plane of the paper and inwards. The conductor CD with current I_2 is situated in this magnetic field. Hence, force on a segment of length l of CD due to magnetic field B_1 is

$$F = B_1 I_2 l$$

Substituting equation

$$F = \frac{\mu_o I_1 I_2 l}{2\pi a}$$

By Fleming's Left Hand Rule, F acts towards left. Similarly, the magnetic induction due to current I_2 flowing in CD at a distance a is

$$B_2 = \frac{\mu_o I_2}{2\pi a}$$

This magnetic field acts perpendicular to the plane of the paper and outwards. The conductor AB with current I_1, is situated in this field. Hence force on a segment of length l of AB due to magnetic field B_2 is

$$F = B_2 I_1 l$$

Substituting equation

$$\therefore \quad F = \frac{\mu_o I_1 I_2 l}{2\pi a}$$

By Fleming's left hand rule, this force acts towards right. These two forces given in equations attract each other. Hence, two parallel wires carrying currents in the same direction attract each other and if they carry currents in the opposite direction, repel each other.

Definition of ampere:

The force between two parallel wires carrying currents on a segment of length l is

$$F = \frac{\mu_o I_1 I_2}{2\pi a} l$$

$\therefore$ Force per unit length of the conductor is

$$\frac{F}{l} = \frac{\mu_o I_1 I_2}{2\pi a}$$

If $I_1 = I_2 = 1A$, $a = 1m$

$$\frac{F}{l} = \frac{\mu_o}{2\pi} \frac{1 \times 1}{1} = \frac{4\pi \times 10^{-7}}{2\pi} = 2 \times 10^{-7}\ Nm^{-1}$$

The above conditions lead the following definition of ampere.

Ampere is defined as that constant current which when flowing through two parallel infinitely long straight conductors of negligible cross section and placed in air or vacuum at a distance of one meter apart, experience a force of 2×10^{-7} Newton per unit length of the conductor.

Force between two moving charges:

What is the Force due to a Magnetic Field?

Magnetic fields can exert a force on electric charge only if it is moving, just as a moving charge produces a magnetic field. This force increases with both an increase in charge and magnetic field strength. Moreover, the force is greater when charges have higher velocities.

The magnetic force, however, always acts perpendicular to the velocity. Thus, this force can never produce work on the charge and cannot impart it any kinetic energy. The magnetic force is given by:

$$\overrightarrow{F_m} = q\,\overrightarrow{v} \times \overrightarrow{B}$$

Where q is the charge, v is the velocity and B is the magnetic field. Notice that the cross product implies that the force always acts perpendicular to both the velocity and magnetic field. Thus, it always acts out of the plane and does not contribute to any work done on the charge. It can merely change the direction of the velocity but cannot change its magnitude. The direction of the force can be easily determined using Fleming's Right-hand Rule.

What is the Force Due to Electric Field?

The force due to the electric field on a charge is built into its definition. It always acts either parallel or anti-parallel to the electric field and is independent of the velocity of the charge. This means it has the ability to do work and impart energy to the charge.

$$\overrightarrow{F_e} = q\,\overrightarrow{E}$$

4.8 Cyclotron (principle, construction, working, limitations and modification):

Cyclotron:

Cyclotron can be defined as a type of particle accelerator in which charged particles accelerate outwards from the centre along a spiral path. These particles are held to a spiral trajectory by a static magnetic field and accelerated by a rapidly varying electric field

Principle-

A charged particle accelerates to very high speed when kept in a the moderate electric field and in uniform, perpendicular magnetic field. The frequency of revolution open charged particle a magnetic field is independent of speed and radius of the Earth.

Construction:

It consists of 2- D shapes dees made up of metals arrange parallel and enclosed under a steel chamber.
To electromagnet with opposite polarity North and South are applied perpendicular to the dees and does are connected to a high frequency oscillator.

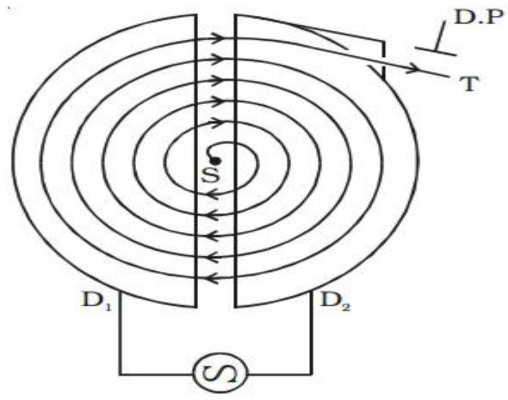


Fig.4.20. Construction of Cyclotron

It consists of a hollow metal cylinder divided into two sections D1 and D2 called Dees, enclosed in an evacuated chamber (Fig). The Dees are kept separated and a source of ions is placed at the centre in the gap between the Dees. They are placed between the pole pieces of a strong electromagnet. The magnetic field acts perpendicular to the plane of the Dees. The Dees are connected to a high frequency oscillator.

Working:

When a positive ion of charge q and mass m is emitted from the source, it is accelerated towards the Dee having a negative potential at that instant of time. Due to the normal magnetic field, the ion experiences magnetic lorentz force and moves in a circular path. By the time the ion arrives at the gap between the Dees, the polarity of the Dees gets reversed. Hence the particle is once again accelerated and moves into the other Dee with a greater velocity along a circle of greater radius. Thus the particle moves in a spiral path of increasing radius and when it comes near the edge, it is taken out with the help of a deflector plate (D.P). The particle with high energy is now allowed to hit the target T.

When the particle moves along a circle of radius r with a velocity v, the magnetic Lorentz force provides the necessary centripetal force.

$Bqv = (vm2) / r$

$v /r = Bq / m = constant$

The time taken to describe a semi-circle

$t = \pi r / v$

Substituting equation

$t = \pi m/ Bq$

It is clear from equation that the time taken by the ion to describe a semi-circle is independent of

(i) the radius (r) of the path and (ii) the velocity (v) of the particle

Hence, period of rotation

$T = 2t$

$T = 2 \pi m / Bq = constant$

So, in a uniform magnetic field, the ion traverses all the circles in exactly the same time. The frequency of rotation of the particle,

$v = 1 /T = Bq / 2 \pi m$

If the high frequency oscillator is adjusted to produce oscillations of frequency as given in equation, resonance occurs.

Cyclotron is used to accelerate protons, deutrons and α - particles.

$$Bq\upsilon = \frac{m\upsilon^2}{r}$$

$$\frac{\upsilon}{r} = \frac{Bq}{m} = \text{constant} \qquad \ldots(1)$$

$$t = \frac{\pi r}{\upsilon} \qquad \ldots(2)$$

$$t = \frac{\pi m}{Bq} \qquad \ldots(3)$$

$$T = \frac{2\pi m}{Bq} = \text{constant} \qquad \ldots(4)$$

$$\upsilon = \frac{1}{T} = \frac{Bq}{2\pi m} \qquad \ldots(5)$$

Limitations of Cyclotron:

- Cyclotron cannot accelerate electrons because electrons are of very small mass.
- A cyclotron cannot be used to accelerate neutral particles.
- It cannot accelerate positively charged particles with large mass due to the relativistic effect.

Cyclotron Modifications:

At first, improvements in cyclotron design were directed at the construction of larger machines that could accelerate particles to greater velocities. Soon, however, a new problem arose. Physical laws state that nothing can travel faster than the speed of light. Thus, adding more and more energy to a particle will not make that particle's speed increase indefinitely. Instead, as the particle's velocity approaches the speed of light, additional energy supplied to it appears in the form of increased mass. A particle whose mass is constantly increasing, however, begins to travel in a path different from that of a particle with constant mass. The practical significance of this fact is that, as the velocity of particles in a cyclotron begins to approach the speed of light, those particles start to fall "out of sync" with the current change that drives them back and forth between dees.

Two different modifications-or a combination of the two-can be made in the basic cyclotron design to deal with this problem. One approach is to gradually change the rate at which the electrical field alternates between the dees. The goal here is to have the sign change occur at exactly the moment that particles have reached a certain point within the dees. As the particles speed up and gain weight, the rate at which electrical current alternates between the two dees slows down to "catch up" with the particles.

In the 1950s, a number of machines containing this design element were built in various countries. Those machines were known as frequency modulated (FM) cyclotrons, synchrocyclotrons, or, in the Soviet Union, phasotrons. The maximum particle energy attained with machines of this design ranged from about 100 MeV to about 1 GeV.

A second solution for the mass increase problem is to alter the magnetic field of the machine in such a way as to maintain precise control over the particles' paths. This principle has been incorporated into the machines that are now the most powerful cyclotrons in the world, the synchrotrons.

A synchrotron consists essentially of a hollow circular tube (the ring) through which particles are accelerated. The particles are actually accelerated to velocities close to the speed of light in smaller machines before they are injected into the main ring. Once they are within the main ring, particles receive additional jolts of energy from accelerating chambers placed at various locations around the ring. At other locations around the ring, very strong magnets control the path followed by the particles. As particles pick up energy and tend to spiral outward, the magnetic fields are increased, pushing particles back into a circular path. The most powerful synchrotrons now in operation can produce particles with energies of at least 400 GeV.

In the 1970s, nuclear physicists proposed the design and construction of the most powerful synchrotron of all, the superconducting super collider (SSC). The SSC was expected to have an accelerating ring 51 mi (82.9 km) in circumference with the ability to produce particles having an energy of 20 TeV. Estimated cost of the SSC was originally set at about $4 billion. Shortly after construction of the machine at Waxahachie, Texas began, however, the United States congress decided to discontinue funding for the project.

Betatron:

A Betatron was developed by D W Kerst to accelerate the electrons to high energies.

Principle:

The principle of the Betatron is same as that of the transformer. In transformer, if an alternating current is passed through the primary coil an alternating magnetic field will appear in the coil. This field produces an induced e.m.f. in the secondary coil. Similarly the changing magnetic flux induces an e.m.f. tangentially along a circular path for the electron which accelerates the electrons to high energies. The electrons is kept accelerating in circular path of constant radius with the help of increasing magnetic field.

Construction:

The Betatron is consists of an evacuated doughnut chamber in which electrons are produced by indirectly heated cathode. The doughnut tube is placed between two strong electromagnet such that, when the a.c current is passed in the electromagnets the flux increases in the centre of doughnut (single coil).

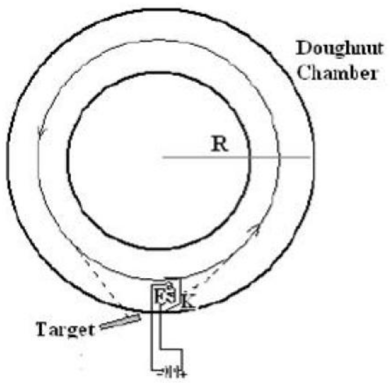


Fig.4.21 Construction of Betatron

Working:

When the electron appears at K (cathode) in doughnut tube and the electromagnets are energized the magnetic field increases, the increasing magnetic field has two effects

(i) Induced e.m.f. is produced in electron orbit by changing magnetic flux that gives an additional energy to electron. According to Faraday's law induced e.m.f is

$$e.m.f = -\frac{d\phi}{dt}$$

(ii) A radial force (magnetic force) is produced by action of magnetic field whose direction

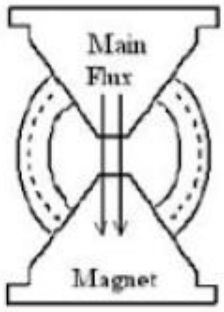
is perpendicular to the electron velocity which keeps the electron moving in circular path.

The force is balanced by

Centripetal force, i.e.,

$$qvB = \frac{mv^2}{r}$$

The particle acceleration occurs only with increasing flux (the duration when the flux increases from zero to a maximum value) i.e., the first quarter of the a.c. cycle (T/4 sec), after this the flux starts decreasing which result in decreasing velocity therefore the electron is kept in the tube only for T/4 sec. As the electrons get faster they need a larger magnetic field to keep moving at a constant radius, which is provided by the increasing field.



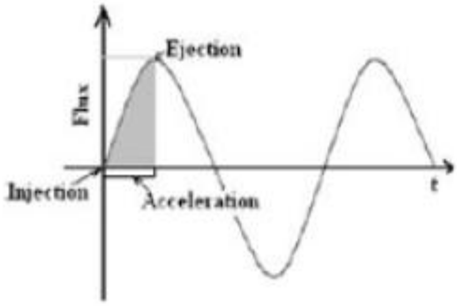


Fig.4.22. Working of Betatron

Betatron Condition:

Induced e.m.f in the coil from Faraday's law of electromagnetic induction

$$e.m.f = -\frac{d\phi}{dt}$$

Work done on an electron in one revolution

$$W = e.m.f \times chargeofelectron$$
$$W = e\frac{d\phi}{dt}$$

Work done = tangential Force 'F' on electron x distance traveled in one revolution

$$W = F \times 2\pi r = e\frac{d\phi}{dt}$$
$$F = \frac{e}{2\pi r}\frac{d\phi}{dt}$$

The electron moves in circular path. The magnetic force is balanced by centripetal force,

i.e.,

$$evB = \frac{mv^2}{r}$$
$$eBr = mv = p$$

From Newtons second law radial force

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt}$$
$$F = \frac{d(eBr)}{dt}$$

In order to maintain path of constant radius (r is constt.)

$$F = er\frac{dB}{dt}$$

Equations are equal, equating both

$$F = er\frac{dB}{dt} = \frac{e}{2\pi r}\frac{d\phi}{dt}$$
$$\frac{d\phi}{dt} = 2\pi r^2\frac{dB}{dt}$$

Integrating the above equation

The relation is known as Betatron condition.

It shows that to ensure that the electron moves in circular path of constant radius, the magnetic flux within the orbit of radius R is always twice what it would have been if magnetic field were uniform throughout the orbit.

Energy Gained by Electron:

The particles have maximum energy when the magnetic field is at its strongest value but the formula used for the cyclotron will not work for Betatron because the electron motion is relativistic. However, if the total energy is much greater than the rest energy then

$$E = pc$$

As the centripetal force is again provided by the Lorentz force,The momentum of the electron will

$$eBr = mv = p$$

and hence Energy

$$E = Berc$$

Number of Revolutions Taken by Electron:

In T/4 seconds if the electron takes N revolutions in circular path of constant radii then the total distance traveled by the electron in gaining the maximum energy E is

$$S = N \times 2\pi r$$
$$N \times 2\pi r = c \times \frac{T}{4}$$
$$N = \frac{c}{4 \times 2\pi f \times r}$$

$$N = c/4\omega r$$

Average Energy Gained per Revolution:

Average energy gained per revolution (Eav) will be given as

$$E_{av} = E/N$$

where

E_{av} =Total energy gained by electron and

N = Number of Revolutions taken

SAQ.3

a) Define the Force between two current carrying conductor
b) What do you mean by Cyclotron and its limitations?
c) What do you mean by Betatron?
d) A nonrelativistic particle with a charge twice that of an electron moves through a uniform magnetic field. The field has strength of $\pi/4$ tesla and is perpendicular to the velocity of the particle. What is the particle's mass if it has a cyclotron frequency of 1,000 hertz?

Examples:

Q.1. Calculate the current through the circuit in which the voltage and resistance be 15V and 3Ω respectively?

Solution: The given parameters are,

V = 15V

R = 3Ω

The equation for current using Ohm's law is,

$I=V/R$

$I = 15/3 = 5A$

Q.2. The voltage and resistance of a circuit are given as 10V and 4Ω respectively. Calculate the current through the circuit?

Solution: The given parameters are,

$V = 10V$

$R = 4\Omega$

The equation for current using Ohm's law is,

$I=V/R$

$I = 10/4 = 2.5A$

Q.3. If the resistance of an electric iron is 50 Ω and a current of 3.2 A flows through the resistance. Find the voltage between two points.

Solution:

If we are asked to calculate the value of voltage with the value of current and resistance given to us, then cover V in the triangle. Now, we are left with I and R or more precisely I × R.

Therefore, we use the following formula to calculate the value of V:

$V = I \times R$

Substituting the values in the equation, we get

$V = 3.2\ A \times 50 = 160\ V$

$V = 160V$

Q.4. If an electric heater consumes electricity at the rate of 500W and the potential difference between the two terminals of electric circuit is 250V, calculate the electric current and resistance through the circuit.

Solution: Given, power input (P) = 500 W

Potential difference (V) = 250 V

Electric current (I) =?

Resistance (R) through the circuit =?

We know that power $(P)=VI(P)=VI$

Or, $500W=250V\times I500W=250V\times I$

Or, $I=500W\div 250V=2AI=500W\div 250V=2A$

We know, resistance $R=VIR=VI$

Or, $R=250V\div 2A=125\Omega$

Q.5. Calculate the electric current passing through the circuit in which the voltage and resistance be 25V and 5 Ω respectively?

Solution: $V = 25\ V$

$R = 5\Omega$

Here, we have to apply ohm's law formula.

The equation for the electric current using Ohm's law is,

$I= V/R$

Putting the known values, we get

$I = 25/5$

$I = 5$ A

Thus the value of electric current is 5 A.

Q.6. Determine the current density when 40 Amperes of current is flowing through the battery in a given area of 10 m^2.

Solution:

It is given that,

$I = 40$ A,

Area = 10 m^2

The current density formula is given by,

$J = I / A$

$= 40 / 10$

$J = 4$ A/m^2.

Q.7. Find the resistance of an electrical circuit that has voltage supply of 10 Volts and current of 5mA.

Solution:

$V = 10$ V, $I = 5$ mA $= 0.005$ A

$R = V / I$

$= 10$ V $/ 0.005$ A

= 2000 Ω = 2 kΩ

Q.8. Calculate the force on the wire given B = 1.50 T, l = 5.00 cm, and I = 20.0 A.

Solution:

The force can be found with the given information by using

$F=IlB\sin\theta$

and noting that the angle θ between I and B is 90º, so that sin θ = 1.

Entering the given values into F = IlB sin θ yields

$F = IlB \sin \theta = (20.0\ A)(0.0500\ m)(1.50\ T)(1).$

The units for tesla are 1 T=N/A·m1 T=N/A·m; thus,

F = 1.50 N.

Q.9. A current of 1.0 A exists in a copper wire of cross section 1.0 mm^2. Assuming one free electron per atom calculate drift speed of free electrons in wire. The density of copper is 9000 kg/m.

Solution:

Current I related to drift velocity vd is given by

$I = n \times e \times A \times |v_d|$

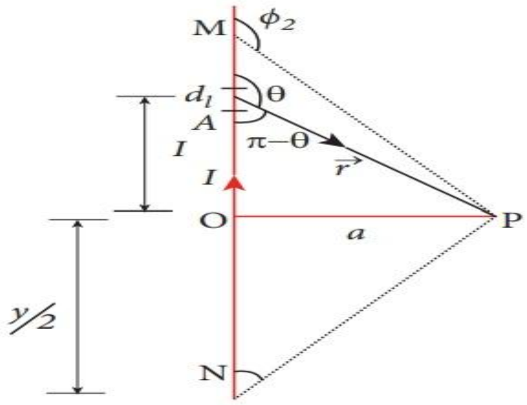
where n is the number density of atoms , e is the charge on electron, A is area of cross section.

gram atomic weight of Copper atom 63.5 ; number density of atoms is given by

$$n = \frac{6.0 \times 10^{23} \times 9000}{63.5 \times 10^{-3}} = 8.5 \times 10^{28}\, m^{-3}$$

$$Hence\ |v_d| = \frac{I}{n \times e \times A} = \frac{1}{8.5 \times 10^{28} \times 1.602 \times 10^{-19} \times 10^{-6}} = 7.3 \times 10^{-5}\, m/s$$

Q.10. Calculate the magnetic field at a point P which is perpendicular bisector to current carrying straight wire as shown in figure.



Solution:

Let the length MN = y and the point P is on its perpendicular bisector. Let O be the point on the conductor as shown in figure.

Therefore, $OM = ON = \frac{y}{2}$, then

$$\cos\varphi_1 = \frac{\frac{y}{2}}{\sqrt{\frac{y^2}{4}+a^2}} = \frac{adjacent\, length}{hypotenuse length}$$

$$= \frac{ON}{PN} = -\frac{\frac{y}{2}}{\sqrt{\frac{y^2}{4}+a^2}} = -\frac{y}{\sqrt{y^2+4a^2}}$$

$$\cos\varphi_2 = \frac{adjacent\, length}{hypotenuse length} = \frac{OM}{PM}$$

$$= -\frac{\frac{y}{2}}{\sqrt{\frac{y^2}{4}+a^2}} = -\frac{y}{\sqrt{y^2+4a^2}}$$

Hence,

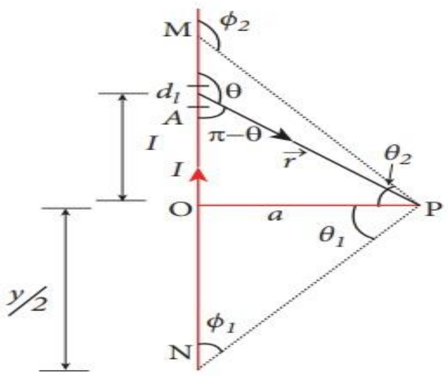
$$\vec{B} = \frac{\mu_\circ I}{4\pi a}\frac{2y}{\sqrt{y^2+4a^2}}\hat{n}$$

For long straight wire, $y \to \infty$,

$$\vec{B} = \frac{\mu_\circ I}{2\pi a}\hat{n}$$

Q.11. Show that for a straight conductor, the magnetic field

$$\vec{B} = \frac{\mu_\circ I}{4\pi a}(\cos\varphi_1 - \cos\varphi_2)\hat{n}$$
$$= \frac{\mu_\circ I}{4\pi a}(\sin\theta_1 + \sin\theta_2)\hat{n}$$



Solution:

In a right angle triangle OPN, let the angle $\angle$ OPN = $\theta 1$ which implies, $\phi 1 = \pi/2 - \theta 1$ and also in a right angle triangle OPM, $\angle$OPM = $\theta 2$ which implies, $\phi 2 = \pi/2 + \theta 2$

Hence,

$$\vec{B} = \frac{\mu_\circ I}{4\pi a}\left(\cos\left(\frac{\pi}{2}-\theta_1\right)-\cos\left(\frac{\pi}{2}+\theta_2\right)\right)\hat{n}$$
$$= \frac{\mu_\circ I}{4\pi a}(\sin\theta_1+\sin\theta_2)\hat{n}$$

Q.12. Two parallel straight wires A and B, of length 10cm are carrying currents of 8A and 5A respectively in the same direction. The distance between the wires is 4cm. Find the force acting on wire A due to wire B. Will the force acting on wire B due to wire A be the same as the above answer?

Solution: Given, L = 0.1m, i_A = 8A, i_B = 5A, r = 0.04m

Force between two straight, parallel current carrying wires is given by:

$F = \mu_o i_A i_B L/(2\pi r) = 4\pi\times10^{-7}\times8\times5\times0.1/(2\pi\times0.04) = 2\times10^{-5}$N

$\therefore F = 2\times10^{-5}$N, directed towards wire B (ans)

Yes, the force acting on wire A due to wire B will be equal to the force on wire B due to A.

Q.13. Determine the magnitude of the magnetic field of a wire loop at the center of the circle with radius R and current I.

Solution:

The magnitude of the magnetic field of the wire loop is given as:

$\frac{\mu_0 I}{2R}$

Q.14. A circular coil of radius 5 × 10^{-2} m and with 40 turns is carrying a current of 0.25 A. Determine the magnetic field of the circular coil at the center.

Solution:

The radius of the circular coil = 5 × 10^{-2} m

Number of turns of the circular coil = 40

Current carried by the circular coil = 0.25 A

Magnetic field is given as:

$$B = \frac{\mu_0 N I}{2a}$$

$$= \frac{4\pi \times 10^{-7} T.m/A(40)0.25A}{2.50 \times 10^{-2} m}$$

$= 1.2 \times 10^{-4}$ T

Q.15. Determine the magnetic field at the center of the semicircular piece of wire with radius 0.20 m. The current carried by the semicircular piece of wire is 150 A.

Solution:

The radius of the semicircular piece of wire = 0.20 m

Current carried by the semicircular piece of wire = 150 A

Magnetic field is given as:

$$B = \frac{\mu_0 NI}{2a}$$

The differential form of Biot-Savart law is given as:

$$dB = \frac{\mu_0 I}{4\pi}\frac{dI sin\theta}{r^2} \; B = \frac{\mu_0}{4\pi} I \int \frac{dI \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\int dI$$

$$= \frac{\mu_0}{4\pi}\frac{I}{r^2}\pi r = \frac{\mu_0 I}{4r} = \frac{4\pi \times 10^{-7} T.m/A(150A)}{4(0.20m)}$$

$=2.4\times10^{-4}$ T

Q.16. The round coil is of 10 turns as well as radius 1m. If a flow of current through it is 5A, then determine the field in the coil from a 2m distance.

Solution:

Number of turns n= 10, Current 5A, Length= 2m, Radius = 1m

The biot savart law statement is given by, B = (μo / 4π) × (2πnI / r)

Then, substitute the above values in the above equation

B = (μo / 4π) × (2 × π × 10 × 5 / 1) = 314.16 × 10-7 T

Q.17. A long straight wire is carrying a current of 50A in the plane of paper in north-south direction. Find the magnitude and direction of magnetic field at a point 2.5meast of the wire.

Solution:

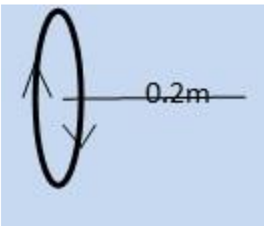
Given: i = 50A, r = 2.5m

The magnetic field due to long wire is given by:

$B = \mu_o i/(2\pi r) = 4\pi\times10^{-7}\times50/(2\pi\times2.5) = 4\times10^{-6}T = 0.04G$ (ans)

The direction will be given by cross product of current element and position vector of point from the current element (idl x r), which isgiven by Fleming's right hand thumb rule, and it will be normal to the plane of paper coming outwards ().

Q.18. A circular coil of wire has 100 turns of radius 8cm, and carrying a current of 0.4A in clockwise direction when viewed from the right side. Find the magnitude and direction of magnetic field: i) at the center of coil, and ii) at a distance of 20cm from the center of coil towards the right and normal to the coil.



Solution: Given, N = 100, r = 0.08m, i = 0.4A, x = 0.2m 0.2m

i) Magnetic field at the center of circular coil is given by:

$B = \mu_o Ni/(2r)$

$B = 4\pi\times10^{-7}\times100\times0.4/(2\times0.08) = 3.14\times10^{-4}T = 3.14G$ (ans)

Direction of magnetic field will be normal to the plane of coil, and from right to leftt side of coil.

ii) Magnetic field at an axial distance from center of coil is given by:

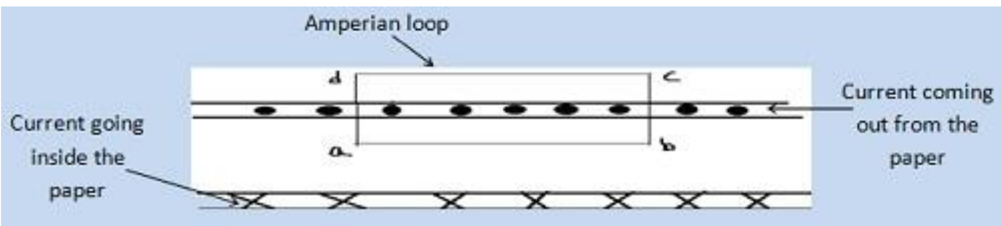
$B = \mu_o NiR^2/(x^2 + R^2)^{(3/2)}$

$B = 4\pi\times10^{-7}\times100\times0.4\times0.0064/(0.0064 + 0.0400)^{(3/2)} = 3.22\times10^{-7}/0.00999$

$\therefore B = 3.22\times10^{-5}T = 0.322G$ (ans)

Q.19.Using Ampere's circuital law, derive the magnetic field inside the solenoid of length L, carrying current i and having N number of turns.

Solution:



Using Ampere's circuital law:

$i_L =$ iNl/LConsidering the Amperian loop abcd of sides l each, current passing through the loop will be:

${}^{b}_{a}\int B.dl + {}^{c}_{b}\int B.dl + {}^{d}_{c}\int B.dl + {}^{a}_{d}\int B.dl = \mu_o Nil/L$

$Blcos0 + Blcos90 + 0lcos180 + Blcos270 = \mu_o Nil/L$

$Bl = \mu_o Nil/L$

$\therefore B = \mu_o Ni/L = \mu_o ni$

Q.20. A closely wound solenoid has length of 80cm, and radius of 0.9cm with 5 layers of windings of 400 turns each. The current flowing through the solenoid is 8A. Find the magnitude of magnetic field inside the solenoid: i) at the center, and ii) at an end of solenoid.

Solution: Given, L = 0.8m, r = 0.009, N = 5×400 = 2000, i = 8A

Number of turns per unit length (n) = 2000/0.8 = 2500/m

i) Magnetic field at the center of solenoid is given by:

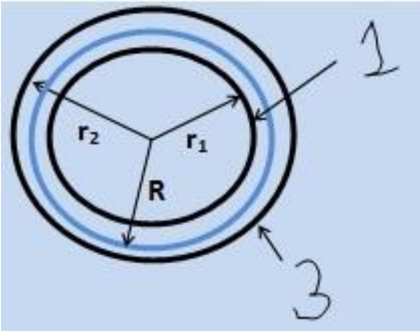
$B = \mu_o ni = 4\pi\times10^{-7}\times2500\times8 = 0.0251T = 251G$ (ans)

ii) Magnetic field at one end of the solenoid is given by:

$B = \mu_o ni/2 = 0.0251/2 = 0.01255T = 125.5G$ (ans)

Q.21. A toroid has inner radius 25cm and outer radius 26cm, with 3500 turns and 11A current flowing through it. Find the magnetic field:i) inside the core of the toroid,ii) outside the toroid and iii) in the empty space surrounded by the toroid.

Solution: Given, r_1= 25cm, r_2 = 26cm, N = 3500, i = 11A



$R = r_1 + (r_2 - r_1)/2$

$= 25 + (26-25)/2$

$=25.5cm$

Number of turns per unit length (n) = $N/(2\pi R)$

$\therefore n = 3500/(2\pi\times0.255) = 2184.5/m$

i) Magnetic field inside the core of toroid is given by:

$B = \mu_o ni = 4\pi\times10^{-7}\times2184.5\times11 = 0.0302T = 302G$ (ans)

ii) Magnetic field outside the toroid is zero because net current through the Amperian loop(1) is zero, hence, by ampere circuital law:

$B = 0$ (ans)

iii) Magnetic field inside the empty space surrounded by toroid is zero because net current through the Amperian loop(3) is zero, hence, by ampere circuital law:

$B = 0$ (ans)

Q.22. A non-relativistic particle with a charge twice that of an electron moves through a uniform magnetic field. The field has strength of $\pi/4$ tesla and is perpendicular to the velocity of the particle. What is the particle's mass if it has a cyclotron frequency of 1,600 hertz?

Solution:

$$F_{\text{lorentz}} = F_{\text{centripetal}}$$

$$qvB = \frac{mv^2}{r}$$

Angular velocity: $\omega = 2\pi f = \frac{v}{r}$

$$qvB = mv2\pi f \rightarrow m = \frac{qB}{2\pi f}$$

$$m = \frac{2(1.6 \times 10^{-19})\pi/4}{2\pi(1,600)} = \frac{1.6 \times 10^{-19}}{(1.6 \times 10^3)4} = 2.5 \times 10^{-23}$$

Q.23. An electron moving perpendicular to a uniform magnetic field 0.500 T undergoes circular motion of radius 2.80 mm. What is the speed of electron?

Solution

Charge of an electron q = -1.60 × 10-19 C

⟹ |q| = 1.60 ×10−19 C

Magnitude of magnetic field B = 0.500 T

Mass of the electron, m = 9.11 × 10-31 kg

Radius of the orbit, r = 2.50 mm = 2.50 × 10-3 m

Velocity of the electron, v = |q| rB/m

$$\text{Velocity of the electron, } v = |q|\frac{rB}{m}$$

$$v = 1.60\times10^{-19}\times\frac{2.50\times10^{-3}\times0.500}{9.11\times10^{-31}}$$

$$v = 2.195\times10^{8}\, m\,s^{-1}$$

v = 2.195 ×108 m s−1

Q.24. A proton moves in a uniform magnetic field of strength 0.500 T magnetic field is directed along the x-axis. At initial time, t = 0 s, the proton has velocity

$$\vec{v} = \left(1.95\times10^{5}\,\hat{i} + 2.00\times10^{5}\,\hat{k}\right) m\,s^{-1}$$

.

Find

(a) At initial time, what is the acceleration of the proton?

(b) Is the path circular or helical?. If helical, calculate the radius of helical trajectory and also calculate the pitch of the helix (Note: Pitch of the helix is the distance travelled along the helix axis per revolution).

Solution

Magnetic field $\vec{B} = 0.500\hat{i}\ T$

Velocity of the particle

$$\vec{v} = \left(1.95\times10^{5}\hat{i} + 2.00\times10^{5}\hat{k}\right)ms^{-1}$$

Charge of the proton $q = 1.60\times10^{-19}C$

Mass of the proton $m = 1.67\times10^{-27}kg$

(a) The force experienced by the proton is

$$\vec{F} = q\left(\vec{v}\times\vec{B}\right)$$
$$= 1.60\times10^{-19}\times\left(\left(1.95\times10^{5}\hat{i} + 2.00\times10^{5}\hat{k}\right)\times\left(0.500\hat{i}\right)\right)$$
$$\vec{F} = 1.60\times10^{-14}N\,\hat{j}$$

Therefore, from Newton's second law,

$$\vec{a} = \frac{1}{m}\vec{F} = \frac{1}{1.67\times10^{-27}}\left(1.60\times10^{-14}\right)$$
$$= 9.58\times10^{12}ms^{-2}$$

(b) Trajectory is helical

Radius of helical path is

$$R = \frac{mv_z}{|q|B} = \frac{1.67\times10^{-27}\times2.00\times10^{5}}{1.60\times10^{-19}\times0.500}$$
$$= 4.175\times10^{-3}m = 4.18\,mm$$

Pitch of the helix is the distance travelled along x-axis in a time T, which is P = vx T

But time,

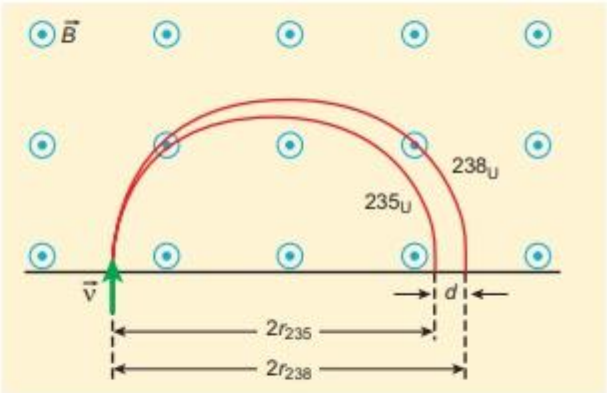
$$T = \frac{2\pi}{\omega} = \frac{2\pi m}{|q|B} = \frac{2\times3.14\times1.67\times10^{-27}}{1.60\times10^{-19}\times0.500}$$
$$= 13.1\times10^{-8}s$$

Hence, pitch of the helix is

$$P = v_x T = \left(1.95\times10^{5}\right)\left(13.1\times10^{-8}\right)$$
$$= 25.5\times10^{-3}m = 25.5\ mm$$

The proton experiences appreciable acceleration in the magnetic field, hence the pitch of the helix is almost six times greater than the radius of the helix.

Q.25. Two singly ionized isotopes of uranium 23592U and 23892U (isotopes have same atomic number but different mass number) are sent with velocity 1.00 × 105 m s-1 into a magnetic field of strength 0.500 T normally. Compute the distance between the two isotopes after they complete a semi-circle. Also compute the time taken by each isotope to complete one semi-circular path. (Given: masses of the isotopes: m235 = 3.90 x 10-25 kg and m238 = 3.95 x 10-25 kg)



Solution

Since isotopes are singly ionized, they have equal charge which is equal to the charge of an electron, q = - 1.6 × 10-19 C. Mass of uranium 23592U and 23892U are 3.90 × 10-25 kg and 3.95 × 10-25 kg respectively. Magnetic field applied, B = 0.500 T. Velocity of the electron is 1.00 × 105 m s-1, then

(a) the radius of the path of 23592U is r235

$$r_{235} = \frac{m_{235}v}{|q|B} = \frac{3.90\times10^{-25}\times1.00\times10^{5}}{1.6\times10^{-19}\times0.500}$$

$$= 48.8\times10^{-2} m$$

$$r_{235} = 48.8 \text{ cm}$$

The diameter of the semi-circle due to $^{235}_{92}U$ is $d_{235} = 2r_{235} = 97.6$ cm

The radius of the path of $^{238}_{92}U$ is r_{238} then

$$r_{238} = \frac{m_{238}v}{|q|B} = \frac{3.90\times10^{-25}\times1.00\times10^{5}}{1.6\times10^{-19}\times0.500}$$

$$= 49.4\times10^{-2} m$$

$$r_{238} = 49.4 \text{ cm}$$

The diameter of the semi-circle due to 23892U is d238 = 2r238 = 98.8 cm

Therefore the separation distance between the isotopes is Δd = d238 − d235 = 1.2cm

(b) The time taken by each isotope to complete one semi-circular path are

$$t_{235} = \frac{\text{magnitude of the displacement}}{\text{velocity}}$$

$$= \frac{97.6\times10^{-2}}{1.00\times10^{5}} = 9.76\times10^{-6} s = 9.76\ \mu s$$

$$t_{238} = \frac{\text{magnitude of the displacement}}{\text{velocity}}$$

$$= \frac{98.8\times10^{-2}}{1.00\times10^{5}} = 9.88\times10^{-6} s = 9.88\ \mu s$$

Q.26. Let E be the electric field of magnitude 6.0 × 106 N C-1 and B be the magnetic field magnitude 0.83 T. Suppose an electron is accelerated with a potential of 200 V, will it show zero deflection?. If not, at what potential will it show zero deflection?

Solution:

Electric field, E = 6.0 × 106 N C-1 and magnetic field, B = 0.83 T.

Then

$$v = \frac{E}{B} = \frac{6.0\times10^{6}}{0.83} = 7.23\times10^{6}\,m\,s^{-1}$$

When an electron goes with this velocity, it shows null deflection. Since the accelerating potential is 200 V, the electron acquires kinetic energy because of this accelerating potential. Hence,

$$\frac{1}{2}mv^{2} = eV \Rightarrow v = \sqrt{\frac{eV}{2m}}$$

Since the mass of the electron, m = 9 .1×10−31 kg and charge of an electron, |q| = e = 1.6 ×10−19 C. The velocity due to accelerating potential 200 V

$$v_{200} = \sqrt{\frac{2\left(1.6\times10^{-19}\right)(200)}{\left(9.1\times10^{-31}\right)}} = 8.39\times10^{6}\,m\,s^{-1}$$

Since the speed v200 > v, the electron is deflected towards direction of Lorentz force. So, in order to have null deflection, the potential, we have to supply is

$$V = \frac{1}{2}\frac{mv^{2}}{e} = \frac{\left(9.1\times10^{-31}\right)\times\left(7.23\times10^{6}\right)^{2}}{2\times\left(1.6\times10^{-19}\right)}$$

$$V = 148.65\ V$$

V =148 65 V

Q.27. Suppose a cyclotron is operated to accelerate protons with a magnetic field of strength 1 T. Calculate the frequency in which the electric field between two Dees could be reversed.

Solution

Magnetic field B = 1 T

Mass of the proton, mp = 1.67 ×10−27 kg

Charge of the proton, q = 1.60 ×10−19 C

$$f = \frac{qB}{2\pi m_p} = \frac{\left(1.60\times10^{-19}\right)(1)}{2(3.14)(1.67\times10^{-27})}$$
$$= 15.3\times10^{6}\,Hz = 15.3\,MHz$$

Q.28. A cyclotron has an oscillator frequency of 10 MHz. What should be the operating magnetic field for accelerating protons? Also, calculate the kinetic energy (in MeV) of the proton beam produced by the accelerator. (e = 1.60 × 10^{-19} C, m_p = 1.67 × 10^{-27} kg, 1 MeV = 1.6 × 10^{-13} J)

Solution:

The oscillator frequency should be equal to the proton's cyclotron frequency.

We use the following formula to find the strength of the magnetic field:

$$B = \frac{2\pi m v}{q}$$

Substituting the values in the above equation, we get

$$B = \frac{6.3\times1.67\times10^{-27}\times10^{7}}{1.6\times10^{-19}} = 0.66\,T$$

The final velocity of the proton can be calculated using the following formula:

$$v = r \times 2\pi v$$

Substituting the values in the above equation, we get

$$v = 0.6 \times 6.3 \times 10^{7} = 3.78 \times 10^{7}\, m/s$$

To find the kinetic energy of the proton, we use the following formula:

$$E = \frac{1}{2}mv^{2}$$

Substituting the values in the above equation, we get

$$E = \frac{1.67\times10^{-27}\times14.3\times10^{14}}{2\times1.6\times10^{-13}} = 7\,MeV$$

The kinetic energy of the proton beam produced by the accelerator is 7 MeV

Summary:

1. An electric current is a flow of electric charge in a circuit. More specifically, the electric current is the rate of charge flow past a given point in an electric circuit. The charge can be negatively charged electrons or positive charge carriers including protons, positive ions or holes.

2. Current density is a quantity related to electric current. As a vector, current density has magnitude and direction. By definition, current density is the product of charge density (ρ) and

velocity (v). The magnitude of current density is also equivalent to the ratio of current (I) to area (A).

3. Ohm's Law states that the current flowing in a circuit is directly proportional to the applied potential difference and inversely proportional to the resistance in the circuit. In other words by doubling the voltage across a circuit the current will also double.

4. Joule's laws are two: first about heat produced by an electric current, and second about how the energy of a gas relates to pressure, volume. Joule's second law says that the internal energy of an ideal gas does not change if volume and pressure change, but does change if temperature changes. Joule's law is important.

5. In physics a drift velocity is the average velocity attained by charged particles, such as electrons, in a material due to an electric field. In general, an electron in a conductor will propagate randomly at the Fermi velocity, resulting in an average velocity of zero.

6. A stationary charge will produce only an electric field in the surrounding space. If the charge is moving, a magnetic field is also produced. An electric field can be produced also by a changing magnetic field.

7. Magnetic fields exert forces on moving charges. ... The direction of the magnetic force on a moving charge is perpendicular to the plane formed by v and B and follows right hand rule–1. The magnitude of the force is proportional to q, v, B, and the sine of the angle between v and B.

8. Current is generally defined as the rate of flow of charge. Magnetic field due to a current-carrying conductor depends on the current in the conductor and distance of the point from the conductor. The direction of the magnetic field is perpendicular to the wire.

9. The Biot Savart Law states that it is a mathematical expression which illustrates the magnetic field produced by a stable electric current in the particular electromagnetism of physics. It tells the magnetic field toward the magnitude, length, direction, as well as closeness of the electric current.

10. A solenoid is a coil of wire with electric current flowing through it, giving it north and south magnetic poles and a magnetic field. ... A solenoid converts electromagnetic energy into motion, providing a burst of power that can move a specific part of a device.

11. If a solenoid is bent in a circular shape and the ends are joined, we get a toroid. Alternatively, one can start with a non-conducting ring and wind a conducting wire closely on it. The magnetic field in such a toroid can be obtained using Ampere's Law.

12. Right Hand Rule: Magnetic fields exert forces on moving charges. This force is one of the most basic known. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by v and B and follows right hand rule–1 (RHR-1) as shown.

13. Lorentz force, the force exerted on a charged particle q moving with velocity v through an electric field E and magnetic field B. The entire electromagnetic force F on the charged particle is

called the Lorentz force (after the Dutch physicist Hendrik A. Lorentz) and is given by $F = qE + qv \times B$.

14. Thus, when two parallel wires carry current in the same direction, they exert equal and opposite attractive forces on each other. Two wires that carry current in opposite directions. Two parallel wires carry current in opposite directions.

15. Magnetic fields exert forces on moving charges, and so they exert forces on other magnets, all of which have moving charges.

16. A cyclotron accelerates charged particles outwards from the center of a flat cylindrical vacuum chamber along a spiral path. The particles are held to a spiral trajectory by a static magnetic field and accelerated by a rapidly varying (radio frequency) electric field.

17. Betatron, a type of particle accelerator that uses the electric field induced by a varying magnetic field to accelerate electrons (beta particles) to high speeds in a circular orbit. ... Modern compact betatron designs are used to produce high-energy X-ray beams for a variety of applications.

Terminal Questions:

1) Define and explain in detail of Electric current and current density.
2) Write short notes on: (i) Ohm's law, (ii) Joule's law, (iii) Drift velocity.
3) Explain the working of Magnetic field around stationary charge.

4) What do you mean by Magnetic field around moving charge and current carrying conductor?
5) Explain the working principle of Biot-Savart law and its application to straight conductor.
6) Discuss the circular loop, solenoid and toriod carrying current using in Biot-Savart law
7) Explain the working principle of Magnetic field due to moving charge.
8) What do you mean by Lorentz force?
9) Discuss the Force between two current carrying conductor and two moving charges.
10) Explain the working principle, construction, limitations of Cyclotron.
11) Explain the working principle of the Betatron.
12) The voltage and resistance of a circuit are given as 20V and 10Ω respectively. Calculate the current through the circuit?
13) Determine the current density when 50 Amperes of current is flowing through the battery in a given area of 150 m^2.
14) An EMF source of 8.0 V is connected to a purely resistive electrical appliance (a light bulb). An electric current of 2.0 A flows through it. Consider the conducting wires to be resistance-free. Calculate the resistance offered by the electrical appliance.
15) An electric geyser consumes electricity at the rate of 1000W. If the potential difference through the electric circuit is 250 V, find the resistance offered by geyser and electric current through the circuit.

16) A current of 2.0 A exists in a copper wire of cross section 4.0 mm^2. Assuming one free electron per atom calculate drift speed of free electrons in wire. The density of copper is 12000 kg/m.

17) Calculate the force on the wire given B = 2.50 T, l = 10.00 cm, and I = 30.0 A.

18) A circular coil of radius 15×10^{-2} m and with 600 turns is carrying a current of 0.45A. Determine the magnetic field of the circular coil at the center.

19) The round coil is of 10 turns as well as radius 1m. If a flow of current through it is 5A, then determine the field in the coil from a 2m distance.

20) Calculate the magnetic field inside a solenoid, when (a) the length of the solenoid becomes twice and fixed number of turns, (b) both the length of the solenoid and number of turns are double, (c) the number of turns becomes twice for the fixed length of the solenoid, Compare the results.

21) A closely wound solenoid has length of 80cm, and radius of 0.15cm with 8 layers of windings of 600 turns each. The current flowing through the solenoid is 15A. Find the magnitude of magnetic field inside the solenoid: i) at the center, and ii) at an end of solenoid.

22) A toroid has inner radius 35cm and outer radius 36cm, with 4500 turns and 20A current flowing through it. Find the magnetic field:i) inside the core of the toroid,ii) outside the toroid and iii) in the empty space surrounded by the toroid.

23) An electron moving perpendicular to a uniform magnetic field 0.800 T undergoes circular motion of radius 3.90 mm. What is the speed of electron?

24) A cyclotron has an oscillator frequency of 30 MHz. What should be the operating magnetic field for accelerating protons? Also, calculate the kinetic energy (in MeV) of the proton beam produced by the accelerator. ($e = 1.60 \times 10^{-19}$ C, $m_p = 1.67 \times 10^{-27}$ kg, 1 MeV = 1.6×10^{-13} J)

Unit 05- Laws of Magnetostatics

Structure:

5.1 Introduction:

The electric lines of force that represent the field of a positive electric charge in space consist of a family of straight lines radiating uniformly in all directions from the charge where they originate. A second positive charge placed in the field would travel radially away from the first charge.

A fundamental feature of magnetic fields that distinguishes them from electric fields is that the field lines form closed loops. We cannot saw the magnet in half to isolate the north and the south poles

Ampere's circuital law states that line integral of magnetic field forming a closed loop around the current(i) carrying wire, in the plane normal to the current, is equal to the μ_o times the net current passing through the close loop.

Ampere's law gives another method to calculate the magnetic field due to a given current distribution. Ampere's law may be derived from the Biot-Savart law and Biot-Savart law may be derived from the Ampere's law. Ampere's law is more useful under certain symmetrical conditions.

According to *Ampere circuital law* the line integral of magnetic field B around any closed path is equal to times total current l enclosed by that closed path. Where l is the steady current. But this *equation* is logically *inconsistent.* Therefore, *Ampere law* is ambiguous as it does not provide *continuity* to current path.

Maxwell found the shortcoming in Ampere's law and he modified Ampere's law to include time-varying electric fields. For Ampere's circuital law to be correct Maxwell assumed that there has to be some current existing between the plates of the capacitor. Outside the capacitor current was due to the flow of electrons.

A Conduction current is due to the flow of electrons in a circuit. It exists even if electrons flow at a uniform rate. Displacement current is due to the time-varying electric field.

In vector calculus, a vector potential is a vector field whose curl is a given vector field. This is analogous to a scalar potential, which is a scalar field whose gradient is a given vector field.

Magnetic vector potential, A, is the vector quantity in classical electromagnetism defined so that its curl is equal to the magnetic field. Together with the electric potential φ, the magnetic vector potential can be used to specify the electric field E as well.

The concept of the magnetic *vector potential*, which is field, but still we will have existence of magnetic field *due* to the current as shown in this. So, if we take a *circular loop* of radius R prime in a plane perpendicular to a so, we're going to now focus on a *straight wire.*

Magnetic Flux Density is amount of magnetic flux through unit area taken perpendicular to direction of magnetic flux. Flux Density (B) is related to Magnetic Field (H) by B=μH. It is measured in Webers per square meter equivalent to Teslas [T]

5.2 Objective:

After studying this unit you should be able to

a) Study and identify Lines of forces, Gauss law in magneto-statics.

b) Explain and identify Ampere circuital law (statement and derivation), its applications to current carrying rod (hollow and solid).
c) Study and identify Inconsistency of Ampere circuital law with equation of continuity.
d) Explain and identify Modification of Ampere circuital law by Maxwell with introducing concepts of displacement currents and its importance. Comparison of displacement current and conduction current.
e) Study and identify Vector potential and its expression due to straight conductor and circular loop.
f) Explain and identify Derivation of magnetic flux density using vector potential for circular loop.

5.3 Lines of forces, Gauss law in magneto-statics:

Lines of Force (or) Magnetic Lines of Force:

Firstly, we shall done Magnetic Lines of Force and understand what they are. As you know, magnets have a force field around them; this is the reason whenever we bring a magnet near a piece of metal, the latter either gets attracted to the magnet or is repelled away. Hence, the movement of the metal either towards the magnet or away from it is due to it being inside the magnetic field of the magnet. This magnetic field is represented by imaginary lines that originate from the North Pole of the magnet and travel in an elliptical fashion towards its south pole where they 'merge' back into the magnet; magnetic lines of force are simply drawn to visualize a magnet’s force field that exists around it. Check the diagram below for a clearer representation.

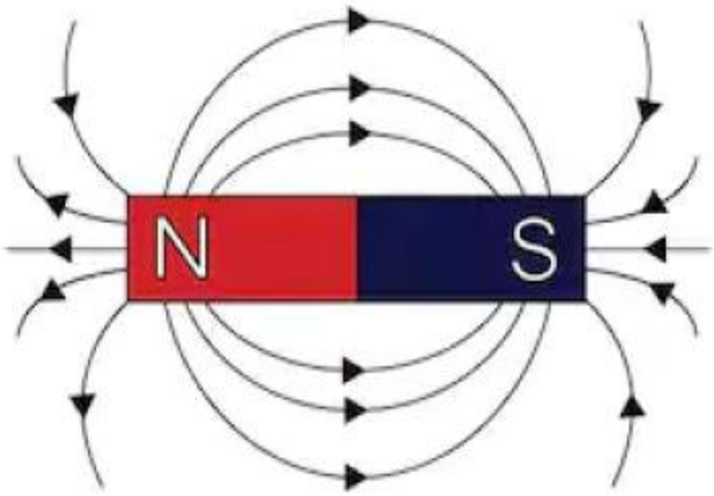


Fig 5.1 Magnetic line of force

Properties of Magnetic Lines of Force:

1. Magnetic lines of force are essentially three-dimensional meaning their effects can be felt in 3D space (along the x, y, and z axes).
2. Magnetic lines of force are typically focused or concentrated at the ends of a magnet; this is because those are the regions (poles) where the magnetic force exerted by the magnet on its surroundings is the highest.
3. The strength of magnetic lines of force (or simply, the magnetic feld) is inversely proportional to their distance from the poles.
4. No two magnetic lines of force cross each other, ever. If they did, a compass at that point would try to point at two directions (one for each field line) which is impossible.

5. The direction of magnetic field lines as they move from the North pole to the South pole is dependent on their ability to track the path that presents the least resistance.
6. Magnetic field lines form a closed elliptical loop (they are imaginary lines, but this explains the presence of the magnetic field around the magnet). The field lines travel inside the magnet from South to North pole and emerge from the North pole once more to travel towards the South.

Magnetic Lines of Force Notes:

1. Magnetic flux density is also known as flux density as well as magnetic induction. All three terms have the same unit which is Wb/m^2 or Tesla (T).
2. Magnetic lines of force are a representation of the magnetic field which is the area surrounding the magnet where the magnetic force exerted by it is felt (by ferromagnetic substances, for example).
3. The magnetic field is a vector quantity because from the North pole of the magnet to its South pole; finding the tangent of any point that lies on any of the magnetic lines of force gives us the direction of the magnetic force at that instance.
4. The magnetic field is measured as Magnetic Flux Density (or simply flux density) and its SI unit is Tesla (T); 1 Tesla (T) = 1 Weber/m^2 (Wb/m^2).
5. Magnetic field lines contract longitudinally and expand transversely.
6. Magnetic flux density is not the same as Magnetic flux; Magnetic flux density is a vector quantity while Magnetic flux is a scalar

quantity. Additionally, magnetic flux is measured in Weber (measured in volt/seconds) while magnetic flux density is measured in Weber/sq. meters.

Gauss's Law for Magnetic Field:

Gauss's law for magnetism states that no magnetic monopoles exist and that the total flux through a closed surface must be zero. This page describes the Time-domain integral and differential forms of Gauss's law for magnetism and how the law can be derived. The frequency-domain equation is also given. At the end of the page, a brief history of the Gauss's law for magnetism is provided.

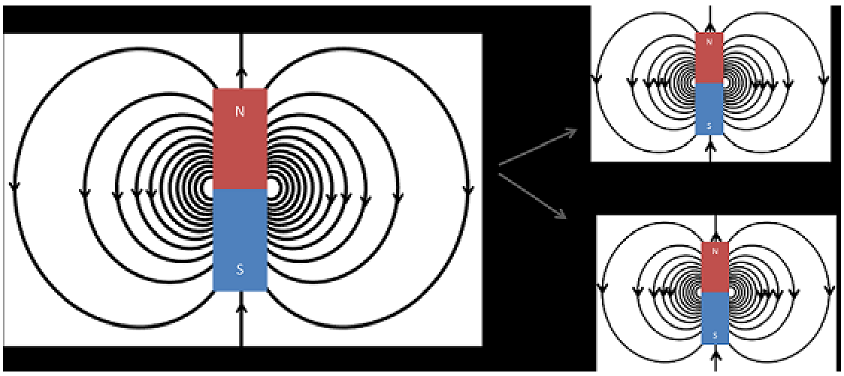


Fig 5.2 When a bar magnet is cut in two, you get two bar magnets

Integral equation:

The Gauss's law for magnetic fields in integral form is given by:

$$\oint_S \mathbf{b} \cdot \mathbf{da} = 0,$$

Where, b is the magnetic flux.

The equation states that there is no net magnetic flux b (which can be thought of as the number of magnetic field lines through an area) that passes through an arbitrary closed surface S. This means the number of magnetic field lines that enter and exit through this closed surface S is the same. This is explained by the concept of a magnet that has a north and a south pole, where the strength of the North Pole is equal to the strength of the South Pole Fig. This is equivalent to saying that a magnetic monopole, meaning a solitary north or south pole, does not exist because for every positive magnetic pole, there must be an equal amount of negative magnetic poles.

Differential equation:

Gauss's law for magnetic fields in the differential form can be derived using the divergence theorem. The divergence theorem states:

$$\int_V (\nabla \cdot \mathbf{f}) dv = \oint_S \mathbf{f} \cdot d\mathbf{a},$$

where f is a vector. The right-hand side looks very similar to above Equation. Using the divergence theorem, above Equation is rewritten as follows:

$$0 = \oint_S \mathbf{b} \cdot d\mathbf{a} = \int_V (\nabla \cdot \mathbf{b}) dv.$$

Because the expression is set to zero, the integrand (∇·b) must be zero also. Thus the differential form of Gauss's law becomes:

$$\nabla \cdot \mathbf{b} = 0.$$

Derivation using Biot-Savart law:

Gauss's law can be derived using the Biot-Savart law, which is defined as:

$$\mathbf{b}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{(\mathbf{j}(\mathbf{r}')dv) \times \,\hat{\underline{\mathbf{r}}}}{|\mathbf{r} - \mathbf{r}'|^2},$$

Where:

b(r) is the magnetic flux at the point r

j(r′) is the current density at the point r′

μ_0 is the magnetic permeability of free space.

Taking the divergence of both sides of Equations yields:

$$\nabla \cdot \mathbf{b}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \nabla \cdot \frac{(\mathbf{j}(\mathbf{r}')dv) \times \,\hat{\underline{\mathbf{r}}}}{|\mathbf{r} - \mathbf{r}'|^2}.$$

To carry through the divergence of the integrand in above Equation, the following vector identity is used:

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}).$$

Thus, the integrand becomes:

$$\left[\mathbf{j}(\mathbf{r}')\cdot\left(\nabla\times\frac{\hat{\mathbf{r}}}{|\mathbf{r}-\mathbf{r}'|^2}\right)\right]-\left[\frac{\hat{\mathbf{r}}}{|\mathbf{r}-\mathbf{r}'|^2}\cdot(\nabla\times\mathbf{j}(\mathbf{r}'))\right]$$

The first part of Equation is zero as the curl of $\frac{\hat{\mathbf{r}}}{|\mathbf{r}-\mathbf{r}'|^2}$ is zero. The second part of Equation becomes zero because jj depends on r′ and ∇ depends only on r. Plugging this back into , the right-hand side of the expression becomes zero. Thus, we see that:

$$\nabla\cdot\mathbf{b}(\mathbf{r})=0,$$

This is Gauss's law for magnetism in differential form.

3.4 Ampere circuital law (statement and derivation), its applications to current carrying rod (hollow and solid).

Statement of Ampere Circuital Law:

Ampere's Circuital Law states the relationship between the current and the magnetic field created by it.

This law states that the integral of magnetic field density (B) along an imaginary closed path is equal to the product of current enclosed by the path and permeability of the medium.

$$\oint \overrightarrow{B}\cdot\overrightarrow{dl}=\mu_0\,I$$

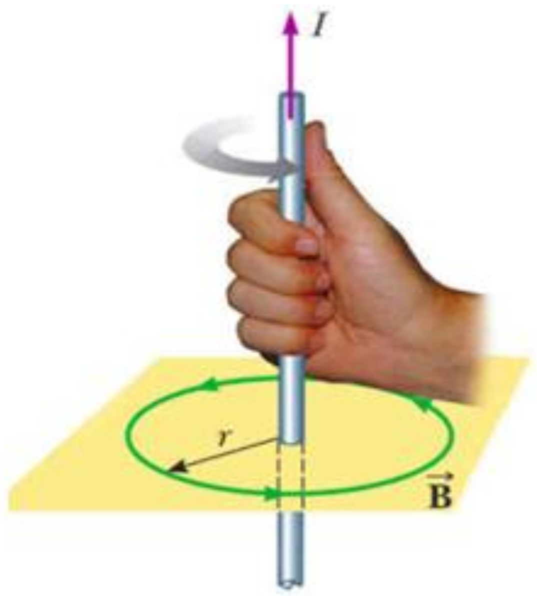


Fig.5.3 Magnetic field B around the conductor

Derivation of Ampere Circuital Law:

Proof:

Consider a straight conductor in which current i is flowing. The current produces the magnetic field B around the conductor. The magnetic field lines are in the form of concentric circles.

Ampere showed that the flux density B at any point near the conductor is directly proportional to the current i and inversely proportional to the distance 'r' from the conductor, so:

$B \propto i$ -------------------(1)

$B \propto \frac{1}{r}$ ---------------------(2)

Combining the above two relations, we have

$B \propto \frac{i}{r}$

$B = \frac{\mu_0 i}{2\pi r}$ -----------------(3)

Here $\mu_o/2\pi$ is the constant of proportionality.

μ_o is called the permeability of free space having value = $4\pi \times 10^{-7}$ WbA^{-1} m^{-1}

$B \times 2\pi = \mu_0 i$ ------------------(4)

Where the length of the path is called the circumference of the circle?

Divide the circle representing the magnetic field line into a large number of small elements each of length dl. The quantity B.dl is calculated for each element as:

$$B.dl = Bdl\cos = Bdl\cos 0 = Bdl$$

Foe complete circle:

$$\oint B.dl = \oint Bdl = \oint \frac{\mu_0}{2\pi r} dl$$

$$\oint B.dl = \frac{\mu_0}{2\pi r} \oint dl = \frac{\mu_0 i}{2\pi r} \times 2\pi r$$

$$\oint B.dl = \mu_0 i$$

Applications of Ampere's law:

a) Current carrying hollow rod:

Consider a conducting hollow cylinder with inner radius r_1 and outer radius r_2. And current I is following through it.

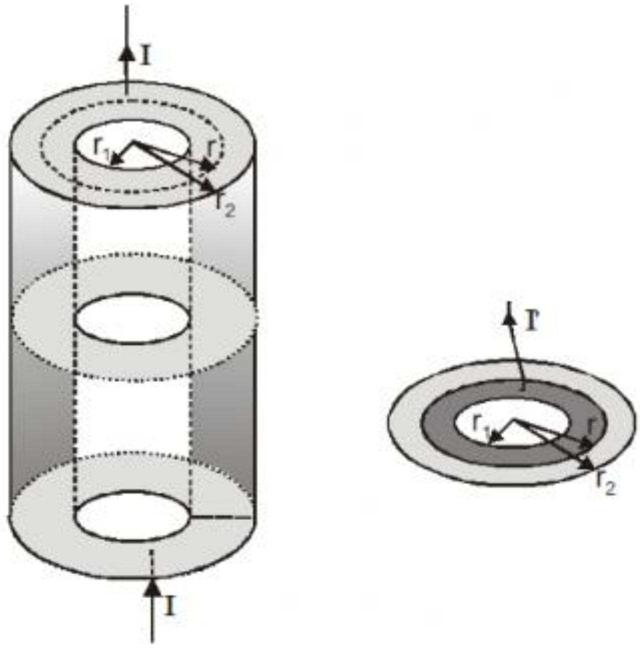


Fig.5.4 Hollow cylinder with inner radius r_1 and outer radius r_2

(I) For r < r1

$\sum I$= 0 and hence B=0

(II) For r1 < r < r2

Now current I is following through $[\pi r_2^2 - \pi r_1^2]$

So, current per unit area = $\frac{I}{\pi(r2^2 - r1^2)}$

Current flowing through area is bet'' r1 < r < r2 is I = $\frac{I}{\pi(r2^2 - r1^2)}$ $(\pi r^2 - \pi r_1^2)$

By using ampere's law for circle of radius r$\oint \vec{B}.\vec{dl}$ = $\mu_0 \sum I$

$$\text{or} \quad \oint B d\ell \cos 0^\circ = \mu_0 \left[\frac{I\left(r^2 - r_1^2\right)}{r_2^2 - r_1^2} \right]$$

$$\text{or} \quad B\oint d\ell = \mu_0 I\left[\frac{r^2 - r_1^2}{r_2^2 - r_1^2}\right]$$

$$\text{or} \quad B = \frac{\mu_0 I}{2\pi r}\left[\frac{r^2 - r_1^2}{r_1^2 - r_1^2}\right]$$

$$\left[\because \oint d\ell = 2\pi r\right]$$

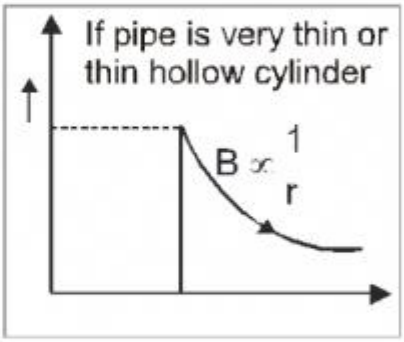


Fig.5.5 Curve for very thin or thin hollow cylinder

(a) For r = r2

$$B = \frac{\mu_0 I}{2\pi r_2}$$

(b) For r> r2

$$B = \frac{\mu_0 I}{2\pi r}$$

b) Current carrying rod Solid:

Along straight rod of radius R carries a steady current I that is uniformly distributed through the cross-section of the wire.

For finding the behavior of magnetic field due to this rod, let us divide the whole region into two parts.

(i) $r \geq R$ and

(ii) $r < R$

r = distance from the center of the wire.

For $r \geq R$: For closed circular path denoted by (1) from symmetry $\vec{B}$ must be constant in magnitude and parallel to $\vec{dl}$ at every point on this circle. Because the total current passing through the plane of the circle is I.

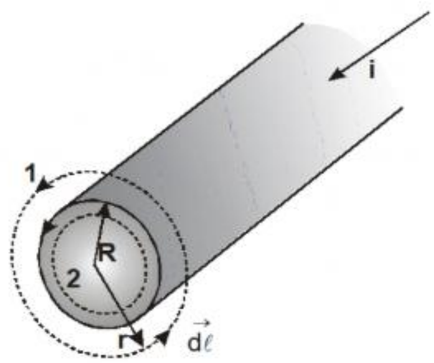


Fig.5.6 Straight rod of radius R carries a steady current I

From Ampere's law

$$\oint \vec{B}.\vec{d\ell} = \mu_0 \, (I_{net})$$

Or

$$B(2\pi r) = \mu_0 I$$

and

$$B = \frac{\mu_0}{2\pi}\frac{I}{r} \qquad (\text{for } r \geq R)$$

For r < R: The current I passing through the plane of circle 2 is less than the total current I. Because the current is uniform over the cross-section of the wire.

Current through unit area

$$= I/\pi R^2$$

So current through area enclosed by circle 2 is

$$r' = I\pi r^2/\pi R^2$$

or $$I' = \left(\frac{r^2}{R^2}\right) I$$

Now we apply Ampere's law for circle 2.

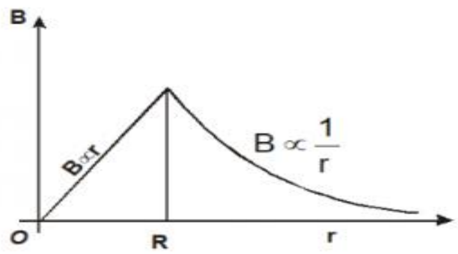


Fig.5.7 Curve for Straight rod

$$\oint \vec{B}.\vec{d\ell} = \mu_0 \, (I_{net})$$

$$B(2\pi r) = \mu_0\left(\frac{r^2}{R^2}\right) I$$

Or

$$B = \left(\frac{\mu_0 I}{2\pi R^2}\right) r \qquad (\text{for } r < R)$$

The magnitude of the magnetic field versus r for this configuration is plotted in figure. Note the inside the wire B →0 as r→0. Note also that $eq^{n.}$ (a) and eq^{n} (b) give the same value of the magnetic field at r =R, demonstrating that the magnetic field is continuous at the surface of the wire.

SAQ.1

a) What do you mean by the Lines of forces?
b) Write the short note on Gauss law in magneto-statics.
c) What do you mean by statement of Ampere circuital law?
d) Discuss the current carrying rod using hollow in Ampere circuital law.
e) The repulsive force between two magnetic poles in air is 9×10^{-3} N. If the two poles are equal in strength and are separated by a distance of 20 cm, calculate the pole strength of each pole.

5.5 Inconsistency of Ampere circuital law with equation of continuity:

- According to Maxwell there was some inconsistency in the Ampere's circuital law.
- This means Ampere's circuital law was correct for some cases but not correct for some.
- Maxwell took different scenarios i.e. he took a capacitor and tried to calculate magnetic field at a specific point in a piece of a capacitor.

- Point P as shown in the figure is where he determined the value of B, assuming some current I is flowing through the circuit.
- He considered 3 different amperial loops as shown in the figs.
- Ampere's circuital law should be same for all the 3 setups.

Case 1: Considered a surface of radius r & dl is the circumference of the surface, then from Ampere's circuital law

$$\int B.dl = \mu_0 I$$

$$\text{or } B(2\pi r) = \mu_0 I$$

$$\text{or } B = \mu_0 I \ / \ 2\pi r$$

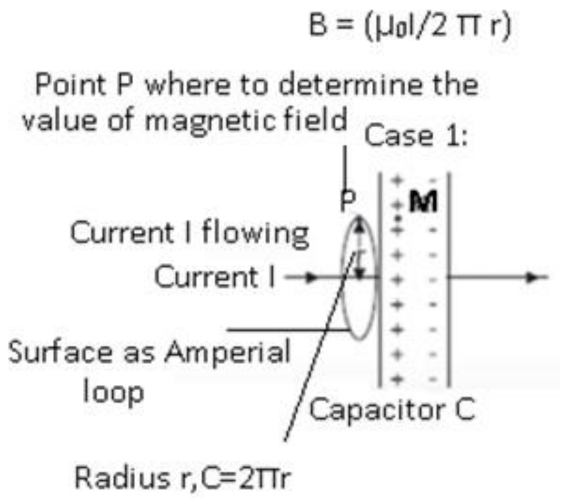


Fig.5.8 Surface of radius r & dl is the circumference of the surface

Case 2: Considering a surface likes a box & its lid is open and applying the Ampere's circuital law

$$\int B.dl = \mu_0 I$$

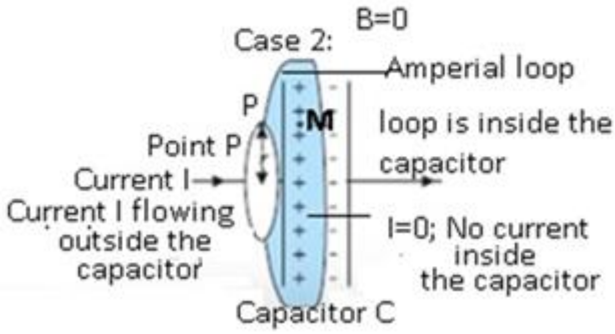


Fig.5.9 Surface likes a box & its lid is open

As there is no current flowing inside the capacitor, therefore I = 0

$$\text{Or} \int B.dl = 0$$

Case 3: Considering the surface between 2 plates of the capacitor, in this case also I=0, so B=0

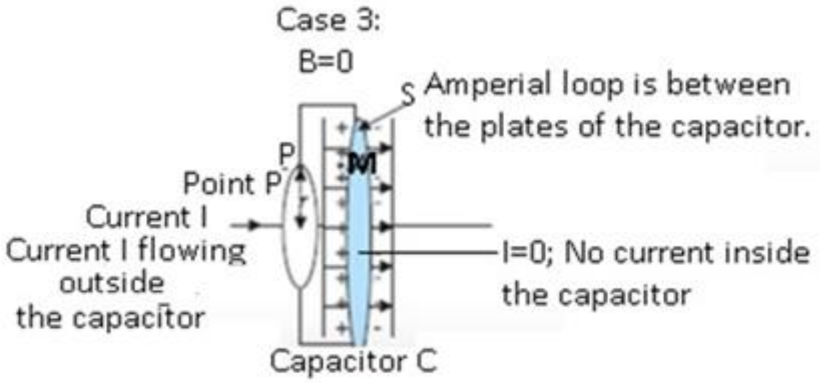


Fig.5.10 Surface between 2 plates of the capacitor

- At the same point but with different amperial surfaces the value of magnetic field is not same. They are different for the same point.
- Maxwell suggested that there are some gaps in the Ampere's circuital law.
- He corrected the Ampere's circuital law. And he made Ampere's circuital law consistent in all the scenarios.

5.6 Modification of Ampere circuital law by Maxwell with introducing concepts of displacement currents and its importance:

Ampere's law states that "the line integral of resultant magnetic field along a closed plane curve is equal to μ_0 time the total current crossing the area bounded by the closed curve provided the electric field inside the loop remains constant".

- Ampere's law is true only for steady currents.
- Maxwell found the shortcoming in Ampere's law and he modified Ampere's law to include time-varying electric fields.
- For Ampere's circuital law to be correct Maxwell assumed that there has to be some current existing between the plates of the capacitor.
- Outside the capacitor current was due to the flow of electrons.
- There was no conduction of charges between the plates of the capacitor.
- According to Maxwell between the plates of the capacitor there is an electric field which is directed from positive plate to the negative plate.
- Magnitude of the electric field $E = (V/d)$

- Where V=potential difference between the plates, d = distance between the plates.
- E = (Q/Cd)
- where Q=charge on the plates of the capacitor, Capacitance of the capacitor=C
- =>= (Q/ (Aε_0d/d))where A =area of the capacitor.
- E=Q/(Aε_0)
- Direction of the electric field will be perpendicular to the selected surface i.e. if considering plate of the capacitor as surface.
- As E =0 outside the plates and E=(Q/(Aε_0)) between the plates.
- There may be some electric field between the plates because of which some current is present between the plates of the capacitor.
- Electric Flux through the surface=Φ_E= (EA) =(QA)/ (Aε_0) =(Q/ ε_0).
- Assuming Q (charge on capacitor i.e. charging or discharging of the capacitor) changes with time current will be get generated.
- Therefore current I_d =(dQ/dt)
- Where I_d =displacement current
- =>DifferentiatingΦ_E =(Q/ ε_0) on both sides w.r.t time,
- (dΦ_E/dt) =(1/ ε_0) (dQ/dt)
- where (dQ/dt) =current

- Therefore $(dQ/dt) = \varepsilon_0 (d\,\Phi_E/dt)$
- =>Current was generated because of change of electric flux with time.
- Electric flux arose because of presence of electric field in the plates of the capacitor.
- $I_d = (dQ/dt)$ = Displacement current
- Therefore Change in electric field gave rise to Displacement current.
- Current won't be 0 it will be I_d.
- There is some current between the plates of the capacitor and there is some current at the surface.
- At certain points there is no displacement current there is only conduction current and vice-versa.
- Maxwell corrected the Ampere's circuital law by including displacement current.
- He said that there is not only the current existed outside the capacitor but also current known as displacement current existed between the plates of the capacitor.
- Displacement current exists due to the change in the electric field between the plates of the capacitor. Magnetic fields are produced both by conduction currents and by time varying fields.

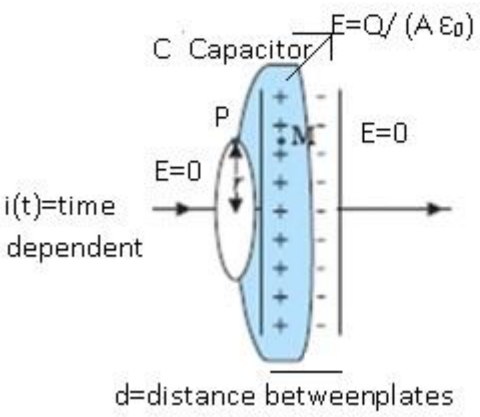


Fig.5.11. Displacement current exists due to the change in the electric field between the plates of the capacitor

Comparison of displacement current and conduction current:

1. Conduction current obeys ohm's law as i=(V/R) but displacement current does not obey ohm's law.

2. Conduction current density is represented by

$$\vec{J}_c = \sigma \vec{E}$$

Whereas displacement current density is given by

$$\vec{J}_d = \frac{\partial \vec{D}}{\partial t} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

3. Conduction current is the actual current whereas displacement current is the apparent current produced by time varying electric field.

SAQ.2

a) What do you mean by Inconsistency of Ampere circuital law with equation of continuity?
b) Write the steps of Modification of Ampere circuital law by Maxwell with introducing concepts of displacement currents.
c) Write the Comparison of displacement current and conduction current.

d) Consider a parallel plate capacitor which is maintained at potential of 150 V. If the separation distance between the plates of the capacitor and area of the plates are 2 and 15 cm^2. Calculate the displacement current for the time in μs.

5.7 Vector potential and its expression due to straight conductor and circular loop:

Vector Potential:

We have seen that the vector potential is not unique and we have a choice of gauge in the matter. The most common gauge in which we work is the Coulomb gauge in which the divergence of the vector potential is chosen to be zero, i.e.

$$\nabla \cdot \vec{A} = 0.$$

We obtained an expression for the vector potential starting with Biot-Savart's law and saw that there exists a much stronger relationship between the vector potential than which exists for the magnetic field itself. In many cases where the direction of the current is constant, the vector potential simply points in the direction of the current. We have the following expression for the vector potential,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\vec{J}(\vec{r'})}{|\vec{r}-\vec{r'}|}$$

Vector Potential for a long straight wire carrying current:

Let the current be in the z direction. The vector potential also points the same way.

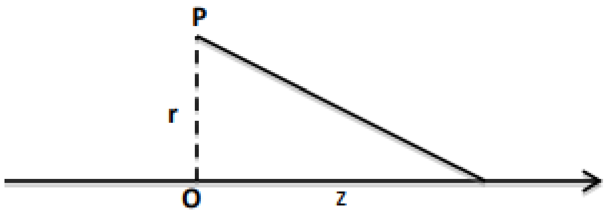


Fig.5.12 Long straight wire carrying current

The current being linear $I\hat{k}$, the vector potential becomes a simple one dimensional integral

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} Ik \int_{-\infty}^{\infty} \frac{dz}{\sqrt{r^2+z^2}} = \frac{\mu_0}{4\pi} I\hat{k}\ \ln(z+\sqrt{r^2+z^2})\Big|_{-\infty}^{\infty}$$

The expression diverges when the limits are evaluated. This is not a very serious issue because we have seen that the vector potential is arbitrary up to a constant which in this case is infinite. For instance, if instead of integrating from $-\infty$ to $+\infty$, we realized that the integrand is even, we could integrate it from zero to infinity and double the result. In that case the integral diverges only in the upper limit

Leaving us with a finite expression in the lower limit. Discarding the infinite constant, we would then have,

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{2\pi} I \ln r\, \hat{k}$$

In this simple case, we can start from our knowledge of the magnetic field and calculate back. We know that the magnetic field has cylindrical symmetry and is directed along the circumferential direction,

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} = \nabla \times \vec{A}$$

Thus the curl of the vector potential only has ø component

$$B_\phi = \left(\nabla \times \vec{A}\right)_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

By symmetry, since the wire is infinite, the derivative with respect to z must be zero and we have

$$-\frac{\partial A_z}{\partial r} = \frac{\mu_0 I}{2\pi r}$$

Which gives

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{2\pi} I \ln r\, \hat{k} + \nabla\psi$$

Where we have explicitly added gradient of an arbitrary scalar field.

There is another trick which is often used to calculate the vector potential which is to relate the line integral of vector potential to the flux. If we take the line integral of the vector potential along any closed loop, we get, using Stoke's theorem,

$$\oint \vec{A} \cdot d\vec{l} = \int_S (\nabla \times \vec{A}) \cdot d\vec{S}$$
$$= \int_S \vec{B} \cdot d\vec{S} = \Phi_B$$

We can then use the symmetry of the problem to find the vector potential.

Vector Potential for a long straight wire circular loop:

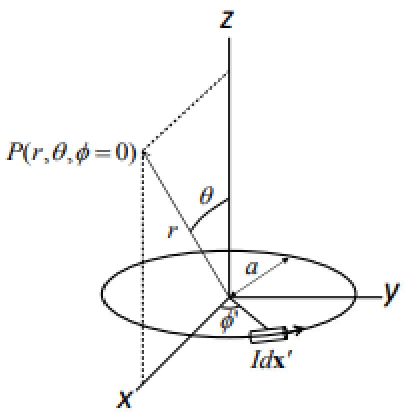


Fig.5.13 Long straight wire circular loop

We consider the problem of a circular loop of radius a, lying in the x-y plane, centered at the origin, and carrying a current I, as shown in Fig. Due to the cylindrical geometry, we may choose the observation point P in the x-z plane (ø = 0) without loss of generality. The expression for the vector potential may be applied to the current circuit by making the substitution:

$$\mathbf{J}d^3x' = Id\mathbf{x}'.$$

Thus

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where

$$d\mathbf{x}' = (-\sin\phi'\,\mathbf{e}_x + \cos\phi'\,\mathbf{e}_y) a\,d\phi'$$

Then, Eq. 1 becomes

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{(-\sin\phi'\,\mathbf{e}_x + \cos\phi'\,\mathbf{e}_y)}{(a^2 + r^2 - 2ar\sin\theta\cos\phi')^{1/2}} d\phi' \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Since the azimuthal integration in Eq. 1 is symmetric about ø’ = 0, the x component vanishes. This leaves only the y component, which is $A_ø$. Therefore

$$A_\phi(r,\theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi'\,d\phi'}{(a^2 + r^2 - 2ar\sin\theta\cos\phi')^{1/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

For a >> r, a << r, or θ << 1,

$$(a^2 + r^2 - 2ar\sin\theta\cos\phi')^{-\frac{1}{2}} \cong \frac{1}{\sqrt{a^2 + r^2}} \left(1 + \frac{ar}{a^2 + r^2} \sin\theta\cos\phi'\right)$$

Hence

$$A_\phi(r,\theta) \cong \frac{\mu_0 I a}{4\pi} \frac{1}{\sqrt{a^2 + r^2}} \int_0^{2\pi} \left(\cos\phi' + \frac{ar}{a^2 + r^2} \sin\theta\cos^2\phi'\right) d\phi'$$

The integration results in

$$A_\phi(r,\theta) = \frac{\mu_0 I a^2 r \sin\theta}{4(a^2 + r^2)^{3/2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The components of magnetic induction,

$$\begin{cases} B_r = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) = \dfrac{\mu_0 I a^2\cos\theta}{2(a^2+r^2)^{3/2}} \\ B_\theta = -\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(rA_\phi\right) = -\dfrac{\mu_0 I a^2\sin\theta}{4(a^2+r^2)^{5/2}}(2a^2-r^2) \\ B_\phi = 0 \end{cases} \qquad \text{............. (5)}$$

The fields far from the loop (for r >> a):

$$\begin{cases} B_r = \dfrac{\mu_0}{2\pi}(I\pi a^2)\dfrac{\cos\theta}{r^3} = \dfrac{\mu_0}{4\pi}\dfrac{2m\cos\theta}{r^3} \\ B_\theta = \dfrac{\mu_0}{4\pi}(I\pi a^2)\dfrac{\sin\theta}{r^3} = \dfrac{\mu_0}{4\pi}\dfrac{m\sin\theta}{r^3} \end{cases} \qquad \text{................... (6)}$$

Where $m = \pi Ia^2 e_z$ is the magnetic dipole moment of the loop. Comparison with the electrostatic dipole fields shows that the magnetic fields are dipole in character.

The fields on the z axis (for θ = 0, z =r):

For $\theta = 0$, $z = r \geq 0$, hence

$$B_z = \frac{\mu_0 I a^2}{2(a^2+z^2)^{3/2}} \qquad \text{.................. (7)}$$

For $\theta = \pi$,

$z = -r \leq 0,\ A_\phi(r, \pi-\theta) = -A_\phi(r,\theta),$

Therefore Eq.7 is valid on any points on the z axis.

5.8 Derivation of magnetic flux density using vector potential for circular loop:

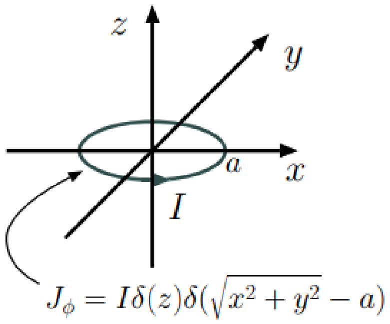


Fig. 5.14 Vector potential for circular loop

Magnetic flux density B of a single current loop I can be calculated after determining its vector potential as follows:

For a loop of radius a on z = 0 plane, we can express the corresponding current density as

$$\mathbf{J}(\mathbf{r}') = I\delta(z')\delta(\sqrt{x'^2+y'^2}-a)\frac{(-y', x', 0)}{\sqrt{x'^2+y'^2}} \quad \text{……………..} (8)$$

Where the ratio on the right is the unit vector $\hat{\phi}'$.

Inserting this into the general solution for vector potential, and performing the integration over z', we obtain

$$\begin{aligned}
\mathbf{A}(\mathbf{r}) &= \frac{\mu_o I}{4\pi}\int \delta(\sqrt{x'^2+y'^2}-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}\sqrt{x'^2+y'^2}}dx'dy' \\
&= \frac{\mu_o I}{4\pi}\int \delta(r'-a)\frac{(-y',x',0)}{\sqrt{(x-x')^2+(y-y')^2+z^2}r'}r'dr'd\phi' \\
&= \frac{\mu_o I}{4\pi}\int_{-\pi}^{\pi} \frac{(-a\sin\phi',a\cos\phi',0)}{\sqrt{(x-a\cos\phi')^2+(y-a\sin\phi')^2+z^2}}d\phi' \equiv \hat{x}A_x(\mathbf{r})+\hat{y}A_y(\mathbf{r})
\end{aligned}$$

……………..

(9)

Given that Az = 0, it can be shown that B = ∇ × A leads to

$$B_x = -\frac{\partial A_y}{\partial z},\quad B_y = \frac{\partial A_x}{\partial z},\quad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

……………..

(10)

From the expected azimuthal symmetry of B about the z-axis, it is sufficient to evaluate these on, say, y = 0 plane — after some algebra, and dropping the primes, we find, on y = 0 plane,

$$B_x = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{z\cos\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi,$$

$$B_y = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{z\sin\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi,$$

…………….. (11&12)

And

$$B_z = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{a - x\cos\phi}{(x^2+a^2+z^2-2ax\cos\phi)^{3/2}}d\phi.$$

……………….. (13)

– We note that By = 0 since the By integrand above is odd in φ and the integration limits are centered about the origin. Hence, the field on y = 0 plane is given as

$$\mathbf{B} = \hat{x}B_x + \hat{z}B_z$$ ……………….. (14)

With Bx and Bz defined above in equation 11 and equation 13.

There are no closed form expressions for the Bx and Bz integrals above for an arbitrary (x, z).

However, it can be easily seen that if x = 0 (i.e., along the z-axis), Bx = 0 (as symmetry would dictate) and

$$B_z = \frac{\mu_o a I}{4\pi}\int_{-\pi}^{\pi}\frac{a}{(a^2+z^2)^{3/2}}d\phi = \frac{\mu_o I a^2}{2(a^2+z^2)^{3/2}}$$

For

$|z| \gg a$

$$B_z \approx \frac{\mu_o I a^2}{2|z|^3}$$

Which is positive and varies with the inverse third power of distance |z|.

Also, Bx and Bz integrals can be performed numerically. Figure in the margin depicts the pattern of $\hat{B}$ on y = 0 plane for a loop of radius a = 1 computed using Mathematical.

SAQ.3

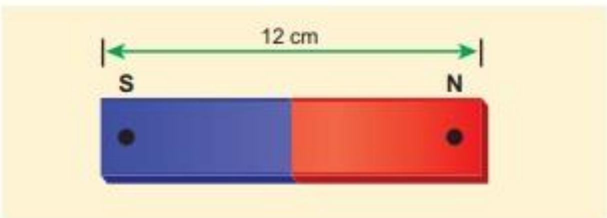
a) What do you mean by Vector potential?

b) Write the expression for the Vector potential due to straight conductor and circular loop.

c) Write the formula of vector potential for circular loop using magnetic flux density using.

d) Calculate magnetic flux density of the magnetic field at the centre of a circular coil of 100 turns, having radius of 0.8m and carrying a current of 5A.

Q.1 Compute the magnetic length of a uniform bar magnet if the geometrical length of the magnet is 12 cm. Mark the positions of magnetic pole points.



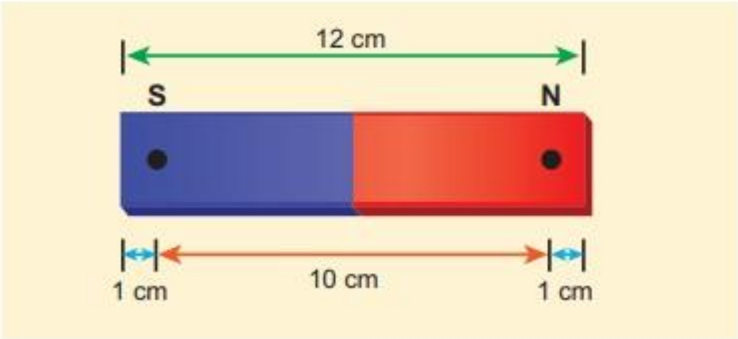
Solution:

Geometrical length of the bar magnet is 12 cm

Magnetic length = 5/6 x (geometrical length)

= 5/6 x 12 = 10cm

In this figure, the dot implies the pole points.



Q.2 The repulsive force between two magnetic poles in air is 9 x 10-3 N. If the two poles are equal in strength and are separated by a distance of 10 cm, calculate the pole strength of each pole.

Solution:

The force between two poles are given by

$$\vec{F} = k\frac{q_{m_A} q_{m_B}}{r^2}\hat{r}$$

The magnitude of the force is

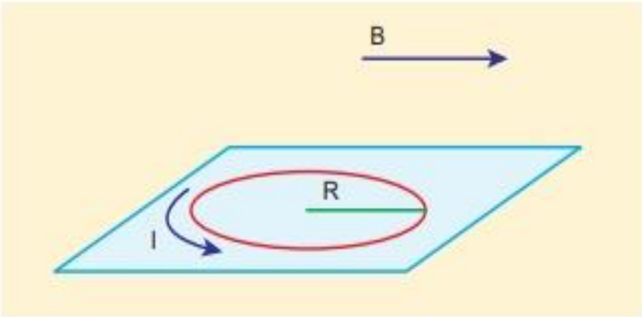
$$F = k\frac{q_{m_A} q_{m_B}}{r^2}$$

F = 9 x 10 Given :-3N, r = 10 cm = 10 x 10-2 m

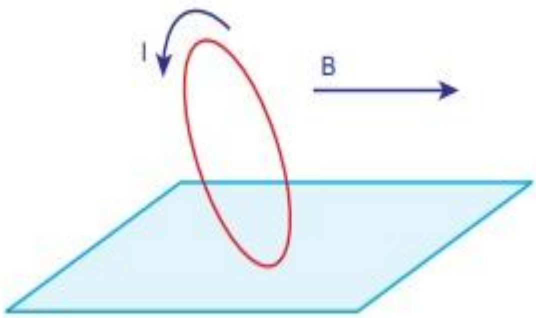
Therefore,

$$9\times10^{-3} = 10^{-7}\times\frac{q_m^2}{\left(10\times10^{-2}\right)^2} \Rightarrow q_m = 30NT^{-1}$$

Q.3 Consider a circular wire loop of radius R, mass m kept at rest on a rough surface. Let I be the current flowing through the loop and $\vec{B}$ be the magnetic field acting along horizontal as shown in Figure. Estimate the current I that should be applied so that one edge of the loop is lifted off the surface?



Solution:



When the current is passed through the loop, the torque is produced. If the torque acting on the loop is increased then the loop will start to rotate. The loop will start to lift if and only if the magnitude of magnetic torque due to current applied equals to the gravitational torque as shown in Figure.

$$\tau_{magnetic} = \tau_{gravitational}$$

$$IAB = mgR$$

$$\text{But } p_m = IA = I\left(\pi R^2\right)$$

$$\pi I R^2 B = mgR$$

$$\Rightarrow I = \frac{mg}{\pi R B}$$

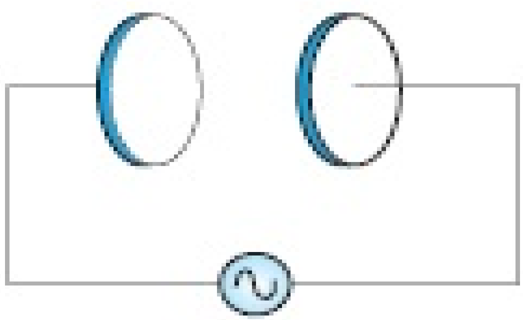
The current estimated using this equation should be applied so that one edge of loop is lifted of the surface.

Q.4 A parallel plate capacitor shown in Fig made of circular plates each of radius R = 6.0 cm has a capacitance C = 100 pF. The capacitor is connected to a 230 V ac supply with a (angular) frequency of 300 rad s^{-1}.

(a) What is the r.m.s. value of the conduction current?

(b) Is the conduction current equal to the displacement current?

(c) Determine the amplitude of B at a point 3.0 cm from the axis between the plates.



Solution: Radius of each circular plate, R = 6.0 cm = 0.06 m

Capacitance of a parallel plate capacitor, C = 100 pF = 100×10^{-12} F

Supply voltage, V = 230 V

Angular frequency, $\omega = 300$ rad s^{-1}

(a) R.m.s. value of conduction current, I

Where,

X_C = Capacitive reactance

$=1/(\omega C)$

Therefore, $I = V \times \omega C$

$= 230 \times 300 \times 100 \times 10^{-12}$

$= 6.9 \times 10^{-6}$ A

$= 6.9$ μA

Hence, the r.m.s. value of conduction current is 6.9 μA.

(b) Yes, conduction current is equal to displacement current.

(c) Magnetic field is given as:

$B = (\mu_0 r)/(2 R^2)\ I_0$ Where,

μ_0 = Free space permeability = $4x\ \pi x 10^{-7} NA^{-2}$

I_0 = Maximum value of current $=\sqrt{2}\ I$

r = Distance between the plates from the axis = 3.0 cm = 0.03 m

Therefore $B = (4x\pi\ x10^{-7}x0.03x\sqrt{2}x6.9x10^{-6})/(2x\pi\ x0.06)^2$

$= 1.63 \times 10^{-11}$ T

Hence, the magnetic field at that point is 1.63×10^{-11} T.

Q.5 How would you establish an instantaneous displacement current of 2 mA in the space between parallel plates having capacitance 2 μF?

Solution:

$C = 2 \times 10^{-6}$ F,

$I_d = 2 \times 10^{-3}$ A

$$I_d = \epsilon_0 \frac{d\phi_E}{dt}$$

$$I_d = \epsilon_0 \frac{d(EA)}{dt}$$

$$I_d = \epsilon_0 A \frac{d(V/d)}{dt}$$

(Since V = E d)

$$I_d = \frac{\epsilon_0 A}{d} \frac{dV}{dt}$$

$$I_d = C \frac{dV}{dt}$$

$$\frac{dV}{dt} = \frac{I_d}{C}$$

$$\frac{dV}{dt} = \frac{2 \times 10^{-3}}{2 \times 10^{-6}}$$

= 1000 V/s

So, by applying a varying potential difference of 500 V/s, we would produce a displacement current of desired value.

Q.6 The direction of the current in a copper wire carrying a current of 6.00 A through a uniform magnetic field with magnitude 2.20T is from the left to right of the screen. The direction of the magnetic field is upward-left, at an angle of θ = 3π/4 radians from the current direction. Determine the magnitude and direction fo the magnetic force acting on a 0.100 m section of the wire?

Solution:

The magnitude of the magnetic force can be found using the formula:

$$\vec{F} = ILB\sin\theta\hat{n}$$

Where,

$\vec{F}$ is the magnetic vector (N),

I is the current magnitude (A),

$\vec{L}$ is the length vector (m),

L is the length of the wire (m),

$\vec{B}$ is the magnetic field vector (T),

B is the magnetic field magnitude (T),

θ is the angle between length and magnetic field vectors (radians)

$\hat{n}$ is the cross product direction vector (unit less)

Substituting the values, we get

$$F = (6.00\,A)(0.100\,m)(2.20\,T)\,\sin(3\pi/4\,radians)$$

$$F = (6.00\,A)(0.100\,m)(2.20\,T)(1/\sqrt{2})$$

$$F = (6.00\,A)(0.100\,m)(2.20\,\frac{kg}{A\cdot s^2})(1/\sqrt{2})$$

$$F = (6.00)(0.100\,m)(2.20\,\frac{kg}{s^2})(1/\sqrt{2})$$

$$F = (6.00)(0.100)(2.20)(1/\sqrt{2})\,kg\cdot m/s^2$$

$$F \simeq 0.933\ kg\cdot m/s^2$$

The magnitude of the force on the 0.100 m section of wire has a magnitude of 0.933 N.

Q.7 Compute the magnitude of the magnetic field of a long, straight wire carrying a current of 1A at distance of 1m from it. Compare it with Earth's magnetic field.

Solution:

Given that 1 = 1 A and radius r = 1 m

$$B_{straightwire} = \frac{\mu_\circ I}{2\pi r} = \frac{4\pi\times10^{-7}\times1}{2\pi\times1} = 2\times10^{-7}\,T$$

But the Earth's magnetic field is $B_{Earth} \approx 10{-}5$ T

So, B straight wire is one hundred times smaller than B_{Earth}.

Q.8 Calculate the magnetic field inside a solenoid, when

(a) The length of the solenoid becomes twice and fixed number of turns.

(b) Both the length of the solenoid and number of turns are double.

(c) The number of turns becomes twice for the fixed length of the solenoid.

Compare the results.

Solution:

The magnetic field of a solenoid (inside) is

$$B_{L,N} = \mu_o \frac{NI}{L}$$

(a) Length of the solenoid becomes twice and fixed number of turns

L→2L (length becomes twice)

N→N (number of turns are fixed)

The magnetic field is

B2L, N = μ NI/2L = 1/2 BL, N

(b) Both the length of the solenoid and number of turns are double

L→2L (length becomes twice)

N→2N (number of turns becomes twice)

The magnetic field is

$$B_{2L,2N} = \mu_\circ \frac{2NI}{2L} = B_{L,N}$$

(c) The number of turns becomes twice but for the fixed length of the solenoid

L→L (length is fixed)

N→2N (number of turns becomes twice)

The magnetic field is

BL ,2 N = μ, 2NI/L = 2BL ,N

From the above results,

BL ,2 N > B2 L ,2 N > B2 L , N

Thus, strength of the magnetic field is increased when we pack more loops into the same length for a given current.

Q.9 Compute the magnetic field of a long straight wire that has a circular loop with a radius of 0.05m. 2amp is the reading of the current flowing through this closed loop.

Solution:

Given

R = 0.05m

I = 2amp

$\mu_0 = 4\pi \times 10^{-7} N/A^2$

Ampere's law formula is

$\oint \vec{B} \, d\vec{l} = \mu_0 I$

In the case of long straight wire

$$\oint \vec{dl} = 2\Pi R = 2 \times 3.14 \times 0.05 = 0.314$$

$$B \oint \vec{dl} = \mu_0 I$$

$$\vec{B} = \frac{\mu_0 I}{2\pi R}$$

$$\vec{B} = \frac{4\pi \times 10^{-7} \times 2}{0.314} = 8 \times 10^{-6} T$$

Q.10 Consider a parallel plate capacitor which is maintained at potential of 200 V. If the separation distance between the plates of the capacitor and area of the plates are 1 and 20 cm^2. Calculate the displacement current for the time in µs.

Solution:

Potential difference between the plates of the capacitor, V = 200 V

The distance between the plates,

$d = 1 \text{ mm} = 1 \times 10^{-3}$ m

Area of the plates of the capacitor,

A = 20 cm^2 = 20 × 10^{-4} m^2

Time is given in micro-second, µs = 10^{-6} s

Displacement current

$$I_d = \varepsilon_\circ \frac{d\Phi_B}{dt} \Rightarrow I_d = \varepsilon_\circ \frac{EA}{t}$$

But electric field, E = V/d

Therefore,

$$I = \frac{V}{d} I_d = \varepsilon_\circ \frac{VA}{td} = 8.85 \times 10^{-12} \times \frac{200 \times 20 \times 10^{-4}}{10^{-6} \times 1 \times 10^{-3}}$$

= 35400 ×10^{-7} = 3 .5 mA

Q.11 A capacitor is made of circular plates, each of radius 12 cm, separated by 5.0 mm. This capacitor is charged with a constant current of 0.15 A, using an external source. (a) Calculate the rate of change of potential difference across the plates. (b) Obtain the displacement current across the plates.

Solution:

Given: $R = 12\ cm = 0.12\ m$

$$d = 5.0\,mm = 5 \times 10^{-3}\,m$$

$$I = 0.15\,A$$

$$\epsilon_0 = 8.854 \times 10^{-12}\,C^2N^{-1}m^{-2}$$

Area of the plates, $A = \pi R^2 = 3.14 \times (0.12)^2\,m^2$

$$C = \frac{A\epsilon_0}{d} = \frac{3.14 \times (0.12)^2 \times 8.854 \times 10^{-12}}{5 \times 10^{-3}}$$

$$= 80.1 \times 10^{-12}F$$

(a) $\boldsymbol{Q} = \mathrm{CV}$ and $I = \frac{dQ}{dt}$

$$\frac{\mathrm{dQ}}{\mathrm{dt}} = \mathrm{C}\frac{\mathrm{dV}}{\mathrm{dt}}$$

$$\therefore \mathrm{I} = \mathrm{C}\frac{\mathrm{dV}}{\mathrm{dt}}$$

Or

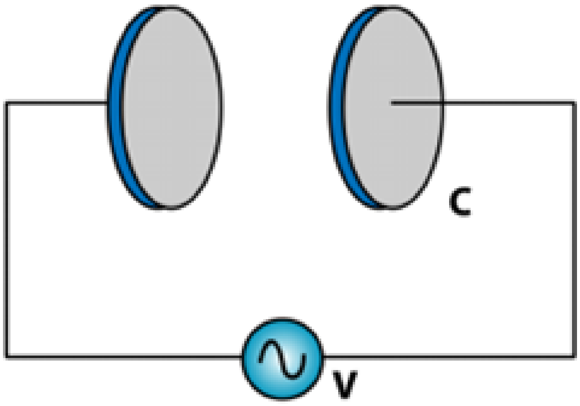
$$\frac{\mathrm{dV}}{\mathrm{dt}} = \frac{\mathrm{I}}{\mathrm{C}} = \frac{0.15}{80.1 \times 10^{-12}} = 1.87 \times 10^9\,\mathrm{Vs}^{-1}$$

(b) The displacement current (i_d) is equal to the conduction current in the charging circuit (i_c)

$$\mathrm{i_d} = \mathrm{i_c}$$

$$= 0.15\,\mathrm{A}$$

Q.12 A parallel plate capacitor made of circular plates, each of radius 6.0 cm, has a capacitor of C = 100 pF. This capacitor is connected to a 230 volt a.c. supply with an angular frequency of =300 rad s^{-1}.



(a) What is the rms value of the conduction current?

(b) Is the conduction current equal to the displacement current?

(c) What is the amplitude of B at a point 3.0 cm from the axis between the plates?

Solution:

Given:

$$R = 6.0 \text{ cm} = 6 \times 10^{-2} \text{ m}$$

$$C = 100 \text{ pF} = 100 \times 10^{-12} \text{F}$$

$$\omega = 300 \text{ rad s}^{-1}$$

$$V = V_{rms} = 230 \text{ volts}$$

$$\text{Capacitive reactance } X_c = \frac{1}{\omega C}$$

(a) rms value of conduction current is:

$$(i_C)_{rms} = I = \frac{V_{rms}}{X_c} = \frac{V}{\frac{1}{\omega C}} = \omega CV$$

$$= 300 \times 100 \times 10^{-12} \times 230$$

$$= 6.9 \times 10^{-6}\ A$$

$$= 6.9\ \mu A$$

$$= \frac{\sqrt{2}}{6} \times 10^{-5} \times 6.9 \times 10^{-6}$$

$$= 1.63 \times 10^{-11}\ \text{tesla}$$

(b) Yes, $i_d = i_c$, even if the current is alternating or varying.

(c) At $r = \frac{R}{2}$, the formula for magnetic field (B),

$B = \frac{\mu_0\, i_d}{2\, \pi R}\left(\frac{r}{R}\right)$ is valid even for alternating i_d.

When this i_d is maximum $\left(\sqrt{2}\, (i_d)_{rms}\right)$, the value of B also becomes maximum.

$$B_{max} = B_{peak} = \frac{\mu_0\, (i_d)_{max}}{2\, \pi\, R}\left(\frac{r}{R}\right)$$

$$= \frac{2 \times 10^{-7} \times \sqrt{2}\, (i_d)_{rms}}{6 \times 10^{-2}}\left(\frac{3 \times 10^{-2}}{6 \times 10^{-2}}\right)$$

Q.13 Calculate magnetic flux density of the magnetic field at the centre of a circular coil of 50 turns, having radius of 0.5m and carrying a current of 5A.

Solution:

Given

n = 50 turns, R = 0.5m, I = 5 A,

According to Bio- Savart Law, Magnetic flux density is give by B

$$B = \frac{\mu_0 nI}{2R}$$

$$B = \frac{4\pi \times 10^{-7} \times 50 \times 5}{2 \times 0.5} = 3.14 \times 10^{-3}\mathrm{T}$$

Summary:

1) Lines of force are the lines that depict the magnetic force that exists in the surrounding of the magnet. As the distance between the poles increases, the density of magnetic lines decreases. The direction of field lines inside the magnet is from the South Pole to the North Pole.
2) A fundamental feature of magnetic fields that distinguishes them from electric fields is that the field lines form closed loops.

3) Ampere's Circuital Law states the relationship between the current and the magnetic field created by it. This law states that the integral of magnetic field density (B) along an imaginary closed path is equal to the product of current enclosed by the path and permeability of the medium.
4) Ampere's Law states that for any closed loop path, the sum of the length elements times the magnetic field in the direction of the length element is equal to the permeability times the electric current enclosed in the loop.
5) According to Ampere circuital law the line integral of magnetic field B around any closed path is equal to times total current l enclosed by that closed path. Therefore, Ampere law is ambiguous as it does not provide continuity to current path.
6) Modified Ampere's Law – This law states that the surface integral of the magnetic field around any closed circuit is equal to times the total current (the sum of conduction and displacement current) threading the closed circuit.
7) During charging or discharging there is a displacement current but not conduction current between plates of capacitor.
8) A vector potential is a vector field whose curl is a given vector field. This is analogous to a scalar potential, which is a scalar field whose gradient is a given vector field.
9) The physical meaning of the electric scalar potential is usually considered to be potential energy per unit charge. The physical meaning of the magnetic vector potential is actually very similar: it's the potential energy per unit element of current.

10) Obtain an *expression* for the *vector potential* at a point *due* to a long current carrying *wire.* Take the *wire* to be along the z-direction, perpendicular to the plane of the page with current flowing in a. Faraday and (Joseph) Henry, however, found that if a *current loop.*

11) Magnetic flux density is defined as the amount of magnetic flux in an area taken perpendicular to the magnetic flux's direction. An example of magnetic flux density is a measurement taken in Tesla.

12) The field intensity H = -Grad(V). Since the given potential is a position vector, the gradient will be 3 and H = -3. Thus the flux density $B = \mu H = 4\pi \times 10^{-7} \times (-3) = -12\pi \times 10^{-7}$ units.

Terminal Questions:

1) Explain the Lines of forces in detail.
2) What do you mean by Gauss law in magneto-statics?
3) Write the statement of Ampere circuital law and derive the expression.
4) Explain and derive the expression for current carrying rod using hollow.
5) Explain and derive the expression for current carrying rod using solid.
6) What do you understand by Inconsistency of Ampere circuital law with equation of continuity?
7) Explain the Modification of Ampere circuital law by Maxwell with introducing concepts of displacement currents and its importance.

8) Write the Comparison between displacement current and conduction current.

9) What do you mean by the Vector potential and its expression due to straight conductor?

10) Derive the expression for Vector potential due to circular loop.

11) Derive the expression of magnetic flux density using vector potential for circular loop.

12) The repulsive force between two magnetic poles in air is 6 x 10^{-3} N. If the two poles are equal in strength and are separated by a distance of 15 cm, calculate the pole strength of each pole.

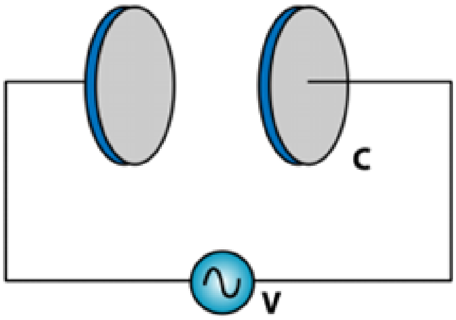
13) Compute the magnetic field of a long straight wire that has a circular loop with a radius of 0.15m. 5amp is the reading of the current flowing through this closed loop.

14) A capacitor is made of circular plates, each of radius 10 cm, separated by 4.0 mm. This capacitor is charged with a constant current of 0.25A, using an external source.

(a) Calculate the rate of change of potential difference across the plates.

(b) Obtain the displacement current across the plates.

15) A parallel plate capacitor made of circular plates, each of radius 4.0 cm, has a capacitor of C = 80 pF. This capacitor is connected to a 230 volt a.c. supply with an angular frequency of =200 rad s^{-1}.



(a) What is the rms value of the conduction current?

(b) Is the conduction current equal to the displacement current?

(c) What is the amplitude of B at a point 3.0 cm from the axis between the plates?

16) Calculate magnetic flux density of the magnetic field at the centre of a circular coil of 200 turns, having radius of 1.5m and carrying a current of 10A.

Unit 06- Magnetic materials

Structure

6.1 Introduction:

Magnetic Properties are the magnetic moment of a system measures the strength and the direction of its magnetism. There are many different magnetic behavior including magnetic flux density B, magnetizing field H, intensity of magnetization Im, susceptibility, relative and absolute permeability.

In classical electromagnetism, magnetization or magnetic polarization is the vector field that expresses the density of permanent or induced magnetic dipole moments in a magnetic material. Physicists and engineers usually define magnetization as the quantity of magnetic moment per unit volume.

The closed loop of B-H curve is called the hysteresis loop and the whole cycle is called a cycle of magnetization. The changes in magnetization in a ferromagnetic material lag behind the variations of the magnetic field applied to it. Thus the intensity of magnetization at every phase lags behind the applied field.

The magnetic flux density that remains in a material when the magnetizing force is zero. Note that residual magnetism and retentivity are the same when the material has been magnetized to the saturation point.

The relation between the tree vectors B H and M points [$B = \mu_0 (H + M)$] in the same direction as that of B or M and M. H points in the same direction as that of B or M. Its unit is Am^{-1}.

In terms of free current, the line integral of the magnetic H-field (in amperes per metre, $A{\cdot}m^{-1}$) around closed curve C equals the free current $I_{f,enc}$ through a surface S.

As no work is done while moving a charge in a closed loop in an electric field, the closed line integral of that field must be zero and hence, curl of the field also must be zero. In other words, the field must be irrotational.

6.2 Objectives:

After studying this unit you should be able to

- Explain and identify Magnetic properties (Magnetic flux density B, Magnetizing field H,
  Intensity of magnetization Im, Susceptibility, Relative and Absolute permeability).
- Study and identify Magnetization, cycle of magnetization, hysteresis loop, Retentivity, residual magnetism.
- Explain and identify three magnetic vectors (B, H, Im), three magnetic currents (free, bound and total).
- Study and identify Curl of intensity of magnetization.

6.3 Magnetic properties:

The magnetic properties of a material are those which determine the ability of material to be suitable for a particular magnetic Application. Some of the typical magnetic properties of materials are listed below-

- Magnetic flux density B,
- Magnetizing field H,
- Intensity of magnetization Im,
- Susceptibility,
- Relative permeability
- Absolute permeability

Magnetic Flux Density (B):

When a substance is subjected to the magnetic field H, then the density of magnetic field lines that pass through the substance per square meter is known as Magnetic Flux Density.

It is given by:

$$B = \mu \text{ X } H \text{ (Tesla or Weber /m}^2\text{)}$$

Where μ is called the Permeability and is defined as the degree to which a substance gets magnetized. The value of permeability in vacuum is given by:

m = 4px 10^{-7}(H/m)

Magnetizing field H:

Magnetic field strength, also called magnetic intensity or magnetic field intensity, the part of the magnetic field in a material that arises from an external current and is not intrinsic to the material itself. It is expressed as the vector H and is measured in units of amperes per meter.

The definition of H is

$$H = B/\mu - M, \text{ and } H=I/L, \text{ where L is diameter of coil}$$

Where B is the magnetic flux density, a measure of the actual magnetic field within a material considered as a concentration of magnetic field lines, or flux, per unit cross-sectional area;

μ is the magnetic permeability; and M is the magnetization.

The magnetic field H might be thought of as the magnetic field produced by the flow of current in wires and the magnetic field B as the total magnetic field including also the contribution M made by the magnetic properties of the materials in the field. When current flows in a wire wrapped on a soft-iron cylinder, the magnetizing field H is quite weak, but the actual average magnetic field (B) within the iron may be thousands of times stronger because B is greatly enhanced by the alignment of the iron's myriad tiny natural atomic magnets in the direction of the field.

Intensity of Magnetization (M or Im**):**

When a material medium is placed in a magnetic field, it gets magnetized. The magnetic moment per unit volume of the material is called the intensity of magnetization M (or simply magnetization).

$$M = \frac{Magnetic\ moment}{Volume}$$

S.I. unit of magnetization is (Am^{-1}). Lines representing intensity of magnetization are called lines of magnetization. For a uniformly magnetized material, each dipole will point in the same direction and M will be constant throughout.

Susceptibility:

Magnetic susceptibility, quantitative measure of the extent to which a material may be magnetized in relation to a given applied magnetic field.

The magnetic susceptibility of a material, commonly symbolized by χ_m, is equal to the ratio of the magnetization *M* within the material to the applied magnetic field strength *H*, or $\chi_m = M/H.$

This ratio, strictly speaking, is the volume susceptibility, because magnetization essentially involves a certain measure of magnetism (dipole moment) per unit volume.

Relative Permeability:

To compare the permeability of any given material with the permeability of free space, it is necessary to use a ratio μ_r which is known as the relative permeability of the material. For air and other non-magnetic materials, μ_r has the value of unity ($\mu_r = 1$).

$$\mu_r = \frac{\mu}{\mu_o}$$

If the non-magnetic core of a solenoid is replaced with a magnetic material, the flux produced by the same number of ampere-turns may be greatly increased. The ratio of the flux produced by the magnetic core to that produced by the non-magnetic core is a direct result of the relative permeability of the magnetic material. For some magnetic materials μ_r can have a value in the thousands.

For any one magnetic material, the relative permeability value can vary considerably, being dependent on the flux density in the material. Relative permeability is higher at low values of flux density.

Absolute Permeability:

To find the absolute permeability of a material, the permeability of free space is multiplied by the relative permeability of the material:

$$\mu=\mu_o\mu_r$$

Where:

- μ = absolute permeability
- μ_o = permeability of free space
- μ_r = relative permeability

6.4 Magnetization:

Magnetization, also termed as magnetic polarization, is a vector quantity that gives the measure of the density of permanent or induced dipole moment in a given magnetic material. As we know, magnetization results from the magnetic moment, which results from the motion of electrons in the atoms or the spin of electrons or the nuclei. The net magnetization results from the response of a material to the external magnetic field, together with any unbalanced magnetic dipole moment that is inherent in the material due to the motion in its electrons as mentioned earlier. The concept of magnetization helps us in classifying the materials on the basis of their magnetic property. In this section, we will learn more about magnetization and the concept of magnetic intensity.

What is Magnetization?

The magnetization of a given sample material M can be defined as the net magnetic moment for that material per unit volume.

Mathematically,

$$M = \frac{m_{net}}{V}$$

Let us now consider the case of a solenoid. Let us take a solenoid with n turns per unit length and the current passing through it be given by I, then the magnetic field in the interior of the solenoid can be given as,

$$B_0 = \mu_0 n I$$

Now, if we fill the interior with the solenoid with a material of non-zero magnetization, the field inside the solenoid must be greater than before. The net magnetic field B inside the solenoid can be given as,

$$B = B_0 + B_m$$

Where B_m gives the field contributed by the core material. Here, B_m is proportional to the magnetization of the material, M. Mathematically,

$$B_m = \mu_0 M$$

Here, μ_0 is the constant of permeability of a vacuum.

Let us now discuss another concept here, the magnetic intensity of a material. The magnetic intensity of a material can be given as,

$$H = \frac{B}{\mu_0} - M$$

From this equation, we see that the total magnetic field can also be defined as,

$$B = \mu_0(H + M)$$

Here, the magnetic field due to the external factors such as the current in the solenoid is given as H and that due to the nature of the core is given by M. The latter quantity, that is M is dependent on external influences and is given by,

$$M = \chi H$$

Where χ is the magnetic susceptibility of the material. It gives the measure of the response of a material to an external field. The magnetic susceptibility of a material is small and positive for paramagnetic materials and is small and negative for diamagnetic materials.

$$B = \mu_0(1 + \chi)H = \mu_0\mu_r H = \mu H$$

Here, the term μ_r is termed as the relative magnetic permeability of a material, which is analogous to the dielectric constants in the case of electrostatics. We define the magnetic permeability as,

$$\mu = \mu_0\mu_r = \mu_0(1 + \chi)$$

The cycle of Magnetization and Hysteresis Loop:

A great deal of information can be learned about the magnetic properties of a material by studying its hysteresis loop. A hysteresis loop shows the relationship between the induced magnetic flux density (B) and the magnetizing force (H). It is often referred to as the B-H loop. An example hysteresis loop is shown below.

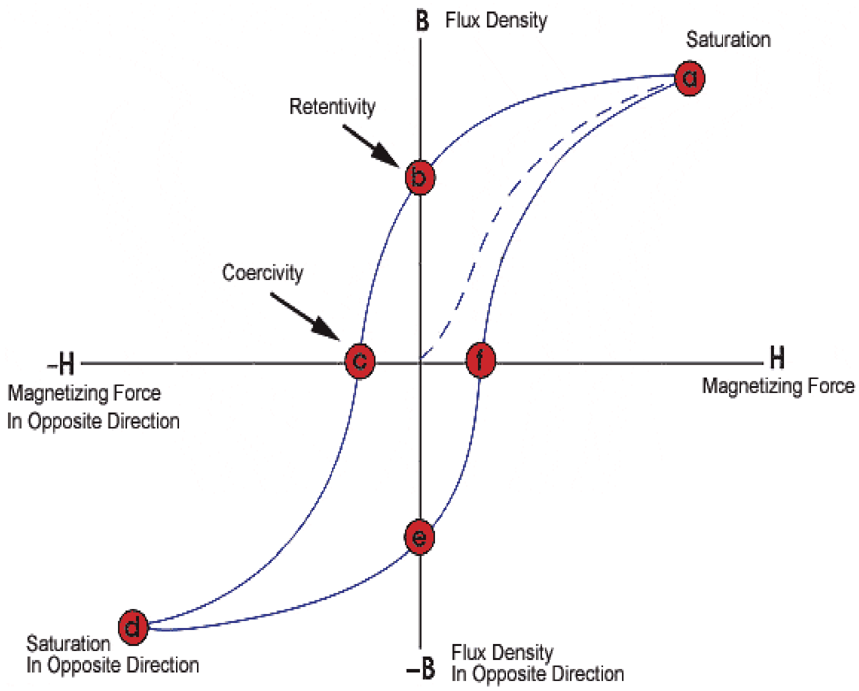


Fig.6.1 Hysteresis loop

The loop is generated by measuring the magnetic flux of a ferromagnetic material while the magnetizing force is changed. A ferromagnetic material that has never been previously magnetized or has been thoroughly demagnetized will follow the dashed line as H is increased. As the line demonstrates, the greater the amount of current applied (H+), the stronger the magnetic field in the component (B+). At point "a" almost all of the magnetic domains are aligned and an additional increase in the magnetizing force will produce very little increase in magnetic flux. The material has reached the point of magnetic saturation. When H is reduced to zero, the curve will move from point "a" to point "b." At this point, it can be seen that some magnetic flux remains in the material even though the magnetizing force is zero. This is referred to as the point of retentivity on the graph and indicates the remanence or level of residual magnetism in the material. (Some of the magnetic domains remain aligned but some have lost their alignment.) As the magnetizing force is reversed, the curve moves to point "c", where the flux has been reduced to zero. This is called the point of coercivity on the curve. (The reversed magnetizing force has flipped enough of the domains so that the net flux within the material is zero.) The force required to remove the residual magnetism from the material is called the coercive force or coercivity of the material.

As the magnetizing force is increased in the negative direction, the material will again become magnetically saturated but in the opposite direction (point "d"). Reducing H to zero brings the curve to point "e." It will have a level of residual magnetism equal to that achieved in the other direction. Increasing H back in the positive direction will return B to zero. Notice that the curve did not return to the origin of the graph because some force is required to remove the residual magnetism. The curve will take a different path from point "f" back to the saturation point where it with complete the loop.

The closed loop ‘abcdefa’ is called the hysteresis loop and the whole cycle is called a hysteresis cycle.

The changes in magnetization in a ferromagnetic material lag behind the variations of the magnetic field applied to it. Thus the intensity of magnetization at every phase lags behind the applied field. This property is called magnetic hysteresis. The area of the hysteresis curve gives the hysteresis loss of energy while a ferromagnetic substance is taken over a complete cycle of magnetization

Advantages of Hysteresis Loop:

1. A smaller region of loop hysteresis is indicative of less loss of hysteresis.

2. Hysteresis loop provides a substance with the importance of retentivity and coercivity.
Therefore the way to selectthe right material to make a permanent magnet is made simpler by the heart of machines.

3. Residual magnetism can be calculated from the B, H graph and it is, therefore, simple to choose material for electromagnets. From the hysteresis loop, a number of primary magnetic properties of a material can be determined.

Retentivity: A measure of the residual flux density corresponding to the saturation induction of a magnetic material. In other words, it is a material's ability to retain a certain amount of residual magnetic field when the magnetizing force is removed after achieving saturation (The value of B at point b on the hysteresis curve).

Residual Magnetism or Residual Flux: The magnetic flux density that remains in a material when the magnetizing force is zero. Note that residual magnetism and retentivity are the same when the material has been magnetized to the saturation point. However, the level of residual magnetism may be lower than the retentivity value when the magnetizing force did not reach the saturation level.

Coercive Force: The amount of reverse magnetic field which must be applied to a magnetic material to make the magnetic flux return to zero (The value of H at point c on the hysteresis curve).

Permeability (m): A property of a material that describes the ease with which a magnetic flux is established in the component.

Reluctance: Is the opposition that a ferromagnetic material shows to the establishment of a magnetic field. Reluctance is analogous to the resistance in an electrical circuit.

SAQ.1

a) What do you mean by Magnetic flux density (B) and Magnetizing field (H)?

b) Define the Relative and Absolute permeability.

c) Explain the cycle of magnetization with the help of hysteresis loop.

d) The magnetic field strength in silicon is 1000 A/m. If the magnetic susceptibility is -0.25×10^{-5}, calculate the magnetization and flux density in silicon.

e) Wire carrying a current of 4A is in the form of a circle. It is necessary to have a magnetic field of induction 10^{-6} T at the center. Find the radius of the circle.

f) Magnetic field and magnetic intensity are respectively 1.8 T and 1000 A/m. Find relative permeability and susceptibility.

6.5 Three magnetic vectors (B, H, Im or M):

Consider a Rowland ring having a toroidal winding of winding of N turns around it. When a current i_0 is sent through the winding, the ring is magnetized along its circumferential length. The current i_0 is the real current which magnetizes the ring.

This magnetization arises due to the alignment of the elementary current-loops (magnetic dipoles) resulting from electronic motions in the materials. The small circles represent the current-loops. These internal tiny circular electron currents tend to cancel each other due to the fact that adjacent current are in opposite directions. As such there is no net current in the outer portions of the outer-most loops remain uncancelled. The numerous tiny localized surface currents can be replaced by a single closed current i_s along the surface. Such a current is called Amperian current.

Let A = area of cross and section and, l = circumferential length of the ring.

Then, volume of the ring = l A.

The ring behaves like a large dipole of magnetic movements i_S A.

Magnetization = M = magnetic moment per unit volume = iNA/lA = iN/l.

The magnetization M, therefore is the surface current per unit lengths of the ring. This is commonly called magnetization currents.

Now, the magnetic induction B with in material of the ring arises due to the free current i0 in the winding, as well as due to the magnetization of the ring itself which can be described in terms of Amperian surface current.

$$B = \mu_0 (Ni_0/l + i_s/L) = \mu_0 (Ni_0 /l + M) \qquad (\because i_s/L = M)$$

Here Ni0/l is the free current per unit and is/l is the Amperian surface current per unit length

$$B/\mu_0 - M = Ni_0 /l$$

The quantity $B/\mu_0 - M$ is called magnetizing field or magnetic field intensity H. i.e.,

$$H = B/\mu_0 - M$$

Or $$B = \mu_0 (H + M$$

This is relation between the tree vectors B H and M points in the same direction as that of B or

M and M. H points in the same direction as that of B or M. Its unit is Am^{-1}.

Above equation can be written as

$$H = Ni_0/l = ni_0$$

Where n is the number of turns per unit length. Thus the value of H depends only on the free current and is independent of the core material.

When no magnetic material is present in the core of the Rowland ring. i.e., there is vacuum in the core, M = 0. Therefore, above equation becomes

$$B_0 = \mu_0 H$$

In vacuum, the magnetic field strength H is related to the magnetic induction B0 by the above relation.

When a magnetic material is placed in an external magnetic field, the specimen is magnetized by producing (or reorienting) magnetic dipoles in the specimen. This will produce additional field. Thus the resultant field B is greater than B0. In such a case, H is related to B by the relation,

$$H = (B/\mu_0) - M.$$

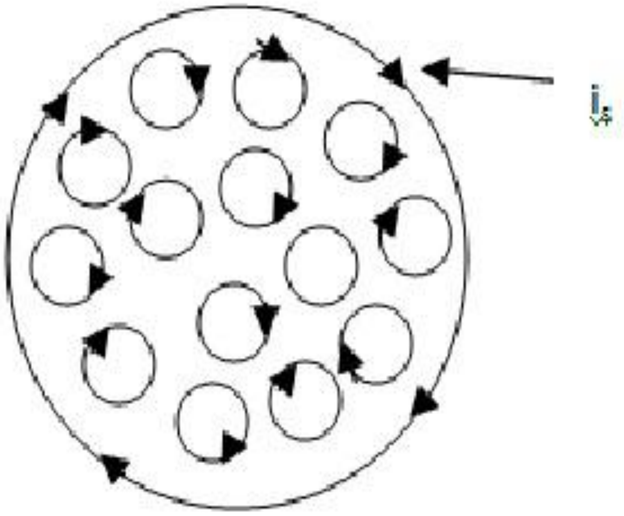


Fig.6.2 Rowland ring having a toroidal winding for Three magnetic vectors (B, H, M)

Three magnetic currents (free, bound and total):

In terms of total current, (which is the sum of both free current and bound current) the line integral of the magnetic B-field (in tesla, T) around closed curve C is proportional to the total current I_{enc} passing through a surface S (enclosed by C). In terms of free current, the line integral of the magnetic H-field (in amperes per metre, $A \cdot m^{-1}$) around closed curve C equals the free current $I_{f,enc}$ through a surface S.

Forms of the original circuital law written in SI units		
	Integral form	Differential form
Using B-field and total current	$\oint_C \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 \iint_S \mathbf{J} \cdot d\mathbf{S} = \mu_0 I_{\mathrm{enc}}$	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

Using H-field and free current	$\oint_C \mathbf{H} \cdot \mathrm{d}\boldsymbol{l} = \iint_S \mathbf{J}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} = I_{\mathrm{f,enc}}$	$\nabla \times \mathbf{H} = \mathbf{J}_\mathrm{f}$

- J is the total current density (in amperes per square metre, A·m^{-2}),
- J_f is the free current density only,
- $\oint_C$ is the closed line integral around the closed curve C,
- $\iint_S$ denotes a 2-D surface integral over S enclosed by C,
- · is the vector dot product,
- dl is an infinitesimal element (a differential) of the curve C (i.e. a vector with magnitude equal to the length of the infinitesimal line element, and direction given by the tangent to the curve C)
- dS is the vector area of an infinitesimal element of surface S (that is, a vector with magnitude equal to the area of the infinitesimal surface element, and direction normal to surface S. The direction of the normal must correspond with the orientation of C by the right hand rule), see below for further explanation of the curve C and surface S.
- ∇ × is the curl operator.

When a material is magnetized (for example, by placing it in an external magnetic field), the electrons remain bound to their respective atoms, but behave as if they were orbiting the nucleus in a particular direction, creating a microscopic current. When the currents from all these atoms are put together, they create the same effect as a macroscopic current, circulating perpetually around the magnetized object. This magnetization current J_M is one contribution to "bound current".

The other source of bound current is bound charge. When an electric field is applied, the positive and negative bound charges can separate over atomic distances in polarizable materials, and when the bound charges move, the polarization changes, creating another contribution to the "bound current", the polarization current J_P.

The total current density J due to free and bound charges is then:

$$\mathbf{J} = \mathbf{J}_{\mathrm{f}} + \mathbf{J}_{\mathrm{M}} + \mathbf{J}_{\mathrm{P}} \ ,$$

with J_f the "free" or "conduction" current density.

All current is fundamentally the same, microscopically. Nevertheless, there are often practical reasons for wanting to treat bound current differently from free current. For example, the bound current usually originates over atomic dimensions, and one may wish to take advantage of a simpler theory intended for larger dimensions.

Curl of intensity:

Let's take the simplest electric field

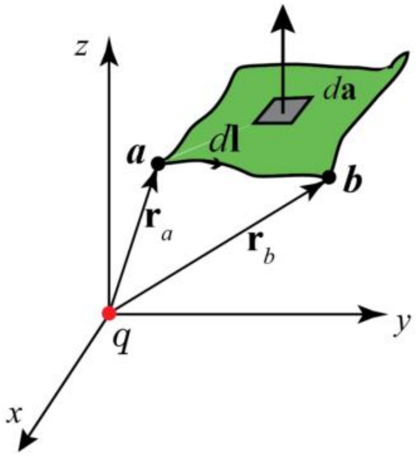


Fig.6.2 Vector diagram of simplest electric field for curl of intensity

Electric field due to a single point charge q is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

We need to find the curl of it

$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\nabla \times \left(\frac{1}{r^2}\hat{\mathbf{r}}\right)$$

Take the area integral

$$\int_{surf} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\int_{surf} \nabla \times \left(\frac{1}{r^2}\hat{\mathbf{r}}\right) \cdot d\mathbf{a}$$

Use Stokes's theorem

$$\int_{surf} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0}\oint \left(\frac{1}{r^2}\hat{\mathbf{r}}\right) \cdot d\mathbf{l} = \frac{q}{4\pi\epsilon_0}\oint \frac{1}{r^2}\,dr$$

Where

$$\boxed{\begin{array}{c}\text{since} \\ d\mathbf{l} = dr\hat{\mathbf{r}} + rd\theta\hat{\boldsymbol{\theta}} \\ \quad +r\sin\theta d\phi\hat{\boldsymbol{\phi}}\end{array}}$$

$$\int_{surf} \nabla \times \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a} = \frac{q}{4\pi\epsilon_0} \times \left.\frac{1}{r}\right|_{r_a}^{r_a} = 0$$

Implies

$$\nabla \times \mathbf{E} = \mathbf{0}$$

Curl of intensity an electric field is zero. We have shown this for the simplest field, which is the field of a point charge. But it can be shown to be true for any electric field, as long as the field is static.

What if the field is dynamic, that is, what if the field changes as a function of time?

Faraday's Law in differential form:

$$\nabla \times \mathbf{E} = -\frac{d\mathbf{B}}{dt}$$

Integrate over a surface

$$\int_{surf} (\nabla \times \mathbf{E}) \cdot d\mathbf{a} = \int_{surf} -\frac{d\mathbf{B}}{dt} \cdot d\mathbf{a}$$

Apply Stokes' theorem

$$\int_{path} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt}\int_{surf} \mathbf{B} \cdot d\mathbf{a}$$

Faraday's Law in integral form

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

Where

ε = EMF

dΦ = Magnetic flux

Curl of magnetization:

Let $\vec{m}_i$ be the magnetic moment of the i-th atom inside a matter. We define magnetization as the net magnetic moment per unit volume

$$\vec{M} = \lim_{\Delta V \to 0} \frac{\sum_i \vec{m_i}}{\Delta V}$$

The quantity is very similar to the polarization in dielectric material.

We will calculate the effect due to the magnetization of the material by calculating the vector potential corresponding to the microscopic currents. We had seen earlier that the magnetic vector potential due to a magnetic moment is given by the expression

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

Where $\vec{r}$ is the position vector of the point of observation with respect to the position of the magnetic moment.

Using this we can write down the expression for the vector potential at a position $\vec{r}$ due to magnetic moments in a magnetized material having a magnetization $\vec{M}$ ($\vec{r}$), where, as before, we have used the primed quantities to indicate the variable to be integrated

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{Vol} \frac{\vec{M}(\vec{r'}) \times (\vec{r} - \vec{r'})}{\left|\vec{r} - \vec{r'}\right|^3} d^3r'$$

As we did the electrostatic case, we can convert this into two integrals, one over the volume and the other over the surface of the material, we can rewrite the vector potential as

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{Vol} \vec{M}(\vec{r'}) \times \nabla' \frac{1}{\left|\vec{r} - \vec{r'}\right|} d^3r'$$

We now use the vector identity for the curl of a product of a scalar with a vector,

$$\nabla \times (\phi\vec{M}) = \nabla\phi \times \vec{M} + \phi\nabla \times \vec{M}$$

Using which we can write,

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi}\int_{Vol} \nabla' \times \left(\frac{\vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|}\right) d^3r' + \frac{\mu_0}{4\pi}\int \frac{1}{|\vec{r} - \vec{r'}|} \nabla' \times \vec{M}\left(\vec{r'}\right) d^3r'$$

The first term can be converted to a surface integral in a manner very similar to the way we converted volume integral of a divergence to a surface integral,

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{4\pi}\int_{S} \hat{n}' \times \left(\frac{\vec{M}(\vec{r'})}{|\vec{r} - \vec{r'}|}\right) dS' + \frac{\mu_0}{4\pi}\int \frac{1}{|\vec{r} - \vec{r'}|} \nabla' \times \vec{M}\left(\vec{r'}\right) d^3r'$$

Thus $\hat{n}$ x $\vec{M}$ takes the role of a surface current. We now identify, as we did in the electrostatic case, a bound volume current and a bound surface current, define by

$$\vec{J}_M^v = \nabla \times \vec{M}$$
$$\vec{J}_M^s = -\hat{n} \times \vec{M}$$

SAQ.2

a) What do you mean by magnetic vectors B, H and Im?
b) Define the magnetic currents for free, bound and total.
c) Explain the Curl of magnetization.

d) A circular coil of wire consisting of 100 turns, each of radius 8.0 *cm* carries a current of 0.40 *A*. What is the magnitude of the magnetic field *B* at the centre of the coil?

e) Find the magnetization of the bar magnet of length 5 cm and cross-sectional area 2 cm^2. The magnetic moment of the magnet is 1 Am^2.

Examples:

Q.1 The magnetic susceptibility of silicon is -0.4×10^{-5}. Calculate the flux density and magnetic moment per unit volume when magnetic field of intensity 5×10^{5} A/m is applied.

Solution: Given: $\chi = -0.4 \times 10^{-5}$

$H = 5 \times 10^{5}$ A /m

B = ? and M = ?

$B = \mu_0(H + M) = \mu_0 H(1 + \chi)$

$= 4\pi \times 10^{-7} \times 5 \times 10^{5} [1 - 0.4 \times 10^{-5}] = 4\pi \times 5 \times 10^{-2} \times 0.9996 = 0.62$ Wb/m^2

$M = \chi H = -0.4 \times 10^{-5} \times 5 \times 10^{5} = -2.0$ A/m.

Q.2 The Dimension of a rectangular loop is 0.50m and 0.60m. B and θ are 0.02T and 45° respectively. Determine the magnetic flux through the surface.

Solution:

Given

Dimensions of rectangular loop = 0.50m and 0.60m,

B = 0.02T

θ = 45°

Magnetic flux formula is given by

$\Phi B = B\ A\ \text{Cos}\theta$

Area, $A = 0.50 \times 0.60$

$= 0.3\ m^2$

$\Phi B = 0.02 \times 0.3 \times \text{Cos } 45$

$\Phi B = 0.00312$ Wb

Q.3 Calculate magnetic flux density of the magnetic field at the centre of a circular coil of 50 turns, having radius of 0.5m and carrying a current of 5A.

Solution:

Given

n=50*turns*, R=0.5*m*, I=5*A*,

According to Bio-sawart Law

$$\Rightarrow B = \frac{\mu_0 n I}{2R}$$

$$\Rightarrow B = \frac{4\pi \times 10^{-7} \times 50 \times 5}{2 \times 0.5} = 3.14 \times 10^{-3} T$$

Q.4 Two circular coils made of similar wires but of radii 20 and 40 *cm* are connected in parallel. Find the ratio of the magnetic fields at their centers.

Solution:

As the coils are connected in parallel so voltage across them should be same. i.e.

$$i_1 R_1 = i_2 R_2 \Rightarrow \frac{i_1}{i_2} = \frac{R_2}{R_1} = \frac{(\rho_2 l_2)/A_2}{(\rho_1 l_1)/A_1}$$

As they are made from same wire so cross section (A) and resistivity are same for both coil i.e.

$A_1 = A_2$ and $\rho_1 = \rho_2$.

thus, $\frac{i_1}{i_2} = \frac{l_2}{l_1} = \frac{2\pi r_2}{2\pi r_1} = \frac{r_2}{r_1}$

The magnetic field at center of a coil of radius r_1 is $B_1 = \frac{\mu_0 i_1}{2r_1}$

The magnetic field at center of a coil of radius r_2 is $B_2 = \frac{\mu_0 i_2}{2r_2}$

$$\frac{B_1}{B_2} = \frac{i_1 r_2}{i_2 r_1} = (\frac{r_2}{r_1})^2 = (40/20)^2 = 4 \quad \text{as } i_1/i_2 = r_2/r_1$$

Q.5 A paramagnetic material has a magnetic field intensity of 10^4 Am^{-1}. If the susceptibility of the material at room temperature is 3.7×10^{-5}. Calculate the magnetization and flux density in the material.

Solution:

Given data:

Magnetic field indensity $H = 10^4 \text{ Am}^{-1}$

Susceptibility $\chi = 3.7 \times 10^{-4}$

1. ***Susceptibility*** $\chi = \dfrac{M}{H}$

$\therefore$ Magnetization $M = \chi H$

$$= 3.7 \times 10^{-3} \times 10^4$$

$$= 3.7 \times 10$$

Magnetization $M = 37 \text{A m}^{-1}$

2. ***Flux density*** $B = m_0 (M + H)$

$$= 4\pi \times 10^{-7} \times (37 + 10^4)$$

$$= 126179.4 \times 10^{-7}$$

$$= 0.0126 \text{ Wb m}^{-2}$$

Magnetization $M = 37\text{A m}^{-1}$

Flux density $B = 0.0126 \text{ Wb m}^{-2}$.

Q.6 A magnetic material has a magnetization of 2300 A m^{-1} and produces a flux density of 0.00314 Wb m^{-2}. Calculate the magnetizing force and the relative permeability of the material.

Solution:

Given data:

Magnetization M = 2300 A m^{-1}

Flux density B = 0.00314 Web m^{-2}.

i) The magnetic flux density

$$B = \mu_0 (M + H)$$

$$\text{The magnetic force} \quad H = \left[\frac{B}{\mu_0} - M\right]$$

$$= \frac{0.00314}{4\pi \times 10^{-7}} - 2300$$

$$H = 198.7326 \text{ A m}^{-1}$$

$$\textbf{\textit{ii) Susceptibility}} \qquad \chi = \frac{M}{H} = (\mu_r - 1)$$

$$\therefore \text{ Relative permeability } \mu_r = \frac{M}{H} + 1$$

$$= \frac{2300}{198.7326} + 1$$

$$\mu_r = 12.573$$

$$\text{Magnetic force } H = 198.7326 \text{ A m}^{-1}$$

$$\text{Relative permeability} \quad \mu_r = 12.573$$

Q.7 A paramagnetic material has FCC structure with a cubic edge of 2.5 A°. If the saturation value of magnetization is 1.8×10^6 A m^{-1}, Calculate the magnetization contributed per atom in Bohr magnetrons.

Solution:

Given data:

The interatomic spacing $a = 2.5 \times 10^{-10}$ m

The magnetization $M = 1.8 \times 10^6$ A m^{-1}

The number of atoms present per unit volume

$$N = \frac{\text{Number of atomas present in an unit cell}}{\text{Volume of the unit cell}}$$

$$= \frac{2}{\left(2.5 \times 10^{-10}\right)}$$

$$N = 1.28 \times 10^{29}\ m^3$$

Total magnetization $M = 1.8 \times 10^6\ A\ m^{-1}$

The magnetization produced per atom

$$= \frac{M}{N}$$

$$= \frac{1.8 \times 10^6}{1.28 \times 10^{29}}$$

$$= 1.4062 \times 10^{-23}\ A\ m^{-2}$$

Bohr magneton $\mu_B = \dfrac{eh}{4\pi m}$

$$= \frac{1.6 \times 10^{-19} \times 6.625 \times 10^{-34}}{4 \times 3.14 \times 9.1 \times 10^{-31}}$$

$$\mu_B = 9.27 \times 10^{-24}\ A\ m^{-2}$$

$\therefore$ Magnetization produced per atom

$$M = \frac{1.40625 \times 10^{-23}}{9.27 \times 10^{-24}}$$

$$M = 1.519 \text{ Bohr magnetons}$$

The average magentization per atom=1.517 Bohr magnetons

Q.8 In a magnetic material the field strength is found to be 10^6 A m^{-1}. If the magnetic susceptibility of the material is 0.5×10^{-5}, calculate the intensity of magnetization and flux density in the material.

Solution:

Given data:

Magnetic field strenght $H = 10^6 \text{ A m}^{-1}$

Susceptibility $\chi = 0.5 \times 10^{-5}$

i) Magnetization

$$M = \chi H$$
$$= 10^6 \times 0.5 \times 10^{-5}$$
$$M = 5 \text{ A m}^{-1}$$

ii) Flux density

$$B = \mu_0 (M + H)$$
$$= 4 \times 3.14 \times 10^{-7} (5 + 10^6)$$
$$B = 1.257 \text{ Wb m}^{-2}$$

Magnetization $M = 5 \text{ A m}^{-1}$

Fluxdensity $B = 1.257 \text{ Wb m}^{-2}$

Q.9 Prove that susceptibility of superconductor is -1 and relative permeability is zero.

Solution:

Given data:

We know, the induced magnetic field $B = \mu_0(M + H)$

In superconductor, $B = 0$

Therefore, $0 = \mu_0(M + H)$

Since $\mu_0 \# 0$, $M = -H$

$$\frac{M}{H} = \chi = -1$$

Also, $\chi = \mu_r - 1$

$$-1 + 1 = \mu_r = 0$$

Therefore the susceptibility $x = -1$

and Relative permeability $\mu_r = 0$

Q.10 A magnetic field of 2000 Amp m^{-1} is applied to a material which has a susceptibility of 1000. Calculate the (i) Intensity and (ii) Flux density.

Solution:

Given data:

Magnetic Field $H = 2000 \text{ Amp m}^{-1}$

Susceptibility $\chi = 1000$

i) Magnetisation

$$M = \chi H$$
$$= 2000 \times 1000$$
$$= 2 \times 10^6 \text{ Amp m}^{-1}$$

ii) Flux density

$$B = \mu_o(M+H)$$
$$= 4 \times 3.14 \times 10^{-7} (2 \times 10^6 + 2000)$$
$$B = 2.514 \text{ wbm}^{-2}$$

Q.11 The magnetic field strength of Silicon is 1500 A m^{-1}. If the magnetic susceptibility is (-0.3×10^{-5}). Calculate the magnetization and flux density in Silicon.

Solution:

Given data:

Magnetic susceptibility $\chi = -0.3 \times 10^{-5}$

Magnetic field strength $H = 1500 \text{ Amp m}^{-1}$

Magnetic susceptibility $\chi = \frac{I}{H}$

$$I = \chi H$$
$$= -0.3 \times 10^{-5} \times 1500$$
$$= -4.5 \times 10^{-3} \text{ Amp m}^{-1}$$

We know $\mu = H + \chi$

$= 1 - 4.5 \times 10^{-3}$

$$\boxed{\mu = 0.999}$$

Flux density $B = \mu H$

$= 0.999 \times 1500$

$$\boxed{B = 1498.5 \text{ web m}^{-2}}$$

Q.12 Compute the intensity of magnetisation of the bar magnet whose mass, magnetic moment and density are 200 g, 2 A m2 and 8 g cm-3, respectively.

Solution:

Density of the magnet is

$$Density = \frac{Mass}{Volume} \Rightarrow Volume = \frac{Mass}{Density}$$

$$Volume = \frac{200 \times 10^{-3} kg}{\left(8 \times 10^{-3} kg\right) \times 10^{6} m^{-3}} = 25 \times 10^{-6} m^{3}$$

Magnitude of magnetic moment $p_m = 2 A m^2$

Intensity of magnetization,

$$I = \frac{Magnetic\,moment}{Volume} = \frac{2}{25 \times 10^{-6}}$$

$$M = 0.8 \times 10^{5} A m^{-1}$$

Q.13 With the help of using the relation $\vec{B} = \mu_0 (\vec{H} + \vec{M})$. Show that $\chi_m = \mu_r - 1$.

Solution:

$\vec{B} = \mu_\circ(\vec{H} + \vec{M})$,

But from equation

$\vec{M} = \chi_m \vec{H}$

Hence, $\vec{B} = \mu_\circ(\chi_m + 1)\vec{H} \Rightarrow \vec{B} = \mu\vec{H}$

where, $\mu = \mu_\circ(\chi_m + 1) \Rightarrow \chi_m + 1 = \frac{\mu}{\mu_\circ} = \mu_r$

$\Rightarrow \chi_m = \mu_r - 1$

Q.14 Two materials X and Y are magnetised, whose intensity of magnetisation are 500 A m-1 and 2000 A m-1, respectively. If the magnetising field is 1000 A m-1, then which one among these materials can be easily magnetized?.

Solution:

The susceptibility of material X is

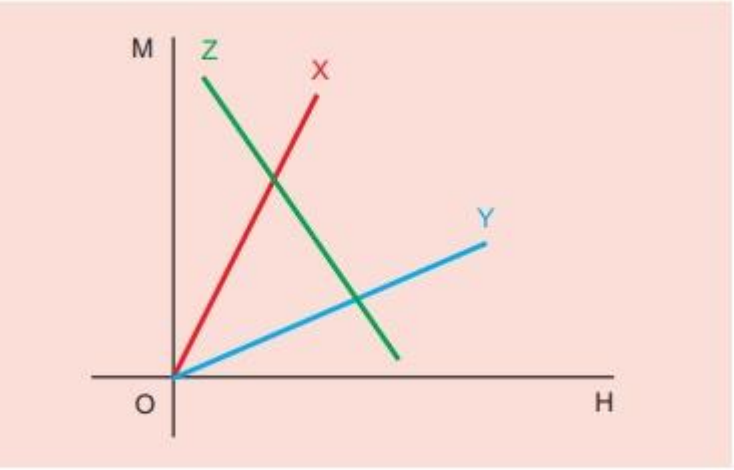
$$\chi_{m,X} = \frac{|\vec{M}|}{|\vec{H}|} = \frac{500}{1000} = 0.5$$

The susceptibility of material Y is

$$\chi_{m,Y} = \frac{|\vec{M}|}{|\vec{H}|} = \frac{2000}{1000} = 2$$

Since, susceptibility of material Y is greater than that of material X, material Y can be easily magnetized than X.

Q.15 The following figure shows the variation of intensity of magnetization with the applied magnetic field intensity for three magnetic materials X, Y and Z. Identify the materials X,Y and Z.



Solution:

The slope of M-H graph measures the magnetic susceptibility, which is

$\chi m = M / H$

Material X: Slope is positive and larger value. So, it is a ferromagnetic material.

Material Y: Slope is positive and lesser value than X. So, it could be a paramagnetic material.

Material Z: Slope is negative and hence, it is a diamagnetic material.

Q.16 Magnetic field and magnetic intensity are respectively 1.6 T and 1000 A/m. Find relative permeability and susceptibility.

Solution:

Given: Magnetic field = B = 1.6 T, Magnetic Intensity = H = 1000 A/m,

$\mu_o = 4\pi \times 10^{-7}$ Wb/Am.

Relative permeability = μ_r = ?, Susceptibility = χ =?

$\mu_r = B/ (\mu_o H) = 1.6/ (4\pi \times 10^{-7} \times 1000) = 1.6/ (4 \times 3.142 \times 10^{-4}) = 1.273 \times 10^3 = 1273$

we have $\mu_r = 1 + \chi$

.: $\chi = \mu_r - 1 = 1273 - 1 = 1272$

Relative permeability = μ_r = 1273, Susceptibility = χ =1272

Q.17 Find the magnetization of the bar magnet of length 10 cm and cross-sectional area 3 cm^2. The magnetic moment of the magnet is 1 Am^2.

Solution:

Given: Length of magnet = l = 10 cm,

Cross- sectional area = A = 3 cm^2,

Magnetic moment = M = 1 Am^2.

To find: Magnetization = M_Z =?

Volume of bar magnet = V = length x cross- sectional area = 10 x 3 = 30 $cm^3 = 30 \times 10^{-6}$ m^3.

$M_z = M/V = 1/(30 \times 10^{-6}) = 3.33 \times 10^4$ A/m

Magnetization = $M_Z = 3.33 \times 10^4$ A/m

Summary:

1. Magnetism is a property of matter and it occurs in different forms and degrees in various Earth materials that act as conductors and insulators. The degree of magnetism is also called magnetization and it is defined as the net magnetic dipole moment of the substance per unit volume.
2. Magnetic flux density (B) is defined as the force acting per unit current per unit length on a wire placed at right angles to the magnetic field.
3. The definition of H is $H = B/\mu - M$, where B is the magnetic flux density, a measure of the actual magnetic field within a material considered as a concentration of magnetic field lines, or flux, per unit cross-sectional area; μ is the magnetic permeability; and M is the magnetization.
4. Intensity of magnetism is defined as the magnetic moment per unit volume of the magnetized material so, $I=M/V$. where M is the total magnetic moment within volume due to the magnetizing field.
5. The definition of susceptibility χ connects the magnetization M of a material (its magnetic moment per unit volume) with the external field Hthat is magnetizing it. For any but ferromagnetic materials, that response is linear, and is described by $M = \chi H$.
6. The term Absolute Permeability (μ_a) of a material is the product of permeability of free space μ_0 ($= 4\pi \times 10^{-7}$ henry/meter) and the Relative Permeability μ_r .The Relative Permeability is a pure numerical number and therefore has no units. As mentioned above, for air and non-magnetic materials, its value is unity.

7. Magnetization or magnetic polarization is the vector field that expresses the density of permanent or induced magnetic dipole moments in a magnetic material.
8. The area of the hysteresis curve gives the hysteresis loss of energy while a ferromagnetic substance is taken over a complete cycle of magnetization.
9. The changes in magnetization in a ferromagnetic material lag behind the variations of the magnetic field applied to it. Thus the intensity of magnetization at every phase lags behind the applied field. This property is called magnetic hysteresis.
10. The ability of a substance to retain or resist magnetization, frequently measured as the strength of the magnetic field that remains in a sample after removal of an inducing field. ‘iron is easily magnetized but has low retentivity’.
11. Remanence or remanent magnetization or residual magnetism is the magnetization left behind in a ferromagnetic material (such as iron) after an external magnetic field is removed.
12. The relation between the tree vectors B H and M points [$B = \mu_0 (H + M)$] in the same direction as that of B or M and M. H points in the same direction as that of B or M. Its unit is Am^{-1}.
13. In terms of free current, the line integral of the magnetic H-field (in amperes per metre, $A{\cdot}m^{-1}$) around closed curve C equals the free current $I_{f,enc}$ through a surface S.
14. As no work is done while moving a charge in a closed loop in an electric field, the closed line integral of that field must be zero and hence, curl of the field also must be zero. In other words, the field must be irrotational.

Terminal Questions:

1) Explain the magnetic properties of magnetic flux density(B) and Magnetizing field(H).
2) What do you understand by the intensity of magnetization (Im) and Susceptibility.
3) Define the terms relative and absolute permeability.
4) Explain the Magnetization of the magnetic materials.
5) Explain the working of hysteresis loop and show the terms retentivity and residual magnetism in its curve.
6) Explain the three magnetic vectors (B, H, Im).
7) What do you understand by three magnetic currents (free, bound and total).
8) Derive the expression for the Curl of intensity.
9) A magnetic material has a magnetization of 3000 Am^{-1} and flux density of 0.044Wb m^{-2}. Calculate the magnetic force and the relative permeability of the material.
10) The magnetic field intensity of a ferric oxide piece is 10^6 Am^{-1}. If the susceptibility of the material at room temperature is 10.5×10^{-3}, calculate the flux density and magnetization of the material.
11) Magnetic field and magnetic intensity are respectively 1.3 T and 900 A/m. Find the permeability, relative permeability and susceptibility.

Uttar Pradesh Rajarshi Tandon

Open University

Bachelor of Science

UGPHS-103

Electromagnetism

Block

3 Electrostatics

UNIT - 7	**Electromagnetic Induction**

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Unit 07- Electromagnetic induction

Structure

7.1 Introduction:

Faraday's law states that a current will be induced in a conductor which is exposed to a changing magnetic field. This phenomenon is known as electromagnetic induction. The working principle of transformers, motors, generators, and inductors are being described by Faraday’s Law of Induction.

Newton's laws of motion relate an object's motion to the forces acting on it. In the first law, an object will not change its motion unless a force acts on it. In the second law, the force on an object is equal to its mass times its acceleration. The third law of motion states that for every action, there is an equal and opposite reaction. This can be observed both in objects at rest and those that are accelerating. For example, a resting box pushes down on the ground due to a gravitational force.

Induction charging is a method used to charge an object without actually touching the object to any other charged object. So that +ve charge which is outside the body (that separate the charges inside the body) is Inducing charge & the negative charge that gets attracted towards +ve charge is Induced Charge.

Induced voltage is an electric potential created by an electric field, magnetic field, or a current. Voltage produced in generator because of moving magnetic field. Voltage generated in secondary of current transformer due to magnetic field of current injected in it's primary.

A current can be induced in a conducting loop if it is exposed to a changing magnetic field. ... In other words, if the applied magnetic field is increasing, the current in the wire will flow in such a way that the magnetic field that it generates around the wire will decrease the applied magnetic field.

Induced power is the power required to maintain enough lift to overcome the force of gravity. One can view this as the force required to accelerate enough air downwards (at speed v_i) to push the bird upwards enough to counteract the force of gravity (mg).

Dynamic Induced emf is generated when a current carrying conductor cuts the magnetic flux using relative motion. The name itself indicates the dynamic, it mean having rotating parts in it, inducing emf with respect to moving parts is known as Dynamic Induced emf.

When this emf is induced in the same circuit in which the current is changing this effect is called Self-induction, (L). However, when the emf is induced into an adjacent coil situated within the same magnetic field, the emf is said to be induced magnetically, inductively or by Mutual induction, symbol (M).

In simple words, faraday's law of electromagnetic induction says that if the magnetic field inside a coil is changing with time, then an emf will be induced across the coil. This emf is proportional to (1) the number of turns in the coil (2) rate of change magnetic flux with time. Hence the term statically induced emf. Self induced emf is that which is induced in a coil, due to the change in its own current or flux. Mutual emf is that induced in a coil due to the neighboring coil's varying current. If the coil is moving (or rotating) and the magnetic field value is constant, then dynamic emf is induced.

Experiments and calculations that combine Ampere's law and Biot-Savart law confirm that the two constants, M_{21} and M_{12} are equal in the absence of material medium between the two coils, $M_{12} = M_{21}$. This property is called reciprocity.

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A transformer is an electrical apparatus designed to convert alternating current from one voltage to another. It can be designed to "step up" or "step down" voltages and works on the magnetic induction principle.

When voltage is introduced to one coil, called the primary, it magnetizes the iron core. Equivalent circuit diagram of a transformer is basically a diagram which can be resolved into an equivalent circuit in which the resistance and leakage reactance of the transformer are imagined to be external to the winding. The equivalent circuit of any electrical system is the circuit representation of a device using standard active and passive elements. It is used to analyze and predict the performance (quantitatively and qualitatively) of that system without actually loading the device.

An ideal transformer is an imaginary transformer which does not have any loss in it, means no core losses, copper losses and any other losses in transformer. Efficiency of this transformer is considered as 100%.

The Efficiency of the transformer is defined as the ratio of useful output power to the input power. The input and output power are measured in the same unit. Its unit is either in Watts (W) or KW.

There are various types of losses in the transformer such as iron loss, copper loss, hysteresis loss, eddy current loss, stray loss, and dielectric loss. Although transformers are very efficient devices, small energy losses do occur in them due to four main causes: Resistance of windings – the low resistance copper wire used for the windings still has resistance and thereby contribute to heat loss. The eddy currents cause heat loss.

7.2 Objectives:

After studying this unit you should be able to

- Explain and identify Faraday's law of electromagnetic induction (statement, integral form, differential form) and analogy with Newton's laws of motion in mechanics.
- Study and identify Condition for existence and depending factors of induced charge, induced voltage, induced current and induced power.
- Explain and identify Dynamic induced EMF and derivation of its expression,
- Explain Self and mutual induction and inductance, static induced EMF (self and mutual).
- Study and identify Reciprocity theorem and Neuman's relation.
- Explain Relation between self and mutual inductance of two coupled coils, energy of coupled circuits.
- Explain and identify Transformer and its equivalent circuit, condition for ideal transformer (expression for efficiency and voltage gain), transformer losses.

7.3 Faraday's law of electromagnetic induction:

What is Faraday's Law?

Faraday's law of electromagnetic induction (referred to as Faraday's law) is a basic law of electromagnetism predicting how a magnetic field will interact with an electric circuit to produce an electromotive force (EMF). This phenomenon is known as electromagnetic induction.

"Current carrying conductor placed in magnetic field an EMF is induced."

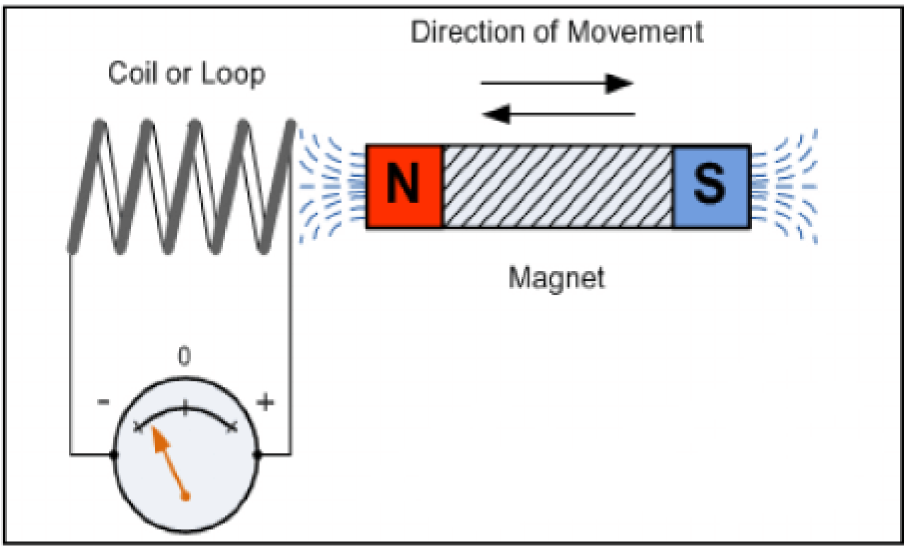


Fig.7.1 Electromagnetic induction

Faraday’s law states that a current will be induced in a conductor which is exposed to a changing magnetic field. Lenz’s law of electromagnetic induction states that the direction of this induced current will be such that the magnetic field created by the induced current opposes the initial changing magnetic field which produced it. The direction of this current flow can be determined using Fleming’s right-hand rule.

Faraday’s law of induction explains the working principle of transformers, motors, generators, and inductors. The law is named after Michael Faraday, who performed an experiment with a magnet and a coil. During Faraday’s experiment, he discovered how EMF is induced in a coil when the flux passing through the coil changes.

Faraday’s Experiment:

In this experiment, a magnet, and a coil are connected with a galvanometer across the coil. At starting, the magnet is at rest, so there is no deflection in the galvanometer i.e. the needle of the galvanometer is at the center or zero position. When the magnet is moved towards the coil, the needle of the galvanometer deflects in one direction.

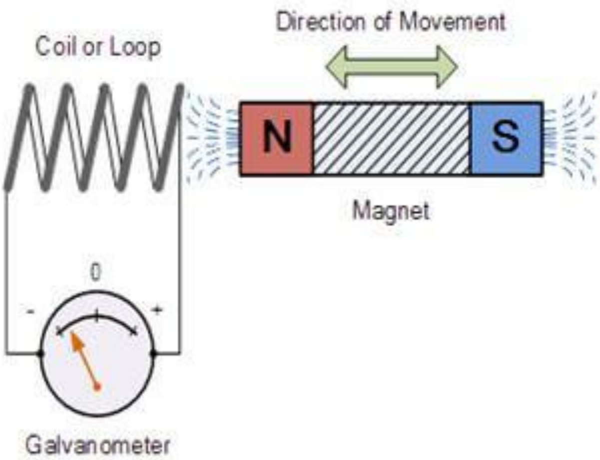


Fig.7.2 Faraday's Experiment for direction

When the magnet is held stationary at that position, the needle of galvanometer returns to zero position. Now when the magnet moves away from the coil, there is some deflection in the needle but opposite direction, and again when the magnet becomes stationary, at that point respect to the coil, the needle of the galvanometer returns to the zero position. Similarly, if the magnet is held stationary and the coil moves away, and towards the magnet, the galvanometer similarly shows deflection. It is also seen that the faster the change in the magnetic field, the greater will be the induced EMF or voltagein the coil.

Position of magnet	**Deflection in galvanometer**
Magnet at rest	No deflection in the galvanometer
Magnet moves towards the coil	Deflection in galvanometer in one direction
Magnet is held stationary at same position (near the coil)	No deflection in the galvanometer
Magnet moves away from the coil	Deflection in galvanometer but in the opposite direction
Magnet is held stationary at the same position (away from the coil)	No deflection in the galvanometer

Conclusion: From this experiment, Faraday concluded that whenever there is relative motion between a conductor and a magnetic field, the flux linkage with a coil changes and this change in flux induces a voltage across a coil.

Michael Faraday formulated two laws on the basis of the above experiments. These laws are called Faraday's laws of electromagnetic induction.

Faraday's First Law:

Any change in the magnetic field of a coil of wire will cause an emf to be induced in the coil. This emf induced is called induced emf and if the conductor circuit is closed, the current will also circulate through the circuit and this current is called induced current. Method to change the magnetic field:

1. By moving a magnet towards or away from the coil
2. By moving the coil into or out of the magnetic field
3. By changing the area of a coil placed in the magnetic field
4. By rotating the coil relative to the magnet

Faraday's Second Law:

It states that the magnitude of emf induced in the coil is equal to the rate of change of flux that linkages with the coil. The flux linkage of the coil is the product of the number of turns in the coil and flux associated with the coil.

Faraday Law Formula:

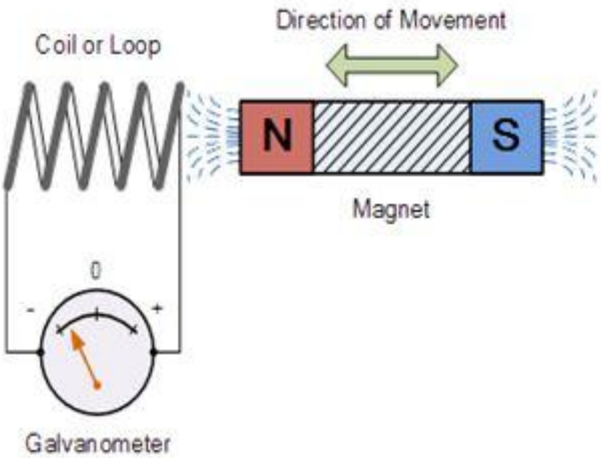


Fig.7.3 EMF induced in the coil

Consider, a magnet is approaching towards a coil. Here we consider two instants at time T_1 and time T_2.

Flux linkage with the coil at time,

$$T_1 = N\phi_1 \ wb$$

Flux linkage with the coil at time,

$$T_2 = N\phi_2 \ wb$$

Change in flux linkage,

$$N(\phi_2 - \phi_1)$$

Let this change in flux linkage be,

$$\phi = (\phi_2 - \phi_1)$$

So, the Change in flux linkage

$$N\phi$$

Now the rate of change of flux linkage

$$\frac{N\phi}{t}$$

Take derivative on right-hand side we will get

$$N\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

The rate of change of flux linkage

$$E = N\frac{d\phi}{dt}$$

But according to Faraday’s law of electromagnetic induction, the rate of change of fluxlinkage is equal to induced emf.

$$E = -\,N\frac{d\phi}{dt}$$

Considering Lenz’s Law

Where:

- Flux Φ in Wb = B.A
- B = magnetic field strength
- A = area of the coil

How to increase EMF Induced in a Coil:

- By increasing the number of turns in the coil i.e N, from the formulae derived above it is easily seen that if the number of turns in a coil is increased, the induced emf also gets increased.

- By increasing magnetic field strength i.e B surrounding the coil- Mathematically, if magnetic field increases, flux increases and if flux increases emf induced will also get increased. Theoretically, if the coil is passed through a stronger magnetic field, there will be more lines of force for the coil to cut and hence there will be more emf induced.

- By increasing the speed of the relative motion between the coil and the magnet – If the relative speed between the coil and magnet is increased from its previous value, the coil will cut the lines of flux at a faster rate, so more induced emf would be produced.

Applications of Faraday's Law:

Faraday law is one of the most basic and important laws of electromagnetism. This law finds its application in most of the electrical machines, industries, and the medical field, etc.

- Power transformers function based on Faraday's law

- The basic working principle of the electrical generator is Faraday's law of mutual induction.

- The Induction cooker is the fastest way of cooking. It also works on the principle of mutual induction. When current flows through the coil of copper wire placed below a cooking container, it produces a changing magnetic field. This alternating or changing magnetic field induces an emf and hence the current in the conductive container, and we know that the flow of current always produces heat in it.

- Electromagnetic Flow Meter is used to measure the velocity of certain fluids. When a magnetic field is applied to an electrically insulated pipe in which conducting fluids are flowing, then according to Faraday’s law, an electromotive force is induced in it. This induced emf is proportional to the velocity of fluid flowing.

- Form bases of Electromagnetic theory, Faraday’s idea of lines of force is used in well known Maxwell’s equations. According to Faraday’s law, change in magnetic field gives rise to change in electric field and the converse of this is used in Maxwell’s equations.

- It is also used in musical instruments like an electric guitar, electric violin, etc.

Integral Form of Faraday's law:

Faraday's law in integral form can be expressed using the following equation:

$$\oint_C \mathbf{e} \cdot \mathbf{dl} = -\int_S \frac{\partial \mathbf{b}}{\partial t} \cdot \hat{\mathbf{n}}\, da$$

Where:

- e is the electric field defined around a closed path C,
- b is the magnetic flux density defined over a closed surface A contoured by C,
- $\hat{n}$ is an outward normal unit vector perpendicular to da,
- dl is a vector element of length along contour C.

Above equation states that the time-dependent rate of change in magnetic flux, through a surface bounded by a closed path, is negatively proportional to the line integral of the electric field it induces over that path.

Differential form of Faraday's law:

The magnetic flux is

$$\Phi_B = \int_S \vec{B} \cdot d\vec{A}$$

where $\vec{A}$ is a vector area over a closed surface S. A device that can maintain a potential difference, despite the flow of current is a source of electromotive force. (EMF) The definition is mathematically

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{s},$$

Where the integral is evaluated over a closed loop C

Faraday's law now can be rewritten

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t}\left(\int \vec{B} \cdot d\vec{A}\right)$$

Using the Stokes' theorem in vector calculus, the left hand side is

$$\oint_C \vec{E} \cdot d\vec{s} = \int_S (\nabla \times \vec{E}) \cdot d\vec{A}$$

Also, note that in the right hand side

$$\frac{\partial}{\partial t}\left(\int \vec{B} \cdot d\vec{A}\right) = \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Therefore, we get an alternative form of the Faraday's law of induction:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}.$$

This is also called a differential form of the Faraday's law.

Analogy with Newton's laws of motion in mechanics:

Newton's First Law of Motion:

Newton's first law states that a body remains in the state of rest or uniform motion in a straight line unless and until an external force acts on it.

Putting Newton's 1st law of motion in simple words, a body will not start moving until and unless an external force acts on it. Once it is set in motion, it will not stop or change its velocity until and unless some force acts upon it once more. The first law of motion is sometimes also known as the law of inertia.

There are two conditions on which the 1st law of motion is dependent:

- **Objects at rest:** When an object is at rest velocity (v= 0) and acceleration (a = 0) are zero. Therefore, the object continues to be at rest.
- **Objects in motion:** When an object is in motion, velocity is not equal to zero ($v \neq 0$) while acceleration (a = 0) is equal to zero. Therefore, the object will continue to be in motion with constant velocity and in the same direction.

Derivation of the equation of motion:

Derivation of the equation of motion is one of the most important topics in Physics. Several important concepts in Physics are based on the equation of motion. In this article, the equation of motion derivations by the graphical method and by the normal method are explained in an easily understandable way for the first, second and third equation of motion.

Derivation:

There are mainly three equations of motion which describe the relationship between velocity, time, acceleration and displacement.

First, consider a body moving in a straight line with uniform acceleration. Then, let the initial velocity be *u*, acceleration be *a*, time period be *t*, velocity be *v*, and the distance travelled be *S*.

The equation of motions derivation can be done in three ways which are:

- Derivation of equations of motion by Simple Algebraic Method
- Derivation of Motion by Graphical Method
- Derivation of Motion by Calculus Method

Below, the equations of motion are derived by all the three methods in a simple and easy to understand way.

Derivation of First Equation of Motion

The first equation of motion is:

$$v = u + at$$

Derivation of First Equation of Motion by Algebraic Method

It is known that the acceleration (a) of the body is defined as the rate of change of velocity.
So, the acceleration can be written as:

$$a = v – ut$$

From this, rearranging the terms, the first equation of motion is obtained, which is:

$$v = u + at$$

Derivation of First Equation of Motion by Graphical Method:

Consider the diagram of the velocity-time graph of a body below

$$a = \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

In this, the body is moving with an initial velocity of *u* at point A. The velocity of the body then changes from A to B in time *t* at a uniform rate. In the above diagram, BC is the final velocity i.e. *v* after the body travels from A to B at a uniform acceleration of *a*. In the graph, OC is the time *t*. Then, a perpendicular is drawn from B to OC, a parallel line is drawn from A to D, and another perpendicular is drawn from B to OE (represented by dotted lines).

Following details are obtained from the graph above:

The initial velocity of the body, u = OA

The final velocity of the body, v = BC

From the graph, BC = BD + DC

So, v = BD + DC

v = BD + OA (since DC = OA)

Finally, $v = BD + u$ (since OA = u) (Eq 1)

Now, since the slope of a velocity-time graph is equal to acceleration *a*,

So,
a = slope of line AB

$a = BD/AD$

Since AD = AC = t, the above equation becomes:

$BD = at$ (Eq 2)

Now, combining Equation 1 & 2, the following is obtained:

$$v = at + u$$

Derivation of First Equation of Motion by Calculus Method:

It is known that,

$$adt = dv$$

Integrating both sides

$$\int_0^t adt = \int_u^v dv$$

$$at = v - u$$

$$\boxed{v = u + at}$$

So,

As we know that the displacement (S) is the product of

Average velocity $\frac{(u+v)}{2}$ and time period (t)

$$S = \frac{(u+v)}{2} \times t \Rightarrow S = \frac{(u+u+at)}{2} \times t$$

$$\Rightarrow S = \left(\frac{2u+at}{2}\right) \times t \Rightarrow S = \left(\frac{2u}{2} + \frac{1}{2}t\right) \times t$$

$$\Rightarrow S = \left(u + \frac{1}{2}t\right) \times t \Rightarrow S = ut + \frac{1}{2}t^2$$

Derivation of Second Equation of Motion:

The second equation of motion is:

$S = ut + ½\, a^2$

Derivation of Second Equation of Motion by Algebraic Method:

Consider the same notations for the derivation of the second equation of motion by simple algebraic method.

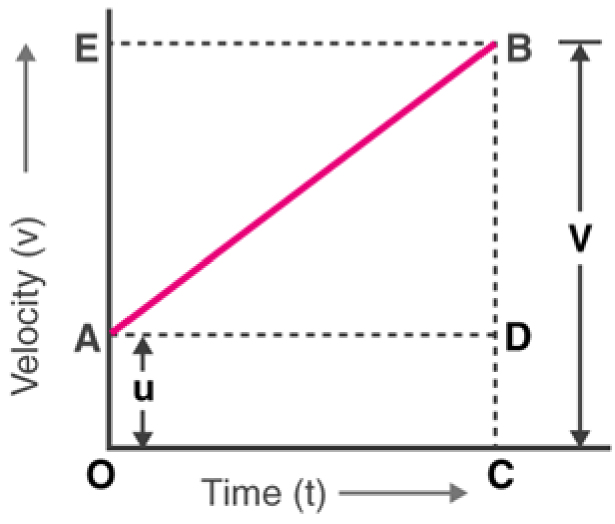


Fig.7.4 Derivation of Second Equation of Motion by Algebraic Method

Derivation of Second Equation of Motion by Graphical Method:

Taking the same diagram used in first law derivation:

$$vdt = dS$$

$$(u + at)dt = dS$$

$$\textit{Integrating both sides}$$

$$\int_0^t (u + at)dt = \int_0^S ds$$

$$ut + \frac{at^2}{2} = S$$

$$\boxed{S = ut + \frac{1}{2}at^2}$$

In this diagram, the distance travelled *(S)* = Area of figure OABC = Area of rectangle OADC + Area of triangle ABD.

Now, the area of the rectangle OADC = OA × OC = *ut*

And, Area of triangle ABD = (1/2) × Area of rectangle AEBD = *(1/2)* at^2 (Since, AD = t and BD = at)

Thus, the total distance covered will be:

$S = ut + (1/2)\ at^2$

Derivation of Second Equation of Motion by Calculus Method:

Velocity is the rate of change of displacement.

Mathematically, this is expressed as

$$v = \frac{ds}{dt}$$

Rearranging the equation, we get

$$ds = vdt$$

Substituting the first equation of motion in the above equation, we get

$$ds = (u + at)dt = (udt + at\,dt)\ \int_0^s ds = \int_0^t u\,dt + \int_0^t at\,dt\ s = ut + \frac{1}{2}at^2$$

Derivation of Third Equation of Motion

The third equation of motion is:

$$v^2 = u^2 + 2aS$$

Derivation of Third Equation of Motion by Algebraic Method:

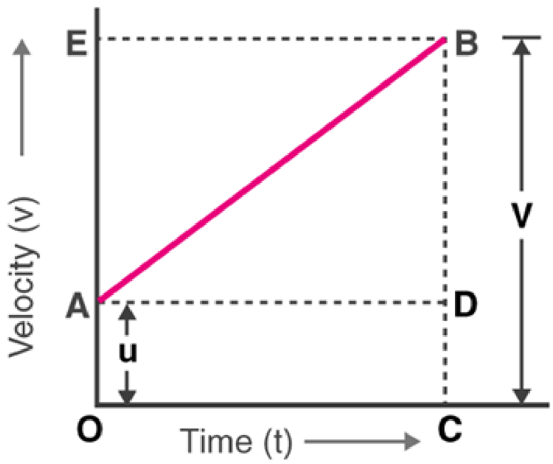


Fig.7.5 Derivation of Third Equation of Motion by Algebraic Method

Derivation of Third Equation of Motion by Graphical Method:

$$a = \frac{dv}{dt} = \frac{dv}{dt} * \frac{dS}{dS} = \frac{vdv}{dS}$$

$$adS = vdv$$

Integrating both sides

$$\int_0^S a = \int_u^v v$$

$$aS = \frac{v^2 - u^2}{2}$$

$$\boxed{v^2 = u^2 + 2aS}$$

The total distance travelled, *S = Area of trapezium OABC.*

So, S= 1/2(Sum of Parallel Sides)×Height

S = (OA+CB)×OC

Since, OA = *u*, CB = *v*, and OC = *t*

The above equation becomes

S= 1/2(u+v)×t

Now, since *t = (v – u)/ a*

The above equation can be written as:

S= 1/2(u+v)×(v-u)/a

Rearranging the equation, we get

S= 1/2(v+u)×(v-u)/a

$S = (v^2-u^2)/2a$

Third equation of motion is obtained by solving the above equation:

$$v^2 = u^2 + 2aS$$

Derivation of Third Equation of Motion by Calculus Method:

It is known that,

As we know that the displacement (S) is the product of

Average velocity $\frac{(u+v)}{2}$ and time period (t)

$$S = \frac{(u + v)}{2} \times t \Rightarrow S = \frac{(u + u + at)}{2} \times t$$

$$\Rightarrow S = \left(\frac{2u + at}{2}\right) \times t \Rightarrow S = \left(\frac{2u}{2} + \frac{1}{2}t\right) \times t$$

$$\Rightarrow S = \left(u + \frac{1}{2}t\right) \times t \Rightarrow S = ut + \frac{1}{2}t^2$$

These were the detailed derivations for equations of motion in the graphical method, algebraic method and calculus method.

Equations of Motion Formula:

Equations of motion	**Formula**
First equation of motion	v=u+at
Second equation of motion	s=ut+1/2at2
Third equation of motion	$v^2 = u^2+2as$

What is an External Force?

An external force is defined as the change in the mechanical energy that is either the kinetic energy or the potential energy in an object. These forces are caused by external agents. Examples of external forces are friction, normal force and air resistance.

Let us Understand First Law of Motion by an Example:

Let us take a block on a smooth surface. By smooth, we mean that there is no friction acting on the surface. The block is at rest, that is, it is not moving.

Now, let us examine the forces acting on the block. The only forces acting on the block are the force of gravity and the normal reaction by the surface. There is no force acting on it in the horizontal direction. Since the forces in the vertical direction are equal to each other in magnitude, they cancel each other out, and hence there is no external force on the block. Since this block is at rest, we can say that it confirms Newton's first law of Motion.

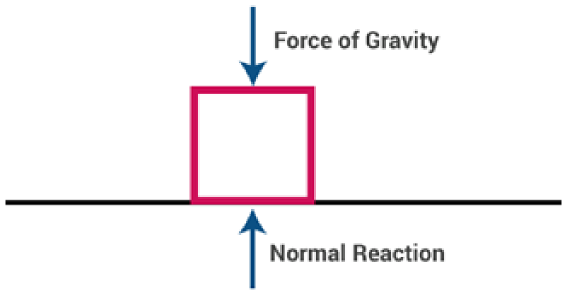


Fig.7.6 No external force on the block

Now, if we apply a constant force F on the block in a horizontal direction, it will start moving with some constant acceleration, in the direction of the applied force.

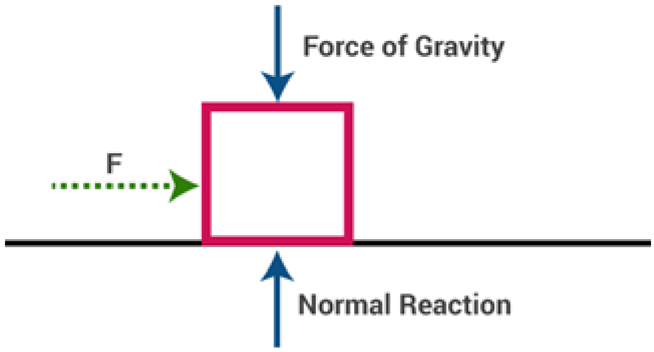


Fig.7.7 Apply a constant force F on the block

Thus, the first law of motion is confirmed again.

Newton's First Law of Motion Examples in Daily Life:

Wearing a seat belt in a car while driving is an example of Newton's 1st law of motion. If an accident occurs, or if brakes are applied to the car suddenly, the body will tend to continue its inertia and move forward, probably proving fatal. To prevent such accidents seat belts are used which stops your body from moving forward in inertia avoiding danger.

Newton's Second Law of Motion:

Force is equal to the rate of change of momentum. For a constant mass, force equals mass times acceleration.

Newton's second law of motion, unlike the first law of motion pertains to the behaviour of objects for which all existing forces are unbalanced. The second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force.

Defining Newton's Second Law of Motion:

Newton's second law states that the acceleration of an object depends upon two variables – the net force acting on the object and the mass of the object. The acceleration of the body is directly proportional to the net force acting on the body and inversely proportional to the mass of the body. This means that as the force acting upon an object is increased, the acceleration of the object is increased. Likewise, as the mass of an object is increased, the acceleration of the object is decreased.

Newton's second law can be formally stated as, "The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object".

This statement is expressed in equation form as,

$$a = \frac{F_{net}}{m}$$

The above equation can be rearranged to a familiar form as

$$F = ma$$

Since force is a vector, Newton's second law can be written as

$$\vec{F} = m\vec{a}$$

The equation shows that the direction of the total acceleration vector points in the same direction as the net force vector.

Deriving Newton's Second Law:

Fig.7.8 Deriving Newton's Second Law

Force = Charge of momentum with change of time

Difference form: $F = \frac{m_1 V_1 - m_o V_o}{t_1 - t_o}$

With constant mass: $F = m\ \frac{V_1 - V_o}{t_1 - t_o}$

Force = mass x acceleration,

t = time, m = mass, V = velocity, X = location

For Changing Mass:

Let us assume that we have a car at a point (0) defined by location X_0 and time t_0. The car has a mass m_0 and travels with a velocity v_0. After being subjected to a force F, the car moves to point 1 which is defined by location X_1 and time t_1. The mass and velocity of the car change during the travel to values m_1 and v_1. Newton's second law helps us determine the new values of m_1 and v_1 if we know the value of the acting force.

Taking the difference between point 1 and point 0, we get an equation for the force acting on the car as follows:

$$F = \frac{m_1 v_1 - m_0 v_0}{t_1 - t_0}$$

Let us assume the mass to be constant. This assumption is good for a car because the only change in mass would be the fuel burned between point "1" and point "0". The weight of the fuel is probably small relative to the weight of the rest of the car, especially if we only look at small changes in time. Meanwhile, if we were discussing the flight of a bottle rocket, then the mass does not remain constant and we can only look at changes in momentum.

For Constant Mass:

For a constant mass, Newton's second law can be equated as follows:

$$F = m\frac{v_1 - v_0}{t_1 - t_0}$$

We know that acceleration is defined as the change in velocity divided by the change in time.
The second law then reduces to a more familiar form as follows:

$$F = ma$$

The above equation tells us that an object will accelerate if it is subjected to an external force and the amount of force is directly proportional to the acceleration and inversely proportional to the mass of the object.

Application of Second Law:

The application of the second law of motion can be seen in identifying the amount of force needed to make an object move or to make it stop. Following are a few examples that we have listed to help you understand this point:

1) **Kicking a ball:** When we kick a ball we exert force in a specific direction, which is the direction in which it will travel. In addition, the stronger the ball is kicked, the stronger the force we put on it and the further away it will travel.

2) **Pushing a cart:** It is easier to push an empty cart in a supermarket than it is to push a loaded one. More mass requires more force to accelerate.

3) **Two people walking:** Among the two people walking, if one is heavier than the other then the one weighing heavier will walk slower because the acceleration of the person weighing lighter is greater.

Newton's Third Law of Motion:

Introduction:

You probably know that when you throw a ball against a wall, the ball exerts a force on the wall. Likewise, the wall puts force on the ball as a result of which the ball bounces off the wall. Similarly, earth pulls you down with gravitational force. What you may not realise is you are also exerting an equal amount of force on the earth. This remarkable fact is a consequence of Newton's third law.

Newton's 3rd Law: If an object A exerts a force on object B, then object B must exert a force of equal magnitude and opposite direction back on object A.

This law signifies a particular symmetry in nature: forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself.

In the next few sections, let us learn Newton's third law in detail.

Newton's Third Law of Motion:

Force is a push or pulls acting on an object resulting in its interaction with another object. Force is a result of an interaction. Force can be classified into two categories: contact force such as frictional force and non-contact force such as gravitational force. According to Newton, when two bodies interact, they exerted force on each other and these forces are known as action and reaction pair which is explained in Newton's third law of motion.

Newton's third law of motion states that "When one body exerts a force on the other body, the first body experiences a force which is equal in magnitude in the opposite direction of the force which is exerted".

The above statement means that in every interaction, there is a pair of forces acting on the interacting objects. The magnitude of the forces are equal and the direction of the force on the first object is opposite to the direction of the force on the second object.

The mathematical representation of Newton's third law of motion is let A be the body exerting force $\vec{F}$ on the body B, then body B too exerts a force $-\vec{F}$ on body A, which is given as:

$$\vec{F}_{AB} = -\vec{F}_{AB}$$

Newton's third law of motion is associated with conservation of momentum. According to the law, for every action there must be an equal and opposite reaction.

Examples of Interaction Force Pairs:

A variety of action-reaction pairs are evident in nature. We have listed a few below and they are as follows:

- A propulsion of fish through water is an example of action-reaction pair. A fish makes use of its fins to push water backwards. This push serves to accelerate the fish forwards. The size of the force on the water equals the size of the force on the fish; the direction of the force on the water (backwards) is opposite the direction of the force on the fish (forwards).

- The flight of the bird is an example of action-reaction pair. The wings of the bird push the air downwards. The air pushes the air upwards.

- A swimmer pushes against the water, while the water pushes back on the swimmer.

- Lift is created by helicopters by pushing the air down, thereby creating an upward reaction force.

- Rock climbers pulling their vertical rope downwards so as to push themselves upwards.

7.4 Condition for existence and depending factors of induced charge, induced voltage, induced current and induced power:

Induced charge:

Induced charges on the Surface in metal objects by a nearby charge. The electrostatic field (lines with arrows) of a nearby positive charge (+) causes the mobile charges in metal objects to separate. Negative charges (blue) are attracted and move to the surface of the object facing the external charge. Positive charges (red) are repelled and move to the surface facing away. These induced surface charges create an opposing electric field that exactly cancels the field of the external charge throughout the interior of the metal. Therefore electrostatic induction ensures that the electric field everywhere inside a conductive object is zero.

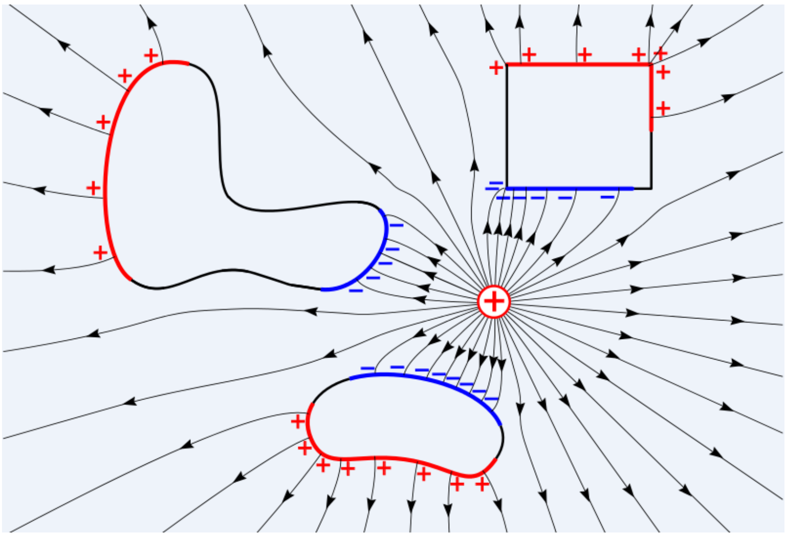


Fig.7.9 Induced charges on the Surface in metal

Induced voltage:

The induced voltage is produced as a product of electromagnetic induction. Electromagnetic induction is the procedure of producing **emf** (induced voltage) by exposing a conductor into a magnetic field. The induced voltage is described by making use of Faraday’s law of induction. The induced voltage of a closed-circuit is described as the rate of change of magnetic flux through that closed circuit. Induced voltage formula is articulated as,

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

Where

ε= Induced voltage

N= Total number of turns of the loop

ΦB **= B.A** (Magnetic flux)

B = Magnetic field

A = Area of the loop

t = time

Induced current:

The current induced in a conducting loop that is exposed to a changing magnetic field is known as induced current. This change may be produced in several ways by:

- change the strength of the magnetic field
- moving the conductor in and out of the field
- altering the distance between the magnet and the conductor, or change the area of a loop located in a stable magnetic field

The strength of the current induced will depend on the changing magnetic flux. The direction of the current is determined by considering Lenz's law, which says that an induced electric current will flow in such a way that it generates a magnetic field that opposes the change in the field that generated it.

Induced Power:

Induced power is the power required to maintain enough lift to overcome the force of gravity. One can view this as the force required to accelerate enough air downwards (at speed v_i) to push the bird upwards enough to counteract the force of gravity (mg). This force must be balanced by a change in the momentum of the air passing through the bird's wing disc (area = $A_{wd}\rho v$, mass/time = $A_{wd}\rho v$), which gives an equation for the induced velocity,

$$v_i = \frac{m\,g}{2\,v\,\rho\,A_{wd}}$$

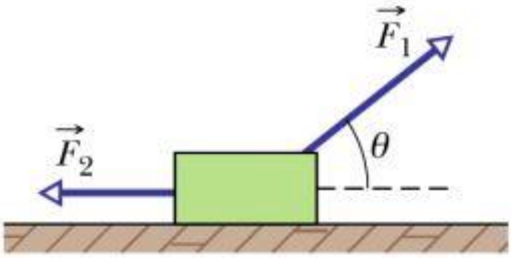
The actual power requirement is the product of the force necessary (mg) and the speed it is applied at (v_i),

$$P_{ind} = K_i \; v_i \; m \; g = \; K_i \; \frac{m^2 \; g^2}{2 \; v \; \rho \; A_{wd}}$$

In the above expression K_i is a correction coefficient to account for the difference between the size of the wing disc and the actual tube of air displaced by the bird's wings.

SAQ.1

a) What do you mean by Faraday's law of electromagnetic induction?
b) Define the Newton's laws of motion in mechanics.
c) What are the Condition for existence and depending factors of induced voltage and induced current?
d) In the figure below, two forces, $\mathbf{F_1}$ and $\mathbf{F_2}$, pull a 50.0 kg crate. The magnitude of F_1 is 215 N and it is applied at a 42.0^o angle. The magnitude of F_2 is 55.0 N. If the crate is accelerating to the right at a rate of 0.500 m/s^2, find the coefficient of kinetic friction between the crate and the floor.



e) **Consider a rectangular coil of 10 turns with a side length 0.8m. This coil reaches the magnetic field 0.4T within 20s. Compute the induced voltage?**

f) Suppose that a sled is accelerating at a rate of 2 m/s^2. If the net force is tripled and the mass is halved, then what is the new acceleration of the sled?

7.5 Dynamic induced EMF and derivation of its expression:

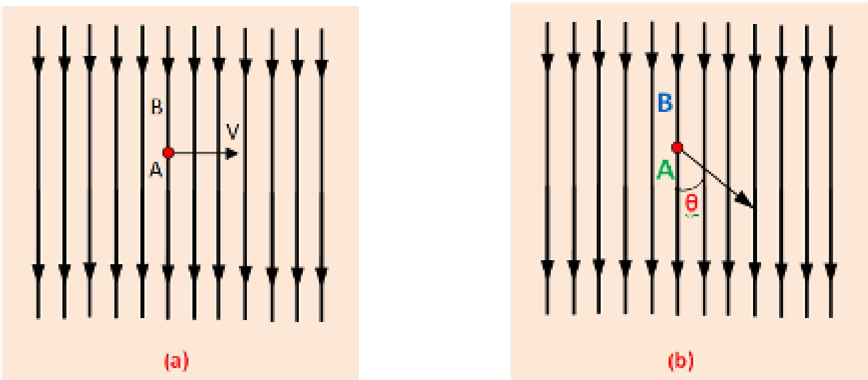


Fig.7.10 (a) Uniform magnetic field, (b) Conductor moving in uniform magnetic field at angle θ

We can see from the figure that a conductor A is lied within a uniform magnetic field whose flux density is a uniform magnetic field and the flux density is B wb ^{3}. In this fig. the movement of the conductor is shown by arrow line. When the conductor A cuts across at right angles to the flux.

Let, 'l'= Length of the conductor lying within the field. And it moves a distance dx in time dt, So, the area swept by the conductor is =ldx. Hence, flux cut by the conductor = l.dx X B, Change in Flux = B.l.dx weber, Time= dt second

According to Faraday's laws. The e.m.f induced in the conductor . And this induced e.m.f is known as dynamically induced e.m.f.

The rate of change of flux linkages = $\frac{B\,l\,dx}{dt} = B\ l\ \frac{dy}{dx} = Blv\ volt$

Where, $\frac{dy}{dx}$ is velocity

If the conductor (A) moves at an angle θ with the direction of flux which is shown in (b).

Then the induced e.m.f. is e=B l v sin θ volts =l$\vec{v}$ x $\vec{B}$

(i.e. as cross product vector $\vec{v}$ and $\vec{B}$

An example, the generator works on the production of dynamically induced e.m.f in the conductors.

7.6 Self Induction and Self Inductance:

Definition of Self Induction:

Self induction is a phenomenon by which a changing electric current produces an induced emf across the coil itself.

Definition of Self Inductance:

Self inductance is the ratio of induced electromotive force (EMF) across a coil to the rate of change of current through the coil. We denote self inductance or coefficient of with English letter L. Its unit is Henry (H).

Since, the induced emf (E) is proportional to the current changing rate, we can write,

$$E \propto \frac{\mathrm{d}i}{\mathrm{d}t} \Rightarrow E = L\frac{\mathrm{d}i}{\mathrm{d}t}$$
$$\Rightarrow L = \frac{E}{\frac{\mathrm{d}i}{\mathrm{d}t}} = self\ inductance$$

But the actual equation is

$$E = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Why there is Minus (-) sign?

According to Lenz's Law, the induced emf opposes the direction of the rate of change of current. So their value is same but sign differs.

Derivation of Inductance:

For the DC source, when the switch is ON, i.e. just at $t = 0^+$, a current starts flowing from its zero value to a certain value and with respect to time, there will be a rate of change in current momentarily. This current produces changing flux (φ) through the coil. As current changes flux (φ) also changes and the rate of change with respect to the time is

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

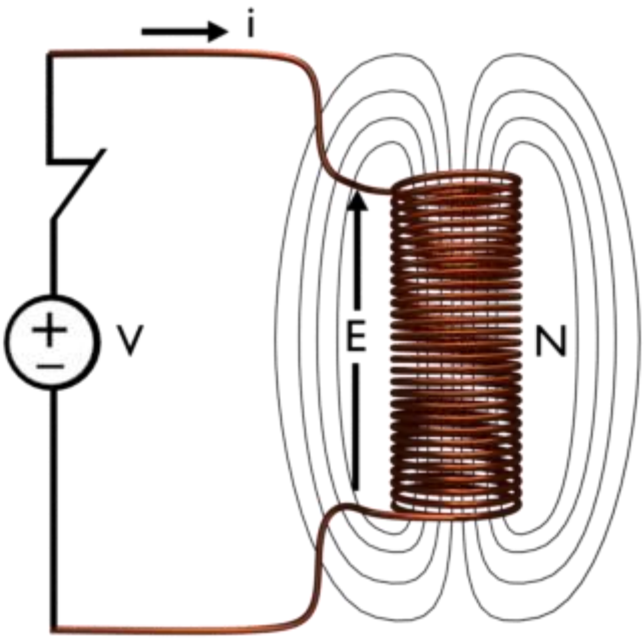


Fig.7.11 Circuit of Self inductance

Now by apply Faraday's Law of Electromagnetic Induction, we get,

$$E = N\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Where, N is the number of turn of the coil and e is the induced EMF across this coil.

Considering Lenz's law we can write the above equation as,

$$E = -N\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Now, we can modify this equation to calculate the value of inductance.

$$E = -N\frac{\mathrm{d}\phi}{\mathrm{d}t} \ \ \& \ \ E = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$
$$\Rightarrow N\ d\phi = L\ di \Rightarrow N\phi = Li$$

So,

$$Li = N\phi = NBA$$

[B is the flux density i.e. B =φ/A, A is area of the coil],

[Nφ or Li is called magnetic flux Linkage and it is denoted by Ψ]

$$Again,\ \ Hl = Ni$$

Where H is the magnetizing force due to which magnetic flux lines flow from south to north pole inside the coil, l (small L) is the effective length of the coil and

$$Again,\ B = \mu H$$

$$Li = NBA \Rightarrow L = \frac{NBA}{i} = \frac{N^2BA}{Ni}$$

$$= \frac{N^2BA}{Hl}\ [\because Ni = Hl] = \frac{N^2\mu HA}{Hl}\ [\because B = \mu H]$$

$$\Rightarrow L = \frac{\mu N^2 A}{l} = \frac{\mu N^2 \pi r^2}{l} \quad [A = \pi r^2]$$

r is the radius of the coil cross-sectional area.

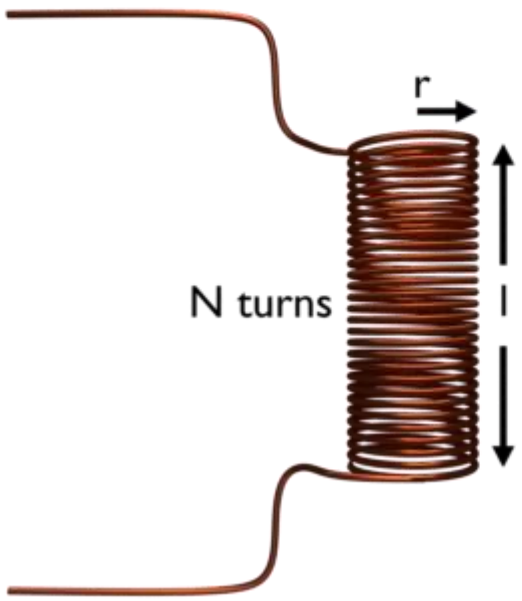


Fig. 7.12 Circuit for self-inductance

Self inductance, L is a geometric quantity; it depends only on the dimensions of the solenoid, and the number of turns in the solenoid. Furthermore, in a DC current when the switch is just closed, then only momentarily effect of self-inductance occurs in the coil. After some time, no effect of **self inductance** remains in the coil because after certain time the current becomes steady.

But in AC circuit, the alternating effect of current always causes the self-induction in the coil, and a certain value of this self-inductance gives the inductive reactance ($X_L = 2\pi fL$) depending on the value of supply frequency.

Mutual Induction and Mutual Inductance:

Definition of Mutual induction:

Mutual induction is a phenomenon when a coil gets induced in EMF across it due to rate of change current in adjacent coil in such a way that the flux of one coil current gets linkage of another coil.

Definition of Mutual Inductance

Mutual Inductance is the ratio between induced emf across a coil to the rate of change of current of another adjacent coil in such a way that two coils are in possibility of flux linkage.

$$M(of\ a\ coil) = \frac{(emf)_{induced\ in\ that\ coil}}{\left(\frac{di(t)}{dt}\right)_{of\ other\ coil}}$$

Mutual Induction:

Whenever there is a time varying current in a coil, the time varying flux will link with the coil itself and will cause self induced emf across the coil. This emf is viewed as a voltage drop across the coil or inductor. But it is not practical that a coil gets linked only with its own changing flux. When a time varying current flows in another coil placed nearby the first one then the flux produced by the second coil may also link the first one. This varying flux linkage from the second coil will also induce emf across the first coil. This phenomenon is called **mutual induction** and the emf induced in one coil due to time varying current flowing in any other coil is called **mutually induced emf.** If the first coil is also connected to the time varying source, the net emf of the first coil is the resultant of self induced and mutually induced emf.

Coefficient of Mutual Induction or Mutual Inductance:

Consider two coils 1 and 2 placed near each other as shown below in the figure

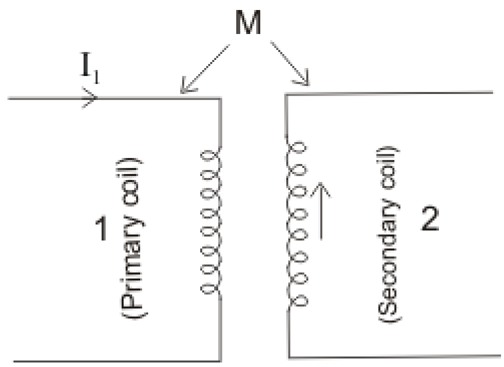


Fig. 7.13 Mutually coupled Two coils placed near each other

Let coil 1 be the primary coil and coil 2 be secondary coil

When current is primary coil changes w.r.t time then the magnetic field produced in the coil also changes with time which causes a change in magnetic flux associated with secondary coil

Due to this change of flux linked with secondary coil an emf is induced in it and this phenomenon is known as mutual induction.

Similarly change in current in secondary coil induces an emf in primary coil.This way as a result of mutual inductance emf is induced in both the coils.

If I_1 is the current in primary coil at any instant ,than the emf induced in secondary coil would be proportional to the rate of change of current in primary coil i.e.

$$\xi_2 \alpha \frac{dI_1}{dt}$$

Or

$$\xi_2 = -M \frac{dI_1}{dt}$$

Where M is a constant known as coefficient of mutual induction and minus sign indicates that direction of induced emf is such that it opposes the change of current in primary coil

Unit of mutual inductance is Henry

We know that a magnetic flux is produced in primary coil due to the flow of current I_1.If this is the magnetic flux associated with secondary coil then from faraday's law of EM induction ,emf induced in secondary coil would be

$$\xi_2 = \frac{-d\phi_{21}}{dt}$$

comparing above two equation we get

$$\phi_{21} = M_{21} I_1$$

Thus coefficient of mutual induction of secondary coil w.r.t primary coil is equal to magnetic flux linked with secondary coil when 1 Ampere of current flows in primary coil and vice-versa.

Similarly, if I_2 is the current in secondary coil at any instant then flux linked with primary coil is

$$\phi_{12} = M_{12} I_2$$

where M_{12} is coefficient of mutual induction of primary coil with respect to secondary coil.

EMF induced in primary coil due to change of this flux is

$$\xi_1 = -M_{12} \frac{dI_2}{dt}$$

For any two circuits

$$M_{12}=M_{21}=M$$

In general mutual inductance of two coil depends on geometry of the coils (shape ,size, number of turns etc),distance between the coils and nature of material on which the coil is wound

Mutual Inductance of two co-axial solenoids:

Consider a long solenoid of length l and area of cross-section A containing N_p turns in its primary coil

Let a shorter secondary coil having N_2 number of turns be wounded closely over the central portion of primary coil as shown below in the figure.

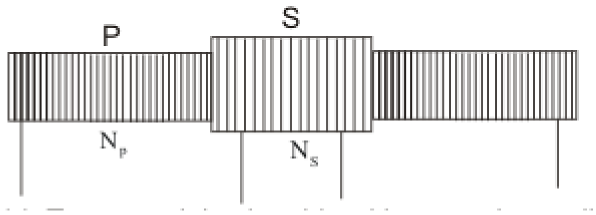


Fig.7.14 Two co-axial solenoid with secondary coil wounded closely over portion of primary coil of length l

If I_p is the current in the primary coil then magnetic field due to primary coil would be

$$B = \frac{\mu_0 N_P I_P}{l}$$

So flux through each turn of secondary coil would be

$$\phi_s = \frac{\mu_0 N_P I_P A}{l}$$

Where A is the area of cross-section of primary coil.

Total magnetic flux through secondary coil is

$$\phi_{s(total} = \frac{\mu_0 N_P N_s I_P A}{l}$$

Emf induced in secondary coil is

$$\xi_s = \frac{-d\phi_{s(total)}}{dt} = \frac{-\mu_0 N_p N_S A}{l}\frac{dI_p}{dt}$$

Thus from equation 24

$$\xi_s = -M\frac{dI_p}{dt}$$

So

$$M = \frac{\mu_0 N_p N_S A}{l}$$

Static Induced EMF:

This type of EMF is generated by keeping the coil and the magnetic field system, stationary at the same time; that means the change in flux linking with the coil takes place without either moving the conductor (coil) or the field system.

This change of flux produced by the field system linking with the coil is obtained by changing the electric current in the field system.

It is further divided in two ways

- **Self-induced electromotive force** (emf which is induced in the coil due to the change of flux produced by it linking with its own turns.)

- **Mutually induced electromotive force** (emf which is induced in the coil due to the change of flux produced by another coil, linking with it.)

Self-Induced EMF:

When the current flowing through the coil is changed, the flux linking with its own winding changes and due to the change in linking flux with the coil, an emf, known as self-induced emf, is induced.

Since according to Lenz's law, an induced emf acts to oppose the change that produces it, a self-induced emf is always in such a direction as to oppose the change of current in the coil or circuit in which it is induced. This property of the coil or circuit due to which it opposes any change of the current in the coil or circuit, is known as self-inductance.

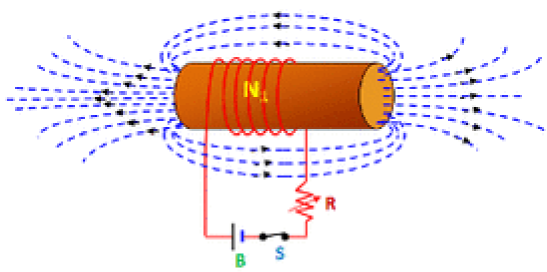


Fig.7.15 Circuit of Statically Self Induced e.m.f.

Consider a Solenoid of N turns, length *l* meters, area of X-section a square meters and of relative permeability μ_r. When the solenoid carries a current of *i* amperes, a magnetic field of flux

$$\frac{\frac{Ni}{l}}{\mu_0 \mu_r a}$$

Weber is set up around the solenoid and links with it.

If the current flowing through the solenoid is changed, the flux produced by it will change and, therefore, an emf will be induced.

self-induced emf,

$$e = -N\frac{d\phi}{dt} = -N\frac{d}{dt}[\frac{\frac{Ni}{l}}{\mu_0\mu_r a}]$$

$$= -N\frac{\frac{N}{l}}{\mu_0\mu_r a}\frac{di}{dt}$$

$$= -\frac{N^2\mu_r\mu_0 a}{l} \times \frac{di}{dt}$$

The quantity $\frac{N^2\mu_r\mu_0 a}{l}$ is a constant for any given coil or circuit and is called coefficient of self-inductance. It is represented by symbol L and is measured in henries.

Hence self-induced e.m.f.

$$e = -L\frac{di}{dt}$$

Where

$$L = \frac{N^2\mu_r\mu_0 a}{l} henrys$$

Coefficient of Self Induction:

The coefficient of self-induction (L) can be determined from any one of the following three relations.

First Method: In case the dimensions of the solenoid are given, the coefficient of self-induction may be determined from the relation

$$L = \frac{N^2 \mu_r \mu_0 a}{l} henrys$$

Second Method: In case the magnitude of induced emf in a coil for a given rate of change of current in the coil is known, self-inductance of the coil may be determined from the following relation.

$$e = L\frac{di}{dt}$$

$$L = \frac{e}{\frac{di}{dt}}$$

Third Method: In case the number of turns of the coil and flux produced per ampere of current in the coil is known, the self-inductance of the coil may be determined from the following relation

$$L = \frac{N\phi}{i}$$

The above relation can be derived as follows:

Magnetic flux produced in a coil of N turns, length *l* meters, area of x-section *a* meters^2 and relative permeability μ_r when carrying a current of I amperes is given by

$$\phi = \frac{\frac{Ni}{l}}{\mu_r \mu_0 a} i$$

and self-inductance of the coil

$$L = \frac{N^2 \mu_r \mu_0 a}{l} = \frac{N}{i}\frac{N \mu_r \mu_0 a}{l} i = \frac{N\phi}{i}$$

From the above relation, it is obvious that the self-inductance of a coil or circuit is equal to weber-turns per ampere in the coil or circuit.

In the above relation if $N\phi$=1Wb-turn and i = 1 A then L = 1 H.

Hence a coil is said to have a self-inductance of one henry if a current of 1 A, when flowing through it, produces flux linkage of I Wb-turn in it.

Mutually Induced EMF:

Consider two coils A and B placed closed together so that the flux created by one coil completely links with the other coil. Let coil A have a battery and switch S and coil B be connected to the galvanometer G.

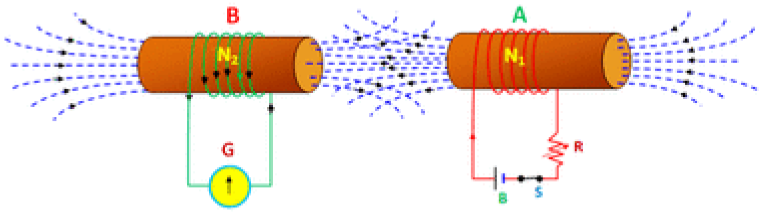


Fig.7.16 Circuit of Statically Mutually Induced e.m.f.

When switch SW1 is opened, no current flows through coil A, so no flux is created in coil A, i.e. no flux links with coil B, therefore, no emf is induced across coil B, the fact is indicated by galvanometer zero deflection. Now when the switch S is closed current in coil. A starts rising from zero value to a finite value, the flux is produced during this period and increases with the increase in current of coil A, therefore, flux linking with the coil B increases and an emf, known as mutually induced emf is produced in coil B, the fact is indicated by galvanometer deflection. As soon as the current in coil A reaches its finite value, the flux produced or flux linking with coil B becomes constant, so no emf is induced in coil B, and galvanometer pointer returns back to zero position. Now if the switch S is opened, current will start decreasing, resulting in decrease influx linking with coil B, an emf will be again induced but in direction opposite to previous one, this fact will be shown by the galvanometer deflection in opposite direction.

Hence whenever the current in coil A changes, the flux linking with coil B changes and an emf, known as mutually induced emf is induced in coil B.

Consider coil A of turns N_1 wound on a core of length *l* meters, area of cross-section *a* square meters and relative permeability μ_r. When the current of i_1 amperes flows through it, a flux of

$$\frac{\frac{N_1 i_1}{l}}{\mu_0 \mu_r a}$$

is set up around the coil A.

Mutually induced emf,

e_m = -Rate of change of flux linkage of coil B

= $-N_2 \times$ rate of change of flux in coil A

$$= -N \frac{d}{dt}[\frac{\frac{N_1 i_1}{l}}{\mu_0 \mu_r a}]$$

$$= -N \frac{N_1 N_2 a \mu_0 \mu_r}{l} \frac{di_1}{dt}$$

The quantity $\frac{N_1 N_2 a \mu_0 \mu_r}{l}$ is called the coefficient of mutual induction of coil B with respect to coil A. It is represented by symbol M and is measured in henrys.

Hence mutually induced emf,

$$e_m = -M\frac{di_1}{dt}$$

Where

$$M = \frac{N_1 N_2 a \mu_0 \mu_r}{l} henrys$$

Coefficient of Mutual Induction:

Mutual inductance may be defined as the ability of one coil or circuit to induce an emf in a nearby coil by induction when the current flowing in the first coil is changed. The action is also reciprocal i.e. the change in current flowing through second coil will also induce an emf in the first coil. The ability of reciprocal induction is measured in terms of the coefficient of mutual induction M.

The coefficient of mutual induction (M) can be determined from any one of the following three relations.

First Method: In case the dimensions of the coils are given, the coefficient of mutual induction may be determined from the relation

$$L = \frac{N_1 N_2 a \mu_0 \mu_r}{l} henrys$$

Second Method: In case the magnitude of induced emf in the second coil for a given rate of change of current in the first coil is known, mutual inductance between the coil may be determined from the following relation

$$e_m = M\frac{di_1}{dt}$$

Or

$$M = \frac{e_m}{\frac{di_1}{dt}}$$

Third Method:

In case the number of turns of the coil and flux linking with this coil per ampere of current in another coil is known, the mutual inductance of the coil may be determined from the following relation

$$M = N_2\frac{\phi_2}{l_1}Henry$$

7.7 Reciprocity theorem and Neuman's relation using Mutual Inductance:

Experiments and calculations that combine Ampere's law and Biot-Savart law confirm that the two constants, M_{21} and M_{12} are equal in the absence of material medium between the two coils.

$$M_{12} = M_{21}$$

This property is called reciprocity and by using reciprocity theorem, we can simply write the mutual inductance between two coils as;

$$M12 = M21 = M$$

What Is Mutual Inductance?

When two coils are brought in proximity with each other the magnetic field in one of the coils tend to link with the other. This further leads to the generation of voltage in the second coil. This property of a coil which affects or changes the current and voltage in a secondary coil is called mutual inductance.

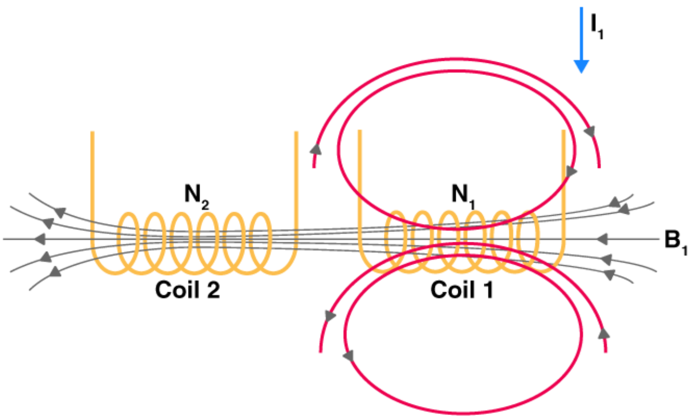


Fig.7.17 Changing I_1 produces changing magnetic flux in coil 2.

In the first coil of N_1 turns, when a current I_1 passes through it, magnetic field B is produced. As the two coils are closer to each other, few magnetic field lines will also pass through coil 2.

$\phi_{21} \rightarrow$ magnetic flux in one turn of coil 2 due to current I_1.

If we vary the current with respect to time, then there will be an induced emf in coil 2.

According to Faraday's law

$$\varepsilon_{ind} = -\frac{d\phi}{dt}$$

$$\varepsilon_{21} = -N_2 \frac{d\phi_{21}}{dt} \;\; \varepsilon_{21} = -N_2 \frac{d}{dt}\left(\overline{B}.\overline{A}\right)$$

The induced emf is coil 2 directly proportional to the current passes through the coil 1.

$$N_2 \phi_{21} \propto I_1 \;\; N_2 \phi_{21} = M_{21} I_1$$

The constant of proportionality is called as mutual inductance. It can be written as

$$M_{21} = \frac{N_2 \phi_{21}}{I_1}$$

The SI unit of inductance is henry (H)

$$1H = \frac{1(\text{Tesla}).1\left(m^2\right)}{1 \; A}$$

In a similar manner, the current in coil 2, I_2 can produce an induced emf in coil 1 when I_2 is varying with respect to time. Then,

$$\varepsilon_{12} = -N_1 \frac{d\phi_{12}}{dt} \;\; N_1 \phi_{12} \propto I_2 \;\; N_1 \phi_{12} = M_{12} I_2$$

$$M_{12} = \frac{N_1 \phi_{12}}{I_2}$$

This constant of proportionality is another mutual inductance.

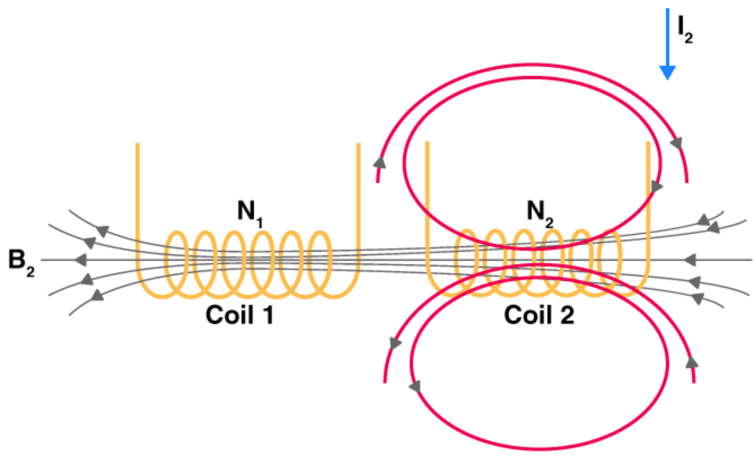


Fig.7.18 Changing I_2 produces changing magnetic flux in coil 1.

EMF of Mutual Inductance:

Considering the mutual inductance between two coil we just discussed, we defined mutual inductance M_{21} of coil 2 with respect to 1 as,

$$\mathrm{M}_{21} = \frac{N_2\phi_{21}}{I_1} \quad \mathrm{M}_{21} I_1 = N_2\phi_{21}$$

If I_1 changes with time,

$$M_{21}\frac{dI_1}{dt} = N_2\frac{d\phi_{21}}{dt}$$

According to Faraday's law of induction,

$$\varepsilon_{ind} = -\frac{d\phi}{dt}$$

Than above equation rewrite

$$M_2\frac{dI_1}{dt} = -\varepsilon_{21}$$

Thus induced emf in coil 2 due to current in coil 1 is given by

$$\varepsilon_2 = -M_{21}\frac{dI_1}{dt}$$

Similarly, induced emf in coil 1 due to changing current in coil 2 can be given as,

$$\varepsilon_1 = -M_{12}\frac{dI_2}{dt}$$

From experiments,

$$M_{21} = M_{12} = M$$

Therefore

$$\varepsilon_1 = -M\frac{dI_2}{dt} \quad \varepsilon_2 = -M\frac{dI_1}{dt}$$

The coefficient of mutual induction – mutual inductance depends only on the geometrical factor of the two coils such as the number of turns, radii of two coils and on the properties of a material medium such as magnetic permeability of the medium surrounding the coils.

Limitations of Reciprocity Theorem:

1. Not applicable to the circuits consisting of any time varying element.
2. Not applicable to the circuits consisting of the dependent source even it is linear.
3. Not applicable to the circuits consisting of non-linear elements like diode, transistor etc.

Application of Reciprocity Theorem:

1. This theorem is applied to analyze Ultrasound Generated by High-Intensity Surface Heating of Elastic Bodies.
2. This theorem is applied to determine line-load-generated surface waves on an inhomogeneous transversely isotropic half-space.

SAQ.2

a) What do you mean by Dynamic induced EMF?
b) Define the Self inductance and mutual inductance.
c) What do mean by self and mutual induced EMF?
d) Define the Reciprocity theorem in the mutual induction.
e) Determine the self-inductance of 5000 turn air-core solenoid of length 2.5m and diameter 0.15 m.
f) The self-inductance of an air-core solenoid is 3.8 mH. If its core is replaced by iron core, then its self-inductance becomes 0.8 H. Find out the relative permeability of iron.
g) A 400 turn coil of radius 3 cm is placed co-axially within a long solenoid of 5 cm radius. If the turn density of the solenoid is 100 turns per cm, then calculate mutual inductance of the coil.
h) A circular wire loop with a radius of 8 cm lies in a plane perpendicular to a uniform magnetic field of magnitude 0.4 T. You reshape the loop into a square in 0.20 seconds. What is the emf induced in the loop?

7.8 Relation between self and mutual inductance of two coupled coils:

Consider two coils of same length l and same area of cross-section placed near each other as shown below in the figure

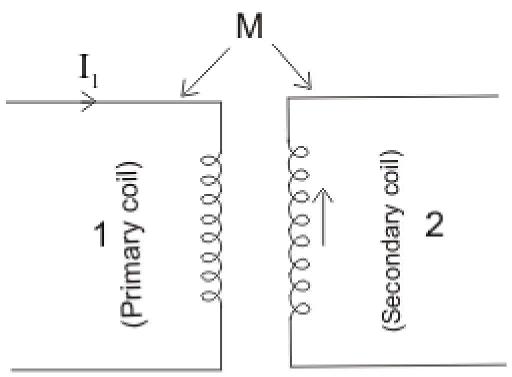


Fig.7.19 Two coil placed near each other

Let there are N_1 number of turns in primary coil and N_2 number of turns in secondary coil,

A current I_1 in the primary coil produces a magnetic field,

$$B = \frac{\mu_0 N_1 I_1}{l}$$

which in turns gives rise to flux?

$$\phi_{11} = BN_1A$$
$$= \frac{\mu_0 N_1^2 A I_1}{l}$$

in primary coil and

$$\phi_{21} = BN_2A$$
$$= \frac{\mu_0 N_1 N_2 A I_1}{l}$$

in the secondary coil due to current in primary coil.

By the definition of self induction

$$\phi_{11} = L_1 I_1$$

So

$$L_1 = \frac{\mu_0 N_1^2 A}{l}$$

and by definition of mutual induction

$$\phi_{21} = M_{21} I_1$$

So

$$M_{21} = \frac{\mu_0 N_1 N_2 A}{l}$$

Reversing the procedure if we first introduce the current I_2 in secondary coil then we get

$$L_2 = \frac{\mu_0 N_2^2 A}{l}$$

And

$$M_{12} = \frac{\mu_0 N_1 N_2 A}{l}$$

So L_1 is the self inductance of primary coil,L_2 is the self induction of secondary coil and M_{21}=M_{12}=M is the mutual inductance between two coils.

Product of L_1 and L_2 is

$$L_1 L_2 = \frac{\mu_0^2 N_1^2 N_2^2 A^2}{l^2} = M^2$$

hence

$$M = \sqrt{L_1 L_2}$$

In practice M is always less than eq due to leakage which gives

$$\frac{M}{\sqrt{L_1 L_2}} = k$$

Where K is called coefficient of coupling and K is always less than 1.

Energy of coupled circuits:

We saw that the energy stored in an inductor is given by

$$w = \frac{1}{2}Li^2$$

We now want to determine the energy stored in magnetically coupled coils.

Consider the circuit in fig. we assume that currents i_1 and i_2 are aero initially, so that stored in the coils is zero. If we let i_1 increase from zero to I_1 while maintaining $i_2 = 0$, the power in coil 1 is

$$p_1(t) = v_1 i_1 = i_1 L_1 \frac{di_1}{dt}$$

and the energy stored in the circuit is

$$w_1 = \int p_1\, dt = L_1 \int_0^{I_1} i_1\, di_1 = \frac{1}{2}L_1 I_1^2$$

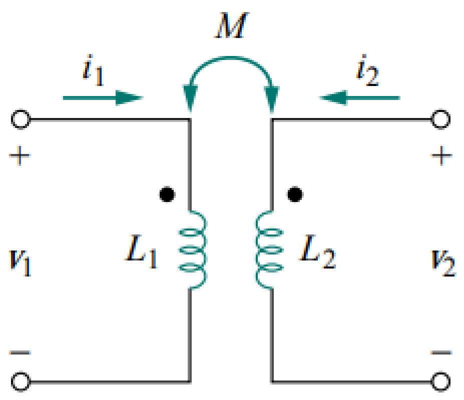


Fig.7.20 Circuit for deriving energy stored in a coupled circuit

If we now maintain $i_1 = I_1$ and increase i_2 from zero to I_2, the mutual voltage induced in coil 1 is $M1_2\ di_2/dt$, while the mutual voltage induced in coil 2 is zero, since i_1 does not change. The power in the coil is now

$$p_2(t) = i_1 M_{12}\frac{di_2}{dt} + i_2 v_2 = I_1 M_{12}\frac{di_2}{dt} + i_2 L_2 \frac{di_2}{dt}$$

and the energy stored in the circuit is

$$\begin{aligned} w_2 = \int p_2\,dt &= M_{12} I_1 \int_0^{I_2} di_2 + L_2 \int_0^{I_2} i_2\,di_2 \\ &= M_{12} I_1 I_2 + \frac{1}{2} L_2 I_2^2 \end{aligned}$$

The total energy in the coils both i_1 and i_2 have reached constant values is

$$w = w_1 + w_2 = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{12} I_1 I_2$$

If we reverse the order by which the currents reach their final values, that is, if we first increase i_2 from zero to I_2 and later increase i_1 from zero to I_1 the total energy stored in the coils is

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M_{21} I_1 I_2$$

Since the total energy stored should br the same regardless of how we reach the final conditions, comparing the above two equation leads us to conclude that

$$M_{12} = M_{21} = M$$

and

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 + M I_1 I_2$$

This equation was derived based on the assumption that the coil currents both entered the dotted terminals. If one current enters one dotted terminal while the other current leaves the other dotted terminal, the mutual voltage is negative, so that the mutual energy $M I_1 I_2$ is also negative. In that case

$$w = \frac{1}{2}L_1 I_1^2 + \frac{1}{2}L_2 I_2^2 - M I_1 I_2$$

Also sinceI_1 and I_2 are arbitrary values, they may be replaced by i_1 and i_2 which gives the instantaneous energy stored in the circuit the general expression

$$\boxed{w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2}$$

7.9 Transformer:

What is a transformer?

A transformer can be defined as a static device which helps in the transformation of electric power in one circuit to electric power of the same frequency in another circuit. The voltage can be raised or lowered in a circuit, but with a proportional increase or decrease in the current ratings. In this article we will be learning about Transformer basics and working principle.

Transformer – Working Principle

The main principle of operation of a transformer is mutual inductance between two circuits which is linked by a common magnetic flux. A basic transformer consists of two coils that are electrically separate and inductive, but are magnetically linked through a path of reluctance. The working principle of the transformer can be understood from the figure below.

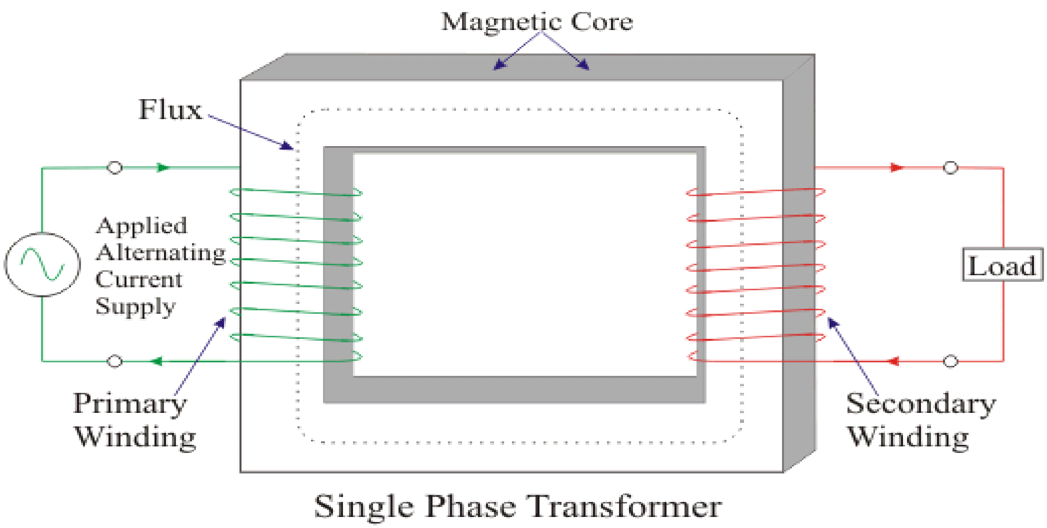


Fig.7.21 Transformer Working

As shown above the electrical transformer has primary and secondary windings. The core laminations are joined in the form of strips in between the strips you can see that there are some narrow gaps right through the cross-section of the core. These staggered joints are said to be ‘imbricated’. Both the coils have high mutual inductance. A mutual electro-motive force is induced in the transformer from the alternating flux that is set up in the laminated core, due to the coil that is connected to a source of alternating voltage. Most of the alternating flux developed by this coil is linked with the other coil and thus produces the mutual induced electro-motive force. The so produced electro-motive force can be explained with the help of Faraday’s laws of Electromagnetic Induction as

$$e = M \frac{dI}{dt}$$

If the second coil circuit is closed, a current flows in it and thus electrical energy is transferred magnetically from the first to the second coil.

The alternating current supply is given to the first coil and hence it can be called as the primary winding. The energy is drawn out from the second coil and thus can be called as the secondary winding.

In short, a transformer carries the operations shown below:

1. Transfer of electric power from one circuit to another.
2. Transfer of electric power without any change in frequency.
3. Transfer with the principle of electromagnetic induction.
4. The two electrical circuits are linked by mutual induction.

Transformer Construction

For the simple construction of a transformer, you must need two coils having mutual inductance and a laminated steel core. The two coils are insulated from each other and from the steel core. The device will also need some suitable container for the assembled core and windings, a medium with which the core and its windings from its container can be insulated.

In order to insulate and to bring out the terminals of the winding from the tank, apt bushings that are made from either porcelain or capacitor type must be used.

In all transformers that are used commercially, the core is made out of transformer sheet steel laminations assembled to provide a continuous magnetic path with minimum of air-gap included. The steel should have high permeability and low hysteresis loss. For this to happen, the steel should be made of high silicon content and must also be heat treated. By effectively laminating the core, the eddy-current losses can be reduced. The lamination can be done with the help of a light coat of core plate varnish or lay an oxide layer on the surface. For a frequency of 50 Hertz, the thickness of the lamination varies from 0.35mm to 0.5mm for a frequency of 25 Hertz.

Types of Transformers:

Types by Design:

The types of transformers differ in the manner in which the primary and secondary coils are provided around the laminated steel core. According to the design, transformers can be classified into two:

1. Core- Type Transformer:

In core-type transformer, the windings are given to a considerable part of the core. The coils used for this transformer are form-wound and are of cylindrical type. Such a type of transformer can be applicable for small sized and large sized transformers. In the small sized type, the core will be rectangular in shape and the coils used are cylindrical. The figure below shows the large sized type. You can see that the round or cylindrical coils are wound in such a way as to fit over a cruciform core section. In the case of circular cylindrical coils, they have a fair advantage of having good mechanical strength. The cylindrical coils will have different layers and each layer will be insulated from the other with the help of materials like paper, cloth, micarta board and so on. The general arrangement of the core-type transformer with respect to the core is shown below. Both low-voltage (LV) and high voltage (HV) windings are shown.

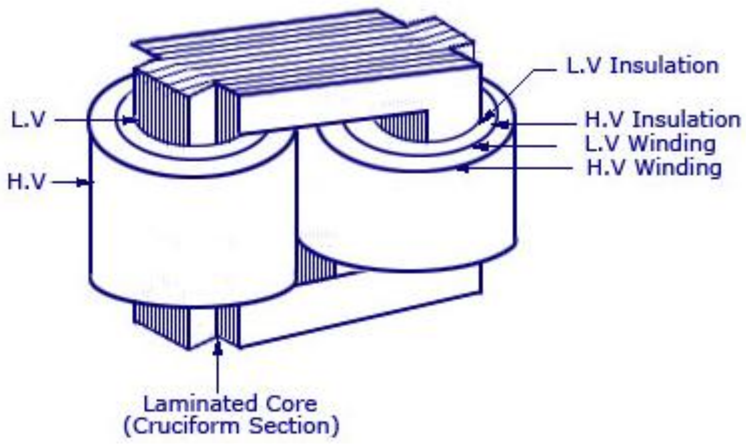


Fig.7.22 Core Type Transformer Cruciform Section

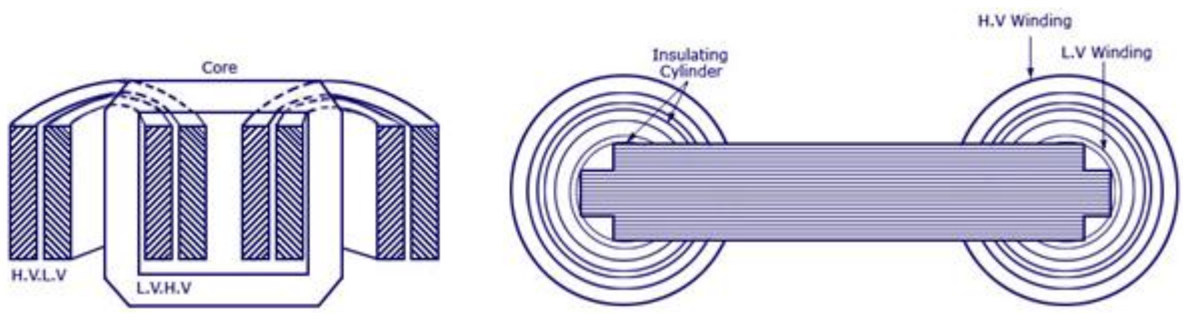


Fig.7.23 Core Type Transformers

The low voltage windings are placed nearer to the core as it is the easiest to insulate. The effective core area of the transformer can be reduced with the use of laminations and insulation.

2. Shell-Type Transformer:

In shell-type transformers, the core surrounds a considerable portion of the windings. The comparison is shown in the figure below.

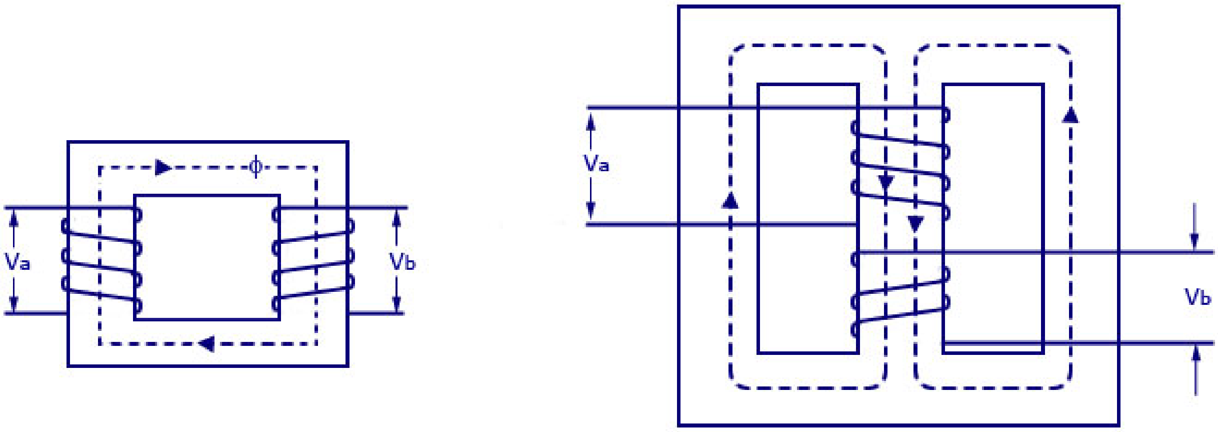


Fig.7.24 Core Type and Shell Type Transformer Winding

The coils are form-wound but are multi layer disc type usually wound in the form of pancakes. Paper is used to insulate the different layers of the multi-layer discs. The whole winding consists of discs stacked with insulation spaces between the coils. These insulation spaces form the horizontal cooling and insulating ducts. Such a transformer may have the shape of a simple rectangle or may also have a distributed form. Both designs are shown in the figure below:

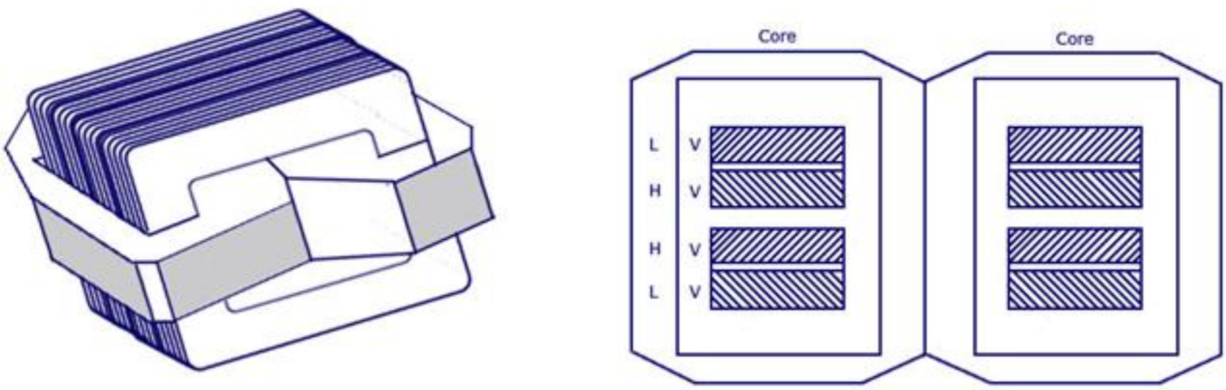


Fig.7.25 Shell Type Transformers Rectangular Form

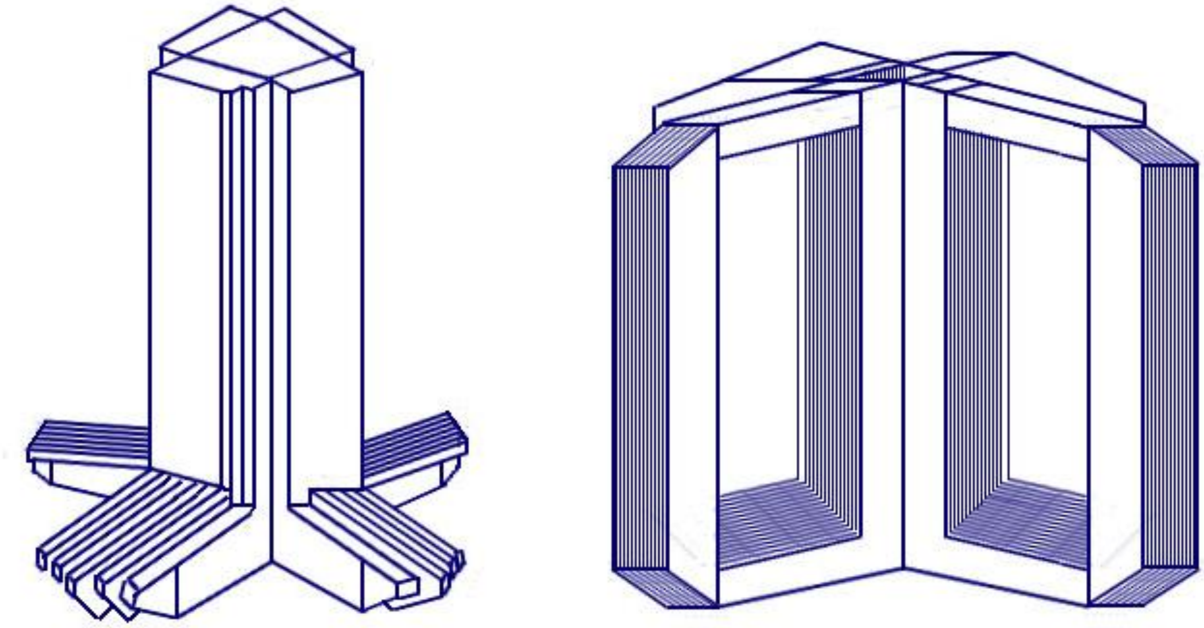


Fig.7.26 Shell Type Transformers Distributed Form

A strong rigid mechanical bracing must be given to the cores and coils of the transformers. This will help in minimizing the movement of the device and also prevents the device from getting any insulation damage. A transformer with good bracing will not produce any humming noise during its working and will also reduce vibration.

A special housing platform must be provided for transformers. Usually, the device is placed in tightly-fitted sheet-metal tanks filled with special insulating oil. This oil is needed to circulate through the device and cool the coils. It is also responsible for providing the additional insulation for the device when it is left in the air.

There may be cases when the smooth tank surface will not be able to provide the needed cooling area. In such cases, the sides of the tank are corrugated or assembled with radiators on the sides of the device. The oil used for cooling purpose must be absolutely free from alkalis, sulphur and most importantly moisture. Even a small amount of moistures in the oil will cause a significant change in the insulating property of the device, as it lessens the dielectric strength of the oil to a great extent.

Mathematically speaking, the presence of about 8 parts of water in 1 million reduces the insulating quality of the oil to a value that is not considered standard for use. Thus, the tanks are protected by sealing them air-tight in smaller units. When large transformers are used, the airtight method is practically difficult to implement. In such cases, chambers are provided for the oil to expand and contract as its temperature increases and decreases.

These breathers form a barrier and resist the atmospheric moisture from contact with oil. Special care must also be taken to avoid sledging. Sledging occurs when oil decomposes due to overexposure to oxygen during heating. It results in the formation of large deposits of dark and heavy matter that clogs the cooling ducts in the transformer.

The quality, durability and handling of these insulating materials decide the life of the transformer. All the transformer leads are brought out of their cases through suitable bushings. There are many designs of these, their size and construction depending on the voltage of the leads. Porcelain bushings may be used to insulate the leads, for transformers that are used in moderate voltages. Oil-filled or capacitive-type bushings are used for high voltage transformers.

The selection between the core and shell type is made by comparing the cost because similar characteristics can be obtained from both types. Most manufacturers prefer to use shell-type transformers for high-voltage applications or for multi-winding design. When compared to a core type, the shell type has a longer mean length of coil turn. Other parameters that are compared for the selection of transformer type are voltage rating, kilo-volt ampere rating, weight, insulation stress, heat distribution and so on.

Transformers can also be classified according to the type of cooling employed. The different types according to these classifications are:

Types of Transformers based on cooling method:

1. Oil Filled Self-Cooled Type:

Oil filled self-cooled type uses small and medium-sized distribution transformers. The assembled windings and core of such transformers are mounted in a welded, oil-tight steel tanks provided with a steel cover. The tank is filled with purified, high quality insulating oil as soon as the core is put back at its proper place. The oil helps in transferring the heat from the core and the windings to the case from where it is radiated out to the surroundings.

For smaller sized transformers the tanks are usually smooth surfaced, but for large size transformers a greater heat radiation area is needed, and that too without disturbing the cubical capacity of the tank. This is achieved by frequently corrugating the cases. Still larger sizes are provided with radiation or pipes.

2. Oil Filled Water Cooled Type:

This type is used for much more economic construction of large transformers, as the above-told self-cooled method is very expensive. The same method is used here as well- the windings and the core are immersed in the oil. The only difference is that a cooling coil is mounted near the surface of the oil, through which cold water keeps circulating. This water carries the heat from the device. This design is usually implemented on transformers that are used in high voltage transmission lines. The biggest advantage of such a design is that such transformers do not require housing other than their own. This reduces the costs by a huge amount. Another advantage is that the maintenance and inspection of this type is only needed once or twice in a year.

3. Air Blast Type:

This type is used for transformers that use voltages below 25,000 volts. The transformer is housed in a thin sheet metal box open at both ends through which air is blown from the bottom to the top.

EMF equation of a transformer and Voltage Transformation Ratio:

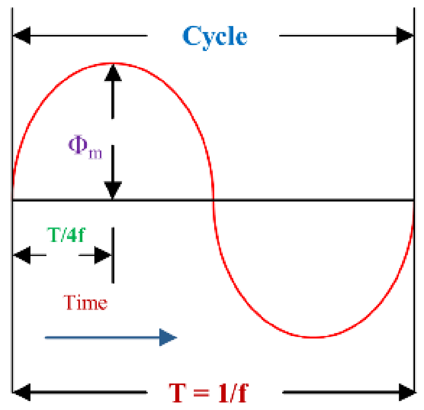


Fig.7.27 Sinusoidal waveform for e.m.f .equation of a transformer

Suppose , N_1 = No. of turns of primary coil &

$\mathbf{N_2}$ = No. of turns of secondary coil of a transformer.

Φ_m = Maximum flux in core (webers)

$= \mathbf{B_m \times A}$

f= frequency of alternating current in Hz

From the figure , it has been seen that the flux Φ increases from its zero value to maximum value $\Phi_m$in one quarter of the cycle i.e in 1/4 f second

$$\therefore \text{average rate of change of flux} = \frac{\Phi m}{1/4f}$$

$$= \mathbf{4}\ f\ \Phi_m\ \mathbf{Wb/s\ or\ volt}$$

Now, rate of change of flux per turn means induced e.m.f in volts.

∴ average e.m.f/ turn = **4** f Φ_m **volt**

If the magnitude of flux Φ varies sinusoidally, then the r.m.s value of induced e.m.f is obtained by multiplying the average value with from factor.

$$\therefore \text{ From factor } = \frac{r.m.s\ value}{average\ value} = 1.11$$

∴ r.m.s value of e.m.f./turn = 1.11 x **4** f Φ_m **= 4.44 f Φ_m volt**

Now, r.m.s value of the induced e.m.f in the primary winding

∴ E_1 = (induced e.m.f/turn) x No. of primary turns

∴ $E_1 = 4.44\ f\ \Phi_m N_1$ (As $\Phi_m = B_m \times A$)

∴ $\mathbf{E_1} = 4.44\ f\ N_1 B_m A$

Similarly, r.m.s value of the e.m.f. induced in secondary is,

∴ E_2 = (induced e.m.f/turn) x No. of Secondary turns

= $4.44 f \Phi_m N_2$ (As $\Phi_m = B_m x A$)

⇒ $E_2 = 4.44 f N_2 B_m A$

It is seen from equation (i) and (ii) that $\mathbf{E_1} / N_1 = E_2 / N_2 = \mathbf{4.44 f} \Phi_m$.

from the above equation it is seen that the e.m.f/ turn is the same in both primary and secondary windings.

Transformation Ratio of Transformer:

The transformation ratio will be given by the equation shown below

$$\frac{V_2}{V_1} = \frac{E_2}{E_1} = \frac{N_2}{N_1} = \frac{I_1}{I_2} = K$$

Constant K is known as voltage transformation ratio.

i) If $N_2 > N_1$ i.e K > 1,then transformer is called step-up transformer.

ii) If $N_2 < N_1$ i.e K < 1,then transformer is called step-down transformer.

Voltage Ratio of Transformer:

This above stated ratio is also known as **voltage ratio of transformer** if it is expressed as ratio of the primary and secondary voltages of transformer.

Turns Ratio of Transformer:

As the voltage in primary and secondary of transformer is directly proportional to the number of turns in the respective winding, the transformation ratio of transformer is sometime expressed in ratio of turns and referred as turns ratio of transformer.

Equivalent Circuit of Transformer Referred to Primary:

For drawing equivalent circuit of transformer referred to primary, first we have to establish general equivalent circuit of transformer then, we will modify it for referring from primary side. For doing this, first we need to recall the complete vector diagram of a transformer which is shown in the figure below.

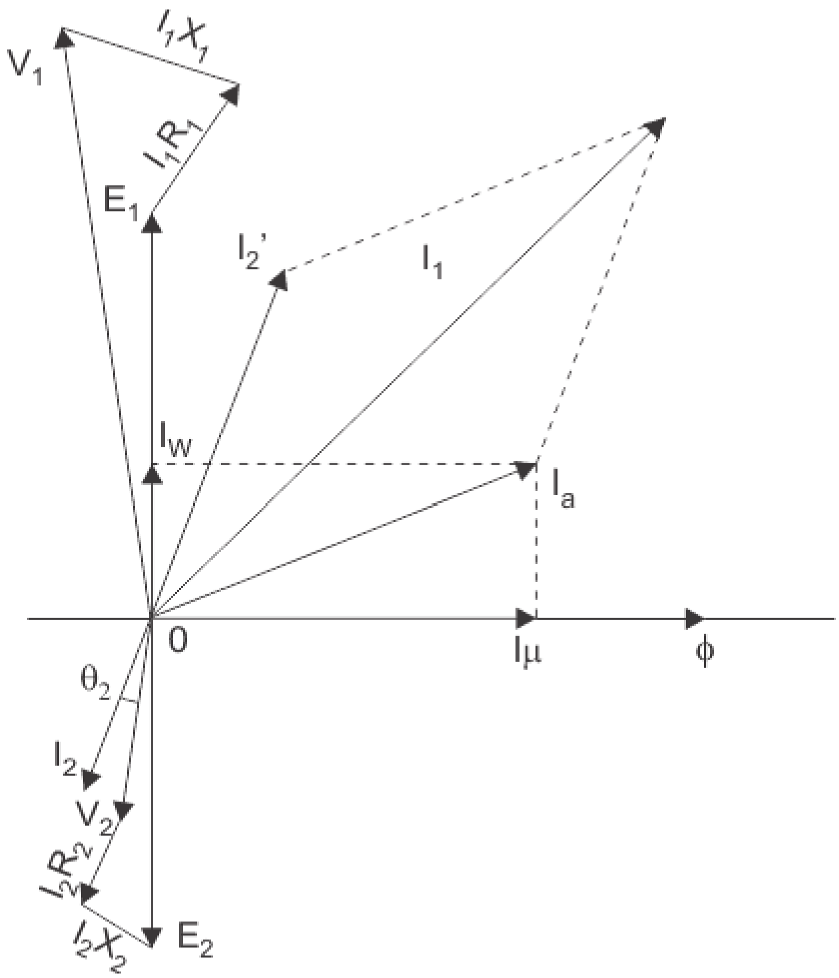


Fig.7.28 Vector diagram of Transformer on load

Let us consider the transformation ratio be,

$$K = \frac{N_1}{N_2} = \frac{E_1}{E_2}$$

In the figure above, the applied voltage to the primary is V_1 and voltage across the primary winding is E_1. Total current supplied to primary is I_1.

So the voltage V_1 applied to the primary is partly dropped by I_1Z_1 or I_1R_1 + $j.I_1X_1$ before it appears across primary winding.

The voltage appeared across winding is countered by primary induced emf E_1. So voltage equation of this portion of the transformer can be written as,

$$V_1 - (I_1R_1 + jI_1X_1) = E_1$$

The equivalent circuit for that equation can be drawn as below,

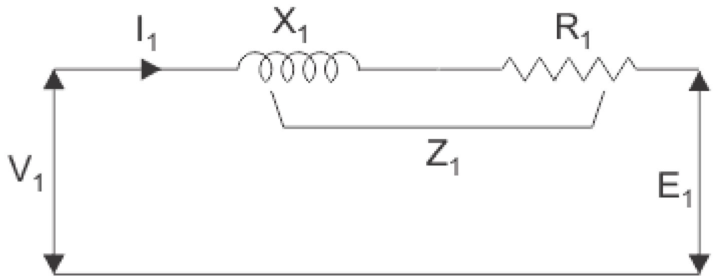


Fig.7.29 Equivalent Circuit

From the vector diagram above, it is found that the total primary current I_1 has two components, one is no – load component I_o and the other is load component I_2'. As this primary current has two components or branches, so there must be a parallel path with primary winding of transformer.

This parallel path of current is known as excitation branch of equivalent circuit of transformer. The resistive and reactive branches of the excitation circuit can be represented as

$$R_0 = \frac{E_1}{I_w} \; and \; X_0 = \frac{E_1}{I_\mu}$$

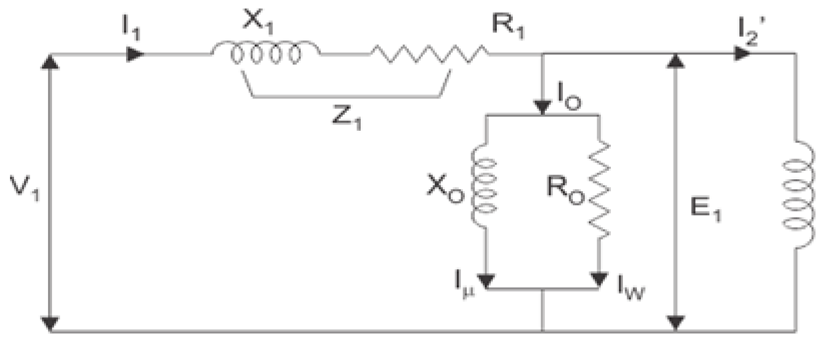


Fig.7.30 Equivalent Circuit of Primary side of Transformer

The load component I_2' flows through the primary winding of transformer and induced voltage across the winding is E_1 as shown in the figure right. This induced voltage E_1transforms to secondary and it is E_2 and load component of primary current I_2' is transformed to secondary as secondary current I_2. Current of secondary is I_2. So the voltage E_2 across secondary winding is partly dropped by I_2Z_2 or $I_2R_2 + j.I_2X_2$ before it appears across load. The load voltage is V_2.
The complete equivalent circuit of transformer is shown below.

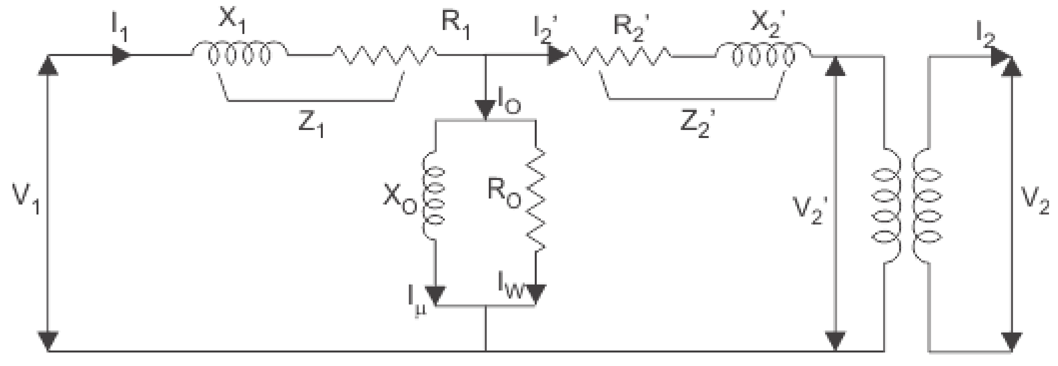


Fig.7.31 Equivalent Circuit of Transformer referred to Primary

Now if we see the voltage drop in secondary from primary side, then it would be 'K' times greater and would be written as $K.Z_2.I_2$.

Again $I_2'.N_1 = I_2.N_2$

$$\Rightarrow I_2 = I_2' \frac{N_1}{N_2}$$

$$\Rightarrow I_2 = KI_2'$$

Therefore,

$$KZ_2 I_2 = KZ_2 KI_2' = K^2 Z_2 I_2'$$

From above equation, secondary impedance of transformer referred to primary is,

$$Z_2' = K^2 Z_2$$

$$Hence,\ R_2' = K^2 R_2\ and\ X_2 = K^2 X_2$$

So, the complete equivalent circuit of transformer referred to primary is shown in the figure below:

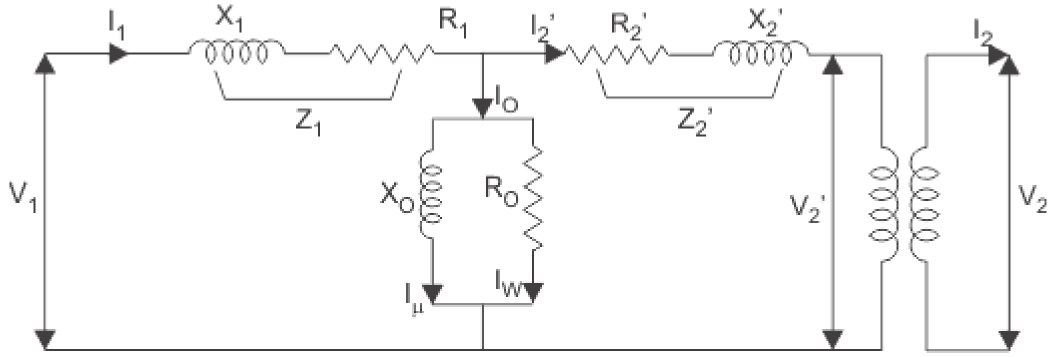


Fig.7.32 Equivalent Circuit of Transformer referred to Primary

Approximate Equivalent Circuit of Transformer:

Since I_o is very small compared to I_1, it is less than 5% of full load primary current, I_o changes the voltage drop insignificantly. Hence, it is good approximation to ignore the excitation circuit in approximate equivalent circuit of transformer. The winding resistance and reactance being in series can now be combined into equivalent resistance and reactance of transformer, referred to any particular side. In this case it is side 1 or primary side.

$$Here,\ V_2' = KV_2$$

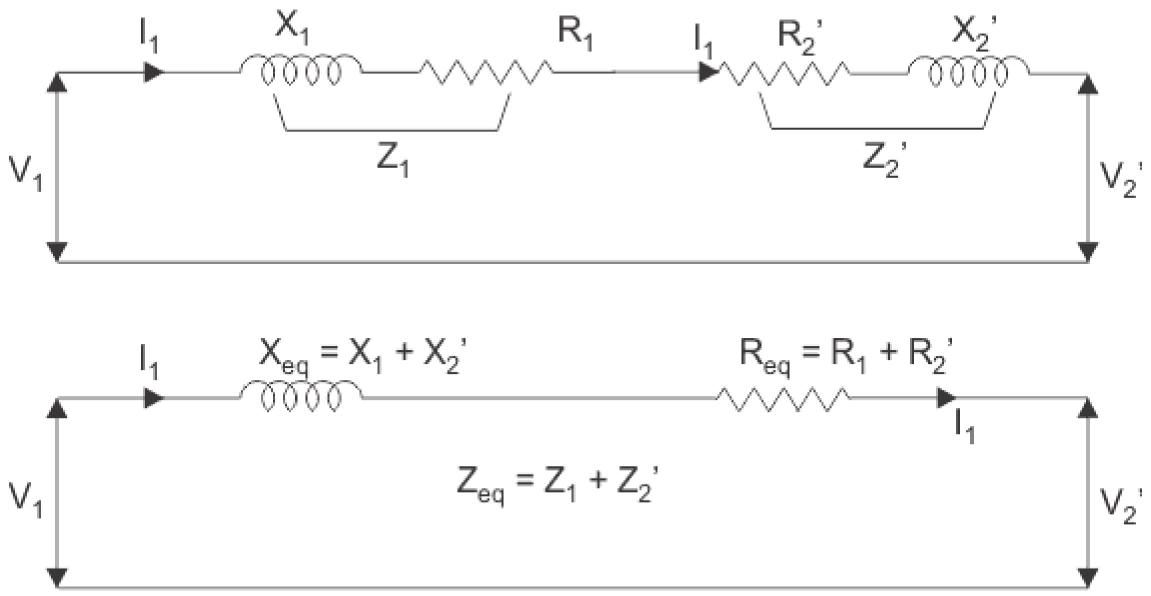


Fig.7.32 Approximate Equivalent Circuit of Transformer referred to Primary

Equivalent Circuit of Transformer Referred to Secondary:

In a similar way, the approximate equivalent circuit of transformer referred to secondary can be drawn. Where equivalent impedance of transformer referred to secondary, can be derived as

$$Z_1 = \frac{Z_1}{K^2}$$

$$Therefore,\ R_1' = \frac{R_1}{K^2}$$

$$X_1' = \frac{X_1}{K^2}$$

$$Here,\ V_1' = \frac{V_1}{K}$$

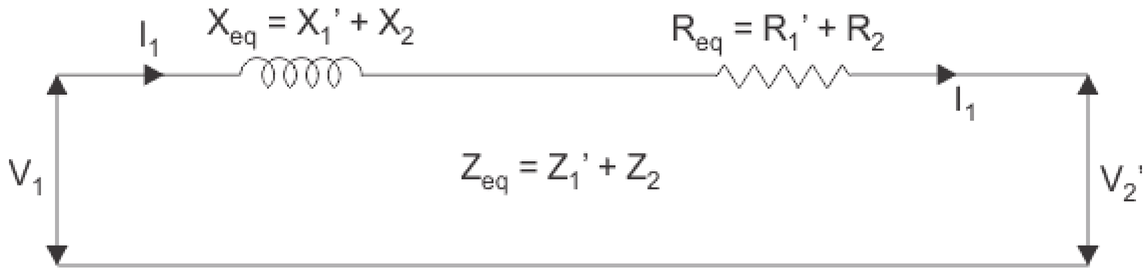


Fig.7.33 Approximate Equivalent Circuit of Transformer referred to Secondary

Condition for ideal transformer:

The transformer which is free from all types of losses is known as an ideal transformer. It is an imaginary transformer that has no core loss, no ohmic resistance, and no leakage flux. The ideal transformer has the following important characteristic.

- The resistance of their primary and secondary winding becomes zero.
- The core of the ideal transformer has infinite permeability. The infinite permeable means less magnetizing current requires for magnetizing their core.
- The leakage flux of the transformer becomes zero, i.e. the whole of the flux induces in the core of the transformer links with their primary and secondary winding.
- The ideal transformer has 100 percent efficiency, i.e., the transformer is free from hysteresis and eddy current loss.

The above mention properties are not possible in the practical transformer. In an ideal transformer, there is no power loss. Therefore, the output power is equal to the input power.

Behavior of Ideal Transformer:

Consider the ideal transformer shown in the figure below:

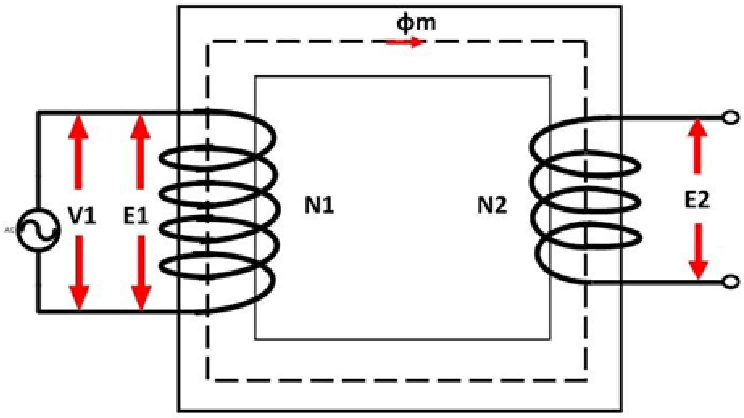


Fig.7.34 Ideal Transformer

The voltage source V_1 is applied across the primary winding of the transformer. Their secondary winding is kept open. The N_1 and N_2 are the numbers of turns of their primary and secondary winding.

The current I_m is the magnetizing current flows through the primary winding of the transformer. The magnetizing current produces the flux φ_m in the core of the transformer.

As the permeability of the core is infinite the flux of the core link with both the primary and secondary winding of the transformer.

The flux link with the primary winding induces the emf E_1 because of self-induction. The direction of the induced emf is inversely proportional to the applied voltage V_1. The emf E_2 induces in the secondary winding of the transformer because of mutual induction.

Phasor Diagram of Ideal Transformer:

The phasor diagram of the ideal transformer is shown in the figure below. As the coil of the primary transformer is purely inductive the magnetizing current induces in the transformer lag 90° by the input voltage V_1.

The E_1 and E_2 are the emf induced in the primary and secondary winding of the transformer. The direction of the induced emf inversely proportional to the applied voltage.

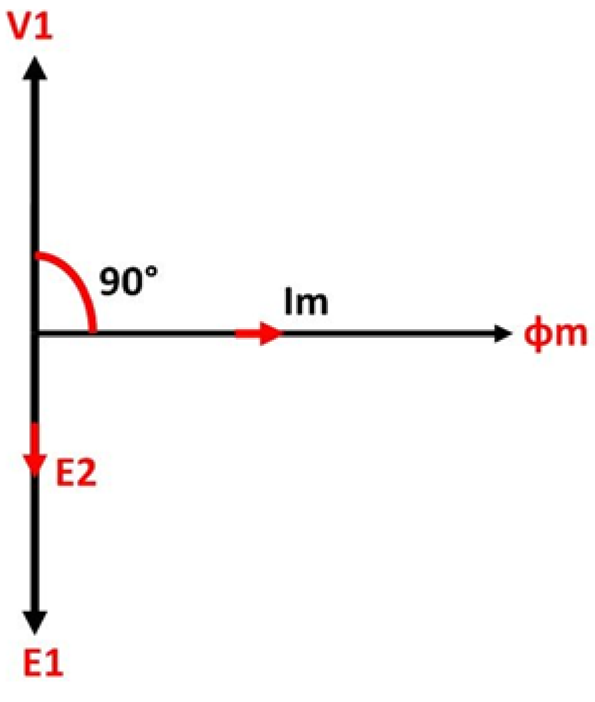


Fig.7.35 Phasor Diagram of an Ideal Transformer

Efficiency of Transformer:

Just like any other electrical machine, efficiency of a transformer can be defined as the output power divided by the input power. That is efficiency = output / input.

Transformers are the most highly efficient electrical devices. Most of the transformers have full load efficiency between 95% to 98.5% . As a transformer being highly efficient, output and input are having nearly same value, and hence it is impractical to measure the efficiency of transformer by using output / input. A better method to find efficiency of a transformer is using,

$$\text{efficiecy} = \frac{(\text{input} - \text{losses})}{\text{input}} = 1 - \frac{\text{losses}}{\text{input}}$$

Condition for maximum efficiency:

Let,

Copper loss = $I_1^2 R1$

Iron loss = W_i

$$\text{efficiency} = 1 - \frac{\text{losses}}{\text{input}} = 1 - \frac{I_1^2 R_1 + W_i}{V_1 I_1 \cos\Phi_1}$$

$$\eta = 1 - \frac{I_1 R_1}{V_1 \cos\Phi_1} - \frac{W_i}{V_1 I_1 \cos\Phi_1}$$

Differentiating above equation with respect to I_1

$$\frac{d\eta}{dI_1} = 0 - \frac{R_1}{V_1 \cos\Phi_1} + \frac{W_i}{V_1 I_1^2 \cos\Phi_1}$$

η will be maximum at

$$\frac{d\eta}{dI_1} = 0$$

Hence efficiency η will be maximum at

$$\frac{R_1}{V_1 \cos\Phi_1} = \frac{W_i}{V_1 I_1^2 \cos\Phi_1}$$

$$\frac{I_1^2 R_1}{V_1 I_1^2 \cos\Phi_1} = \frac{W_i}{V_1 I_1^2 \cos\Phi_1}$$

$$I_1^2 R_1 = W_i$$

Hence, efficiency of a transformer will be maximum when copper loss and iron losses are equal.
That is Copper loss = Iron loss.

All day efficiency of transformer:

As we have seen above, ordinary or commercial efficiency of a transformer can be given as

$$\text{ordinary efficiency} = \frac{\text{output (in watts)}}{\text{input (in watts)}}$$

But in some types of transformers, their performance cannot be judged by this efficiency. For example, distribution transformers have their primaries energized all the time. But, their secondary's supply little load all no-load most of the time during day (as residential use of electricity is observed mostly during evening till midnight).

That is, when secondary's of transformer are not supplying any load (or supplying only little load), then only core losses of transformer are considerable and copper losses are absent (or very little). Copper losses are considerable only when transformers are loaded. Thus, for such transformers copper losses are relatively less important. The performance of such transformers is compared on the basis of energy consumed in one day.

$$\text{All day efficiency} = \frac{\text{output (in kWh)}}{\text{input (in kWh)}} \quad \text{(for 24 hours)}$$

All day efficiency of a transformer is always less than ordinary efficiency of it.

Applications of Single Phase Transformer:

The advantages of three single-phase units are transportation, maintenance, and spare unit availability. The single-phase transformers are widely used in commercial low voltage application as electronic devices.

They operate as a step-down voltage transformer and decrease the home voltage value to the value suitable for electronics supplying. On the secondary side, rectifier is usually connected to convert a AC voltage to the DC voltage which is used in electronics application.

Voltage gain in Transformer using Step-Up Transformer:

What is Step-up Transformer?

In a Step Up Transformer, there are more turns on the secondary coil compared to the primary coils. The current which flows across the Primary coil is much higher compared to secondary coil. It basically converts low voltage, high-current to high voltage-low current i.e. the voltage has been Stepped Up. Hence the name, Step-up Transformer.

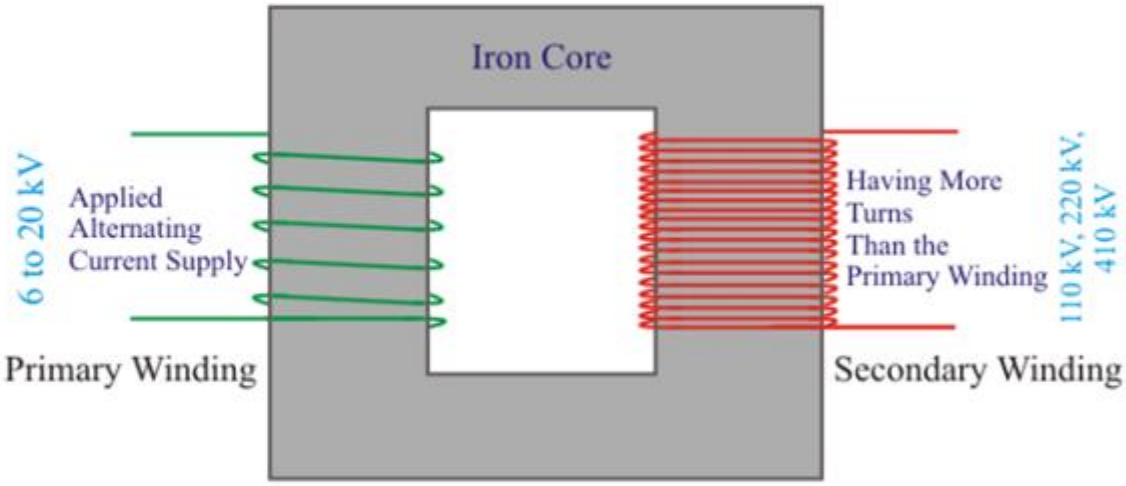


Fig.7.36 Step-up Transformer

Construction of Step-up Transformers

Construction of Step Up Transformers include the building up of the winding, the enclosures and other accessories along with the Core of the Transformer. Below is the detailed procedure of building of the Step Up Transformer.

- Step Up Transformer Core
- Winding(s)

Step Up Transformer Core:

- To build the core of the Transformer, a high penetrable material is used. To form the Core, thin Silicon Steel is assembled and tightly clamped which is laminated. The preamble material which is used in forming of the core is designed to let the magnetic flux to flow with less loss.
- The characteristic of the Core restricts the magnetic field lines in the air which in turn increases the efficiency of the Transformer.
- Less coercive materials are preferred such as Silicon Steel to build the Core. If the core is built with other Ferro-magnetic materials it might result in Hysteresis Loss and Eddy current Loss.

Winding(s):

- The Winding(s) help transfer the currents which are wound to the Transformers. The winding(s) are designed to cool the transformers and withstand the operational and test conditions.

- The wire on the Primary winding is thick with less number of turns. While the wire on the Secondary winding is thinner and has large number of turns. This is mainly designed in such a way that the primary winding can carry low power voltage compared to the secondary winding which carry higher voltage power.

- The material used in the winding is Copper and Aluminium. Copper being the expensive material increases the life of Step Up Transformer when compared to Aluminium which is less expensive.

- Lamination of Core reduces Eddy Currents. They are of many types. Most common Laminations are E-E Type and E-I Type to which Primary and Secondary winding is fixed and they are stacked to minimize the air gaps as shown in the Fig. 3 (a) and (b). Primary and Secondary Winding on the Laminated Core is shown in the Fig. 3 (c)

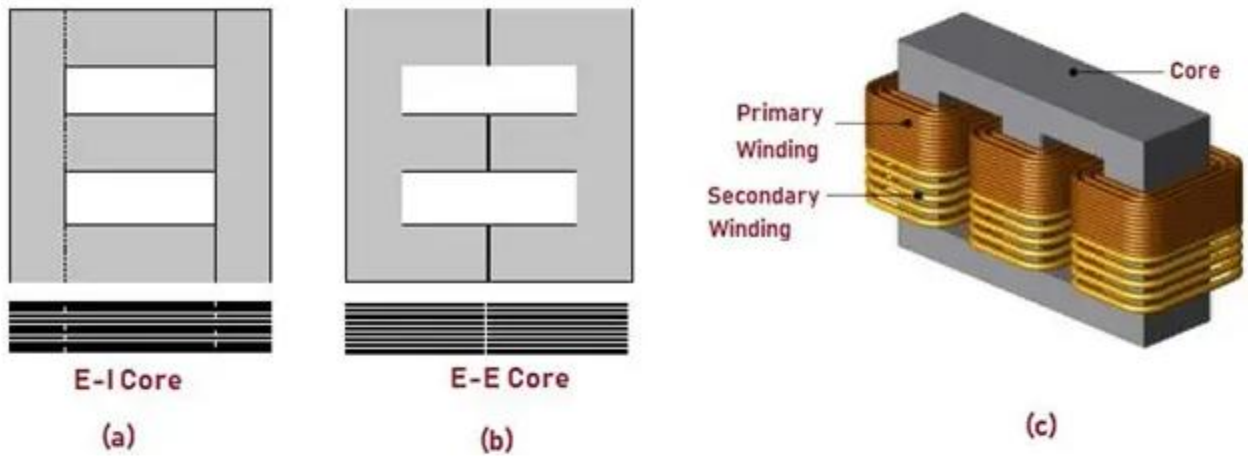


Fig.7.37 (a) E-I Core (b) E-E Core (c) Winding on Laminated Core

How Does Step Up Transformer Work?

A Step Up Transformer has been explained in a more detailed manner with a schematic diagram as shown in Fig. 4. Here V_1 and V_2 are the input and output Voltages respectively. T1 and T2 are the Turns on the Primary and the Secondary windings. Primary winding is the input winding to a Transformer and the Secondary winding is the output winding to a Transformer. If there are more turns of wire on the Secondary than on the Primary, the output voltage will be higher than the input voltage.

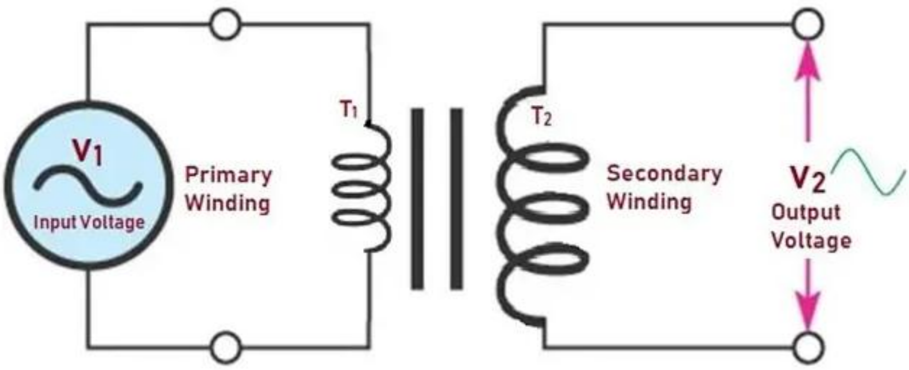


Fig.7.38 Schematic diagram Step Up Transformer

As the current flowing in a Transformer is Alternating Current, it flows in one direction, stops, then reverses and flows in the other direction. The flow of electricity creates a magnetic field around the wire or winding. The north and south poles of the magnetic field gets reversed when the flow of current reverses.

The magnetic field induces voltage into the wire. Similarly voltage will be induced in the second coil when it is placed in a moving magnetic field. This phenomenon is called as Mutual Induction. Hence we can conclude that, Alternating current in Primary winding produces a moving magnetic field which induces voltage in Secondary winding.

The relationship between the voltage and the number of Turns in each coil is given by the equation:

$$\frac{Voltage\ in\ secondary\ coil}{Voltage\ in\ primary\ coil} = \frac{Turns\ on\ secondary\ coil}{Turns\ on\ primary\ coil}$$

$$\frac{V2}{V1} = \frac{T2}{T1}$$

Applications of Step-up Transformer:

Applications of Step-up Transformers include:

- Step-up transformers are found in the electronic devices such as Inverters and Stabilizers where in the Transformers help in stabilizing the low voltage to the higher voltage.
- It is also used in the Electrical Distribution of Power

Advantages of Step up Transformers:

Step Up Transformers are the need for the hour in most of the commercial and residential places. The advantages are mentioned below.

Power Transmitter:

The Step Up transformers are the ones which transmit the electricity at lower prices for a longer distance. The voltage of the currents is increased which has to be transmitted whereby the resistance is reduced on the line. This helps in decreasing the losses along the way and make efficient use of the power supplied across.

Continuous Working:

Step Up transformers has the capability and the capacity to work nonstop without any breaks unlike most of the electrical instruments. This creates a huge advantage which helps in the power distribution system.

Maintenance:

Apart from being a system to work without any break, Step Up Transformers also is a low maintenance device. The Step Up Transformer requires only a minimal maintenance such as the oil check, replacement or repair of damaged pieces etc.

Quick Start:

Once installed, the Transformer is fast to start up process without any delays or time consuming procedure.

Efficiency:

As the technologies have been upgraded along the years, the efficiency level of the Step Up Transformer has also increased. There is less wastage along the lines hence keeping the efficiency level above 95%.

Disadvantages of Step-up Transformers:

As previously stated, there is no 100% efficiency level. Hence there are some disadvantages along the way of Step-up Transformers.

Cooling System:

As the Step Up Transformer continuously performs its task without any break, it needs a cooling system. Since the Step Up Transformer cannot be shut down to cool, there has to be a provision to attach a round the clock cooling system to the Transformers.

Huge in Size:

As the voltage capacity increases bigger the transformer size which will also include a bigger cooling system. This creates a bulky and huge Transformer occupying a larger space.

Works for AC (Alternate Current):

The Transformers are used only for stepping up AC voltages or the Alternating Currents. They do not work on the DC or the Direct Current. The limitations are only for the applications related to the AC operations.

Losses in transformer:

There are various types of losses in the transformer such as iron loss, copper loss, hysteresis loss, eddy current loss, stray loss, and dielectric loss. The hysteresis losses occur because of the variation of the magnetization in the core of the transformer and the copper loss occurs because of the transformer winding resistance.

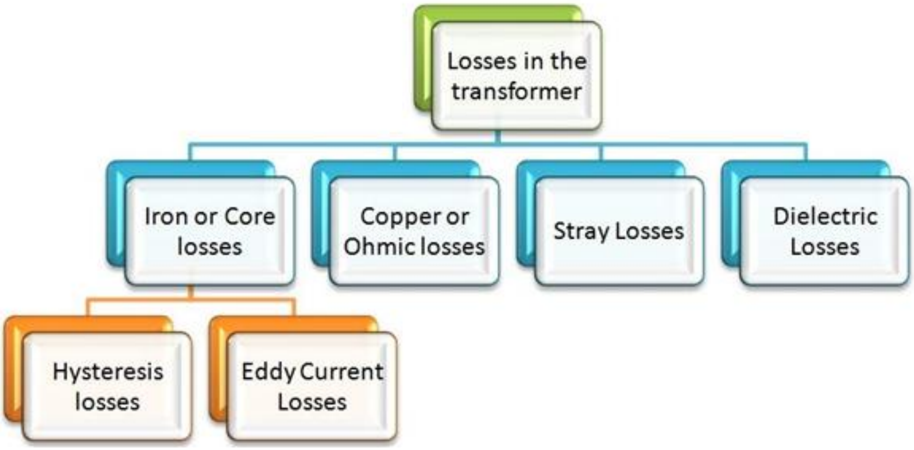


Fig.7.39 Types of Losses in the Transformer

1. Iron Losses:

Iron losses are caused by the alternating flux in the core of the transformer as this loss occurs in the core it is also known as **Core loss**. Iron loss is further divided into hysteresis and eddy current loss.

(a) Hysteresis Loss:

The core of the transformer is subjected to an alternating magnetizing force, and for each cycle of emf, a hysteresis loop is traced out. Power is dissipated in the form of heat known as hysteresis loss and given by the equation shown below:

$$P_h = K\eta B_{max}^{1.6} f\,V \quad \text{watts}$$

Where

- $K\eta$ is a proportionality constant which depends upon the volume and quality of the material of the core used in the transformer,
- f is the supply frequency,
- Bmax is the maximum or peak value of the flux density.

The iron or core losses can be minimized by using silicon steel material for the construction of the core of the transformer.

(b) Eddy Current Loss:

When the flux links with a closed circuit, an emf is induced in the circuit and the current flows, the value of the current depends upon the amount of emf around the circuit and the resistance of the circuit.

Since the core is made of conducting material, these EMFs circulate currents within the body of the material. These circulating currents are called Eddy Currents. They will occur when the conductor experiences a changing magnetic field. As these currents are not responsible for doing any useful work, and it produces a loss (I^2R loss) in the magnetic material known as an Eddy Current Loss.

The eddy current loss is minimized by making the core with thin laminations.

The equation of the eddy current loss is given as:

$$P_e = K_e B_m^2 t^2 f^2 V \qquad \text{watts}$$

Where,

- K_e – coefficient of eddy current. Its value depends upon the nature of magnetic material like volume and resistivity of core material, the thickness of laminations
- B_m – maximum value of flux density in wb/m^2
- T – thickness of lamination in meters
- F – frequency of reversal of the magnetic field in Hz
- V – the volume of magnetic material in m^3

2. Copper Loss Or Ohmic Loss:

These losses occur due to ohmic resistance of the transformer windings. If I_1 and I_2 are the primary and the secondary current. R_1 and R_2 are the resistance of primary and secondary winding then the copper losses occurring in the primary and secondary winding will be $I_1^2R_1$ and $I_2^2R_2$ respectively.

Therefore, the total copper losses will be

$$P_c = I_1^2R_1 + I_2^2R_2$$

These losses varied according to the load and known hence it is also known as variable losses. Copper losses vary as the square of the load current.

3. Stray Loss:

The occurrence of these stray losses is due to the presence of leakage field. The percentages of these losses are very small as compared to the iron and copper losses so they can be neglected.

4. Dielectric Loss:

Dielectric loss occurs in the insulating material of the transformer that is in the oil of the transformer, or in the solid insulations. When the oil gets deteriorated or the solid insulation gets damaged, or its quality decreases, and because of this, the efficiency of the transformer gets affected.

SAQ.3

a) What are the Relation between self and mutual inductance of two coupled coils?
b) Explain the working principal of 1- phase Transformer.
c) What are the conditions for ideal transformer?
d) What do you mean by voltage gain in transformer?
e) A 200 turn coil of radius 2 cm is placed co-axially within a long solenoid of 3 cm radius. If the turn density of the solenoid is 90 turns per cm, then calculate mutual inductance of the coil.
f) A 200/50V, 50 Hz single phase transformer is connected to a 200 V, 50 Hz supply with secondary winding open. Primary winding has 400 turns.

   (i) What is the value of maximum flux through the core, if the primary winding has 400 turns?

   (ii) What is the peak value of flux if the primary voltage is 200V, 25 Hz?

   (iii) What happens to no-load current?
g) The efficiency of a 1000 kVA, 110/220 V, 50 Hz, single phase transformer is 98.5 % at half full-load at 0.8 pf leading and 98.8 % at full-load upf. Determine (a) core loss, (b) full-load copper loss, (c) maximum efficiency at unity p.f.

Examples:

Q.1 How much horizontal net force is required to accelerate a 1000 kg car at 4 m/s^2?

Solution:

Newton's 2nd Law relates an object's mass, the net force on it, and its acceleration:

Therefore, we can find the force as follows:

$\mathbf{F_{net} = ma}$

Substituting the values, we get

1000 kg × 4 m/s^2 = 4000 N

Therefore, the horizontal net force is required to accelerate a 1000 kg car at 4 m/s_2 is 4000 N.

Newton's second law is applied in daily life to a great extent. For instance, in Formula One racing, the engineers try to keep the mass of cars as low as possible. Low mass will imply more acceleration, and the more the acceleration, the chances to win the race are higher.

Q.2 If there is a block of mass 2kg, and a force of 5N is acting on it in the positive x-direction, and a force of 3N in the negative x-direction, then what would be its acceleration?

Solution: To calculate its acceleration, we first have to calculate the net force acting on it.

Fnet = 5N – 3N = 2N

Mass = 2kg

$\mathbf{F_{net} = ma}$

a = F_{net} / m

∴ Acceleration = 2/2 = 1 m/s2

Q.3 Consider a rectangular coil of 5 turns with a side length 0.5m. This coil reaches the magnetic field 0.3T within 10s. Compute the induced voltage?

Solution**:**

Known values are,

N = 5

l = 0.5m

B = 0.3T

dt = 10s

The formula for induced voltage is articulated as,

$$\varepsilon = -N\frac{d\Phi_B}{dt}$$

$$\Phi_B = B.A = 0.3 \times 0.5^2$$

$$= 0.3 \times 0.25 = 0.075Tm^2$$

$$So, \varepsilon = -5 \times \frac{0.075}{10} = -0.0375V$$

Q.4 A copper disc 20 cm in diameter rotates with an angular velocity of 60 rev s^{-1} about its axis. The disc is placed in a magnetic field of induction 0.2 T acting parallel to the axis of rotation of the disc. Calculate the magnitude of the e.m.f. induced between the axis of rotation and the rim of the disc.

Solution:

As the disc rotates, any of its radii cuts the lines of force of magnetic field.

Area swept by radius vector during one revolution

$= \pi r^2$

$= \pi(10 \text{ cm})^2$

$= 100\pi \text{ cm}^2$

$= \pi\times10^{-2} \text{ m}^2$

Area swept in one second,

A = (area swept in one revolution) × (Number of revolutions per second)

$= \pi\times10^{-2}\times60$

$A = 0.6\ \pi m^2$

Rate of change of magnetic flux

$= d\phi_B/dt$

$= BA$

$= 0.2\times0.6\pi$

$= 0.12\pi$ Wb

According to Faraday's law, magnitude of induced e.m.f is,

$E = d\phi_B/dt$

Therefore, magnitude of e.m.f. is

$E = 0.12\,\pi V$

Or, $E = 0.377$ V

Thus from the above observation we conclude that, the magnitude of the e.m.f. induced between the axis of rotation and the rim of the disc would be 0.377 V.

Q.5 A rectangular loop of N turns of area A and resistance R rotates at a uniform angular velocity ω about Y-axis. The loop lies in a uniform magnetic field B in the direction of X-axis. Assuming that at $t = 0$, the plane of the loop is normal to the lines of force, find an expression for the peak value of the emf and current Induced in the loop. What is the magnitude of torque required on the loop to keep it moving with constant ω?

Solution:

As ϕ is maximum at $t = 0$,

$\phi(t) = BA \cos \omega t$

Magnitude of induced emf $= N|d\phi/dt|$

$= BA\omega N\ |\sin\omega t|$

Magnitude of induced current = $[BA\omega N/R]\ |\sin\omega t|$

So, peak value of emf = $BA\omega N$

peak value of induced current = $BA\omega N/R$

To obtain the magnitude of torque required on the loop to keep it moving with constant ω, we have to equate power input is equal to heat dissipation per second.

So, power input = heat dissipation per second

Or, $\tau\omega = I^2R$

Or, $\tau = [(BA\omega N)^2/R]\ |\sin^2\omega t|$

From the above observation we conclude that, the magnitude of torque required on the loop to keep it moving with constant ω would be $[(BA\omega N)^2/R]\ |\sin^2\omega t|$.

Q.6 An alternating emf 200 virtual volts at 50 Hz is connected to a circuit of resistance 1W and inductance 0.01 H. What is the phase difference between the current and the emf in the circuit. Also find the virtual current in the circuit.

Solution:

In case of an ac, the voltage leads the current in phase by angle,

$\phi = \tan^{-1}(X_L/R)$

Here, $X_L = \omega L$

$= (2\pi fL)$

$= (2\pi)\ (50)\ (0.01)$

$= \pi\Omega$

and

$R = 1\Omega$

So, $\phi = \tan^{-1}(\pi)$

$\approx 72.3°$

Further, $i_{rms} = V_{rms}/|Z|$

$= V_{rms}/\sqrt{R^2+X_L^2}$

Substituting the values we have,

$i_{rms} = 200/\sqrt{(1)^2+(\pi)^2}$

$= 60.67$ amp

From the above observation we conclude that, the virtual current in the circuit would be 60.67 amp.

Q.7 A long solenoid of length 1 m, cross sectional area 10 cm^2, having 1000 turns has wound about its centre a small coil of 20 turns. Compute the mutual inductance of the two circuits. What is the emf in the coil when the current in the solenoid changes at the rate of 10 Amp/s?

Solution:

Let N_1 = number of turns in solenoid

N2 = number of turns in coil

A_1 nad A_2 be their respective areas of cross-section.

($A_1 = A_2$ in this problem)

Flux ϕ_2 through coil crated by current i_1 in solenoid is $\phi_2 = N_2(B_1A_2)$

$\phi_2 = N_2\ (\mu_0 i_1 N_1/l)A_2$

Or, $\phi_2 = (\mu_0 N_1 N_2 A_2/l)\ i_2$

Comparing with $\phi_2 = Mi_1$, we get,

Mutual inductance, $M = \mu_0 N_1 N_2 A_2/l$

$= [4\pi\times10^{-7}\times1000\times20\times10\times10^{-4}]/1$

$= 2.51\times10^{-5}$ H

So magnitude of induced emf = $E_2 = M|di_1/dt|$

Q.8 The magnetic flux passing through a coil perpendicular to its plane is a function of time and is given by ΦB = (2t3 + 4t2 + 8t + 8) Wb. If the resistance of the coil is 5 Ω, determine the induced current through the coil at a time t = 3 second.

Solution:

Given data:

$\phi_B = (2\,t^3 + 4\,t^2 + 8t + 8)$ Wb

t = 3 s ; R = 5Ω ; induced current = ?

emf = $\frac{d\phi_B}{dt}$ = $6t^2 + 8t + 8$

at time t = 3s

emf = $6(3)^2 + 8(3) + 8 = 54 + 24 + 8$

emf = 86 V

We know that emf = I R

$I = \frac{emf}{R} = \frac{86}{5} = 1.72$ A

Q.9 An induced current of 2.5 mA flows through a single conductor of resistance 100 **Ω**. Find out the rate at which the magnetic flux is cut by the conductor.

Solution:

Given data:

I = 2.5 mA

R = 100 Ω

$\frac{d\phi}{dt}$ =?

emf = IR = $2.5 \times 10^{-3} \times 100$ = 250 mV

emf $\propto \frac{d\phi}{dt}$

$\therefore \frac{d\phi}{dt} = 250\ \text{mWbs}^{-1}$

Q.10 Determine the self-inductance of 4000 turn air-core solenoid of length 2m and diameter 0.04 m. (Ans: 12.62 mH)

Solution:

Given data:

N = 4000

$\ell = 2$ m

d = 0.04 m

r = 0.02 m = 2×10^{-2} m

L = ?

$$L = \frac{\mu_o N^2 A}{\ell} = \frac{\mu_o N^2 (\pi r^2)}{\ell}$$

$$= \frac{4\pi \times 10^{-7} \times (4000)^2 \times (\pi \times 2 \times 10^{-2} \times 2 \times 10^{-2})}{2}$$

$$= \frac{256}{2} \times 9.86 \times 10^{-5} = 12.62 \times 10^{-3}\ \text{H}$$

L = 12.62 mH

Q.11 A 50 cm long solenoid has 400 turns per cm. The diameter of the solenoid is 0.04 m. Find the magnetic flux of a turn when it carries a current of 1 A.

Solution:

Given data:

ℓ = 50 cm = 0.5 m

N = 400 turns/cm = 40000 turns/m

d = 0.04m

r = 0.02 m = 2 $\times 10^{-2}$ m

I = 1A

ϕ =?

$$\phi = BA = \frac{\mu_o NIA}{\ell} = \frac{4\pi\times10^{-7}\times40000\times1\times\pi\times(2\times10^{-2})^2}{0.5}$$

= 1.26 × 10^{-3}

ϕ = 1.26 mWb

Q.12 A coil of 200 turns carries a current of 0.4 A. If the magnetic flux of 4 mWb is linked with the coil, find the inductance of the coil.

Solution:

Given data:

N = 200

ϕ = 4 mWb = 4 × 10^{-3} Wb

I = 0.4 A

L = ?

$$L = \frac{N\phi}{I} = \frac{200\times4\times10^{-3}}{0.4} = 2$$

L = 2H

Q.13 A long solenoid having 400 turns per cm carries a current 2A. A 100 turn coil of cross-sectional area 4 cm2 is placed co-axially inside the solenoid so that the coil is in the field produced by the solenoid. Find the emf induced in the coil if the current through the solenoid reverses its direction in 0.04 sec.

Solution:

Given data:

$\left(\frac{N_1}{\ell}\right)$ n_1 = 400 turns per cm = 4×10^4 per M

I_1 = 2A $\qquad$ I_2 = -2A

N_2 = 100

$A = 4 \times 10^{-4}\ m^2$

t = 0.04 s

emf =?

$$emf = -M\frac{dI}{dt}$$

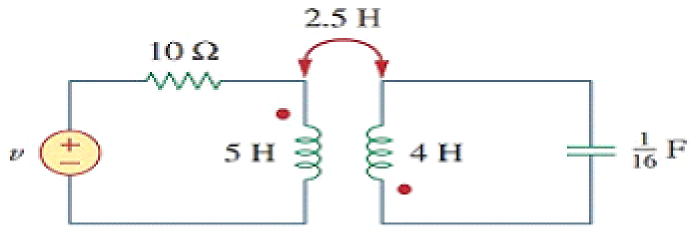
$$\frac{dI}{dt} = \frac{4}{0.04} = 100\ As^{-1}$$

$$M = \frac{\mu_o N_1 N_2 A}{\ell} = 4{\times}3.14 \times 10^{-7} \times 4{\times}10^4 \times 100 \times 4{\times}10^{-4}$$

$$M \simeq 2 \times 10^{-3}\ H$$

$$emf = 2\times 10^{-3} \times 100 = 0.2\ V$$

Q.14 Consider the circuit in Figure. Determine the coupling coefficient. Calculate the energy stored in the coupled inductors at time t = 1 s if v = 60 cos (4 t + 30◦) V.



Solution:

The coupling coefficient is

$$k = \frac{M}{\sqrt{L_1 L_2}} = \frac{2.5}{\sqrt{20}} = 0.56$$

indicating that the inductors are tightly coupled. To find the energy stored, we need to obtain the frequency-domain equivalent of the circuit.

$$\begin{aligned}
60\cos(4t + 30^\circ) \quad &\Rightarrow \quad 60\underline{/30^\circ}, \quad \omega = 4 \text{ rad/s} \\
5 \text{ H} \quad &\Rightarrow \quad j\omega L_1 = j20\ \Omega \\
2.5 \text{ H} \quad &\Rightarrow \quad j\omega M = j10\ \Omega \\
4 \text{ H} \quad &\Rightarrow \quad j\omega L_2 = j16\ \Omega \\
\frac{1}{16}\text{ F} \quad &\Rightarrow \quad \frac{1}{j\omega C} = -j4\ \Omega
\end{aligned}$$

The frequency-domain equivalent is shown in Figure. We now apply mesh analysis. For mesh 1,

$$(10 + j20)\mathbf{I}_1 + j10\mathbf{I}_2 = 60\underline{/30^\circ}$$

For mesh 2

$$j10\mathbf{I}_1 + (j16 - j4)\mathbf{I}_2 = 0$$

or

$$\mathbf{I}_1 = -1.2\mathbf{I}_2$$

Substituting this to Equation yields

$$\mathbf{I}_2(-12 - j14) = 60\underline{/30^\circ} \quad \Rightarrow \quad \mathbf{I}_2 = 3.254\underline{/160.6^\circ}\text{ A}$$

and

$$\mathbf{I}_1 = -1.2\mathbf{I}_2 = 3.905\underline{/-19.4^\circ} \text{ A}$$

In the time-domain,

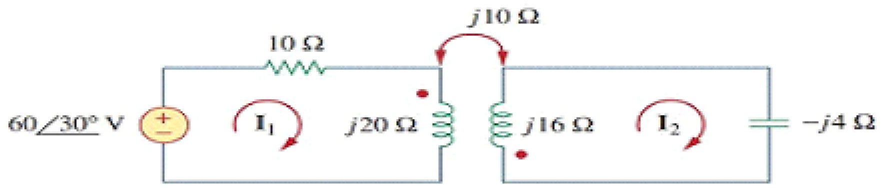
$$i_1 = 3.905 \cos(4t - 19.4^\circ), \qquad i_2 = 3.254 \cos(4t + 160.6^\circ)$$

At time t = 1 s, 4t = 4 rad = 229.2°, and

$$i_1 = 3.905 \cos(229.2^\circ - 19.4^\circ) = -3.389 \text{ A}$$
$$i_2 = 3.254 \cos(229.2^\circ + 160.6^\circ) = 2.824 \text{ A}$$

The total energy stored in the coupled inductors is

$$w = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 + M i_1 i_2$$
$$= \frac{1}{2}(5)(-3.389)^2 + \frac{1}{2}(4)(2.824)^2 + 2.5(-3.389)(2.824) = 20.73 \text{ J}$$



Frequency-domain Equivalent circuit

Q.15 It is desired to have a 4.13 mWb maximum core flux in a transformer at 110V and 50 Hz. Determine the required number of turns in the primary.

Solution: EMF induced in primary, E_1 = 110V

Supply frequency, f = 50Hz

Maximum core flux, ϕ_{max} = 4.13 mWb

$= 4.13 \times 10^{-3}$ Wb

Required number of turns on primary,

$$N_1 = \frac{E_1}{4.44 f \phi_{max}}$$

$$= \frac{110}{4.44 \times 50 \times 4.13 \times 10^{-3}} = 120$$

Q.16 The emf per turn of a single phase 10 kVA, 2200/220V, 50 Hz transformer is 10V. Calculate (i) the number of primary and secondary turns, (ii) the net cross-sectional area of core for a maximum flux density of 1.5T.

Solution: EMF per turn = 10V

Primary induced emf, $E_1 = V_1 = 2{,}200$ V

Secondary induced emf, $E_2 = V_2 = 220$V

Supply frequency, f = 50 Hz

Maximum flux density, $B_{max} = 1.5$ T

For (i)

Number of primary turns,

$$N_1 = \frac{E_1}{EMF\ per\ turn} = \frac{2,200}{10} = 220$$

Number of secondary turns,

$$N_2 = \frac{E_2}{EMF\ per\ turn} = \frac{220}{10} = 22$$

Maximum value of flux,

$$\phi_{max} = \frac{EMF\ per\ turn}{4.44 \times f} = \frac{10}{4.44 \times 50} = 0.045Wb$$

For (ii)

Net cross-sectional area of core,

$$a = \frac{\phi_{max}}{B_{max}} = \frac{0.045}{1.5} = 0.03m^2$$

Q.17 A single phase transformer has 350 primary and 1,050 secondary turns. The net cross-sectional area of the core is 55 cm^2. If the primary winding be connected to a 400 V, 50 Hz single phase supply, calculate (i) maximum value of the flux density in the core and (ii) the voltage induced in the secondary winding.

Solution: Net cross-section area of core,

$$a = 55^2 = 0.0055m^2$$

Maximum value of flux,

$$\phi_{max} = \frac{E_1}{4.44 \times f \times N_1} = \frac{400}{4.44 \times 50 \times 350} = 5.148 \times 10^{-3}Wb$$

For (i)

peak value of flux density in the core,

$$B_{max} = \frac{\phi_{max}}{a} = \frac{5.148 \times 10^{-3}}{0.0055} = 0.936T$$

For (ii)

Voltage induced in the secondary winding,

$$E_2 = E_1 \times \frac{N_2}{N_1} = 400 \times \frac{1050}{350} = 1200V$$

Q.18 A 25 kVA, single phase transformer has 250 turns on the primary and 40 turns on the secondary winding. The primary is connected to 1500 V, 50 Hz mains calculate (i) secondary emf (ii) primary and secondary current on full load (iii) maximum flux in the core.

Solution:

Supply voltage V_i = 1500 V

Primary induced emf, $E_1 = V_i$ = 1500 V

For (i)

Secondary emf,

$$E_2 = \frac{E_1 \times N_2}{N_1} = \frac{1500 \times 40}{250} = 240V$$

For (ii)

Appropriate value of primary current on full load,

$$I_1 = \frac{Rated\ kVA\ 1000}{V_i} = \frac{24 \times 1000}{1500} = 16{;}667A$$

Appropriate value of secondary current on full load,

$$I_2 = \frac{Rated\ kVA\ \times 1000}{E_2\ or\ V_2} = \frac{25 \times 1000}{240} = 104.167A$$

For (iii)

Maximum value of flux in the core,

$$\phi_{max} = \frac{E_1}{4.44fN_1} = \frac{1500}{4.44 \times 50 \times 250} = 0.027Wbor27mWb$$

Q.19 A 200/50V, 50 Hz single phase transformer is connected to a 200 V, 50 Hz supply with secondary winding open. Primary winding has 400 turns.

a) What is the value of maximum flux through the core, if the primary winding has 400 turns?
b) What is the peak value of flux if the primary voltage is 200V, 25 Hz?
c) What happens to no-load current?

Solution:

For (i)

Maximum value of flux,

$$\phi_{max} = \frac{E_1}{4.44 \times f \times N_1} = \frac{200}{4.44 \times 50 \times 400} = 2.25 \times 10^{-3} = 2.25mWb$$

For (ii)

When primary voltage is 200 V and supply frequency is 25 Hz

$$\phi_{max1} = \frac{200}{4.44 \times 25 \times 400} = 4.5mWb$$

For (iii)

The no-load primary current I_0, called the exciting current, is very small in comparison to the full-load primary current (2-5% of full load primary current). This current is made up of a relatively larger quadrature or magnetizing component I_m and a comparatively small in-phase or energy component I_e.

When supply frequency is reduced, keeping supply voltage constant, maximum value of flux is increased in inverse ratio of supply frequencies, and therefore, the magnetizing component of no-load current and thus the no-load current.

Q.20 A 400 kVA transformer has a primary winding resistance of 0.5 ohm and a secondary winding resistance of 0.001 ohm. The iron loss is 2.5 Kw and the primary and secondary voltages are 5 kV and 320 V respectively. If the power factor of the load is 0.85, determine the efficiency of the transformer (i) on full load and (ii) on half load.

Solution

Rated output = 400 kVA = $400 x 10^3$ kVA

Full load secondary current, I_2 = Rated output/V_2 = 1250 A

Total resistance referred to secondary, $r_{e2} = r_2 + r_1(V_2/V_1)^2$ = 0.033 ohm Full load copper loss, $P_c = I_2^2\, r_{e2}$ = 51.5625 Kw

Iron loss, Pi = 2.5 $x10^3$ watts

(i) Transformer efficiency at full load and 0.85 pf

$$\eta = \frac{x.V_2 I_2 \cos\theta_2}{xV_2 x_2 \cos\theta_2 + P_i + P_c} x100$$
$$= 86.2\%$$

= 86.2%

(ii) Transformer efficiency at half load and 0.85 pf

x=1/2

$$\eta = \frac{1/2.V_2 I_2 \cos\theta_2}{1/2.V_2 x_2 \cos\theta_2 + P_i + P_c} x100$$
$$= 91.69\%$$

= 91.69%

Q.21 Find all day efficiency of a transformer having maximum efficiency of 98% at 15 KVA at unity power factor. Compare it's all day efficiencies for the following load cycles: (a) Full load of 20 KVA, 12 hours per day and no load rest of the day. (b) Full load, 4 hours per day and 0.4 full load rest of the day. Assume the load to operate on unity power factor all day.

Solution:

$$\eta_{max} = \frac{P_0}{P_0 + 2P_i}$$

$P_i = 0.153$ kW

$$k^2 = \frac{P_i}{P_c}$$

$P_c = 0.272$ kW

(a)

P_0	Time, h	W_0
20	12	240
0	12	0
Total		240 kwh

$P_{in} = P_0+P_i+k^2P_c$	W_{in}
20+0.153+0.272 = 20.425	245.1
0+0.153 = 0.153	1.8
Total	246.9 kwh

$$\text{All day efficiency} = \frac{\Sigma W_0}{\Sigma W_{in}} = 97.2\%$$

(b)

P_0	Time, h	W_0
20	4	80
8	20	160
Total		240 kwh

$P_{in} = P_0+P_i+k^2P_c$	W_{in}
20+0.153+0.272 = 20.425	81.7
$8+0.153+(8/20)^2$x0.272 = 8.196	163.9
Total	245.6 kwh

$$\text{All day efficiency} = \frac{\Sigma W_0}{\Sigma W_{in}} = 97.7\%$$

Q.22 A 200 kVA single-phase transformer is in circuit throughout 24 hours. For 8 hours in a day, the load is 150 kW at 0.8 power factor lagging and for 7 hours, the load is 90 kW at 0.9 power factor. Remaining time or the rest period, it is at no-load condition. Full-load Cu loss is 4 kW and the iron loss is 1.8 kW. Calculate the all-day efficiency of the transformer.

Solution:

Full-load output = 200 kVA, Full-load Cu loss = 4 kW, Iron loss = 1.8 kW.

$$\text{Copper loss for 24 hours} = \left(\frac{\frac{150}{0.9}}{200}\right)^2 \times 4 \times 8 + \left(\frac{\frac{90}{1}}{200}\right)^2 \times 4 \times 7 = 27.89 \text{ kWh}$$

$$\text{Iron loss for 24 hours} = 1.8 \times 24 = 43.2 \text{ kWh}$$

$$\text{All-day output} = (150 \times 8) + (90 \times 7) = 1{,}830 \text{ kWh}$$

$$\text{All-day input} = 1{,}830 + 27.89 + 43.2 = 1{,}901.09 \text{ kWh}$$

$$\text{All-day efficiency } (\eta_{\text{all-day}}) = \frac{\text{All-day output}}{\text{All-day input}}$$

$$= \frac{1{,}830}{1{,}901.09} = 0.9626 \text{ p.u.} = 96.26\%$$

Summary:

1) Faraday's law of induction (briefly, Faraday's law) is a basic law of electromagnetism predicting how a magnetic field will interact with an electric circuit to produce an electromotive force (EMF)—a phenomenon known as electromagnetic induction.
2) Faraday's Law. Now that we have a basic understanding of the magnetic field, we areready to define Faraday's Law of Induction. It states that the induced voltage in a circuit is proportional to the rate of change over time of the magnetic flux through that circuit.
3) Faraday's First Law of Electromagnetic Induction- Whenever a conductor is placed in a varying magnetic field, an electromotive force is induced. If the conductor circuit is closed, a current is induced which is called induced current.
4) Faraday's law of induction is one of the important concepts of electricity. It looks at the way changing magnetic fields can cause current to flow in wires. Basically, it is a formula/concept that

describes how potential difference (voltage difference) is created and how much is created.

5) Newton's first law is demonstrated by the act of exerting a force. The car remains at rest until the mass is expelled, producing a force. The car then moves. The action force exerted on the car produces an equal and opposite reaction force.
6) Newton's first law of motion - sometimes referred to as thelaw of inertia. Newton's first law of motion is often stated as. An object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force.
7) Newton's Second Law of Motion states that force is equal to the change in momentum per change in time. For a constant mass, force equals mass timesacceleration, i.e. F = m*a. Newton's Third Law of Motion states that for every action there is an equal and opposite reaction.
8) Induced voltage is an electric potential created by an electric field, magnetic field, or a current. Voltage produced in generator because of moving magnetic field. Voltage generated in secondary of current transformer due to magnetic field ofcurrent injected in it's primary.
9) Induction charging is a method used to charge an object without actually touching the object to any other charged object. An understanding of charging by inductionrequires an understanding of the nature of a conductor and an understanding of the polarization process.

10) Induced power is the power required to maintain enough lift to overcome the force of gravity. One can view this as the force required to accelerate enough air downwards (at speed v_i) to push the bird upwards enough to counteract the force of gravity (mg).

11) In dynamically induced electromotive force the magnetic field system is kept stationary, and the conductor is moving, or the magnetic field system is moving, and the conductor is stationary. Thus by following either of the two process the conductor cuts across the magnetic field and the emf is induced in the coil.

12) The process in which a changing current in one coil induces emf in another coil, is called mutual induction. While the phenomenon in which a changing current in a coil induces an emf in itself is called self-induction.

13) In self inductance the change in the strength of current in the coil is opposed by the coil itself by inducing an e.m.f. whereas in mutual inductance out of the two coils one coil opposes change in the strength of the current flowing in the other coil.

14) This emf is proportional to (1) the number of turns in the coil (2) rate of change magnetic flux with time. Statically induced emf is that emf which is produced due to pulsation of flux inside a coil, without any relative movement between coil and magnetic field. Hence the term Statically induced emf.

15) Self induced emf is that which is induced in a coil, due to the change in its own current or flux. Mutual emf is that induced in a coil due to the neighbouring coil's varying current.

16) Experiments and calculations that combine Ampere’s law and Biot-Savart law confirm that the two constants, M_{21} and M_{12} are

equal in the absence of material medium between the two coils, $M_{12} = M_{21.}$ This property is called reciprocity.

17) When this emf is induced in the same circuit in which the current is changing this effect is called Self-induction, (L). However, when the emf is induced into an adjacent coil situated within the same magnetic field, the emf is said to be induced magnetically, inductively or by Mutual induction, symbol (M).

18) After learning what is the mutual inductance and dot convention, we will move on how to calculate the energy in a coupled electric circuit. We can call an electric circuit as a coupled circuit if the circuit has a mutual inductance from two coils or inductors.

19) A transformer is an electrical apparatus designed to convert alternating current from one voltage to another. It can be designed to "step up" or "step down" voltages and works on the magnetic induction principle. When voltage is introduced to one coil, called the primary, it magnetizes the iron core.

20) Transformers generally have one of two types of cores: Core Type and Shell Type. These two types are distinguished from each other by the manner in which the primary and secondary coils are place around the steel core. Core type - With this type, the windings surround the laminated core.

21) Equivalent Circuit diagram of single phase Transformer. Equivalent circuit diagram of a transformer is basically a diagram which can be resolved into an equivalent circuit in which the resistance and leakage reactance of the transformer are imagined to be external to the winding.

22) In an ideal transformer, it is assumed that entire amount of flux get linked with secondary winding (that is, no leakage flux). 100% efficiency: An ideal transformer does not have any losses like hysteresis loss, eddy current loss etc. So, the outputpower of an ideal transformer is exactly equal to the input power.

23) The Efficiency of the transformer is defined as the ratio of useful output power to the input power. The input and output power are measured in the same unit. Its unit is either in Watts (W) or KW. Transformer efficiency is denoted by Ƞ.

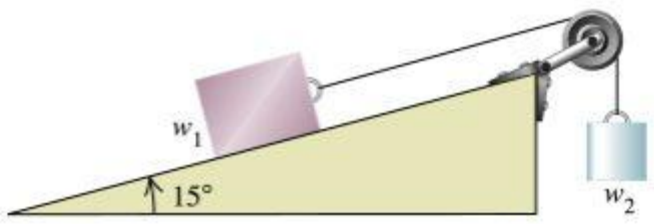
24) There are various types of losses in the transformer such as iron loss, copper loss, hysteresis loss, eddy current loss, stray loss, and dielectric loss.

Terminal Questions:

1) Explain the working of Faraday's law of electromagnetic induction in differential form.
2) What do mean by analogy with Newton's laws of motion in mechanics?
3) Define all the terms for Condition for existence and depending factors of induced charge, induced voltage, induced current and induced power.
4) Explain the working of Dynamic induced EMF and derivation of its expression.
5) What do you mean by self and mutual induction and inductance?
6) Define the static induced EMF with the help of self and mutual induced EMF.

7) Explain the Reciprocity theorem and its Neuman's relation.
8) What do you mean by Relation between self and mutual inductance of two coupled coil.
9) Explain and derive the expression for energy of coupled circuits.
10) Explain the working principal of 1-phase Transformer and draw its equivalent circuit.
11) Derive the expression for efficiency of 1-phase Transformer and voltage.
12) Explain the working principal of voltage gain in Transformer.
13) Explain the various types of transformer losses.
14) A circular wire loop with a radius of 5 cm lies in a plane perpendicular to a uniform magnetic field of magnitude 0.2 T. You reshape the loop into a square in 0.10 seconds. What is the emf induced in the loop?

15) In the figure below, a block of weight $w_1 = 100.0$ N on a frictionless inclined plane of angle 15^o is connected by a cord over a massless, frictionless pulley to a second block of weight $w_2 = 30.0$ N. (a) What are the magnitude and direction of the acceleration of each block? (b) What is the tension in the cord?



16) Determine the accelerations that result when a 12-N net force is applied to a 3-kg object and then to a 6-kg object.

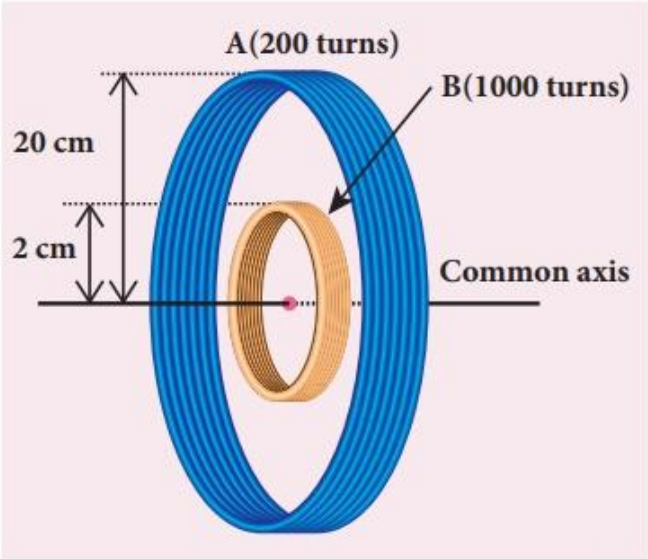
17) A net force of 15 N is exerted on an encyclopedia to cause it to accelerate at a rate of 5 m/s^2. Determine the mass of the encyclopedia.

18) A straight metal wire crosses a magnetic field of flux 4 mWb in a time 0.4 s. Find the magnitude of the emf induced in the wire.

19) A closely wound coil of radius 0.02 m is placed perpendicular to the magnetic field. When the magnetic field is changed from 8000 T to 2000 T in 6 s, an emf of 44 V is induced. Calculate the number of turns in the coil.

20) **A very small circular loop of area 5×10^{-4} m^2, resistance 2 ohm and negligible self inductance initially coplanar and concentric with a much larger fixed circular loop of radius 0.1 m. A constant current of 1.0 A is passed through the bigger loop. The smaller loop is rotated with constant angular velocity ω rad/sec about it's diameter. Calculate the (a) induced emf and (b) the induced current through the smaller loop as a function of time.**

21) Consider two coplanar, co-axial circular coils A and B as shown in figure. The radius of coil A is 20 cm while that of coil B is 2 cm. The number of turns is 200 and 1000 for coils A and B respectively. Calculate the mutual inductance of coil B with respect to coil A. If the current in coil A changes from 2 A to 6 A in 0.04 sec, determine the induced emf in coil B and the rate of change of flux through the coil B at that instant.

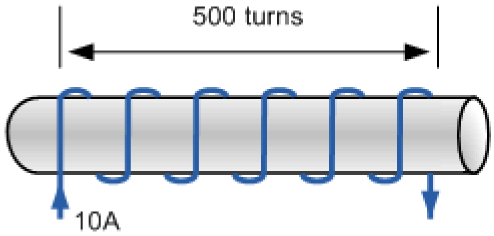


22) A rectangular coil of area 6 cm2 having 3500 turns is kept in a uniform magnetic field of 0.4 T. Initially, the plane of the coil is perpendicular to the field and is then rotated through an angle of 180º. If the resistance of the coil is 35 Ω, find the amount of charge flowing through the coil.

23) A fan of metal blades of length 0.4 m rotates normal to a magnetic field of 4 ×10−3T . If the induced emf between the centre and edge of the blade is 0.02 V, determine the rate of rotation of the blade.

24) A bicycle wheel with metal spokes of 1 m long rotates in Earth's magnetic field. The plane of the wheel is perpendicular to the horizontal component of Earth's field of 4×10^{-5} T. If the emf induced across the spokes is 31.4 mV, calculate the rate of revolution of the wheel.

25) A hollow air cored inductor coil consists of 500 turns of copper wire which produces a magnetic flux of 10mWb when passing a DC current of 10 amps. Calculate the self-inductance of the coil in milli-Henries.



26) Two air core solenoids have the same length of 80 cm and same cross–sectional area 5 cm2. Find the mutual inductance between them if the number of turns in the first coil is 1200 turns and that in the second coil is 400 turns.

27) A closed coil of 40 turns and of area 200 cm2, is rotated in a magnetic field of flux density 2 Wb m-2. It rotates from a position where its plane makes an angle of 30º with the field to a position perpendicular to the field in a time 0.2 sec. Find the magnitude of the emf induced in the coil due to its rotation.

28) The self-inductance of an air-core solenoid is 4.8 mH. If its core is replaced by iron core, then its self-inductance becomes 1.8 H. Find out the relative permeability of iron.

29) The current flowing in the first coil changes from 2 A to 10 A in 0.4 sec. Find the mutual inductance between two coils if an emf of 60 mV is induced in the second coil. Also determine the induced emf in the second coil if the current in the first coil is changed from 4 A to 16 A in 0.03 sec. Consider only the magnitude of induced emf.

30) A circular metal of area 0.03 m^2 rotates in a uniform magnetic field of 0.4 T. The axis of rotation passes through the centre and perpendicular to its plane and is also parallel to the field. If the disc completes 20 revolutions in one second and the resistance of the disc

is 4 Ω, calculate the induced emf between the axis and the rim and induced current flowing in the disc.

31) The emf per turn of a single phase 25 kVA, 2200/220V, 50 Hz transformer is 20V. Calculate (i) the number of primary and secondary turns, (ii) the net cross-sectional area of core for a maximum flux density of 2.4T.

32) An ideal transformer has 460 and 40,000 turns in the primary and secondary coils respectively. Find the voltage developed per turn of the secondary if the transformer is connected to a 230 V AC mains. The secondary is given to a load of resistance 104Ω. Calculate the power delivered to the load.

33) A 460/2400V transformer has a series leakage reactance of 37.2 Ω as referred to the high-voltage side. A load connected to the low-voltage side is observed to be absorbing 25 kW, unity power factor, and the voltage is measured to be 450 V. Calculate the corresponding voltage and power factor as measured at the high-voltage terminals.

34) A single-phase 10 kVA, 2400/240 V, 50 Hz distribution transformer has the following characteristics: Core loss at full voltage = 100 W, Copper loss at half load = 60 W

(a) Determine the per-unit rating at which the transformer efficiency is maximum.

(b) The transformer has the following load cycle: No load for 6 hours, 70 % full load for 10 hours at 0.8 pf, 90 % full load for 8 hours at 0.9 pf .

Determine the all-day efficiency of the transformer.

Uttar Pradesh Rajarshi Tandon

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Bachelor of Science

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Electromagnetism

Block

4 Electromagnetic Theory

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Unit 08-Fundamental equations

Structure

8.1 Introduction:

In this chapter we will discuss Four Maxwell's equations statement and physical significance. Maxwell was the first person to calculate the speed of propagation of electromagnetic waves which was same as the speed of light and came to the conclusion that EM waves and visible light are similar.

Maxwell’s equations are the basic equations of electromagnetism which are a collection of Gauss’s law for electricity, Gauss’s law for magnetism, Faraday’s law of electromagnetic induction and Ampere’s law for currents in conductors.

We discuss Maxwell’s equations and features of their general plane wave solution in source free space and simple dielectrics.

We also discuss Differential equation and velocity for electromagnetic waves in source free space and dielectric medium.

Maxwell's derivation of the electromagnetic wave equation has been replaced in modern physics education by a much less cumbersome method involving combining the corrected version of Ampere’s circuital law with Faraday's law of induction.

The inherent characteristic of electromagnetic waves is its frequency. According to Maxwell, varying the electric field gives rise to a magnetic field. An accelerated charge produces a time-varying magnetic field which in turn produces a time-varying electric field.

The "opposition" to the wave or wave impedance or impedance of a medium to a wave is caused by characteristics of the medium analogous to the resistance, capacitance and inductance. While resistance is a pretty generic term, applicable to different types of waves, the energy storing characteristics, capacitance and inductance, could be generalized as compliance or stress and inertia or motion.

Refractive index is defined as the speed of light in a medium depends on the properties of the medium. In electromagnetic waves, the speed is dependent on the optical density of the medium.

When an AC current is applied to a conductor, the current concentrates near the surface of the conductor and its strength decreases as you go towards the center of the conductor. The depth till which current flows in a conductor is called as Skin Depth.

8.2 Objectives:

After studying this unit you should be able to

- Explain and identify Four Maxwell's equations (statement and physical significance).
- Study and identify Maxwell's equations and features of their general plane wave solution in source free space.
- Explain and identify Maxwell's equations and features of their general plane wave solution in simple dielectrics.
- Study and identify Differential equation and velocity for electromagnetic waves in source free space and dielectric medium.
- Explain and identify Characteristics of electromagnetic waves, impedance and refractive index.
- Study and identify Skin depth and its importance.

8.3 Four Maxwell's equations (statement and physical significance):

Four Maxwell's equations statement:

Maxwell was the first person to calculate the speed of propagation of electromagnetic waves which was same as the speed of light and came to the conclusion that EM waves and visible light are similar.

These are the set of partial differential equations that form the foundation of classical electrodynamics, electric circuits and classical optics along with Lorentz force law. These fields highlight modern communication and electrical technologies.

Maxwell's equations integral form explains how the electric charges and electric currents produce magnetic and electric fields. The equations describe how the electric field can create a magnetic field and vice versa.

$$\nabla \cdot \mathbf{D} = \rho_V$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

We have discussed the equations of Maxwell individually up until now.

1. Gauss' Law:

$$\nabla \cdot \mathbf{D} = \rho_V$$

Gauss' Law is equivalent to the Force Equation for Electric Charges: like charges repel each other and opposite charges (i.e. positive and negative charge) attract.

Gauss' Law also says that Electric Field lines diverge away from Electric Charges. This means that positive charge acts as a source of Electric Fields (like the way a faucet is a source of water). Gauss' Law means that negative charges acts as a sink for Electric Fields (the way water drains or exits a region via a sink hole). This means Electric Field lines start and stop on Electric Charge.

2. <u>Gauss' Law for Magnetism</u>:

$$\nabla \cdot \mathbf{B} = 0$$

Maxwell's Second Equation says that magnetic monopoles do not exist. While we have Electric Charges (Electric Monopoles), we have never found the magnetic equivalent - magnetic charge or a magnetic monopoles. This equation states that the magnetic field tends to wrap around things - since the divergence is zero the fields tend to form closed loops.

3. <u>Faraday's Law</u>:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's Law tells us that a magnetic field that is changing in time will give rise to a circulating E-field. This means we have two ways of generating E-fields - from Electric Charges (or flowing electric charge, current) or from a magnetic field that is changing.

4. <u>Ampere's Law</u>:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

Ampere's Law tells us that a flowing electric current gives rise to a magnetic field that circles the wire. In addition to this, it also says that an Electric Field that is changing in time gives rise to a magnetic field that encircles the E-field - this is the Displacement Current term that Maxwell himself introduced.

This means there are 2 ways to generate a solenoidal (circulating) H-field - a flowing electric current or a changing Electric Field. Both give rise to the same phenomenon.

Physical Significance of Maxwell's Equations:

By means of Gauss and Stoke's theorem we can put the field equations in integral form of hence obtain their physical significance

1. Maxwell's first equation is $\nabla . D = \rho$.

Integrating this over an arbitrary volume V we get

$$\int_V \nabla . D \, dV = \int_V \rho \, dV.$$

But from Gauss Theorem, we get

$$\int_S D . dS = \int_V \rho \, dV = q$$

Here, q is the net charge contained in volume V. S is the surface bounding volume V. Therefore, Maxwell's first equation signifies that:
The total electric displacement through the surface enclosing a volume is equal to the total charge within the volume.

2. Maxwell's second equations is $\nabla . B = 0$

Integrating this over an arbitrary volume V, we get

$$\int_V \nabla . B = 0.$$

Using Gauss divergence theorem to change volume integral into surface integral, we get

$\int_s B.dS = 0.$

Maxwell's second equation signifies that:

The total outward flux of magnetic induction B through any closed surface S is equal to zero.

3. Maxwell's third equation is $\nabla \times E = - \partial B/\partial t \,.\, dS$

Converting the surface integral of left hand side into line integral by Stoke's theorem, we get

$\Phi_c E.\, dI = - \int_s \partial B/\partial t.\, dS.$

Maxwell's third equation signifies that:

The electromotive force (e.m.f. $e = \int_C E.dI$) around a closed path is equal to negative rate of change of magnetic flux linked with the path (since magnetic flux $\Phi = \int_s B.dS$).

4. Maxwell's fourth equation is

$\nabla \times H = J + \partial D/\partial t$

Taking surface integral over surface S bounded by curve C, we obtain

$\int_s \nabla \times H.\, dS = \int_s (J + \partial D/\partial t)\, dS$

Using Stoke's theorem to convert surface integral on L.H.S. of above equation into line integral, we get

$\Phi_c H.dI = \int_s (J + \partial D/\partial t).dS$

Maxwell's fourth equation signifies that:

The magneto motive force (m.m.f. = $\Phi_c H.\, dI$) around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.

8.4 Maxwell's equations and features of their general plane wave solution in source free space:

Maxwell's equations are the basic equations of electromagnetism which are a collection of Gauss's law for electricity, Gauss's law for magnetism, Faraday's law of electromagnetic induction and Ampere's law for currents in conductors. Maxwell equations give a mathematical model for electric, optical, and radio technologies, like power generation, electric motors, wireless communication, radar, and, Lenses, etc. These Equations explain how magnetic and electric fields are produced from charges.

These equations are part of the comprehensive and symmetrical theory of electromagnetism, which is essential to understand electromagnetic waves, optics, radio and TV transmission, microwave ovens and magnetically levitated trains.

The four of Maxwell's equations for free space are:

The First Maxwell's equation (Gauss's law for electricity):

The Gauss's law states that flux passing through any closed surface is equal to $1/\varepsilon_0$ times the total charge enclosed by that surface.

Integral form of Maxwell's 1st equation

$$\phi_E = \frac{q}{\varepsilon_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

and

$$\emptyset_e = \int \vec{E}.d\vec{A} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

Comparing equation (1) and (2) we have

$$\int \vec{E}.d\vec{A} = \frac{q}{\epsilon 0} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

It is the integral form of Maxwell's 1st equation.

Maxwell's first equation in differential form

The value of total charge in terms of volume charge density is $q=\int \rho\, dv$. So equation (3) becomes

$$\int \vec{E}.d\vec{A} = \frac{1}{\epsilon 0} \int \rho dv$$

Applying divergence theorem on left hand side of above equation we have

$$\int (\vec{\nabla}.\vec{E}) d.V = \frac{1}{\epsilon 0} \int \rho dv$$

$$\int (\vec{\nabla}.\vec{E}) d.V - \frac{1}{\epsilon 0} \int \rho dv = 0$$

$$\int [(\vec{\nabla}.\vec{E}) - \frac{\rho}{\epsilon 0}]\, dv = 0$$

$$(\vec{\nabla}.\vec{E}) - \frac{\rho}{\epsilon 0} = 0$$

$$(\vec{\nabla}.\vec{E}) = \frac{\rho}{\epsilon 0}$$

It is called the differential form of Maxwell's 1st equation.

The Second Maxwell's equation (Gauss's law for magnetism):

The Gauss's law for magnetism states that net flux of the magnetic field through a closed surface is zero because monopoles of a magnet do not exist.

$$\int \vec{B}.\overrightarrow{dA}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

It is the integral from of Maxwell's second equation.

Applying divergence theorem

$$\int (\vec{\nabla}.\vec{B})dV=0$$

This implies that:

$$\vec{\nabla}.\vec{B}=0$$

It is called differential from of Maxwell's second equation.

The Third Maxwell's equation (Faraday's law of electromagnetic induction):

According to Faraday's law of electromagnetic induction

$$\varepsilon = -N\frac{d\emptyset_m}{dt} \quad \ldots\ldots\ldots\ldots\ldots (5)$$

Since emf is related to electric field by the relation

$$\varepsilon = \int \vec{E}.\overrightarrow{dA}$$

Also

$$\emptyset_m = \int \vec{B}.\overrightarrow{dA}$$

Put these values in equation (5) we have

$$\int \vec{E}.\overrightarrow{dA} = -N \int \vec{E}.\overrightarrow{dA} \int \vec{B}.\overrightarrow{dA}$$

For N=1, we have

$$\int \vec{E}.\overrightarrow{dA} = -\frac{d}{dt} \int \vec{B}.\overrightarrow{dA} \quad \text{...............(6)}$$

It is the integral form of Maxwell's 3rd equation.

Applying stokes theorem on L.H.S. of equation (6) we have

$$\int (\vec{\nabla} \times \vec{E}) d\vec{A} = -\frac{d}{dt} \int \vec{B}.\overrightarrow{dA}$$

$$\int (\vec{\nabla} \times \vec{E}) d\vec{A} + \frac{d}{dt} \int \vec{B}.\overrightarrow{dA} = 0$$

$$(\vec{\nabla} \times \vec{E}) + \frac{d\vec{B}}{dt} = 0$$

$$(\vec{\nabla} \times \vec{E}) = -\frac{d\vec{B}}{dt}$$

It is the differential form of Maxwell's third equation.

The Fourth Maxwell's equation (Ampere's law):

The magnitude of the magnetic field at any point is directly proportional to the strength of the current and inversely proportional to the distance of the point from the straight conductors is called Ampere's law.

$$\int \vec{B}.\overrightarrow{ds} = \mu_0 i \quad \ldots\ldots\ldots\ldots\ldots (7)$$

It is the integral from of Maxwell's 4th equation.

The value of current density

$$i = \int \vec{j}.\overrightarrow{dA}$$

Now the equation (7) becomes

$$\int \vec{B}.\overrightarrow{ds} = \mu_0 \int \vec{j}.\overrightarrow{dA}$$

Applying Stoke's theorem on L.H.S. of above equation, we have

$$\int (\vec{\nabla} \times \vec{B}) d\vec{A} = \mu_0 \int \vec{j}.\overrightarrow{dA}$$

$$\int [(\vec{\nabla} \times \vec{B}) d\vec{A} - \mu_0 j]. d\mathbf{A} = 0$$

$$(\vec{\nabla} \times \vec{B}) = \mu_0 \boldsymbol{j}$$

Third Maxwell's equation says that a changing magnetic field produces an electric field. But there is no clue in fourth Maxwell's equation whether a changing electric field produces a magnetic field? To overcome this deficiency, Maxwell's argued that if a changing magnetic flux can produce an electric field then by symmetry there must exist a relation in which a changing electric field must produce a changing magnetic flux.

SAQ.1

a) Define Four Maxwell's equations with statement.

b) What do you mean by physical significance for all four Maxwell's equations?

c) Explain the general plane wave solution in source free space for Maxwell's equations.

8.5 Maxwell's equations and features of their general plane wave solution in simple dielectrics:

Consider a region of space filled by "simple dielectrics" materials, that is:

- Linear: μ and ε are constants;
- Isotropic: These is full rotational symmetry (no preferred or special direction in space);
- Homogeneous: There is translational symmetry in all directions (no space location in space);
- Source free: The charge density ρ is zero;

Non-conducting: The conductivity σ is zero, and hence the current density $\vec{J} = \sigma \vec{E}$ is also zero;

In our "simple Dielectric" material, Maxwell's equations take the from:

$$\begin{aligned} \nabla \cdot \vec{E} &= 0 && (1) \\ \nabla \cdot \vec{B} &= 0 && (2) \\ \nabla \times \vec{B} &= \mu\varepsilon \dot{\vec{E}} && (3) \\ \nabla \times \vec{E} &= -\dot{\vec{B}} && (4) \end{aligned}$$

If we take the curl of equation (4) we find, using a vector density:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla \times \dot{\vec{B}} \qquad (5)$$

Using equation (1), this becomes:

$$\nabla^2 \vec{E} = \nabla \times \dot{\vec{B}} \qquad (6)$$

Taking the time derivative of equation (3) gives:

$$\nabla \times \dot{\vec{B}} = \mu\varepsilon \ddot{\vec{E}} \qquad (7)$$

Then combining equation (6) and (7) we find:

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \qquad (8)$$

In a “simple Dielectric” Material, the electric field satisfies equation (8):

$$\nabla^2 \vec{E} - \mu\varepsilon \ddot{\vec{E}} = 0 \qquad (9)$$

Similarly, we find that the magnetic field satisfies:

$$\nabla^2 \vec{B} - \mu\varepsilon \ddot{\vec{B}} = 0 \qquad (10)$$

These are the equation for wave travelling with speed v, given by

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (11)$$

Experimentally, we find that v is the speed of light in the material. In a vacuum, the speed of the wave is c, given by:

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \qquad (12)$$

Light is an electromagnetic wave. The existence of such waves, and derive their properties (including their speed) from Maxwell’s equations.

The operator:

$$\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2} \qquad (13)$$

Is called the Hertzian operator. The wave equations (8) and (10) are:

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2}\right)\left\{\begin{array}{c}\vec{E}(\vec{r},t)\\ \vec{B}(\vec{r},t)\end{array}\right\} = 0 \qquad (14)$$

These equations are necessary, but not sufficient, constraints on the possible functions $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ we must always check that solution to the wave equations (8) and (10) also satisfy Maxwell's equations.

We find that the wave equation (8) is solved by:

$$\vec{E}(\vec{r},t) = \vec{E}_0 \cos\left(\omega t - \vec{k}\cdot\vec{r} + \phi_0\right) \qquad (15)$$

Where $\vec{E}_O$ and $\vec{k}$ are constant vectors, and ω and Φ_0 are constant scalars. Equation (15) is the equation of a plane wave of frequency ω. The fact that only a single frequency (i.e. a single value of ω) is present in the wave means that the wave is monochromatic.

The field (15) is a valid solution of the wave equation (8) if $\vec{k}$ and ω satisfy:

$$\frac{\omega^2}{\vec{k}^2} = v^2 = \frac{1}{\mu\varepsilon} \qquad (16)$$

Equation (16) is known as a dispersion relation: it relates the frequency of the wave ω to the wave vector $\vec{k}$.

Maxwell's equations impose further constraints.

By writing the vectors $\vec{E}_O$ and $\vec{k}$ terms of components:

$$\vec{E}_0 = (E_{0x}, E_{0y}, E_{0z}) \qquad (17)$$
$$\vec{k} = (k_x, k_y, k_z) \qquad (18)$$

We find that for the field given by equation (15):

$$\nabla \cdot \vec{E} = \vec{k} \cdot \vec{E}_0 \sin\left(\omega t - \vec{k} \cdot \vec{r} + \phi_0\right) \qquad (19)$$

Maxwell's equations (with zero charge density):

$$\nabla \cdot \vec{E} = 0 \qquad (20)$$

Is only satisfied for all positions $\vec{r}$and times t, if $\vec{k}$ and $\vec{E}_O$

$$\vec{k} \cdot \vec{E}_0 = 0 \qquad (21)$$

Consider our solution (15) to the wave equation:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos\left(\omega t - \vec{k} \cdot \vec{r} + \phi_0\right) \qquad (22)$$

At fixed position $\vec{r}$ the field strength $\vec{E}$ varies sinusoidally, with angular frequency ω.

At fixed time t, the field strength varies sinusoidally in the direction of $\vec{k}$, with wavelength 2π/k. Along planes perpendicular to $\vec{k}$, the field is at the same phase. (Consider $\vec{r} \to \vec{r} + \vec{r}'$ where $\vec{k} \cdot \vec{r}' = 0$

Therefore, the electric field $\vec{E}(\vec{r}, t)$ takes the from of a plane wave advancing in the direction of $\vec{k}$. Since $\vec{k}.\vec{E}_O = 0$ the field is perpendicular to the direction of motion: it is transverse wave.

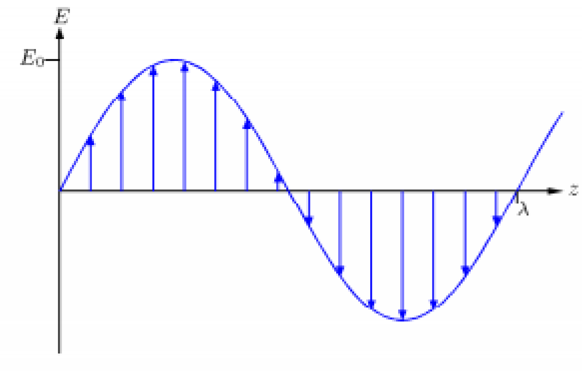


Fig.8.1 Transverse wave

The wave equation (10) for the magnetic field $\vec{B}$ has a similar solution:

$$\vec{B}(\vec{r},t) = \vec{B}_0 \cos\left(\omega t - \vec{k}\cdot\vec{r} + \phi_0\right) \qquad (23)$$

Where there are the same constraints on $\vec{B}_0$, $\vec{k}$ and ω.

Despite the fact that we derived independent wave equations for $\vec{E}$ and $\vec{B}$, Maxwell's equations tell us that the electric and magnetic field s are not independent . in particular, we must satisfy:

$$\begin{aligned} \nabla \times \vec{E} &= -\dot{\vec{B}} && (24) \\ \nabla \times \vec{B} &= \mu\varepsilon\dot{\vec{E}} && (25) \end{aligned}$$

Substituting in the solution (15) and (23), we find that $\vec{k}$, ω and Φ_0 must be the same foe both $\vec{E}$ and $\vec{B}$, furthermore we must have:

$$\begin{aligned} \vec{k} \times \vec{E}_0 &= \omega\vec{B}_0 && (26) \\ \vec{k} \times \vec{B}_0 &= -\mu\varepsilon\omega\vec{E}_0 && (27) \end{aligned}$$

Maxwell's equations demand that the electric and magnetic fields satisfy (26) and (27):

$$\begin{aligned}\vec{k} \times \vec{E}_0 &= \omega \vec{B}_0 \\ \vec{k} \times \vec{B}_0 &= -\mu\varepsilon\omega \vec{E}_0\end{aligned}$$

These constrains can be satisfied if the vectors $\vec{k}$, $\vec{E}_0$ and $\vec{B}_0$ are mutually perpendicular.

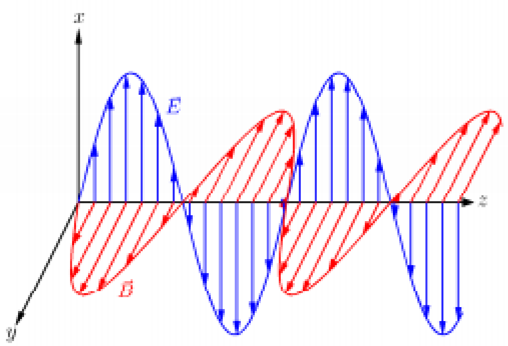


Fig.8.2 Waveform for magnitudes of the electric and magnetic field

The magnitudes of the electric and magnetic field satisfy:

$$\frac{E_0}{B_0} = \frac{1}{\sqrt{\mu\varepsilon}} = v \qquad (28)$$

Where v is the phase velocity.

8.6 Differential equation and velocity for electromagnetic waves in source free space:

Maxwell's derivation of the electromagnetic wave equation has been replaced in modern physics education by a much less cumbersome method involving combining the corrected version of Ampère's circuital law with Faraday's law of induction.

To obtain the electromagnetic wave equation in a vacuum using the modern method, we begin with the modern 'Heaviside' form of Maxwell's equations. In a vacuum- and charge-free space, these equations are:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

These are the general Maxwell's equations specialized to the case with charge and current both set to zero. Taking the curl of the curl equations gives:

$$\begin{aligned}
\nabla \times (\nabla \times \mathbf{E}) &= \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \\
\nabla \times (\nabla \times \mathbf{B}) &= \nabla \times \left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}
\end{aligned}$$

We can use the vector identity

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla (\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$$

where V is any vector function of space. And

$$\nabla^2 \mathbf{V} = \nabla \cdot (\nabla \mathbf{V})$$

where ∇V is a dyadic which when operated on by the divergence operator ∇ · yields a vector. Since

$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{B} = 0$$

Then the first term on the right in the identity vanishes and we obtain the wave equations:

$$\frac{1}{c_0^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$
$$\frac{1}{c_0^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} - \nabla^2 \mathbf{B} = 0$$

where

$$c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.99792458 \times 10^8 \text{ m/s}$$

is the speed (i.e. phase velocity) of light in free space.

Differential equation and velocity for electromagnetic waves in dielectric medium:

Consider a plane electromagnetic wave, linearly polarized in the x-direction that propagates in the z-direction through a transparent dielectric medium, such as glass or water. As is well-known, the electric component of the wave causes the neutral molecules making up the medium to polarize: that is, it causes a small separation to develop between the mean positions of the positively and negatively charged constituents of the molecules (i.e., the atomic nuclei and the orbiting electrons). [Incidentally, it can be shown that the magnetic component of the wave has a negligible influence on the molecules, provided the wave amplitude is sufficiently small that the wave electric field does not cause the electrons and nuclei to move with relativistic velocities (ibid.).] If the mean position of the positively charged constituents of a given molecule, of net charge (+q), develops a vector displacement (d) with respect to the mean position of the negatively charged constituents, of net charge -q , in response to a wave electric field E , then the associated electric dipole moment is P = q d , where d is generally parallel to E (ibid.). Furthermore, if there are N such molecules per unit volume then the electric dipole moment per unit volume is written P = N q d. In a linear, isotropic, dielectric medium (ibid.),

$$\mathbf{P} = \epsilon_0 (\epsilon - 1)\, \mathbf{E},$$

Where ε > 1 is a dimensionless quantity, known as the *relative dielectric constant*, that is a property of the medium in question. In the presence of a dielectric medium,

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0} \frac{\partial H_y}{\partial z},$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0} \frac{\partial E_x}{\partial z},$$

Above Equations generalize to give

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon_0}\left(\frac{\partial P_x}{\partial t} + \frac{\partial H_y}{\partial z}\right),$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}.$$

When combined with Equation these expressions yield

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\epsilon\,\epsilon_0}\frac{\partial H_y}{\partial z},$$

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\frac{\partial E_x}{\partial z}.$$

It can be seen that the previous equations are just like the corresponding vacuum equations except that ε_o has been replaced by $\varepsilon\ \varepsilon_o$. It immediately follows that the phase velocity of an electromagnetic wave propagating through a dielectric medium is

$$v = \frac{1}{\sqrt{\epsilon\,\epsilon_0\,\mu_0}} = \frac{c}{n},$$

Where $c = 1 / (\varepsilon_o\ \mu_o)^{1/2}$ is the velocity of light in vacuum, and the dimensionless quantity

$$n = \sqrt{\epsilon}$$

is known as the refractive index of the medium. Thus, an electromagnetic wave propagating through a transparent dielectric medium does so at a phase velocity that is less than the velocity of light in vacuum by a factor n (where n > 1). The dispersion relation of the wave is thus

$$\omega = k\,v = \frac{k\,c}{n}$$

Furthermore, the impedance of a transparent dielectric medium becomes

$$Z = \sqrt{\frac{\mu_0}{\epsilon\,\epsilon_0}} = \frac{Z_0}{n},$$

Where Z_0 is the impedance of free space

Incidentally, the signal that travels down a transmission line is a form of guided electromagnetic wave. It follows that if the space between the two conductors that constitute the line is filled with dielectric material of relative dielectric constant ε then the signal propagates down the line at the reduced phase velocity

$$v = \frac{c}{\sqrt{\epsilon}}.$$

This occurs because the dielectric material increases the capacitance per unit length of the line by a factor ε, but leaves the inductance per unit length unchanged. For the same reason, the presence of the dielectric material decreases the impedance of the line by a factor$\sqrt{\varepsilon}$. Hence, the impedance of a dielectric filled co-axial cable is

$$Z = \frac{1}{2\pi\sqrt{\epsilon}} \ln\left(\frac{b}{a}\right) Z_0.$$

Here, “a” and “b” are the radii of the inner and outer conductors, respectively.

SAQ.2

a) What do you mean by general plane wave solution in simple dielectrics for Maxwell’s equations?

b) Write the Differential equation and velocity for electromagnetic waves in source free.

c) Write the Differential equation and velocity for electromagnetic waves in dielectric medium.

8.7 Characteristics of electromagnetic waves:

The inherent characteristic of electromagnetic waves is its frequency. According to Maxwell, varying the electric field gives rise to a magnetic field. An accelerated charge produces a time-varying magnetic field which in turn produces a time-varying electric field. Thus, an electromagnetic wave consists of sinusoidal time-varying electric and magnetic fields, and both the fields are perpendicular to each other.

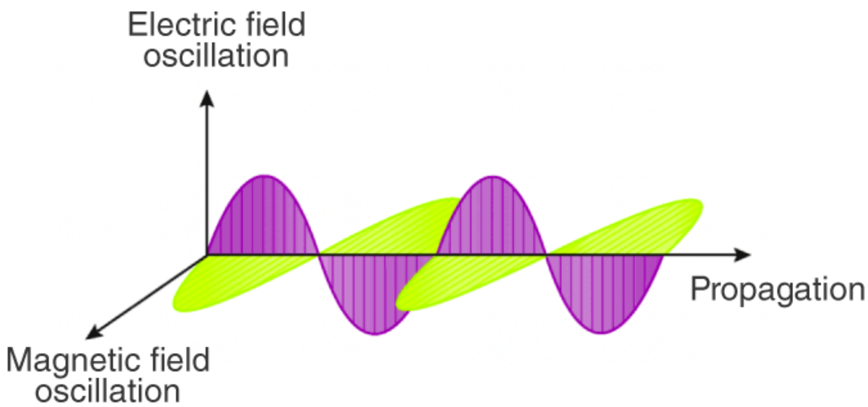


Fig.8.3 Waveform for characteristic of electromagnetic waves

Listed below are some important characteristics and properties of electromagnetic waves.

- Electromagnetic waves are transverse in nature as they propagate by varying the electric and magnetic fields such that the two fields are perpendicular to each other.
- Accelerated charges are responsible to produce electromagnetic waves.
- Electromagnetic waves have constant velocity in vacuum and it is nearly equal to $3\times10^8 ms^{-1}$ which is denoted by $C = 1/\sqrt{\mu_o \epsilon_o}$.
- Electromagnetic wave propagation does not require any material medium to travel.
- The inherent characteristic of an electromagnetic wave is its frequency. Their frequencies remain unchanged but its wavelength changes when the wave travels from one medium to another.
- The refractive index of a material is given by: $n = \sqrt{\mu_r \epsilon_r}$
- Electromagnetic wave follows the principle of superposition.
- The light vector (also known as the electric vector) is the reason for the optical effects due to an electromagnetic wave.
- In an electromagnetic wave, the oscillating electric and magnetic fields are in the same phase and their magnitudes have a constant ratio. The ratio of the amplitudes of electric and magnetic fields is equal to the velocity of the electromagnetic wave. $C = E_0/B_0$
- The energy is carried by the electric and magnetic fields of electromagnetic waves are equal, i.e. the electric energy (u_E) and the magnetic energy (u_M) are equal; $u_E = u_M$.

- There is a vector quantity S, called the Poynting vector which represents the energy transferred by electromagnetic waves per second per unit area.

$$\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B}$$

Characteristics of impedance:

The "opposition" to the wave or wave impedance or impedance of a medium to a wave is caused by characteristics of the medium analogous to the resistance, capacitance and inductance. While resistance is a pretty generic term, applicable to different types of waves, the energy storing characteristics, capacitance and inductance, could be generalized as compliance or stress and inertia or motion.

When a wave is propagated, it energizes the medium and the speed of the propagation is reduced or opposed by the resistance of the medium and by its ability to store energy. So both high capacitance or compliance and high inductance or inertia of the medium act to slow down the wave or, we can say, it takes more time and energy to energize a medium with high capacitance and inductance. This for instance, is reflected in a formula for the wave propagation speed in an ideal transmission line, $V_p = \frac{1}{\sqrt{LC}}$ where both capacitance and inductance contribute symmetrically to oppose or slow down the wave.

The impedance, on the other hand, characterizes the tendency of a medium to oppose the motion component of the wave at a given stress level or, in electrical domain, the tendency to oppose the current or the magnetic field at a given level of voltage or electric field. This is reflected in a formula for the characteristic impedance of an ideal transmission line, $Z_0 = \sqrt{\frac{L}{C}}$. . Here, capacitance and inductance are not contributing symmetrically: high capacitance encourages the current flow, while high inductance impedes it.

For EM wave in space, the formulas for the propagation speed and impedance, $V = \frac{1}{\sqrt{\mu\epsilon}}$ and $Z = \sqrt{\frac{\mu}{\epsilon}}$, have similar meaning and underling mechanisms. The propagation is opposed or slowed down by both greater magnetic permeability (inductance) and electric permittivity (capacitance) of the medium. On the other hand, the impedance (to the motion component of the wave) is increased with the magnetic permeability and decreased with its electrical permittivity.

Characteristics of refractive index:

Refractive index is defined as the speed of light in a medium depends on the properties of the medium. In electromagnetic waves, the speed is dependent on the optical density of the medium. Optical density is the tendency of the atoms in a material to restore the absorbed electromagnetic energy. The more optically dense material is, the slower the speed of light. One such indicator of the optical density of a medium is the refractive index.

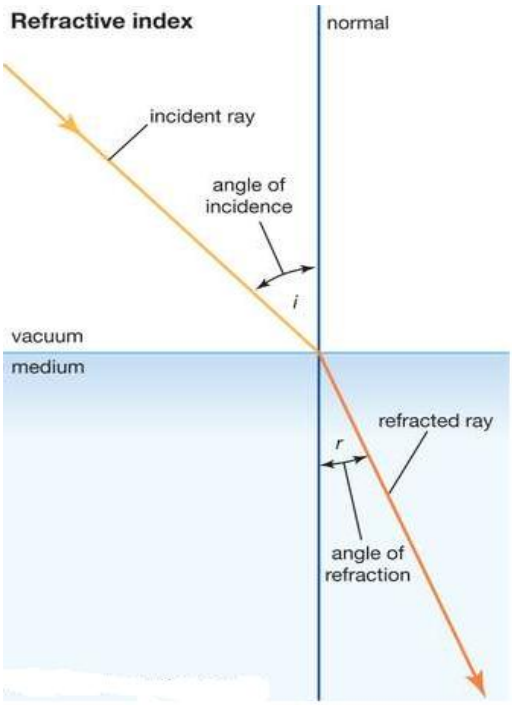


Fig.8.4 Refractive index diagram of a light ray being refracted

Refractive Index Formula:

The refractive index is dimensionless. It is a number that indicates the number of times slower than a light wave would be in the material than it is in a vacuum. The refractive index, represented by symbol n, is the velocity of light in vacuum divided by the velocity of light in a medium. The formula of the refractive index is as follows:

$n = c/v$

Where,

- n is the refractive index
- c is the velocity of light in a vacuum (3×10^8 m/s)
- v is the velocity of light in a substance

The vacuum has a refractive index of 1. The refractive index of other materials can be calculated from the above equation. Higher the refractive index, the higher the optical density and slower is the speed of light. The table below lists the refractive index of different media.

Material	Refractive Index
Air	1.0003
Water	1.333
Diamond	2.417
Ice	1.31
Ethyl Alcohol	1.36

Refractive Index Example:

The refractive index of glass n_g is 1.52 and that of water n_w is 1.33. Since the refractive index of glass is higher than the water, the speed of light in water is faster than the speed of light through glass. If the refractive index of a medium is greater than that of another, then the first medium is said to be optically denser. Most of the substances we know have a positive refractive index having value more than zero. The material will have a negative refractive index when it has negative permittivity and permeability.

The refractive index provides a measure of the relative speed of light in different media. Knowing the refractive indices of different media helps the student to identify the direction in which way the light would bend while passing from one medium to another.

Why is high refractive index important for optical polymers?

Optical polymers with high refractive index allow light rays to bend more within the material, which helps in lowering the profile of the lens. Also, as the refractive index increases, the thickness of the lens decreases, resulting in less weight.

What is refractive index gradient?

- The refractive index gradient is defined as the rate of change of refractive index with respect to distance in the material. Distance refers to the slope of the refractive index profile at any point.
- The refractive index gradient is expressed in terms of reciprocal of a unit of distance.
- An example of a refractive index gradient are the rate of change of refractive index at any point with respect to distance.
- The refractive index gradient is a vector point function.

How does the refractive index vary with wavelength?

According to the definition of the refractive index, the speed of light is the product of frequency and wavelength. The frequency of the light wave remains unchanged irrespective of the medium. Whereas the wavelength of the light wave changes based on refraction. Hence, the refractive index varies with wavelength.

8.8 Skin depth and its importance:

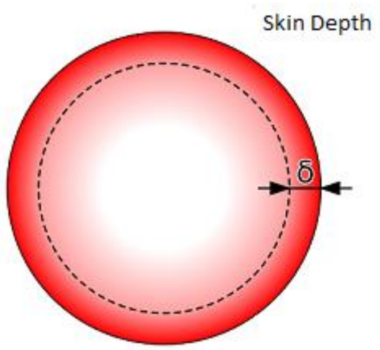


Fig.8.5 Center of the conductor for Skin depth

When an AC current is applied to a conductor, the current concentrates near the surface of the conductor and its strength decreases as you go towards the center of the conductor. The depth till which current flows in a conductor is called as Skin Depth. The figure shows the cross section of a cylindrical conductor, the intensity of the red color represents the intensity of the current in a cylindrical conductor.

The Skin Depth is dependent on the frequency of the current/signal and the resistivity of the material. It inversely proportional to the frequency and directly proportional to the resistivity.

The Skin Depth can be calculated using the following formula:

$$Skin\ Depth = \delta = \sqrt{\frac{\rho}{\pi f \mu}} = \sqrt{\frac{\rho}{\pi f \mu_r \mu_o}}$$

Where,

ρ = Resistivity of the Material

f = Frequency

μ_r = Relative Permeability (usually 1)

μ_o = Permeability Constant = $4\pi \times 10^{-7}$

Everything RF has created a calculator which enables you to easily calculate Skin Depth for a particular material at a particular frequency.

Importance of skin depth:

1) It conveys you that resistive element of line increases, in turn voltage drop, with decrease in depth.

2) As skin depth reduces, it will have less power handling capacity.

3) Skin depth conveys you how much material (inner) is not required in building transmission line. This will save material (in turn cost) and it reduces the weight (easy handling).

4) Decrease in skin depth creates power concentration increase on surface and thus nearby area you need more space to avoid discharge and sparking as well as give rise to capacitive effect.

5) It will increases loss tangent as leakage will increase

SAQ.3

a) What do you mean by Characteristics of electromagnetic waves.

b) Define the impedance and refractive index.

c) Define the Skin depth and its importance.

d) The speed of light in an unknown medium is 1.52×10^8 m/s. Calculate the refractive index of the medium.

e) Find the refractive index of the medium whose critical angle is 25°.

Examples:

Q.1 (a) Show that Maxwell's equations are consistent with the conservation of electric charge.
(b) Show that the power P injected into a circuit by an electric field is given by $\int \mathbf{j}\cdot\mathbf{E}\, dV$. Verify that in steady state this reproduces the Ohmic heat loss in a "thin wire" approximation.

Solution:

- Concepts:
  Maxwell's equations

- Reasoning:
  (a) If we have a conserved quantity in a volume V, then the rate at which it flows out of the volume must equal the rate it decreases inside the volume.
  For charge this is expressed as $\int_{closed\_A} \mathbf{j}\cdot\mathbf{n}dA = -(\partial/\partial t)\int_V \rho dV$ for any volume V.
  Gauss' theorem then yields $\int_V \nabla\cdot\mathbf{j}dV = -(\partial/\partial t)\int_V \rho dV$ for any volume

V.

Therefore $\nabla\cdot j = -\partial\rho/\partial t$.

- Details of the calculation:

  Maxwell's equations relate the sources and the fields.

  $\nabla\times B = \mu_0 j + (1/c^2)\partial E/\partial t$ (SI units), $\nabla\cdot(\nabla\times B) = 0 \dashrightarrow \mu_0\nabla\cdot j + (1/c^2)(\partial/\partial t)\nabla\cdot E = 0$.

  $\nabla\cdot E = \rho/\varepsilon_0$, $\mu_0\nabla\cdot j + (1/(\varepsilon_0 c^2))(\partial\rho/\partial t) = 0$.

  $\mu_0\varepsilon_0 = 1/c^2$, $\nabla\cdot j + (\partial\rho/\partial t) = 0$.

  Maxwell's equations are consistent with the conservation of electric charge.

  (b) For a single charge, the rate of doing work by external fields B and E is $qv\cdot E$, in which v is the velocity of the charge. Let $\rho dV = dq$ be the amount of charge in a volume element dV. $dW/dt = dq\, v\cdot E = \rho v\cdot E\, dV = j\cdot E\, dV$ = rate at which work is done by the field on the charges in dV. The power P injected into a circuit is obtained by integration dW/dt over the volume of the circuit, $P = \int j\cdot E\, dV$. If we consider a steady current to a thin wire, then for an element of length dl and cross section dA of the wire, $dV = dl\cdot dA$. In this approximation j is parallel to dl and E does not vary appreciably over the cross section of the wire. Hence $\int j\cdot E\, dV = \int j\cdot dA \int E\cdot dl = IV = I^2R$ which is the Ohmic heat loss.

Q.2 Do the fields $E = i\, E_0\cos(\omega t - kx)$, $B = 0$ satisfy Maxwell's equations? If a special condition for ρ and j is needed, what is it?

Solution:

- Concepts:

  Maxwell's equations

  $\nabla\cdot E = \rho/\varepsilon_0$, $\nabla\times E = -\partial B/\partial t$, $\nabla\cdot B = 0$, $\nabla\times B = \mu_0 j + (1/c^2)\partial E/\partial t$

- Reasoning

  We are supposed to check if $E = i\, E_0 \cos(\omega t - kx)$ is consistent with Maxwell's equations.

- Details of the calculation:

  $\nabla\cdot E = \partial[E_0\cos(\omega t - kx)]/\partial x = kE_0\sin(\omega t - kx) \neq 0$, $\rho \neq 0$.

  $\rho = k\varepsilon_0 E_0\sin(\omega t - kx)$.

  Since $B = 0$, $\nabla\cdot B = 0$.

  $\nabla\times E = 0$, $\partial B/\partial t = 0$, $B = 0$ is a possible solution.

  The equation of continuity follows from Maxwell's equations.

  $\nabla\cdot j = -(\partial\rho/\partial t) = -\omega k\varepsilon_0 E_0\cos(\omega t - kx)$.

  Since $B = 0$, $\nabla\times B = 0$. This requires $j = (-1/\mu_0 c^2)\partial E/\partial t = (\omega/\mu_0 c^2)E_0\sin(\omega t - kx)i$.

  $\nabla\cdot j = \partial[(\omega/\mu_0 c^2)E_0\sin(\omega t - kx))]/\partial x = (-k\omega/\mu_0 c^2)E_0\cos(\omega t - kx) = -\omega k\varepsilon_0 E_0\cos(\omega t - kx)$.

  The fields $E = i\, E_0\cos(\omega t - kx)$, $B = 0$ satisfy Maxwell's equations with $\rho = k\varepsilon_0 E_0\sin(\omega t - kx)$ and $j = (\omega/\mu_0 c^2)E_0\sin(\omega t - kx)i$.

Q.3 (a) Write down Maxwell's equations in vacuum for a charge density and current density free medium ($\rho = 0$ and $j = 0$).

(b) Show that the electric field and the magnetic field satisfy a wave equation.

(c) Write down plane-wave solutions of the wave equations for the electric field and magnetic field. How are they related to each other? Find the group velocity and phase velocity of the electromagnetic waves.

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  Maxwell's equations in free space yield the wave equation for both E and B.

- Details of the calculation:

  (a) $\nabla\cdot E = 0$, $\nabla\times E = -\partial B/\partial t$, $\nabla\cdot B = 0$, $\nabla\times B = (1/c^2)\partial E/\partial t$.

  (b) $\nabla\times(\nabla\times E) = \nabla(\nabla\cdot E) - \nabla^2 E = -(\partial/\partial t)(\nabla\times B) = -\mu_0\varepsilon_0\partial^2 E/\partial t^2$.

  $\nabla^2 E = \mu_0\varepsilon_0\partial^2 E/\partial t^2$.

  $\nabla\times(\nabla\times B) = \nabla(\nabla\cdot B) - \nabla^2 B = \mu_0\varepsilon_0(\partial/\partial t)(\nabla\times E) = -\mu_0\varepsilon_0\partial^2 B/\partial t^2$.

  $\nabla^2 B = \mu_0\varepsilon_0\partial^2 B/\partial t^2$.

  (c) Plane wave solutions $E = E((k/k)\cdot r - vt)$, $B = B((k/k)\cdot r - vt)$ exist.

  ($v = (\mu_0\varepsilon_0)^{-½} = c$) All plane wave solutions are linear superpositions of harmonic waves of the form $\sin(k\cdot r - \omega t)$ and $\cos(k\cdot r - \omega t)$, with $\omega/k = c$.

  The phase velocity is $\omega/k = c$, the group velocity is $d\omega/dk = c$. There is no dispersion for EM waves in free space.

Q.4 Consider an electromagnetic traveling wave with electric and magnetic fields given by

$E_x = E_0\cos(kz - \omega t) + \varphi)$, and $B_y = B_0\cos(kz - \omega t) + \varphi)$.

Using Maxwell's equations show that B_0 can be written in terms of E_0.

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  Maxwell's equations in free space yield the wave equation for both E and B. They can also be used to show that $E \perp B$, $E \perp k$, $B \perp k$, $B = (\mu_0\varepsilon_0)^{1/2}(k/k)\times E$.

- Details of the calculation:

  From Maxwell's equations:

  $\nabla\times E = -\partial B/\partial t$.

  $\nabla\times E = \partial E_x/\partial z\, j = -kE_0\sin(kz - \omega t + \varphi)\, j$, $-\partial B/\partial t = -\omega B_0\sin(kz - \omega t + \varphi)\, j$.

  Therefore $kE = \omega B_0$, $B_0 = (k/\omega)E_0$.

  In free space $k/\omega = 1/c$.

Q.5 Use Maxwell's equations to find the magnetic field of an EM wave in vacuum for which the electric field is given by $E = (E_{0x}i + E_{0y}j)\sin(\omega t - kz + \varphi)$.

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  Maxwell's equation in vacuum are

  $\nabla\cdot E = 0$, $\nabla\times E = -\partial B/\partial t$, $\nabla\cdot B = 0$, $\nabla\times B = (1/c^2)\partial E/\partial t$.

  Using these equations we can derive the homogeneous wave equation for E and B and show that

$E \perp B$, $E \perp k$, $B \perp k$, $B = (1/c^2)\partial(k/k)\times E$.

- Details of the calculation:

  For the given plane wave:

  $(\nabla\times E)_x = \partial E_z/\partial y - \partial E_y/\partial z = k\ E_{0y}\cos(\omega t - kz + \varphi) = -\partial B_x/\partial t$

  Therefore $B_x = (k/\omega)E_{0y}\sin(\omega t - kz + \varphi) = -(E_{0y}/c)\sin(\omega t - kz + \varphi)$

  $(\nabla\times E)_y = \partial E_x/\partial z - \partial E_z/\partial x = -k\ E_{0x}\cos(\omega t - kz + \varphi) = -\partial B_y/\partial t$

  Therefore $B_y = (k/\omega)E_{0x}\sin(\omega t - kz + \varphi) = (E_{0x}/c)\sin(\omega t - kz + \varphi)$

  $(\nabla\times E)_z = \partial E_y/\partial x - \partial E_x/\partial y = -\partial B_z/\partial t = 0$, therefore $B_z = 0$.

  (We are not interested in constant fields.)

  $B = (1/c)(E_{0x}j - E_{0y}i)\sin(\omega t - kz + \varphi)$.

Q.6 Starting with Maxwell's equations:

(a) Derive the wave equations for a light wave in vacuum. Write out solutions for these equations for E and B.

(b) Show that the electric and magnetic fields are in phase, perpendicular to each other and perpendicular to the direction of motion.

(c) Determine the relative magnitude of the E and B fields.

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  In regions where ρ and j are zero Maxwell's equations lead to the homogeneous wave equation for E and B. All solutions can be viewed as linear superpositions of sinusoidal plane wave solutions.

Inserting these solutions into Maxwell's equations we derive (b) and (c).

- Details of the calculation:

  (a) Maxwells equations in SI units are

  $\nabla \cdot E = \rho/\varepsilon_0$, $\nabla \times E = -\partial B/\partial t$, $\nabla \cdot B = 0$, $\nabla \times B = \mu_0 j + (1/c^2)\partial E/\partial t$

  Assume ρ and j are zero in the medium.

  $\nabla \times(\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = -(\partial/\partial t)(\nabla \times B) = -\mu_0\varepsilon_0\partial^2 E/\partial t^2$.

  $\nabla^2 E = \mu_0\varepsilon_0\partial^2 E/\partial t^2$.

  $\nabla \times(\nabla \times B) = \nabla(\nabla \cdot B) - \nabla^2 B = \mu_0\varepsilon_0(\partial/\partial t)(\nabla \times E) = -\mu_0\varepsilon_0\partial^2 B/\partial t^2$.

  $\nabla^2 B = \mu_0\varepsilon_0\partial^2 B/\partial t^2$.

  $\mu_0\varepsilon_0 = 1/c^2$.

  (b) Each Cartesian component of E and B satisfies the 3-dimensional, homogeneous wave equation.

  Sinusoidal plane wave solutions $E(r,t) = E_0 \exp(i(k_i \cdot r - \omega t))$, $B(r,t) = B_0 \exp(i(k_i \cdot r - \omega t))$ exist.

  $\nabla \cdot E = \partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z = ik\cdot E = 0$

  $\nabla \cdot E = 0$ requires that $E\cdot k = 0$ for radiation fields, i.e. that E is perpendicular to k.

  Similarly, $\nabla \cdot B = 0$ requires that $B\cdot k = 0$ for radiation fields.

  $\nabla \times E = -\partial B/\partial t$ requires that $ik\times E = i\omega B$, i.e. B is perpendicular to E and k.

  (c) $B = (k/\omega)E = E/c$.

Q.7 A time-dependent, vacuum electromagnetic field in three dimensions (x, y, z) at time, t = 0, is shown in the figure.

It has the following form:

$E(r, t = 0) = i\ E_0\exp(-(z/a)^2)$, $B(r, t = 0) = 0$.

(a) Evaluate $\partial E/\partial t$ at $t = 0$.

(b) Evaluate $\partial B/\partial t$ at $t = 0$.

(c) Evaluate $\partial^2 E/\partial t^2$ at $t = 0$ and show that E satisfies the wave equation at $t = 0$.

(d) What are the values of the fields $E(r, t)$ and $B(r, t)$ for a general time t, satisfying the inequality $ct/a >> 1$.

(e) Sketch in a single diagram the fields found in (d).

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  Maxwell's equations relate partial derivatives of E and B with respect to space and time and yield the wave equation

- Details of the calculation:

  $\nabla \cdot E = 0$, $\nabla \cdot B = 0$,

  $\nabla \times E = -\partial B/\partial t$, $\nabla \times B = (1/c^2)\partial E/\partial t$

  (a) Here $E(r, t = 0) = i\ E_0\exp(-(z/a)^2)$, $B(r, t = 0) = 0$.

At $t = 0$, $\nabla \times \mathbf{B} = (1/c^2)\partial \mathbf{E}/\partial t = 0$, $\partial \mathbf{E}/\partial t = 0$.

(b) $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t = \mathbf{j}\ (\partial E_x/\partial z)$, $\partial \mathbf{B}/\partial t = \mathbf{j}\ (2z/a^2)\ E_0\exp(-(z/a)^2)$.

(c) $\partial^2\mathbf{E}/\partial t^2 = c^2(\partial/\partial t)(\nabla \times \mathbf{B}) = c^2\nabla \times \partial \mathbf{B}/\partial t$.

At $t = 0$, $\partial^2\mathbf{E}/\partial t^2 = c^2\nabla \times(\mathbf{j}\ (2z/a^2)\ E_0\exp(-(z/a)^2))$.

$\partial^2\mathbf{E}/\partial t^2 = -c^2\mathbf{i}\ (\partial/\partial z)[(2z/a^2)\ E_0\exp(-(z/a)^2))] = c^2\mathbf{i}\ (\partial^2/\partial z^2)[E_0\exp(-(z/a)^2))] = c^2\nabla^2\mathbf{E}$.

$\nabla^2\mathbf{E} -(1/c2)\partial^2\mathbf{E}/\partial t^2 = 0$.

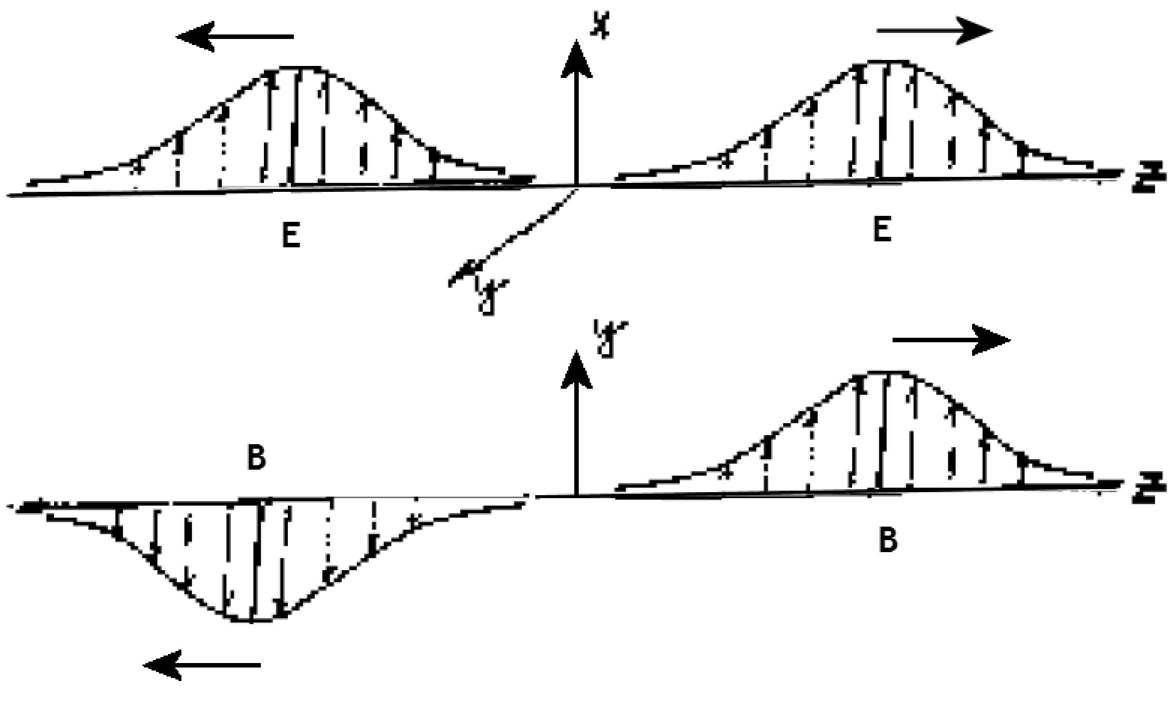
E satisfies the wave equation at $t = 0$.

(d) The given E can only satisfy the wave equation if it is the sum of a function of z - ct and another function of c + zt. The same holds for B.

$\mathbf{E}(\mathbf{r}, t) = \mathbf{i}\ [(E_0/2)(\exp(-((z- ct)/a)^2) + \exp(-((z + ct)/a)^2)]$,

$\mathbf{B}(\mathbf{r}, t) = \mathbf{j}\ [(E_0/(2c))(\exp(-((z - ct)/a)^2) - \exp(-((z + ct)/a)^2)]$.

(e) We have two Gaussian-shaped electromagnetic pulses one traveling into the positive and one in the negative z-direction.



Q.8 In unbounded free space the electric and magnetic fields satisfy

$\nabla \cdot E = \nabla \cdot B = 0$, $\nabla \times E = -\partial B/\partial t$, $\nabla \times B = (1/c^2)\partial E/\partial t$, and therefore the homogeneous wave equation.

Assume that at t = 0 E(r, t = 0) = j f(x) and B(r, t = 0) = 0.

Find E(r,t) and B(r, t) for t > 0.

Solution:

- Concepts:

  Maxwell's equations

- Reasoning:

  Maxwell's equations relate partial derivatives of E and B with respect to space and time and yield the wave equation

- Details of the calculation:

  E and B must satisfy the homogeneous wave equation, $\nabla^2 E = (1/c^2)\partial^2 E/\partial t^2$.

  We have E ⊥ B, E, B ⊥ direction of propagation, E×B pointing in the direction of propagation.

  E is perpendicular to the xz-plane. $\nabla \times E = k\, \partial E_y/\partial x = k\, \partial f(x)/\partial x$.

  B points in the ±z-direction. The EM wave therefore propagates along the ±x-direction.

  The given E can only satisfy the wave equation if it is the sum of a function of x - ct and another function of x + ct. The same holds for B.

  E(r, t) = j ½(f(x - ct) + f(x + ct)),

  B(r, t) = k ½(f(x + ct) - f(x + ct)).

We have two electromagnetic pulses, one traveling into the positive and one in the negative x-direction.

Q.9 What is the refractive index of the medium in which the speed of light is 1.5×10^8m/s?

Solution: The refractive index of the medium can be calculated using the formula:

$n = c/v$

Substituting the values in the equation, we get

$n = 3 \times 10^8$ m/s/1.5×10^8 m/s = 2

The refractive index of the medium is 2.

Q.10 The speed of light in an unknown medium is 1.76×10^8 m/s. Calculate the refractive index of the medium.

Solution: The refractive index of a medium is calculated by the formula:

$n = c/v$

where c is the speed of light in vacuum

v is the speed of light in the medium

Substituting the values in the above equation, we get

$n = (3 \times 10^8)/(1.76 \times 10^8) = 1.7045$

Q.11 An optical fibre made up the glass with refractive index $n_1 = 1.5$ which is surrounded by another glass of glass with refractive index n_2. Find the refractive index n_2 of the cladding such that the critical angle between the two cladding is 80°.

Solution:

Critical angle, $\theta = 80°$

Refractive index, $n_1 = 1.5$

Refractive index $n_2 = ?$

Using the below formula, we can calculate n_2:

$$sin\theta = \frac{n_2}{n_1}$$

$$sin\ 80^\circ = \frac{n_2}{1.5}\ n_2 = 1.5 \times sin80^\circ\ n_2 = 1.48$$

Q.12 Find the refractive index of the medium whose critical angle is 40°.

Solution:

Critical angle, $\theta = 40°$

Refractive index of the medium, $\mu = ?$

$$\mu = \frac{1}{sin\ \theta}\ \mu = \frac{1}{sin\ 40^\circ}\ \mu = \frac{1}{0.65}$$

$\mu = 1.6$

Summary:

1) Maxwell was the first person to calculate the speed of propagation of electromagnetic waves which was same as the speed of light and came to the conclusion that EM waves and visible light are similar.

2) Maxwell’s equations integral form explains how the electric charges and electric currents produce magnetic and electric fields.
3) Maxwell’s equations are the basic equations of electromagnetism which are a collection of Gauss’s law for electricity, Gauss’s law for magnetism, Faraday’s law of electromagnetic induction and Ampere’s law for currents in conductors.
4) To obtain the electromagnetic wave equation in a vacuum using the modern method, we begin with the modern 'Heaviside' form of Maxwell's equations.
5) Consider a region of space filled by “simple dielectrics” materials, that is: Linear, Isotropic, Homogeneous, Source free and Non-conducting.
6) If the mean position of the positively charged constituents of a given molecule, of net charge (+q), develops a vector displacement (d) with respect to the mean position of the negatively charged constituents, of net charge -q , in response to a wave electric field E , then the associated electric dipole moment is P = q d , where d is generally parallel to E (ibid.). Furthermore, if there are N such molecules per unit volume then the electric dipole moment per unit volume is written P = N q d. In a linear, isotropic, dielectric medium (ibid).
7) The inherent characteristic of electromagnetic waves is its frequency. According to Maxwell, varying the electric field gives rise to a magnetic field. An accelerated charge produces a time-varying magnetic field which in turn produces a time-varying electric field.

8) The "opposition" to the wave or wave impedance or impedance of a medium to a wave is caused by characteristics of the medium analogous to the resistance, capacitance and inductance.
9) Refractive index is defined as the speed of light in a medium depends on the properties of the medium. In electromagnetic waves, the speed is dependent on the optical density of the medium.
10) When an AC current is applied to a conductor, the current concentrates near the surface of the conductor and its strength decreases as you go towards the center of the conductor. The depth till which current flows in a conductor is called as Skin Depth.

Terminal Questions:

1) Explain the Four Maxwell's equations with statement and physical significance.

2) Explain the Maxwell's equations for general plane wave solution in source free space.

3) Explain the Maxwell's equations for general plane wave solution in simple dielectrics.

4) What do you mean by Differential equation and velocity for electromagnetic waves in source free space?

5) Define the Differential equation and velocity for electromagnetic waves in dielectric medium.

6) Write short notes on: (i) Characteristics of electromagnetic waves,(ii) Impedance, (iii) Refractive index.

7) Explain the Skin depth and its importance.

8) The speed of light in an unknown medium is 1.87×10^8 m/s. Calculate the refractive index of the medium.

9) Find the refractive index of the medium whose critical angle is 60°.

10) An optical fibre made up the glass with refractive index $n_1 = 2.5$ which is surrounded by another glass of glass with refractive index n_2. Find the refractive index n_2 of the cladding such that the critical angle between the two cladding is 60°.

Unit 09- Energy and momentum of an electromagnetic wave

Structure

9.1 Introduction:

A plane wave is a constant-frequency wave, whose value at any moment is constant over any plain that is perpendicular to fixed directs of spouse. The speed of any periodic wave is the product of its wavelength and frequency.

$v = \lambda f$. The speed of any electromagnetic waves in free space is the speed of light $c = 3*10^8$ m/s. Electromagnetic waves can have any wavelength λ or frequency f as long as $\lambda f = c$.

Electromagnetic waves are ubiquitous in nature (i.e., light) and used in modern technology—AM and FM radio, cordless and cellular phones, garage door openers, wireless networks, radar, microwave ovens, etc. These and many more such devices use electromagnetic waves to transmit data and signals.

Every form of electromagnetic radiation, including visible light, oscillates in a periodic fashion with peaks and valleys, and displaying a characteristic amplitude, wavelength, and frequency that defines the direction, energy, and intensity of the radiation.

Electromagnetic waves are transverse waves, similar to water waves in the ocean or the waves seen on a guitar string. This is as opposed to the compression waves of sound. As you learned in Wave Motion, all waves have amplitude, wavelength, velocity and frequency.

The Poynting theorem should read rate of change of energy in the fields = negative of work done by the fields on the charged particles minus the Poynting vector term. The compensating change in momentum and energy would occur in the bodies holding the electric and magnetic fields. The vector obtained in the direction of a right-hand screw from the cross-product (vector product) of the electric field vector rotated into the magnetic field vector of an electromagnetic wave.

The energy stored in a magnetic field is equal to the work needed to produce a current through the inductor. Energy is stored in a magnetic field.

Momentum is a vector quantity; i.e., it has both magnitude and direction. Isaac Newton's second law of motion states that the time rate of change of momentum is equal to the force acting on the particle.

The Maxwell stress tensor (named after James Clerk Maxwell) is a symmetric second-order tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum. ... The latter describes the density and flux of energy and momentum in space time.

9.2 Objectives:

After studying this unit you should be able to

- Explain and identify Differential equation of plane electromagnetic waves in conducting media and its solution.
- Study and identify Behavior and property of electromagnetic waves for good dielectric and good conductors.
- Explain and identify Poynting theorem (statement and derivation).
- Study and identify Expression for electromagnetic energy density.
- Explain and identify Momentum density vector and its importance
- Study and identify Maxwell’s stress tensor (statement and derivation).

9.3 Differential equation of plane electromagnetic waves in conducting media and its solution:

In a “simple” dielectric material, we derived the wave equations:

$$\nabla^2\vec{E} - \mu\varepsilon\ddot{\vec{E}} = 0 \qquad (1)$$
$$\nabla^2\vec{B} - \mu\varepsilon\ddot{\vec{B}} = 0 \qquad (2)$$

To derive these equations, we used Maxwell’s equations with the assumptions that the charge density ρ and current density J were zero, and that the permeability μ and permittivity ε were constants. We found that the above equations had plane- wave solutions, with phase velocity:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \qquad (3)$$

Maxwell’s equations imposed additional of the electric and magnetic fields. How are the equations (and their solutions) modified for the case of electrically conducting media? We shall restrict our analysis to the case ohmic conductors, which are defined by

$$\vec{J} = \sigma\vec{E} \qquad (4)$$

where σ is a constant, the conducting of the material.
All we need to do is substitute from equation (4) into Maxwell’s equations, then proceed as for the case of a dielectric.
In our “simple” conductor, Mexwell’s equation take the from:

$$\nabla \cdot \vec{E} = 0 \qquad (5)$$
$$\nabla \cdot \vec{B} = 0 \qquad (6)$$
$$\nabla \times \vec{E} = -\dot{\vec{B}} \qquad (7)$$
$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\vec{J} \qquad (8)$$

Where $\vec{J}$ is the current density. Assuming an ohmic conductor,
We can write:

$$\vec{J} = \sigma\vec{E} \qquad (9)$$

So equation (8) becomes:

$$\nabla \times \vec{B} = \mu\varepsilon\dot{\vec{E}} + \mu\sigma\vec{E} \qquad (10)$$

Taking the curl of equation (7) and making appropriate substitutions as before, we arrive at the wave equation:

$$\nabla^2 \vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \qquad (11)$$

The wave equation for the electric field in a conducting material is (11):

$$\nabla^2 \vec{E} - \mu\sigma\dot{\vec{E}} - \mu\varepsilon\ddot{\vec{E}} = 0 \qquad (12)$$

Let us try a solution of the same form as before:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \qquad (13)$$

Remember that to find the physical field; we have to take the real part. Substituting (13) into the wave equation (11) gives the dispersion relation:

$$-\vec{k}^2 - j\omega\mu\sigma + \omega^2\mu\varepsilon = 0 \qquad (14)$$

Compared to the dispersion relation for a dielectric, the new feature is the presence of an imaginary term in σ. This means the relationship between the wave vector $\vec{k}$ and the frequency ω is a little more complicated than before.

From the dispersion relation (14), we can expect the wave vector $\vec{k}$ to have real and imaginary parts. Let us write:

$$\vec{k} = \vec{\alpha} - j\vec{\beta} \qquad (15)$$

For parallel real vectors $\vec{\alpha}$ and $\vec{\beta}$. .

Substituting (15) into the dispersion relation (14) and taking real and imaginary parts, we find:

$$\alpha = \omega\sqrt{\mu\varepsilon}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{\sigma^2}{\omega^2\varepsilon^2}}\right]^{1/2} \qquad (16)$$

and

$$\beta = \frac{\omega\mu\sigma}{2\alpha} \qquad (17)$$

Equations (16) and (17) give the real and imaginary parts of the vector $\vec{k}$ in the frequency ω, and the material properties μ, ε and σ.

Using equation (15) the solution (13) to the wave equation in a conducting material can be written:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{\alpha}\cdot\vec{r})} e^{-\vec{\beta}\cdot\vec{r}} \qquad (18)$$

The first exponential factor, $e^{j(\omega t - \vec{\alpha}\cdot\vec{r})}$ given the usual plane-wave variation of the field with position $\vec{r}$ and time t; note the conductivity of the material affects the wavelength for a given frequency.

The second exponential factor, $e^{-\vec{\beta}\cdot\vec{r}}$ given an exponential decay in the amplitude of the wave.

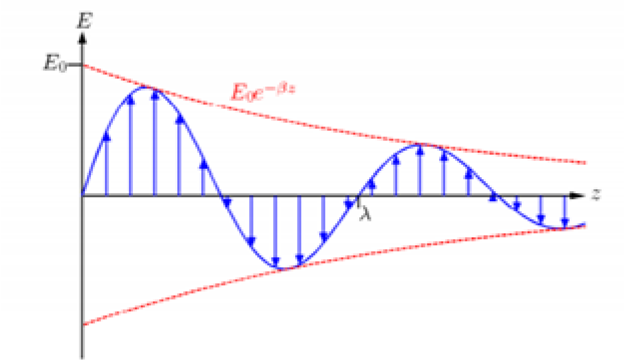


Fig.9.1 Plane monochromatic wave in a conducting Material

In a “simple” non-conducting material there is no exponential decay of the amplitude: electromagnetic waves can travel for ever, without any loss of energy.

If the wave enters an electrical conductor, however, we can expect very different behavior. The electrical field in the wave will cause currents to flow in the conductor. When a current flows in a conductor (assuming it is not a superconductor) there will be some energy changed into heat. This energy must come from the wave. Therefore, we expect the wave gradually to decay.

The varying electric field must have a magnetic field associated with it. Presumably, the magnetic field has the same wave vector and frequency as the electric field: this is the only way we can satisfy Maxwell's equations for all positions and times.

Therefore, we try a solution of the from:

$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \tag{19}$$

Now we use Maxwell's equation (7):

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{20}$$

which gives:

$$\vec{k} \times \vec{E}_0 = \omega \vec{B}_0 \tag{21}$$

or:

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{22}$$

The magnetic field in a wave in a conducting material is related to the electric field by (22)

$$\vec{B}_0 = \frac{\vec{k}}{\omega} \times \vec{E}_0 \tag{23}$$

As in a non-conducting material, the electric and magnetic fields are perpendicular to the direction of motion (the wave is a transverse wave) and are perpendicular to each other.

But there is a new feature, because the wave vector is complex.

In a non-conducting material, the electrical and magnetic fields were in phase: the expressions for the fields both had the same phase angle ϕ_0. In complex notation, the complex phase angles of field amplitudes $\vec{E}_0$ and $\vec{B}_0$ were the same.

In a conductor, the complex phases of $\vec{k}$ gives a phase difference between the electric and magnetic fields.

In a conducting material, there is a difference between the phase angles of $\vec{E}_0$ and $\vec{B}_0$, given by the phase angle ϕ of $\vec{k}$.

This is:

$$\tan\phi = \frac{\beta}{\alpha} \qquad (24)$$

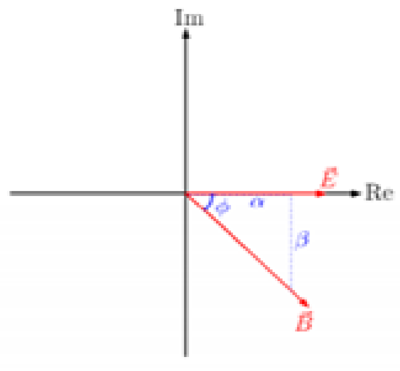


Fig.9.2 Difference between the phase angles

Plane monochromatic wave in a poor conductor:

Let us consider the special case of a good insulator. In this case:

$$\sigma \ll \omega\varepsilon \qquad (25)$$

From equation (16), we then have:

$$\alpha \approx \omega\sqrt{\mu\varepsilon} \qquad (26)$$

and from equation (17) we have:

$$\beta \approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} = \frac{\alpha}{2}\frac{\sigma}{\omega\varepsilon} \qquad (27)$$

It follows that $\beta << \alpha$. We recover the same situation as in the case of a non-conducting material. The decay of the wave is very slow (in terms of the number of wavelengths); the magnetic and electric components of the wave are approximately in phase ($\phi \approx 0$), and are related by:

$$B_0 \approx \frac{\alpha}{\omega}E_0 \approx \frac{E_0}{v_p} \qquad (28)$$

where the phase velocity v_p is, as before, given by $v_p = 1/\sqrt{\mu\varepsilon}$.

9.4 Behavior and property of electromagnetic waves for good dielectric and good conductors:

(i) Behavior and property of Plane monochromatic electromagnetic wave in a good conductor:

Let us consider the special case of a very good conductor. In this case:

$$\sigma \gg \omega\varepsilon \qquad (29)$$

From equation (16), we then have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} \qquad (30)$$

And from equation (17) we have:

$$\beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \approx \alpha \qquad (31)$$

In the case of a very good conductor, the real and imaginary parts of the wave vector $\vec{k}$ become equal. This means that the decay of the wave is very fast in terms of the number of wavelengths. Note that the vectors $\vec{\alpha}$ and $\vec{\beta}$ have the same units as $\vec{k}$, i.e. meters^{-1}.

Phase velocity in a good conductor:

The electric field in the wave varies as (18):

$$\vec{E}(\vec{r},t) = \overrightarrow{E_0} e^j (wt - \vec{\alpha}.\vec{r})_e - \vec{\beta}.\vec{r}$$

…………………….. (32)

The phase velocity is the velocity of a point that stays in phase with the wave. Consider a wave moving in the +z direction:

$$\vec{E}(\vec{r},t) = \overrightarrow{E_0} e^j (wt - \alpha z)_e - \beta_2$$

…………………….. (33)

For a point staying at a fixed phase, we must have:

$$\omega t - \alpha z(t) = \text{constant} \qquad (34)$$

So the phase velocity is given by:

$$v_p = \frac{dz}{dt} = \frac{\omega}{\alpha}$$

…………………….. (35)

But note that in a good conductor, α is itself a function of ω.

For a poor conductor (σ << ωε), we have:

$$\alpha \approx \omega\sqrt{\mu\epsilon}$$

…………………….. (36)

So the phase velocity in a poor conductor is:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}} \qquad (37)$$

If μ and ε are constants (i.e. are independent of ω) then the phase velocity is independent of the frequency: there is no dispersion.

However, in a good conductor (σ >> ωε), we have:

$$\alpha \approx \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\mu\varepsilon}\sqrt{\frac{\omega\sigma}{2\varepsilon}} \qquad (38)$$

Then the phase velocity is given by:

$$v_p = \frac{\omega}{\alpha} \approx \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \qquad (39)$$

The phase velocity depends on the frequency: there is dispersion!

The presence of dispersion means that the group velocity v_g (the velocity of a wave pulse) can differ from the phase velocity v_p (the velocity of a point staying at a fixed phase of the wave).

To understand what this means, consider the superposition of two waves with equal amplitudes, both moving in the +z direction, and with similar wave numbers:

$$E_x = E_0 \cos\left(\omega_+ t - [k_0 + \Delta k]\, z\right) + E_0 \cos\left(\omega_- t - [k_0 - \Delta k]\, z\right) \quad (40)$$

Using a trigonometric identity:

$$\cos A + \cos B \equiv 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \quad (41)$$

The electric field can be written:

$$E_x = 2E_0 \cos\left(\omega_0 t - k_0 z\right)\cos\left(\Delta\omega\, t - \Delta k\, z\right) \quad (42)$$

where :

$$\omega_0 = \frac{1}{2}\left(\omega_+ + \omega_-\right) \qquad \Delta\omega = \omega_+ - \omega_- \quad (43)$$

We have written the total electric field in our superposed waves as (42):

$$E_x = 2E_0 \cos\left(\omega_0 t - k_0 z\right)\cos\left(\Delta\omega\, t - \Delta k\, z\right) \quad (44)$$

Assuming that $\Delta k << k_0$, the first trigonomatric factor represents a wave of (short) wavelength $2\pi/ k_0$ and phase velocity:

$$v_p = \frac{\omega_0}{k_0} \quad (45)$$

While the second trigonometric factor represents a modulation of (long) wavelength 2π/ Δk, which travels with velocity:

$$v_g = \frac{\Delta\omega}{\Delta k} \qquad (46)$$

v_g is called the group velocity. Since Δω represents the change in frequency that corresponds to a change Δk in wave number, we can write:

$$v_g = \frac{d\omega}{dk} \qquad (47)$$

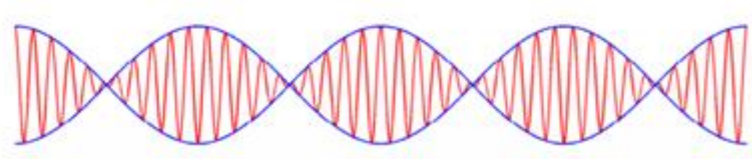


Fig.9.3 Waveform representation of phase velocity with group velocity

The red wave moves with the phase velocity v_p; the modulation (represented by the blue line) moves with group velocity v_g.

Since the energy in a wave depends on the local amplitude of the wave, the energy in the wave is carried at the group velocity v_g.

If ther is no dispertion, then the phase velocity is independent of frequency:

$$v_p = \frac{\omega}{k} = \text{constant} \qquad (48)$$

and the group velocity is equal to the phase velocity:

$$v_g = \frac{d\omega}{dk} = v_p \qquad (49)$$

in the absence of dispersion, a modulation resulting from the superposition of two waves with similar frequencies will travel at the same speed as the waves themselves.

However, if there is dispersion, then the group velocity can differ from the phase velocity.

Group velocity of an EM wave in a good conductor:

The dispersion relation for an electromagnetic wave in a good conductor is, from (38):

$$\omega = \frac{1}{\mu\varepsilon}\frac{2\varepsilon}{\sigma}\alpha^2 \qquad (50)$$

Where α is the real part of the wave vector. The group velocity is then:

$$\begin{aligned} v_g &= \frac{d\omega}{d\alpha} \\ &\approx \frac{1}{\mu\varepsilon}\frac{4\varepsilon}{\sigma}\alpha \\ &\approx \frac{2}{\sqrt{\mu\varepsilon}}\sqrt{\frac{2\omega\varepsilon}{\sigma}} \end{aligned} \qquad (51)$$

Comparing with equation (39) for the phase velocity of an electromagnetic wave in a good conductor, we find that:

$$v_g \approx 2v_p \qquad (52)$$

In other words, the group velocity is approximately twice the phase velocity.

The skin depth of a good conductor:

Skin depth is inversely proportional to square root of frequency.

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\mu_o \sigma \omega}}$$

The real part, α, of the wave vector k in a conductor gives the wavelength of the wave. B measures the distance that the wave travels before its amplitude falls to 1/e of its original value. Let, the direction in a good conductor as:

$$\vec{E}(\vec{r}, t) = \overrightarrow{E'_o}(\vec{r}) e^{j}(wt - \vec{\alpha}.\vec{r})$$

…………………….. (53)

where:

$$\overrightarrow{E'_o}(\vec{r}) = \overrightarrow{E_o} e^{-\vec{\beta}.\vec{r}}$$

…………………….. (54)

The amplitude of the wave falls by a factor 1/e in a distance 1/β. We define the skin depth δ:

$$\delta = \frac{1}{\beta} \qquad (55)$$

From equation (31), we see that for a good conductor (σ >> ωε), the skin depth is given by:

$$\delta \approx \sqrt{\frac{2}{\omega \mu \sigma}} \qquad (56)$$

For example, consider silver, which has conductivity $\sigma \approx 6.30 \times 10^{7}\ \Omega^{-1}\ m^{-1}$, and permittivity $\varepsilon \approx \varepsilon_0 \approx 8.85 \times 10^{-12}$ Fm^{-1}.

For radiation of frequency 10^{10} Hz, the "good conductor" condition is satisfied, and the skin depth of the radiation is approximately 0.6 micron (0.6×10^{-6} m).

Note that in vacuum, the wavelength of radiation of frequency 10^{10} Hz is about 3 cm; but in silver, the wavelength is:

$$\lambda = \frac{2\pi}{\alpha} \approx 2\pi\delta \approx 4 \text{ micron} \qquad (57)$$

The phase difference between the electric and magnetic fields in a good conductor is given by:

$$\tan\phi = \frac{\beta}{\alpha} \approx 1 \qquad (58)$$

So the phase difference is approximately 45^o.

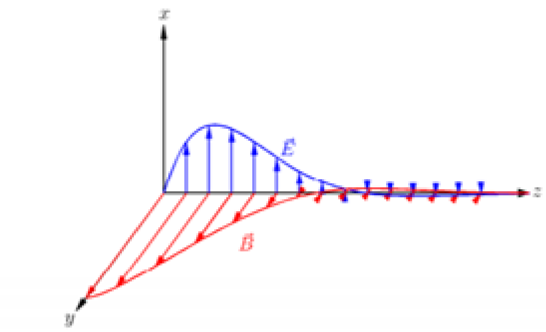


Fig.9.4 Phase difference between the plane

EM wave impedance in a good conductor:

Using the plane wave solutions:

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \qquad (59)$$
$$\vec{B}(\vec{r},t) = \vec{B}_0 e^{j(\omega t - \vec{k}\cdot\vec{r})} \qquad (60)$$

in Maxwell's equation:

$$\nabla \times \vec{E} = -\dot{\vec{B}} \qquad (61)$$

and using also the relation $\vec{B} = \mu\vec{H}$, we find the relation between the electric field and magnetic intensity:

$$\vec{k} \times \vec{E}_0 = \omega\mu\vec{H}_0 \qquad (62)$$

The vector $\vec{k}$, $\vec{E}_0$ and $\vec{H}_0$ are mutually perpendicular. Therefore, we can write for the wave impedance:

$$Z = \frac{E_0}{H_0} = \frac{\omega\mu}{\alpha - j\beta} \qquad (63)$$

EM wave impedance in a good conductor:

In a good conductor (σ >> ωε), we have (31):

$$\alpha \approx \beta \approx \sqrt{\frac{\omega\mu\sigma}{2}} \qquad (64)$$

It the follows that the wave impedance (63) in a good conductor is given by:

$$Z \approx \frac{1}{1-j}\sqrt{\frac{2\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}} \qquad (65)$$

Note that the impedance is now a complex number. As we shall see later, the behavior of waves on a boundary depends on the impedances of the media on either side of the boundary.

The complex phase of the impedance will tell us about the phases of the waves reflected from and transmitted across the boundary.

Energy densities in an EM wave in a good conductor:

The time averaged energy densities in the electric and magnetic fields are:

$$\langle U_E \rangle_t = \frac{1}{2}\varepsilon \langle \vec{E}^2 \rangle_t = \frac{1}{4}\varepsilon E_0^2 e^{-2\vec{\beta}\cdot\vec{r}} \qquad (66)$$

$$\langle U_H \rangle_t = \frac{1}{2}\mu \langle \vec{H}^2 \rangle_t = \frac{1}{4}\mu H_0^2 e^{-2\vec{\beta}\cdot\vec{r}} \qquad (67)$$

The ratio is:

$$\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} = \frac{\varepsilon}{\mu}\frac{E_0^2}{H_0^2} = \frac{\varepsilon}{\mu}|Z|^2 \qquad (68)$$

In a good conductor, the square of the square of the magnitude of the impedance is:

$$|Z|^2 \approx \frac{\omega\mu}{\sigma} \qquad (69)$$

Hence, in a good conductor, most of the energy is in the magnetic field:

$$\frac{\langle U_E \rangle_t}{\langle U_H \rangle_t} \approx \frac{\omega\varepsilon}{\sigma} \ll 1 \qquad (70)$$

(ii) Behavior and property of electromagnetic waves for good dielectric (Propagation of EM Waves in Different Mediums):

In electromagnetic fields, the materials are classified as conductors, dielectrics and lossy dielectric. The electrical parameters such as μ, ϵ and σ are the variable parameters that decide the type of medium. Different materials affect the materials differently.

Suppose if we pass through a tunnel or under the bridge, our radio ceases to receive the signals and also compared to the day, during the night time, we will experience a better reception of radio signals. Thus the waves are affected by the materials or environmental conditions.

So it is necessary to know the propagation of electromagnetic waves in order to choose the appropriate values of frequency, power, type of wave and other parameters needed for the design of applications include transmission lines, antennas, waveguides, etc.

Consider the waves obtained for a medium from the above equations

$$\nabla^2 \bar{E} = \mu \sigma \times \partial \bar{E} / \partial t + \mu \epsilon (\partial^2 \bar{E} / \partial t^2)$$

$$\nabla^2 \bar{H} = \mu \sigma (\partial \bar{H} / \partial t) + \mu \epsilon (\partial^2 \bar{H} / \partial t^2)$$

Both electric and magnetic fields are varying with time for a uniform plane wave. Then, the partial derivative with respective time can be replaced by jw. Thus, the electric and magnetic fields can be written as

$$\nabla^2 \bar{E} = \mu \sigma \times (j\omega \bar{E}) + \mu \epsilon (j\omega)^2 \bar{E}$$

$$\nabla^2 \bar{E} = [j\omega\mu (\sigma + j\omega \epsilon)] \bar{E}$$

Similarly

$$\nabla^2 \bar{H} = [j\omega\mu (\sigma + j\omega \epsilon)] \bar{H}$$

The above two equations are called as wave equation in a waveform. In the above equations the terms inside the bracket is same and properties of the medium through which the wave is propagating is represented by this term. This term is equal to the square of a propagation constant ɣ. Then the wave equations becomes

$$\nabla^2 \bar{E} = \gamma^2 \bar{E}$$

$$\nabla^2 \bar{H} = \gamma^2 \bar{H}$$

In terms of the properties of the medium, the propagation constant is given as

$$\gamma = \sqrt{[j\omega\mu\ (\sigma + j\omega\ \epsilon)]} = \alpha + j\ \beta$$

In general, the wave gets attenuated when it travels through a medium, hence the amplitude of the wave get attenuated. This is represented by the real part of the propagation constant and it is given by

$$\alpha = \omega \sqrt{}\ ((\mu\ \epsilon / 2) \sqrt{}\ (1 + (\sigma / \omega\ \epsilon)\ 2)) - 1)$$

Similarly, the phase change occurs in a wave when it propagates through a medium. This phase change is represented as imaginary part of the propagation constant and is given as

$$\beta = \omega \sqrt{}\ ((\mu\ \epsilon / 2) \sqrt{}\ (1 + (\sigma / \omega\ \epsilon)\ 2)) + 1$$

And also the intrinsic impedance of a medium can be expressed as

$$\eta = \sqrt{[(j\omega\mu) / (\sigma + j\omega\ \epsilon)]}$$

Uniform Plane Wave in Free Space

For free space J = 0, σ = 0, $\epsilon = \epsilon_o$ and $\mu = \mu_o$ then the properties of the propagation constant are

$\alpha = 0$ and

$$\beta = \omega \sqrt{(\mu_o \epsilon_o)}$$

Therefore the propagation constant is purely imaginary for free space.

Uniform Plane Wave in Lossless Dielectric:

For a perfect or lossless dielectric the properties are given as, $\sigma = 0$, $\epsilon = \epsilon_o \epsilon_r$ and $\mu = \mu_o \mu_r$. In both free space medium and lossless dielectric medium $\sigma = 0$, so the analysis of the wave propagation is much similar in both cases. But as the permeability and permittivity values are different then expression in both cases gets varied.

The Velocity of propagation,

$$v = (1/\sqrt{(\mu \epsilon)})$$

$$= (1/\sqrt{(\mu_o \mu_r \epsilon_o \epsilon_r)}) = 1/(\sqrt{(\mu_o \epsilon_o)} \sqrt{(\mu_r \epsilon_r)})) = 1/(\sqrt{(\mu_o \epsilon_o)}/ \sqrt{(\mu_r \epsilon_r)}))$$

Therefore

$$v = c/ \sqrt{(\mu_r \epsilon_r)} \text{ m/s}$$

The propagation constant,

$$\gamma = \sqrt{[j\omega\mu (\sigma + j\omega \epsilon)]} \text{ m-1}$$

By substituting $\sigma = 0$, $\epsilon = \epsilon_o \epsilon_r$ and $\mu = \mu_o \mu_r$ in the above equation for a perfect or lossless dielectric, we get

$$\gamma = +/- j\omega \sqrt{(\mu\epsilon)} \text{ m-1}$$

And also attenuation constant, $\alpha = 0$

The phase constant,

$$\beta = \omega \sqrt{(\mu\, \epsilon)} \text{ rad/m}$$

Intrinsic Impedance,

$$\eta = \sqrt{[(j\omega\mu) / (\sigma + j\omega\, \epsilon)]} \text{ ohms}$$

$$= \sqrt{(\mu_o / \epsilon_o)}\, \sqrt{(\mu_r / \epsilon_r)}$$

$$= \eta_o \sqrt{(\mu_r / \epsilon_r)}$$

$$\eta = 377\sqrt{(\mu_r / \epsilon_r)} \text{ ohms}$$

Uniform Plane Wave in Lossy Dielectric:

A lossy dielectric is a poor insulator, in which free charges conducts up to some extent in partial conducting medium. It is an imperfect conductor and imperfect dielectric (which is a partial conducting medium) with $\sigma \neq 0$.

The propagation constant is given as

$$\gamma = \sqrt{[j\omega\mu\, (\sigma + j\omega\, \epsilon)]}$$

Rearranging the terms, we get

$$\gamma = \sqrt{[j\omega\, \epsilon\, (1 + (\sigma / j\omega\, \epsilon))\, j\omega\mu]}$$

Therefore,

$$\gamma = \alpha + j\, \beta = j\omega \sqrt{\mu\epsilon} \sqrt{(1 - j\, (\sigma / \omega\, \epsilon))}$$

The above equation gives the propagation constant for lossy dielectric medium which is different from lossless dielectric medium due to the presence of radical factor. The attenuation constant α and phase constant are calculated by substituting the values of ω, μ, ϵ, and σ in the above equation.

The attenuation constant α indicates the certain loss of the wave signal in the medium and hence this type of medium is called as lossy dielectric.

And also due to $\sigma \neq 0$, the intrinsic impedance becomes a complex quantity and is given as

$$\eta = \sqrt{[(j\omega\mu) / (\sigma + j\omega\,\epsilon)]}$$

$$\eta = |\eta| \angle \Theta_n \text{ Ohms.}$$

Because of the complex quantity, η is represented in polar form as shown in the above equation where Θn is the phase angle difference between electric and magnetic fields. Thus, in lossy dielectric medium there exist a phase difference between the electric and magnetic fields.

The intrinsic impedance can be expressed as

$$\eta = \sqrt{[(j\omega\mu) / (\sigma + j\omega\,\epsilon)]}$$

$$= \sqrt{[(j\omega\mu) / j\omega\epsilon\,(1 + (\sigma/ j\omega\,\epsilon)]}$$

$$\eta = (\sqrt{(\mu/ \epsilon)})\,(1 / \sqrt{(1 - j\,(\sigma/ \omega\,\epsilon)})\text{ ohms}$$

And the angle Θ_n is given as

$$\Theta_n = \tfrac{1}{2}\,[(\pi/2) - \tan^{-1}(\omega\,\epsilon/ \sigma)]$$

This angle depends on the frequency of the signal as well as properties of the lossy dielectric medium. Then, w becomes very small for a low frequency signal. Thus, the phase angle is given as

$$\Theta_n = (\pi/4)$$

For very high frequency signal, w becomes very large then,

$$\Theta_n = 0$$

So the range of Θn of a lossy dielectric for complete frequency range is 0 [$\Theta_n = (\pi/4)$].

Applications of Electromagnetic Waves:

In general, a wave phenomenon constitutes both time varying electric and magnetic fields. Some of the applications where the electromagnetic waves can be encountered are given below. In addition to the below application areas there are many other applications where the knowledge of the electromagnetic waves is profoundly used.

Transmission Lines:

In case of power transmission at low frequencies, electrical parameters like resistance, capacitance, inductance, etc are enough to characterize the complete electric circuit. In such circuit analysis, the physical size of electrical components is not considered and simple Kirchoff's laws are enough to analyze the circuit.

However, if the frequency is increased, the size of the physical parameters must be considered and also space starts playing a role in the analysis of the circuit.

In such transmission the voltages and currents are exists in the form of waves. This type of approach for analysing the circuit with inclusion of space consideration is called as a transmission line approach.

Antennas:

An antenna is one of most important devices in the communication system, although it appears as a passive looking device. It can efficiently launch and receive electromagnetic waves. Several types of antennas have been in use for serving different applications.

With the advancements in the mobile communications, compact, multi-frequency and efficient antennas are developed during recent years. By using the power ranging from a few watts to Mega watts, the communication is established by these antennas.

Mobile Communications:

The understanding of radio environment requires the knowledge of electromagnetic wave propagation. In a cellular system, different frequency reuse schemes are employed depends on the signal strength variation as a function of distance. One of the major important aspects of the mobile communication is fading. Thus, for correctly predicting the behavior of the fading, signal processing algorithms need the knowledge of the radio environment. Hence the electromagnetic waves and its analysis plays key role in mobile communication systems.

Fiber Optic Communication:

A high speed and efficient long haul communication use a variety of fiber optic devices which are developed by employing the complex phenomena of electromagnetic waves. This communication is the modern form of guided wave communication.

For the investigation light propagation in the optical fibers electromagnetic theory is used. Due to the direct consequence of the direct consequence of the wave nature of light results a modal propagation inside an optical fiber. Also for analysing the photo and laser detectors, the electromagnetic wave theory is very important.

Electromagnetic Interference (EMI) and Compatibility:

In general, an electric circuit tends to give electromagnetic radiation, especially when they are switching heavy currents. This radiation may interfere with other parts or elements in the network, thereby affect the overall circuit performance.

Example case is SMPS and high speed digital circuits produce a considerable electromagnetic interference. Mostly shielding circuits are used for protecting the circuits from EMI. Thus the proper design of such EMI shields requires the knowledge of electromagnetic waves.

Radio Astronomy:

The radio astronomy is a combination of physics and electronics engineering. It is one of the major important areas where understanding of the electromagnetic waves is necessary. In astronomy, the observations of the sky are carried out at radio frequencies.

These RF signals are very weak in nature and thus sate of art communication receivers and antennas are used to detect such signals.

Therefore, in radio astronomy all aspects of electromagnetic waves are employed.

9.5 Poynting theorem (statement and derivation):

The Poynting Theorem is in the nature of a statement of the conservation of energy for a configuration consisting of electric and magnetic fields acting on charges. Consider a volume V with a surface S. Then the time rate of change of electromagnetic energy within V plus the net energy flowing out of V through S per unit time is equal to the negative of the total work done on the charges within V.

Consider first a single particle of charge q traveling with a velocity vector v. Let E and B be electric and magnetic fields external to the particle; i.e., E and B do not include the electric and magnetic fields generated by the moving charged particle. The force on the particle is given by the Lorentz formula

$$F = q(E + v \times B)$$

The work done by the electric field on that particle is equal to $qv \cdot E$. The work done by the magnetic field on the particle is zero because the force due to the magnetic field is perpendicular to the velocity vector v.

For a vector field of current density J the work done on the charges within a volume V is

$$\int_V J \cdot E dV$$

For a single particle of charge q traveling with velocity v the above quantity reduces to $q\mathbf{v}\cdot\mathbf{E}$.

One form of the Ampere-Maxwell's Law says that

$$J = (c/4\pi)\nabla\times H - (1/4\pi)(\partial D/\partial t)$$

When the RHS of the above is substituted for J the work done by the external fields on the charges within a volume V is

$$(1/4\pi)\int_V[cE\cdot(\nabla\times H) - E\cdot(\partial D/\partial t)]dV$$

There is a vector identity

$$\nabla\cdot(A\times B)=B\cdot(\nabla\times A)-A\cdot(\nabla\times B)$$

which can be rewritten as

$$A\cdot(\nabla\times B) = -[\nabla\cdot(A\times B)] + B\cdot(\nabla\times A)$$

This means that

$$E\cdot(\nabla\times H) = -\nabla\cdot(E\times H) + H\cdot(\nabla\times E)$$

When this expression is substituted into the expression for the rate at which work is being done the result is

$$\int_V J\cdot E dV = (1/4\pi)\int_V[-c\nabla\cdot(E\times H) - E\cdot(\partial D/\partial t) + cH\cdot(\nabla\times E)]dV$$

Faraday's law states that

$$\nabla\times E = -(1/c)(\partial B/\partial t)$$

When Faraday's law is taken into account the previous equation can be expressed as:

$$\int_V J\cdot E dV = (-1/4\pi)\int_V[c\nabla\cdot(E\times H) + E\cdot(\partial D/\partial t) + H\cdot(\partial B/\partial t)]dV$$

The total energy density U of the fields at a point is

$$U = (1/8\pi)(E\cdot D + B\cdot H)$$

where $D=\varepsilon E$ and $H=(1/\mu)B$ and ε and μ, called the dielectric and permabiity, respectively, are properties of the material in which the fields are located. The dielectric and permability are independent of the location.

This means that

$$U=(1/8\pi)(\varepsilon E\cdot E+(1/\mu)B\cdot B)$$

and thus

$$(\partial U/\partial t) = (1/4\pi)(\varepsilon E\cdot(\partial E/\partial t) + (1/\mu)B\cdot(\partial B/\partial t))$$

which is equivalent to

$$(\partial U/\partial t) = (1/4\pi)(E\cdot(\partial D/\partial t) + B\cdot(\partial H/\partial t))$$

The RHS of this latter expression occurs in a previous expression so that

$$-\int_V J\cdot E dV = \int_V[(\partial U/\partial t) + (c/4\pi)\nabla\cdot(E\times H)]dV$$

It is convenient to define a vector P, known as the Poynting vector for the electrical and magnetic fields, such that

$$P = (c/4\pi)(E\times H)$$

The previous equation then becomes

$$-\int_V J\cdot E dV = \int_V[(\partial U/\partial t) + \nabla\cdot P]dV$$

By Gauss' Divergence Theorem

$$\int_V(\nabla\cdot P)dV = \int_S n\cdot P dS$$

where S is the surface of the volume V and n is the unit normal to the surface element dS. The vector P has the dimensions of energy×time per unit area. Thus $\int_S n\cdot P dS$ is the net flow of energy out of the volume V.

The above means that work done by the electric and magnetic fields on the charges within a volume must match the rate of decrease of the energy of the fields within that volume and the net flow of energy into the volume. The big question is what does the net flow of energy into the volume correspond to physically. One possibility is that it might correspond to electromagnetic radiation. The above equation can also be stated as the negative of the work done on the charges within a volume must be equal to the increase in the energy of the electric and magnetic fields within the volume plus the net flow of energy out of the volume.

There is a major problem with the Poynting vector P; it is independent of the charges involved. It is the same whether there is one charge or one hundred million charges, or for that matter, zero charges. It can change with time but only as a result of the changes in the electric and magnetic fields.

Usually any difference between the change in energy and the work done is the energy of radiation. This is what is universally presumed in the case of the Poynting theorem, but the empirical evidence is that this cannot be so. If the Poynting vector corresponded to radiation then if a permanent magnet was placed in the vicinity of a body charged with static electricity the combination should glow and is that is not the case.

The Poynting vector is completely independent of the charges and their velocities in the volume being considered. In a word it is exogenous. The charges and their velocities are also exogenous. It is the rate of change of the energy stored in the fields that is endogenous. The Poynting theorem should read rate of change of energy in the fields = negative of work done by the fields on the charged particles minus the Poynting vector term.

However in the case of a permanent magnet and static electric charge the fields cannot change. Charged particles impinging upon an electric and magnetic field would experience work of them. The compensating change in momentum and energy would occur in the bodies holding the electric and magnetic fields. The charged particles hitting the electric and magnetic fields would induce a reaction as though they hit the magnet and charged body which creates the fields.

The dimensions of the Poynting vector term are energy per unit area per unit time. This is what would be expected if there were radiation generated in the volume. But the fact that the Poynting vector is exogenous means that without any charged particles at all being involved there would be radiation generated. The amount of radiation generated is fixed and no matter how many charged particles are injected into the volume at whatever velocities the same amount of radiation would be generated.

So the Poynting vector term apparently does not correspond to radiation. It is a puzzle as to what it does correspond to but there is no possibility that it corresponds to radiation.

The Differential Form of the Poynting Theorem

Since the volume element is arbitrary the above equation implies that

$$(\partial U/\partial t) + \nabla\cdot P = -E\cdot J$$

The interpretation of the term ∇·P is also problematical. It has a sign but it does not have a direction. It also is independent of the charge distribution, in this case J. In another study the case will be made that ∇·P is the time rate of change of the energy resulting from the interaction of the electrical and magnet field.

SAQ.1

a) What do you mean by Differential equation of plane electromagnetic waves in conducting media?
b) Discuss the property of electromagnetic waves for good dielectric.
c) What is the statement of Poynting theorem?
d) In a wave if $E_0 = 1000\ V\ m^{-1}$. Then find the magnitude of Poynting vector.

9.6 Expression for electromagnetic energy density:

Energy density refers to the total amount of energy in a system per unit volume. (Even though generally energy per unit mass is also mentioned as energy density, the proper term for the same is specific energy. The term density usually measures the amount per unit spatial extension).

Energy density is denoted by letter U.

Magnetic and electric field can also store the energy.

In the case of electric field or capacitor, the energy density is given by

$$U = \frac{1}{2}\varepsilon_0 E^2$$

The energy density in the case of magnetic field or inductor is given by,

$$U = \frac{1}{2\mu_0} B^2$$

For electromagnetic wave, both magnetic and electric field are equally involved in contributing to energy density. Therefore, the energy density is the sum of the energy density of electric and magnetic fields.

i.e.,

$$U = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$$

9.7 Momentum density vector and its importance:

We have seen that electromagnetic waves carry energy. It turns out that they also carry momentum. Consider the following argument, due to Einstein. Suppose that we have a railroad car of mass M and length L which is free to move in one dimension. Suppose that electromagnetic radiation of total energy E is emitted from one end of the car, propagates along the length of the car, and is then absorbed at the other end. The effective mass of this radiation is $m = E / c^2$ (from Einstein's famous relation $E = m c^2$). At first sight, the process described above appears to cause the centre of mass of the system to spontaneously shift. This violates the law of momentum conservation (assuming the railway car is subject to no external forces). The only way in which the centre of mass of the system can remain stationary is if the railway car moves in the opposite direction to the direction of propagation of the radiation. In fact, if the car moves by a distance x then the centre of mass of the system is the same before and after the radiation pulse provided that

$$M\,x = m\,L = (E / c^2)L$$

It is assumed that $m \ll M$ in this derivation.

But, what actually causes the car to move? If the radiation possesses momentum P then the car will recoil with the same momentum as the radiation is emitted. When the radiation hits the other end of the car then the car acquires momentum P in the opposite direction, which stops the motion. The time of flight of the radiation is L/c. So, the distance traveled by a mass M with momentum P in this time is

$$x = v\,t = (P/M)(L/c)$$

giving

$$P = M\,x\,(c / L) = E / c$$

Thus, the momentum carried by electromagnetic radiation equals its energy divided by the speed of light. The same result can be obtained from the well-known relativistic formula

$$E^2 = P^2c^2 + m^2c^4$$

relating the energy E, momentum P, and mass m of a particle. According to quantum theory, electromagnetic radiation is made up of massless particles called photons. Thus,

$$P = E/c$$

For individual photons, so the same must be true of electromagnetic radiation as a whole. If follows from Eq. (that the momentum density g of electromagnetic radiation equals its energy density over c, so

$$g = U/c$$

$$= \frac{|\mathbf{u}|}{c^2} = \frac{\epsilon_0\, E^2}{c}$$

It is reasonable to suppose that the momentum points along the direction of the energy flow (this is obviously the case for photons), so the **vector momentum density** (which gives the direction, as well as the magnitude, of the momentum per unit volume) of electromagnetic radiation is

$G = u/c^2$

Thus, the momentum density equals the energy flux over c^2.

Importance:-

- The momentum density equals the energy flux over. Of course, the electric field associated with an electromagnetic wave oscillates rapidly, which implies that the previous expressions for the energy density, energy flux, and momentum density of electromagnetic radiation are also rapidly oscillating.

- Thus Momentum is a vector quantity. A vector quantity possesses both a magnitude and direction. A scalar quantity possesses only a magnitude and no direction. Mass is an example of a scalar quantity (mass doesn't point in any direction!) whereas velocity is a vector quantity.
- Thomson, who speculated as to electro- magnetic mass/momentum, and continued with a concept of momentum stored in the electromagnetic field, with the field-momentum density being the Poynting vector divided by c^2, pEM = S/c^2, where c is the speed of light in vacuum.

9.8 Maxwell's stress tensor (statement and derivation):

Here we'll look at a purely classical, non-relativistic form of the tensor in electromagnetism. In doing so, we'll look only at the spatial components of the tensor, so it becomes a 3×3 matrix. The derivation starts with a calculation of the total force due to electromagnetic fields on the charges and currents within some volume V. From the Lorentz force law, we have

$$\begin{aligned}\mathbf{F} &= \int_{\mathcal{V}} \rho(\mathbf{E}+\mathbf{v}\times\mathbf{B})\,d^3\mathbf{r} \\ &= \int_{\mathcal{V}} (\rho\mathbf{E}+\mathbf{J}\times\mathbf{B})\,d^3\mathbf{r}\end{aligned}$$

We can think of the integrand as a force density, or force per unit volume f:

$$\mathbf{f} \equiv \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

We can express this entirely in terms of fields by using Maxwell's equations:

$$\begin{aligned} \rho &= \epsilon_0 \nabla \cdot \mathbf{E} \\ \mathbf{J} &= \frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

So we get

$$\mathbf{f} = (\epsilon_0 \nabla \cdot \mathbf{E})\mathbf{E} + \left[\frac{1}{\mu_0} \nabla \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right] \times \mathbf{B}$$

We now need to do a bit of vector calculus gymnastics. From the product rule

$$\frac{\partial}{\partial t}(\mathbf{E} \times \mathbf{B}) = \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} + \mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t}$$

and from Faraday's law

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Combining these two we get

$$\begin{aligned}\frac{\partial \mathbf{E}}{\partial t}\times\mathbf{B} &= \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\mathbf{E}\times\frac{\partial \mathbf{B}}{\partial t}\\ &= \frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})+\mathbf{E}\times(\nabla\times\mathbf{E})\end{aligned}$$

We can insert this into 6 and while we're at it, we can add on a term 1 μ_0 $(\nabla\cdot B)B$. This is always zero because $\nabla\cdot B = 0$, but it gives the equation a symmetry that will be useful in a minute. We get for the force density:

$$\mathbf{f}=\epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}(\nabla\cdot\mathbf{B})\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}-\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}$$
$$=\epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E}+\frac{1}{\mu_0}(\nabla\cdot\mathbf{B})\mathbf{B}+\frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B}-$$
$$\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})-\epsilon_0\mathbf{E}\times(\nabla\times\mathbf{E})$$

Now another identity from vector calculus says

$$\nabla(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}$$

If A = B = E, we get

$$\nabla\left(E^2\right)=2\mathbf{E}\times(\nabla\times\mathbf{E})+2(\mathbf{E}\cdot\nabla)\mathbf{E}$$

So

$$\begin{aligned}\mathbf{E}\times(\nabla\times\mathbf{E}) &= \frac{1}{2}\nabla\left(E^2\right)-(\mathbf{E}\cdot\nabla)\mathbf{E}\\ \mathbf{B}\times(\nabla\times\mathbf{B}) &= \frac{1}{2}\nabla\left(B^2\right)-(\mathbf{B}\cdot\nabla)\mathbf{B}\end{aligned}$$

Putting this into above we

$$
\begin{aligned}
\mathbf{f} &= \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \\
& \frac{1}{\mu_0} \mathbf{B} \times (\nabla \times \mathbf{B}) - \epsilon_0 \mathbf{E} \times (\nabla \times \mathbf{E}) \\
&= \epsilon_0 (\nabla \cdot \mathbf{E}) \mathbf{E} + \frac{1}{\mu_0} (\nabla \cdot \mathbf{B}) \mathbf{B} - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B}) - \\
& \frac{1}{2} \nabla \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) + \epsilon_0 (\mathbf{E} \cdot \nabla) \mathbf{E} + \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \\
&= \epsilon_0 [(\nabla \cdot \mathbf{E}) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{E}] + \frac{1}{\mu_0} [(\nabla \cdot \mathbf{B}) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{B}] - \\
& \frac{1}{2} \nabla \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \epsilon_0 \frac{\partial}{\partial t} (\mathbf{E} \times \mathbf{B})
\end{aligned}
$$

It might not seem that we're making any progress, since the equations just get longer with each alteration. However, we can now introduce the Maxwell stress tensor ←→T which is a 3×3 matrix with components defined by

$$
\boxed{T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)}
$$

Note that the tensor is symmetric: Tij = Tji. If we define the scalar product of the tensor with an ordinary vector to be another vector:

$$
\left[\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}} \right]_j = \sum_i a_i T_{ij}
$$

where the subscript j indicates the jth component of the resulting vector, then the divergence is

$$
\begin{aligned}
\left[\nabla\cdot\overleftrightarrow{\mathbf{T}}\right]_j &= \sum_i \partial_i T_{ij} \\
&= \epsilon_0 \sum_i \left(\left(\partial_i E_i\right) E_j + E_i\left(\partial_i E_j\right) - \frac{1}{2}\delta_{ij}\partial_i E^2\right) + \\
&\quad \frac{1}{\mu_0}\sum_i \left(\left(\partial_i B_i\right) B_j + B_i\left(\partial_i B_j\right) - \frac{1}{2}\delta_{ij}\partial_i B^2\right) \\
&= \epsilon_0 \left(\left(\nabla\cdot\mathbf{E}\right) E_j + \left(\mathbf{E}\cdot\nabla\right) E_j - \frac{1}{2}\partial_j E^2\right) + \\
&\quad \frac{1}{\mu_0}\left(\left(\nabla\cdot\mathbf{B}\right) B_j + \left(\mathbf{B}\cdot\nabla\right) B_j - \frac{1}{2}\partial_j B^2\right)
\end{aligned}
$$

Comparing this with above equation, we see that we can write f in terms of ←→T and the Poynting vector as

$$
\boxed{\mathbf{f} = \nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}}
$$

The total force on the volume is then

$$
\begin{aligned}
\mathbf{F} &= \int_{\mathcal{V}} \mathbf{f} d^3\mathbf{r} \\
&= \int_{\mathcal{V}} \left(\nabla\cdot\overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0\frac{\partial\mathbf{S}}{\partial t}\right) d^3\mathbf{r}
\end{aligned}
$$

From the formula in above equation for the divergence, we can see that the vector resulting from the divergence has as its components the divergences of each column of ←→T . Therefore we can apply the divergence theorem to the first term in the integrand to get

$$
\boxed{\mathbf{F} = \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a} - \epsilon_0\mu_0\frac{\partial}{\partial t}\int_{\mathcal{V}} \mathbf{S} d^3\mathbf{r}}
$$

where S is any surface that encloses only the charges and currents within V.

Example. We can revisit the problem of finding the magnetic force between the two halves of a spherical shell of surface charge density σ rotating with angular velocity ω = ωzˆ. In our earlier solution we used the Biot-Savart law and integrated over each differential ring in the rotating sphere. Using the stress tensor, we can integrate over any volume that encloses the upper half of the sphere, so we can choose the half space consisting of all space

above the xy plane (we're assuming that the centre of the sphere is at the origin, so the xy plane contains the sphere's equator). Since the distribution of charges and currents is finite, all fields will go to zero at infinity, so we need to integrate only over the xy plane. We saw earlier that the magnetic field inside and outside the sphere is

$$B = \begin{cases} \frac{2\mu_o R\omega 6}{3}\left(cos\theta\hat{r} - sin\theta\hat{\theta}\right) = \frac{2\mu_0 R\omega 6}{2}\hat{z}\; r < R \\ \frac{\mu_o R^4\omega 6}{3}\frac{1}{r^3}\left(2cos\theta\hat{r} + sin\theta\hat{\theta}\right)\; r > R \end{cases}$$

In the xy plane, θ = π/2 so the field is

$$\mathbf{B} = \begin{cases} \frac{2\mu_0 R\omega\sigma}{3}\hat{\mathbf{z}} & r < R \\ -\frac{\mu_0 R^4\omega\sigma}{3}\frac{1}{r^3}\hat{\mathbf{z}} & r > R \end{cases}$$

Since we're interested only in the magnetic field, we can ignore E here, although there is a repulsive force between the two hemispheres due to the electric field as well. Also, as the currents are steady, ∂S/∂t = 0. From the symmetry of the problem, the force is in the z direction, so we need to work out only h←→T · da i z . We get Txz = Tyz = 0 because Bx = By = 0 on the xy plane, so we're left with just Tzz:

$$T_{zz} = \frac{1}{2\mu_0} B_z^2 = \begin{cases} \frac{2}{9}\mu_0\sigma^2\omega^2 R^2 & r < R \\ \frac{1}{18}\mu_0\sigma^2\omega^2 \frac{R^8}{r^6} & r > R \end{cases}$$

The total force is then (the minus sign is because Tzz > 0 and da points towards −z):

$$\begin{aligned} \mathbf{F} &= \int_S \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} \\ &= -\hat{\mathbf{z}} \left[\frac{2}{9}\mu_0\sigma^2\omega^2 R^2 2\pi \int_0^R r\,dr + \frac{2\pi}{18}\mu_0\sigma^2\omega^2 R^8 \int_R^\infty \frac{r\,dr}{r^6} \right] \\ &= -\hat{\mathbf{z}} \left(\frac{2\pi}{9}\mu_0\sigma^2\omega^2 R^4 + \frac{\pi}{36}\mu_0\sigma^2\omega^2 R^4 \right) \\ &= -\frac{\pi}{4}\mu_0\sigma^2\omega^2 R^4 \hat{\mathbf{z}} \end{aligned}$$

This agrees with the result we got earlier using the Biot-Savart law.

SAQ.2

a) Define the expression for electromagnetic energy density.
b) What do you mean by Momentum density vector?
c) Write the short note on Maxwell’s stress tensor.
d) In a certain region of space, the magnetic field was a value of 2.0 X 10^{-2} T, and the electric field has a value of 4.0 X 10^{6} Vm^{-1}. Find the combined energy density of the electric and magnetic fields.
e) Calculate the momentum of a bullet having mass of 25g thrown using hand with a velecity of 10 m/s.

Examples:

Q.1. In a wave if $E_0 = 100\ V\ m^{-1}$. Then find the magnitude of Poynting vector.

Solution:

$$B = \frac{E}{c} \Longrightarrow |\vec{p}| = \frac{EB}{\mu_0} = \frac{E^2}{c\mu_0} = \frac{10^4}{3 \times 10^8 \times 4\pi \times 10^{-7}} = 26.5\mathrm{W\,m^{-2}}$$

Q.2. In a certain region of space, the magnetic field was a value of 1.0 X 10^{-2} T, and the electric field has a value of 2.0 X 10^6 Vm^{-1}. Find the combined energy density of the electric and magnetic fields.

Solution:

$E = 2.0 \text{ X } 10^6\ Vm^{-1;}$ $B = 1.0 \text{ X } 10^{-2}$ T

For the electric field, the energy density is:

$U_E = ½(\varepsilon_o E^2) =(½)\ \text{X } 8.85 \text{ X } 10^{-12}(2.0 \text{ X } 10^6)^2 = 18\ Jm^{-3}$

For the magnetic field, the energy density is:

$U_B = ½\ (B^2/\ \mu_O) = (1/2) \text{ X } ((1.0 \text{ X } 10^{-2}) / (4\pi \text{ X } 10^{-7})) = 40\ Jm^{-3}$

The net energy density is the sum of the energy density due to the electric field and the energy density due to the magnetic field:

$U = U_E + U_B = 18 + 40 = 58\ Jm^{-3}$

Q.3. A linearly polarized laser beam propagating in air has an intensity of $10^6\ W/m^2$. Calculate the magnitude of the electric and magnetic fields within the electromagnetic wave.

Solution: Given data

Intensity of the laser beam I = 1 x 10^6 W/m^2

Speed of light in vacuum c = 3.00 x 10^8 m/s

Permittivity of free space ε_o = 8.854 x 10^{-12} F/m

Let E_0, B_0 be the electric field amplitude and magnetic field amplitude.

The intensity of the electromagnetic wave can be expressed as

$$I = \frac{1}{2} c\epsilon_0 E_0^2$$

Therefore the magnitude of the electric field

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}}$$

$$E_0 = \sqrt{\frac{2 \times 10^6 \text{ W/m}^2}{3.0 \times 10^8 \text{ m/s} \times 8.854 \times 10^{-12} \text{ F/m}}}$$

$$E_0 \approx 2.7 \times 10^4 \text{ V/m}$$

The magnetic field

$$B_0 = \frac{E_0}{c}$$

$$B_0 = \frac{2.7 \times 10^4 \text{ V/m}}{3.0 \times 10^8 \text{ m/s}}$$

$$B_0 = 9.0 \times 10^{-5} \text{ T}$$

Q.4. Microwave ovens emit microwave energy with a wavelength of 12.0 cm. What is the energy of exactly one photon of this microwave radiation?

Solution: Determine the energy, E, of one photon of the microwave radiation. We do this by applying the equation,

$$E = \frac{hc}{\lambda}$$

Where h is Planck's constant, c is the speed of light, and λ is the wavelength. We use the following values for the variables:

- $h = 1.056 \times 10^{-34}\ J\ s$
- $c = 3.0 \times 10^{8}\ m/s$
- $\lambda = 12.0\ cm = 0.12\ m$

We simply plug in the given values to determine the answer

$$\begin{aligned} E &= \frac{hc}{\lambda} \\ &= \frac{(1.056 \times 10^{-34}\ J\ s)(3.0 \times 10^{8}\ m/s)}{0.12\ m} \\ &\approx \boxed{2.6 \times 10^{-25}\ \mathrm{J}} \end{aligned}$$

Q.5 Find the energy density of a capacitor if its electric field, E = 5 V/m.

Solution: Given,

E = 5V/m

We know that,

$\epsilon 0 = 8.8541 \times 10^{-12}$F/m

The energy density formula of the capacitor is given by

$U = ½ (\varepsilon_O E^2)$

$= (1 \times 8.8541 \times 10^{-12} \times 5^2)/2$

$U = 1.10 \times 10^{-10}\ FV^2/m^3$

Q.6. What will be the momentum of a stone having mass of 10 kg when it is thrown with a velocity of 2m/s?

Solution:

Mass (m) = 10kg,

Velocity (v) = 2m/s,

Momentum (ρ) =?

We know that, momentum (ρ) = Mass (m) x Velocity (v)

Therefore, p = 10kg x 2m/s = 20kgm/s

Thus the momentum of the stone = 20kgm/s

Q.7 Calculate the momentum of a bullet of 25g when it is fired from a gum with a velocity of 100m/s.

Solution:

Given, velocity of the bullet (v) = 100m/s

Mass of the bullet (m) = 25g = (25/1000) kg = 0.025kg

Momentum (ρ) =?

We know that, momentum (ρ) = Mass(m) x Velocity(v)

Therefore p = 0.025kg x 100m/s

Or, p = 2.5kg m/s

Thus the momentum of the bullet = 2.5kg m/s

Q.8 Calculate the momentum of a bullet having mass of 25g thrown using hand with a velecity of 0.1m/s.

Solution:

Given, velocity of the bullet (v) = 0.1m/s

Mass of the bullet (m) = 25g = (25/1000) kg = 0.025kg

Momentum (ρ) =?

We know that, momentum (ρ) = Mass(m) x Velocity(v)

Therefore p = 0.025kg x 0.1m/s

Or, p = 0.0025kg m/s

Thus the momentum of the bullet = 0.0025kg m/s

Q.9 find all elements of the Maxwell stress tensor for a monochromatic plane wave travelling in the z direction and linearly polarized in the x direction, i.e.

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{x}\,, \quad \mathbf{B}(z,t) = \frac{E_0}{c}\cos(kz - \omega t + \delta)\hat{y}\,.$$

Common on the form of your answer (remember that $\overleftrightarrow{\mathbf{T}}$ represents the momentum flux density)

Solution:

First, we recall that the components of Maxwell's stress tensor are given by

$$T_{ij} = \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij}\mathbf{E}^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij}\mathbf{B}^2 \right)$$

Since only Ex and By are nonzero, it is clear that all the off- diagonal components vanish. A quick calculation then shows:

$$\begin{aligned}
T_{xx} &= \epsilon_0 \left(E_x E_x - \frac{1}{2}\mathbf{E}^2 \right) + \frac{1}{\mu_0} \left(-\frac{1}{2}\mathbf{B}^2 \right) = 0 \\
T_{yy} &= \epsilon_0 \left(-\frac{1}{2}\mathbf{E}^2 \right) + \frac{1}{\mu_0} \left(B_y B_y - \frac{1}{2}\mathbf{B}^2 \right) = 0 \\
T_{zz} &= \epsilon_0 \left(-\frac{1}{2}\mathbf{E}^2 \right) + \frac{1}{\mu_0} \left(-\frac{1}{2}\mathbf{B}^2 \right) = -\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) = -u
\end{aligned}$$

Where u is the energy density. That only the Tzz component is non-vanishing is consistent with the fact that the momentum of the field points in the z direction, and it is being transported in the z direction a well.

Summary:

1) In this chapter derive the Differential equation of plane electromagnetic waves in conducting media and find the solution.
2) Discuss the Behavior and property of electromagnetic waves for good dielectric and good conductors and find the expression with proper solution.
3) The Poynting vector, named after John Henry Poynting, is used in order to demonstrate the energy flux density of an EM field. Per definition, the Poynting vector is the result of the vector product of the field's electric and magnetic components.
4) The energy density of an electromagnetic wave is proportional to the square of the amplitude of the electric (or magnetic) field.
5) Thus, the momentum density equals the energy flux over . Of course, the electric field associated with an electromagnetic wave oscillates rapidly, which implies that the previous expressions

for the energy density, energy flux, and momentum density of electromagnetic radiation are also rapidly oscillating.

6) The Maxwell stress tensor is a symmetric second-order tensor used in classical electromagnetism to represent the interaction between electromagnetic forces and mechanical momentum.

Terminal Questions:

1) Explain and derive the Differential equation of plane electromagnetic waves in conducting media and its solution.
2) Explain the Behavior and property of electromagnetic waves for good dielectric and good conductors.
3) Explain the working of Poynting theorem with proper statement and derive the derivation.
4) Discuss the expression for electromagnetic energy density.
5) Explain the Momentum density vector and its importance
6) Explain the statement of Maxwell's stress tensor and derive the its derivation.
7) In a wave if $E_0 = 10^6$ V m^{-1}. Then find the magnitude of Poynting vector.
8) A linearly polarized laser beam propagating in air has an intensity of $10x10^6$ W/m^2. Calculate the magnitude of the electric and magnetic fields within the electromagnetic wave.
9) Find the energy density of a capacitor if its electric field, E = 15V/m.
10) Calculate the momentum of a bullet having mass of 25g thrown using hand with a velecity of 0.01m/s.

Unit 10- Fresnel's equation

Structure

10.1 Introduction:

Displacement boundary conditions derived from symmetry of the specimen geometry and the applied load with respect to the planes x1 = 0 and x2 = 0 were applied together with a uniform displacement condition at the boundary x2 = L/2, where L is the specimen length.

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. These discontinuities can be described mathematically as *boundary conditions* and used to constrain solutions for the associated electromagnetic quantities. In this section, we derive boundary conditions on the electric field intensity E.

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these interfaces.

In this section, we derive the boundary condition on the magnetic flux density B at a smooth interface between two material regions.

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At a boundary between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these boundaries. In this section, we derive boundary conditions on the magnetic field intensity H.

Reflection is the abrupt change in the direction of propagation of a wave that strikes the boundary between two different media. At least some part of the incoming wave remains in the same medium. Assume the incoming light ray makes an angle θ_i with the normal of a plane tangent to the boundary. Then the reflected ray makes an angle θ_r with this normal and lies in the same plane as the incident ray and the normal.

Refraction is the change in direction of propagation of a wave when the wave passes from one medium into another, and changes its speed. Light waves are refracted when crossing the boundary from one transparent medium into another because the speed of light is different in different media. Assume that light waves encounter the plane surface of a piece of glass after traveling initially through air as shown in the figure to the right.

A perpendicular polarized wave incident at angle θ_i a dielectric medium 2. Snell's Law states that a reflected wave will be at the same angle $\theta_r = \theta_i$, and the transmitted wave in medium 2 at angle θ_t can be calculated using this law. The amplitude of the reflected and transmitted waves can be determined by applying the continuity of the tangential components of E & H at the boundary.

Plane waves are not normally incident, so now we must consider the general problem of a plane wave propagating along a specified axis that is arbitrarily relative to a rectangular coordinate system. The most convenient way is in terms of the direction cosines of the uniform plane wave, the equi-phase surfaces are planes perpendicular to the direction of propagation.

Determine the angle of reflection θ_r and the amplitude of the reflected electric field $\hat{E}_m^r$ by using the boundary conditions at z = 0. This also includes zero values of the tangential electrical field E and the normal component of the magnetic field H.

The unknown amplitudes of the reflected and transmitted electric fields $E_{\parallel}^{r}$ *and* $E_{\parallel}^{t}$ can be determined by simply applying the boundary conditions at the dielectric interface. The electric fields $E_{\parallel}^{r}$ *and* $E_{\parallel}^{t}$ will now be used in the analysis to emphasize the case of parallel polarization, instead of using the electric fields E_{m}^{r} *and* E_{m}^{t}.

An incident wave polarized with the E field in the plane of incidence and the power flow in the direction of β_i at angle θ_i with respect to the normal to the surface of the perfect conductor.

Total internal reflection, in physics, complete reflection of a ray of light within a medium such as water or glass from the surrounding surfaces back into the medium. The phenomenon occurs if the angle of incidence is greater than a certain limiting angle, called the critical angle.

Brewster's law, relationship for light waves stating that the maximum polarization (vibration in one plane only) of a ray of light may be achieved by letting the ray fall on a surface of a transparent medium in such a way that the refracted ray makes an angle of 90° with the reflected ray.

Degree of polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized. A perfectly polarized wave has a DOP of 100%, whereas an unpolarized wave has a DOP of 0%. A wave which is partially polarized, and therefore can be represented by a superposition of a polarized and unpolarized component, will have a DOP somewhere in between 0 and 100%. DOP is calculated as the fraction of the total power that is carried by the polarised component of the wave.

Plasma waves the physical description of an electromagnetic wave propagating in a given medium necessitates a self-consistent handling of the particles comprising the medium (and their mutual interactions) on one hand, and of the electromagnetic field on the other hand.

Metals have high reflectivity, reflecting almost all wavelengths in the visible region of the spectrum. Therefore, the Reflectance (R) of a material can be defined as the efficiency of a material to reflect incident light.

Dispersion occurs when pure plane waves of different wavelengths have different propagation velocities, so that a wave packet of mixed wavelengths tends to spread out in space.

10.2 Objectives:

After studying this unit you should be able to

- Explain and identify Boundary conditions at discontinuity for D, E, B and H.
- Study and identify Reflection and refraction at normal and oblique incidence of electric vectors perpendicular to boundary.
- Explain and identify Reflection and refraction at normal and oblique incidence of electric vectors parallel to boundary.
- Study and identify Total internal reflection, Brewster's law, degree of polarization.
- Explain and identify Plane wave propagation in plasma and its properties (qualitative), metallic reflection.
- Study and identify Elementary theory of dispersion.

10.3 Boundary conditions at discontinuity for D, E, B and H:

(i) Boundary conditions at discontinuity for D:

We define the displacement vector as

$$\vec{D} = \epsilon \vec{E} = \epsilon_0 \epsilon_r \vec{E}$$

This description of the displacement vector as it includes permittivity of the medium is helpful while discussing the behavior of the electrostatic field within the medium. Here ε_o is the absolute permittivity of the free space and ε_r is the relative permittivity of the corresponding medium.

Now we apply the integral form of the Curl theorem at the boundary of two dielectric mediums as shown in the diagram below

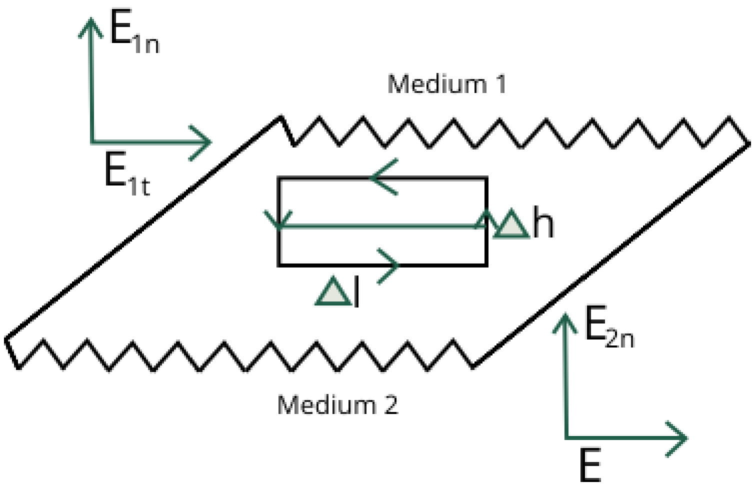


Fig.10.1 Curl theorem at the boundary of two dielectric mediums

$$\vec{E_1} \cdot \vec{\Delta l} + \vec{E_1} \cdot \frac{\vec{\Delta h}}{2} + \vec{E_2} \cdot \frac{\vec{\Delta h}}{2} + \vec{E_2} \cdot \vec{\Delta l} + \vec{E_2} \cdot \frac{\vec{\Delta h}}{2} + \vec{E_1} \cdot \frac{\vec{\Delta h}}{2} = 0$$

To calculate the boundary condition we make the height of the loop to be infinitesimally small or $\Delta h \to 0,$

$$\vec{E_1} \cdot \vec{\Delta l} = \vec{E_2} \cdot \vec{\Delta l}$$

$$E_{1t} = E_{2t}$$

where subscript t denotes the tangential component. Thus at the boundary of two medium, tangential component of the electric field is continuous.

$$E_{1t} = E_{2t}$$

$$\frac{D_{1t}}{\epsilon_{1r}} = \frac{D_{2t}}{\epsilon_{2r}}$$

Thus at the boundary, the tangential component of $\vec{D}$ is discontinuous.

Tangential component of the electric field is continuous across the boundary however the displacement vector is discontinuous.

We will now use the integral form of the divergence theorem of the displacement vector at the boundary of two medium to figure out the fate of the normal component of fields.

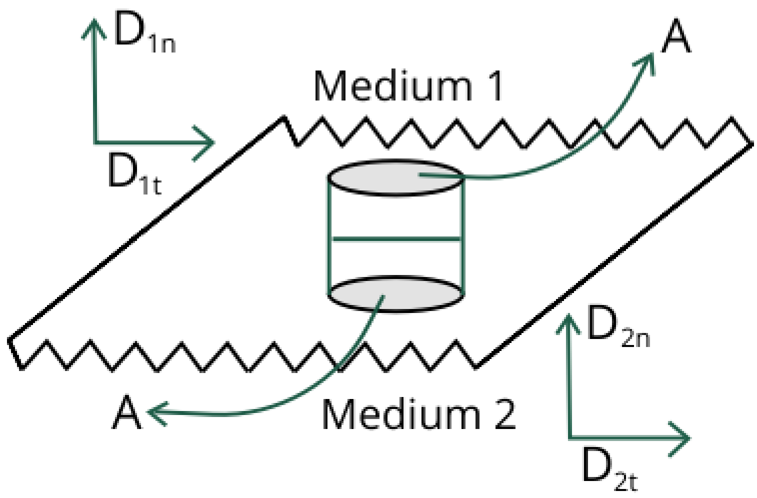


Fig.10.2 Displacement vector at the boundary of two medium

$$\oint \vec{D}.\vec{da} = Q_{free} = \sigma_{free} A$$

$$D_{1n} - D_{2n} = \sigma_{free}$$

Again at the boundary, we can reduce the height of the pillbox to be infinitesimally small. Here subscript n denotes the normal component of the field. If there is no free surface charge, we will have

$$D_{1n} = D_{2n}$$

Thus the normal component of the displacement vector is continuous in absence of any free charge at the surface.

$$\epsilon_{1r} E_{1n} = \epsilon_{2r} E_{2n}$$

Thus the normal component of the electric vector is discontinuous at the boundary. Normal component of the electric field is discontinuous across the boundary, however, the displacement vector is continuous in absence of any surface charge.

(ii) Boundary conditions at discontinuity for E:

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. These discontinuities can be described mathematically as *boundary conditions* and used to constrain solutions for the associated electromagnetic quantities. In this section, we derive boundary conditions on the electric field intensity E.

To begin, consider a region consisting of only two media that meet at an interface defined by the mathematical surface S, as shown in Figure.

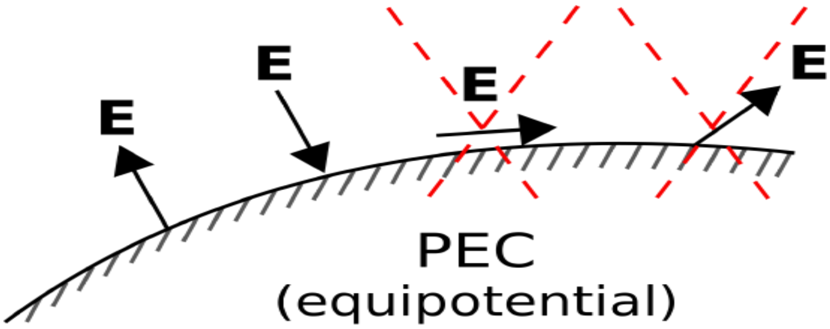


Fig.10.3 At the surface of a perfectly-conducting region, E may be perpendicular to the surface (two leftmost possibilities), but may not exhibit a component that is tangent to the surface (two rightmost possibilities).

If either one of the materials is a perfect electrical conductor (PEC), then SS is an equi-potential surface; i.e., the electric potential V is constant everywhere on S. Since E is proportional to the spatial rate of change of potential (recall $E=-\nabla V$ we find:

The component of E that is tangent to a perfectly- conducting surface is zero.

This is sometimes expressed informally as follows:

$$\boxed{E_{tan} = 0 \text{ on PEC surface}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 1$$

where "Etan" is understood to be the component of E that is tangent to S. Since the tangential component of E on the surface of a perfect conductor is zero, the electric field at the surface must be oriented entirely in the direction perpendicular to the surface, as shown in Figure

The following equation expresses precisely the same idea, but includes the calculation of the tangential component as part of the statement:

$$\mathbf{E} \times \hat{\mathbf{n}} = 0 \text{ (on PEC surface)} \quad \ldots\ldots\ldots\ldots\ldots 2$$

where n^ is either normal (i.e., unit vector perpendicular to the surface) to each point on S. This expression works because the cross product of any two vectors is perpendicular to either vector or any vector which is perpendicular to n^ is tangent to SS.

We now determine a more general boundary condition that applies even when neither of the media bordering SS is a perfect conductor. The desired boundary condition can be obtained directly from Kirchhoff's Voltage Law

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 3$$

Let the closed path of integration take the form of a rectangle centered on S, as shown in Figure

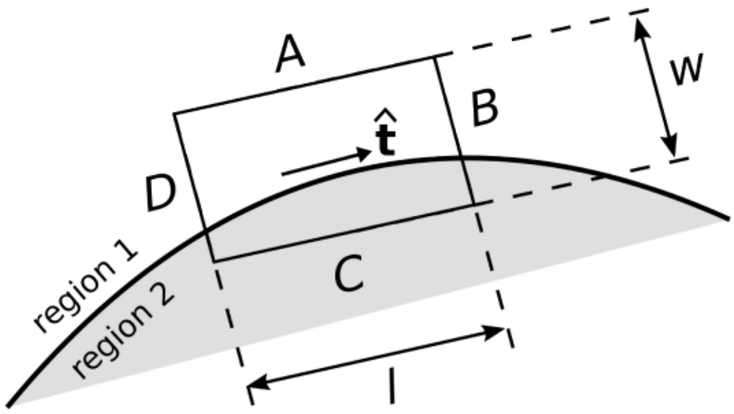


Fig.10.4 Use of KVL to determine the boundary condition on EE

Let the sides A, B, C, and D be perpendicular or parallel to the surface, respectively. Let the length of the perpendicular sides be w, and let the length of the parallel sides be l. From KVL we have

$$\begin{aligned}\oint_C \mathbf{E}\cdot d\mathbf{l} = \int_A \mathbf{E}\cdot d\mathbf{l} \\ + \int_B \mathbf{E}\cdot d\mathbf{l} \\ + \int_C \mathbf{E}\cdot d\mathbf{l} \\ + \int_D \mathbf{E}\cdot d\mathbf{l} \quad &= 0\end{aligned}$$

Now, let us reduce w and ll together while (1) maintaining a constant ratio w/l≪1 and (2) keeping CC centered on S. In this process, the contributions from the B and D segments become equal in magnitude but opposite in sign; i.e.,

$$\int_B \mathbf{E}\cdot d\mathbf{l} + \int_D \mathbf{E}\cdot d\mathbf{l} \to 0$$

……………..4

This leaves

$$\oint_C \mathbf{E}\cdot d\mathbf{l} \to \int_A \mathbf{E}\cdot d\mathbf{l} + \int_C \mathbf{E}\cdot d\mathbf{l} \to 0 \qquad \text{……………..5}$$

Let us define the unit vector t^ ("tangent") as shown in Figure 2. When the lengths of sides A and C become sufficiently small, we can write the above expression as follows:

$$\mathbf{E}_1 \cdot \hat{\mathbf{t}}\Delta l - \mathbf{E}_2 \cdot \hat{\mathbf{t}}\Delta l \to 0 \qquad \text{……………..6}$$

where E_1 and E_2 are the fields evaluated on the two sides of the boundary and Δl→0 is the length of sides A and C while this is happening. Note that the only way Equation (6) can be true is if the tangential components of E_1 and E_2 are equal. In other words:

The tangential component of E must be continuous across an interface between dissimilar media.

Note that this is a generalization of the result we obtained earlier for the case in which one of the media was a PEC – in that case, the tangent component of E on the other side of the interface must be zero because it is zero in the PEC medium.

As before, we can express this idea in compact mathematical notation. Using the same idea used to obtain Equation (2) we have found

$$\mathbf{E}_1 \times \hat{\mathbf{n}} = \mathbf{E}_2 \times \hat{\mathbf{n}} \quad \text{on } \mathcal{S} \qquad \text{……………7}$$

or, as it is more commonly written:

$$\boxed{\hat{\mathbf{n}} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad \text{on } \mathcal{S}} \qquad \text{……………….8}$$

We conclude this section with a note about the broader applicability of this boundary condition:

Above equation (8) is the boundary condition that applies to E for both the electrostatic and the general (time-varying) case.

Although a complete explanation is not possible without the use of the Maxwell-Faraday Equation the reason why this boundary condition applies in the time-varying case can be disclosed here. In the presence of time-varying magnetic fields, the right-hand side of Equation (3) may become non-zero and is proportional to the area defined by the closed loop. However, the above derivation requires the area of this loop to approach zero, in which case the possible difference from Equation (3) also converges to zero. Therefore, the boundary condition expressed in Equation (8) applies generally.

(iii) Boundary conditions at discontinuity for B:

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these interfaces.

In this section, we derive the boundary condition on the magnetic flux density B at a smooth interface between two material regions, as shown in Figure 1. The desired boundary condition may be obtained from Gauss' Law for Magnetic Fields (GLM):

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

………………1

where S is any closed surface. Let S take the form of cylinder centered at a point on the interface, and for which the flat ends are parallel to the surface and perpendicular to n^, as shown in Figure.1 Let the radius of this cylinder be a, and let the length of the cylinder be 2h.

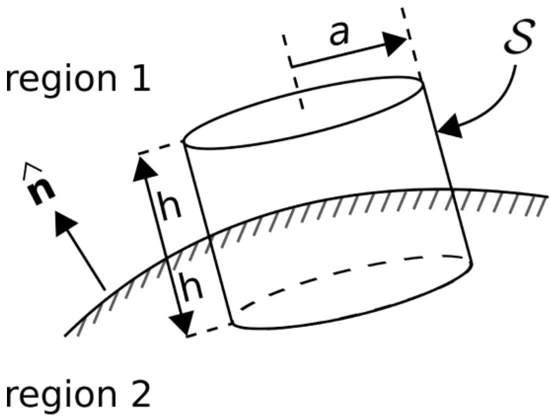


Fig.10.5 Determination of the boundary condition on B at the interface between material regions

From GLM, we have

$$\begin{aligned}\oint_S \mathbf{B} \cdot d\mathbf{s} = &\int_{top} \mathbf{B} \cdot d\mathbf{s} \\ &+ \int_{side} \mathbf{B} \cdot d\mathbf{s} \\ &+ \int_{bottom} \mathbf{B} \cdot d\mathbf{s} \qquad = 0\end{aligned}$$

Now let us reduce h and a together while (1) maintaining a constant ratio h/a≪1 and (2) keeping SS centered on the interface. Because h≪a, the area of the side can be made negligible relative to the area of the top and bottom. Then, as h→0, we are left with

$$\int_{top} \mathbf{B} \cdot d\mathbf{s} + \int_{bottom} \mathbf{B} \cdot d\mathbf{s} \to 0$$ ……………………….2

As the area of the top and bottom sides become infinitesimal, the variation in B over these areas becomes negligible. Now we have simply:

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} \Delta A + \mathbf{B}_2 \cdot (-\hat{\mathbf{n}}) \, \Delta A \to 0$$ …………………..3

where B1 and B2 are the magnetic flux densities at the interface but in regions 1 and 2, respectively, and ΔA is the area of the top and bottom sides. Note that the orientation of n^ is important – we have assumed n^ points into region 1, and we must now stick with this choice. Thus, we obtain

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0}$$ ……………4

where, as noted above, n^ points into region 1.

(iv) Boundary conditions at discontinuity for H:

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At a boundary between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these boundaries. In this section, we derive boundary conditions on the magnetic field intensity H.

To begin, consider a region consisting of only two media that meet at a smooth boundary as shown in Figure 1 The desired boundary condition can be obtained directly from Ampere's Circuital Law (ACL):

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_{encl}$$

where C is any closed path and I_{encl} is the current that flows through the surface bounded by that path in the direction specified by the "right-hand rule" of Stokes' theorem.

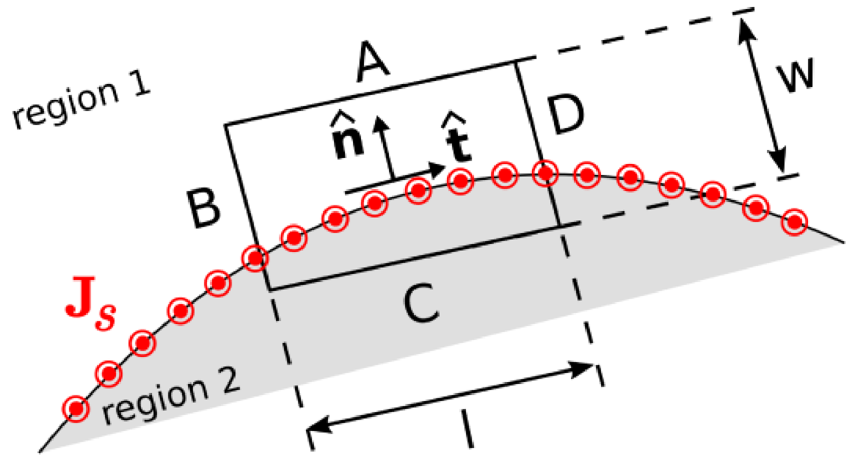


Fig.10.6 Determining the boundary condition on H at the smooth boundary between two material regions.

Let C take the form of a rectangle centered on a point on the boundary as shown in fig. perpendicular to the direction of current flow at that location. Let the sides A, B, C, and D be perpendicular and parallel to the boundary. Let the length of the parallel sides be l, and let the length of the perpendicular sides be w. Now we apply ACL. We must integrate in a counter-clockwise direction in order to be consistent with the indicated reference direction for Js.

Thus:

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_A \mathbf{H} \cdot d\mathbf{l} + \int_B \mathbf{H} \cdot d\mathbf{l} + \int_C \mathbf{H} \cdot d\mathbf{l} + \int_D \mathbf{H} \cdot d\mathbf{l} \qquad = I_{encl}$$

Now we let w and l become vanishingly small while (1) maintaining the ratio l/w and (2) keeping C centered on the boundary. In this process, the contributions from the B and D segments become equal in magnitude but opposite in sign; i.e.,

$$\int_B \mathbf{H} \cdot d\mathbf{l} + \int_D \mathbf{H} \cdot d\mathbf{l} \to 0 \qquad \text{......................2}$$

This leaves

$$\int_A \mathbf{H} \cdot d\mathbf{l} + \int_C \mathbf{H} \cdot d\mathbf{l} \to I_{encl} \qquad \text{.................3}$$

Let us define the unit vector t^ ("tangent") as shown in Figure.1. Now we have simply:

$$\mathbf{H}_1 \cdot \left(-\hat{\mathbf{t}}\Delta l\right) + \mathbf{H}_2 \cdot \left(+\hat{\mathbf{t}}\Delta l\right) = I_{encl} \qquad \text{.................4}$$

where H_1 and H_2 are the fields evaluated on the two sides of the boundary, and Δl→0 is the length of sides A and C.

As always, I_{encl} (units of A) may be interpreted as the flux of the current density Js (units of A/m) flowing past a line on the surface having length Δl (units of m) perpendicular to t^×n^, where n^ is the normal to the surface, pointing into Region 1. Stated mathematically:

$$I_{encl} \rightarrow \mathbf{J}_s \cdot \left(\Delta l\, \hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) \quad \ldots\ldots\ldots\ldots\ldots.5$$

Before proceeding, note this is true regardless of the particular direction we selected for t^; it is only necessary that t^ be tangent to the boundary. Thus, t^×n^ need not necessarily be in the same direction as Js. Now Equation.4 can be written:

$$\mathbf{H}_2 \cdot \hat{\mathbf{t}} \Delta l - \mathbf{H}_1 \cdot \hat{\mathbf{t}} \Delta l = \mathbf{J}_s \cdot \left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) \Delta l \quad \ldots\ldots\ldots\ldots.6$$

Eliminating the common factor of Δl and arranging terms on the left:

$$\left(\mathbf{H}_2 - \mathbf{H}_1\right) \cdot \hat{\mathbf{t}} = \mathbf{J}_s \cdot \left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) \quad \ldots\ldots\ldots\ldots\ldots7$$

The right side may be transformed using a vector identity (A·(B×C)=B·(C×A)=C·(A×B)) to obtain:

$$\left(\mathbf{H}_2 - \mathbf{H}_1\right) \cdot \hat{\mathbf{t}} = \hat{\mathbf{t}} \cdot \left(\hat{\mathbf{n}} \times \mathbf{J}_s\right) \quad \ldots\ldots\ldots\ldots\ldots8$$

Equation(8) is the boundary condition we seek. We have found that the component of $H_2 - H_1$ (the difference between the magnetic field intensities at the boundary) in any direction tangent to the boundary is equal to the component of the current density flowing in the perpendicular direction n^×Js. Said differently:

A discontinuity in the tangential component of the magnetic field intensity at the boundary must be supported by surface current flowing in a direction perpendicular to this component of the field.

An important consequence is that:

If there is no surface current, then the tangential component of the magnetic field intensity is continuous across the boundary.

It is possible to obtain a mathematical form of the boundary condition that is more concise and often more useful than Equation (8). This form may be obtained as follows. First, we note that the dot product with respect to t^ on both sides of Equation (8)means simply "any component that is tangent to the boundary." We need merely to make sure we are comparing the same tangential component on each side of the equation. For example n^×($H_2 - H_1$) is tangential to the boundary, since n^ is perpendicular to the boundary and therefore any cross product involving n^ will be perpendicular to n^. The corresponding component of the current density is n^×(n^×Js) , so Equation (8) may be equivalently written as follows:

$$\hat{\mathbf{n}} \times (\mathbf{H}_2 - \mathbf{H}_1) = \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{J}_s \quad \ldots\ldots\ldots\ldots\ldots..9$$

Applying a vector identity (A×(B×C)=B(A·C)−C(A·B)) to the right side of Equation (9) we obtain:

$$\begin{aligned}\hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{J}_s &= \hat{\mathbf{n}}\left(\hat{\mathbf{n}} \cdot \mathbf{J}_s\right) - \mathbf{J}_s\left(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}\right) \\ &= \hat{\mathbf{n}}\left(0\right) - \mathbf{J}_s\left(1\right) \\ &= -\mathbf{J}_s\end{aligned}$$

Therefore:

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right) = -\mathbf{J}_s \quad \ldots\ldots\ldots\ldots\ldots 10$$

The minus sign on the right can be eliminated by swapping H_2 and H_1 on the left, yielding

$$\boxed{\hat{\mathbf{n}} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{J}_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 11$$

This is the form in which the boundary condition is most commonly expressed.

It is worth noting what this means for the magnetic field intensity B. Since B=μH:

In the absence of surface current, the tangential component of B across the boundary between two material regions is discontinuous if the permeabilities are unequal.

10.4 Reflection and refraction at normal and oblique incidence of electric vectors perpendicular to boundary:

Reflection

Reflection is the abrupt change in the direction of propagation of a wave that strikes the boundary between two different media. At least some part of the incoming wave remains in the same medium. Assume the incoming light ray makes an angle θ_i with the normal of a plane tangent to the boundary. Then the reflected ray makes an angle θ_r with this normal and lies in the same plane as the incident ray and the normal.

$$\text{Law of reflection: } \theta_i = \theta_r$$

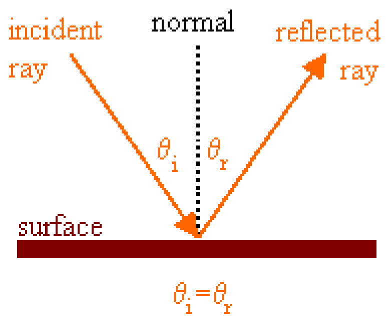


Fig.10.7 Reflection wave that strikes the boundary between two different media

Specular reflection occurs at smooth, plane boundaries. Then the plane tangent to the boundary is the boundary itself. Reflection at rough, irregular boundaries is diffuse reflection. The smooth surface of a mirror reflects light specularly, while the rough surface of a wall reflects light diffusely. The reflectivity or reflectance of a surface material is the fraction of energy of the oncoming wave that is reflected by it. The reflectivity of a mirror is close to 1.

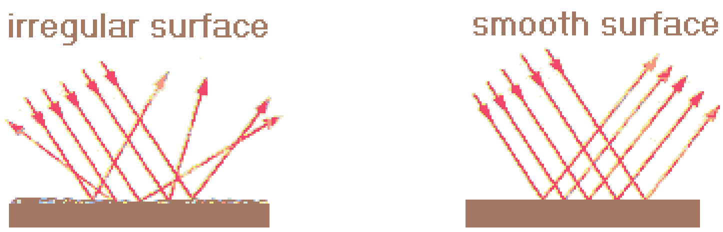


Fig.10.8

Refraction

Refraction is the change in direction of propagation of a wave when the wave passes from one medium into another, and changes its speed. Light waves are refracted when crossing the boundary from one transparent medium into another because the speed of light is different in different media. Assume that light waves encounter the plane surface of a piece of glass after traveling initially through air as shown in the figure to the right.

What happens to the waves as they pass into the glass and continue to travel through the glass? The speed of light in glass or water is less than the speed of light in a vacuum or air. The speed of light in a given substance is $v = c/n$, where n is the index of refraction of the substance. Typical values for the index of refraction of glass are between 1.5 and 1.6, so the speed of light in glass is approximately two-thirds the speed of light in air. The distance between wave fronts will therefore be shorter in the glass than in air, since the waves travel a smaller distance per period T.

If f is the frequency of the wave and $T = 1/f$ is the period, i.e. the time interval between successive crests passing a fixed point in space, then $\lambda_1 = v_1T = cT/n_1$ and $\lambda_2 = v_2T = cT/n_2$, or $\lambda_1/\lambda_2 = n_2/n_1$.

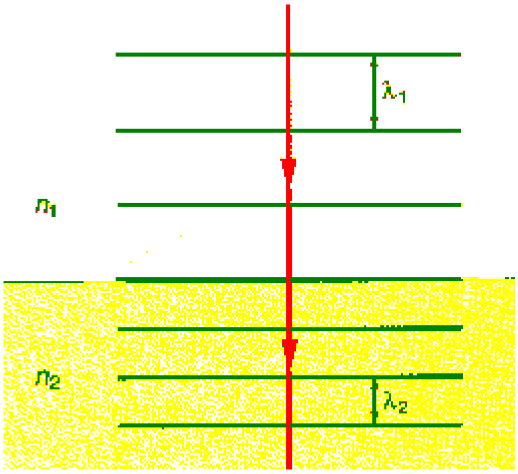


Fig.10.9

Now consider wave fronts and their corresponding light rays approaching the surface at an angle. We can see that the rays will bend as the wave passes from air to glass. The bending occurs because the wave fronts do not travel as far in one cycle in the glass as they do in air. As the diagram shows, the wave front halfway into the glass travels a smaller distance in glass than it does in air, causing it to bend in the middle. Thus, the ray, which is perpendicular to the wave front, also bends. The situation is like a marching band marching onto a muddy field at an angle to the edge of the field. The rows bend as the speed of the marchers is reduced by the mud. The amount of bending depends on the angle of incidence and on the indices of refraction of glass and air, which determine the change in speed. From the figure we can see that $\lambda_1/\lambda_2 = \sin\theta_1/\sin\theta_2$. But $\lambda_1/\lambda_2 = n_2/n_1$. Therefore $n_2/n_1 = \sin\theta_1/\sin\theta_2$, or $n_1\sin\theta_1 = n_2\sin\theta_2$.

This is Snell's law, or the law of refraction.

$$n_i\sin\theta_i = n_t\sin\theta_t.$$

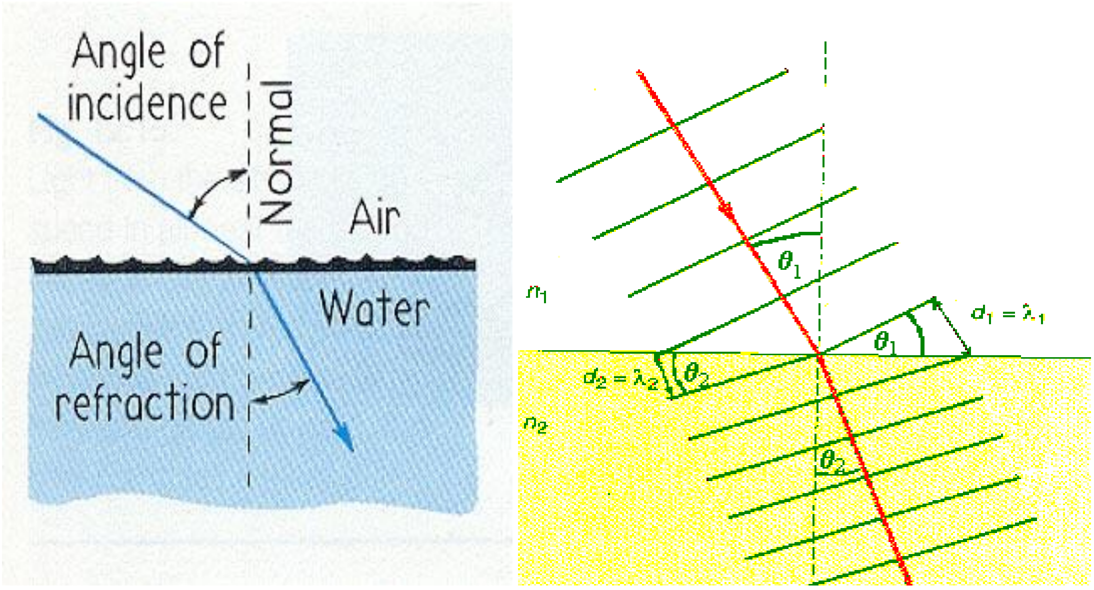


Fig.10.10 Law of refraction

When light passes from one transparent medium to another, the rays are bent toward the surface normal if the speed of light is smaller in the second medium than in the first. The rays are bent away from this normal if the speed of light in the second medium is greater than in the first. The picture on the right shows a light wave incident on a slab of glass.

One part of the wave is reflected, and another part is refracted as it passes into the glass. The rays are bent towards the normal. At the second interface from glass into air the light passing into the air is refracted again. The rays are now bent away from the normal.

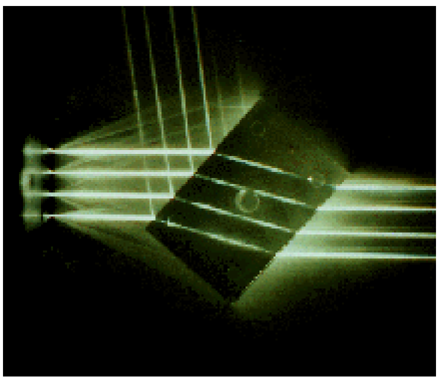


Fig.10.11 Second interface from glass into air the light passing into the air is refracted again

Perpendicular Polarization case – E Normal to Plane of Incidence:

As shown in figure is a perpendicular polarized wave incident at angle θ_i a dielectric medium 2. Snell's Law states that a reflected wave will be at the same angle $\theta_r = \theta_i$, and the transmitted wave in medium 2 at angle θ_t can be calculated using this law. The amplitude of the reflected and transmitted waves can be determined by applying the continuity of the tangential components of E & H at the boundary.

This is given by –

$$\hat{H}^i_\perp \cos\theta_i - \hat{H}^r_\perp \cos\theta_i = \hat{H}^t_\perp \cos\theta_t$$

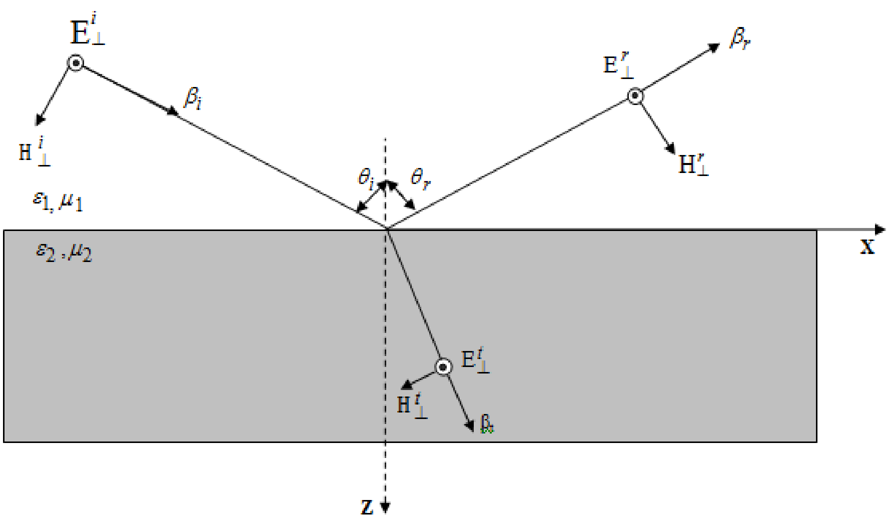


Fig.10.12 E Normal to Plane of Incidence

Since E & H are related by η,

$$\frac{\hat{E}^i_\perp}{\eta_1}\cos\theta_i - \frac{\hat{E}^r_\perp}{\eta_1}\cos\theta_i = \frac{\hat{E}^t_\perp}{\eta_2}\cos\theta_t$$

$$\hat{E}_{\perp}^{i} + \hat{E}_{\perp}^{r} = \hat{E}_{\perp}^{t} \qquad \text{At } z = 0$$

*Note: The exponential factors were canceled after substituting z = 0 and using Snell's Laws in the above two equations.

$$\hat{\Gamma}_{\perp} = \frac{\hat{E}_{\perp}^{r}}{\hat{E}_{\perp}^{i}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

And for nonmagnetic materials, $\mu_1 = \mu_2 = \mu_\circ$,

$$\hat{\Gamma}_{\perp} = \frac{\cos\theta_t - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}{\cos\theta_t + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}$$

at z = 0,

$$\hat{T}_{\perp} = \frac{\hat{E}_{\perp}^{t}}{\hat{E}_{\perp}^{i}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

For nonmagnetic material,

$$\hat{T}_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}$$

Oblique incidence of electric vectors perpendicular to boundary:

Plane waves are not normally incident, so now we must consider the general problem of a plane wave propagating along a specified axis that is arbitrarily relative to a rectangular coordinate system. The most convenient way is in terms of the direction cosines of the uniform plane wave, the equiphase surfaces are planes perpendicular to the direction of propagation.

Definitions:

Uniform planes wave generally have inform or constant properties in plane perpendicular to their direction of propagation. A free space plane wave at an infinite distance from the generator, having constant amplitude electric and magnetic field vectors over the equi-phase surfaces.

Equi-phase surface – any surface in a wave over which the field vectors of a particular instant have either 0° or 180° phase difference.

For a plane wave propagating along the +z axis

$$\hat{E}(z) = \hat{E}_m e^{-j\beta z} a_x \qquad (1)$$

Equation (6.1) states that each z equal to a constant plane will represent an equi-phase surface with no spatial variation in the electric or magnetic fields. In other words,

$$\frac{\partial}{\partial_x} = 0 = \frac{\partial}{\partial_y} \quad \Rightarrow \text{for a uniform plane wave}$$

It will be necessary to replace **z** for a plane wave traveling in an arbitrary direction with an expression when put equal to a constant (βz = constant), that will result in equi-phase surfaces.

The equation of an equi-phase plane is given by

$$\beta \cdot r = \beta n_{\beta} \cdot r$$

The radial vector (r) from the origin to any point on the plane, and β is the vector normal to the plane is shown in Figure.

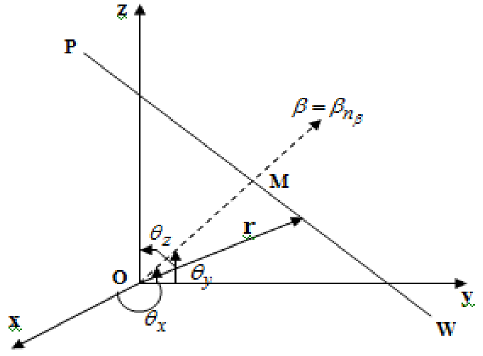


Fig.10.13

As you can see from figure, the plane perpendicular to the vector β is seen from its side appearing as a line P-W. The dot product $n_\beta \cdot r$ is the projection of the radial vector r along the normal to the plane and will have the constant value OM for all points on the plane. The equation $\beta \cdot r$ = constant is the characteristic property of a plane perpendicular to the direction of propagation β.

The equiphase equation is

$$\beta \cdot r = \beta_x x + \beta_y y + \beta_z z$$

$$= \beta\,(\cos \theta_x x + \cos \theta_y y + \cos \theta_z z)$$

= constant

$\mathbf{r} = x a_x + y a_y + z a_z$

$\beta = \beta_x a_x + \beta_y a_y + \beta_z a_z$

θ_x, θ_y, θ_z, are the angles the β vector makes with x, y, and z axes, respectively.

Definition:

Transverse electromagnetic wave (TEM) – electromagnetic wave having electric field vectors and magnetic field vectors perpendicular to the direction of propagation.

H is perpendicular to E, and both E and H are perpendicular to the direction of propagation β. The expressions for $\hat{\mathrm{E}}$ *and* $\hat{\mathrm{H}}$ are

$$\hat{\mathrm{E}} = \hat{\mathrm{E}}_m e^{-j\beta \cdot r} \qquad (2)$$

$$\hat{\mathrm{H}} = \frac{n_\beta \wedge \hat{\mathrm{E}}}{\eta}$$

The unit vector $\mathbf{n}_\beta$ along **β** and **η** is the wave impedance in the propagation medium. See Figure for the illustration of orthogonal relations between $\hat{\mathrm{E}}$ *and* $\hat{\mathrm{H}}$ and the direction of propagation.

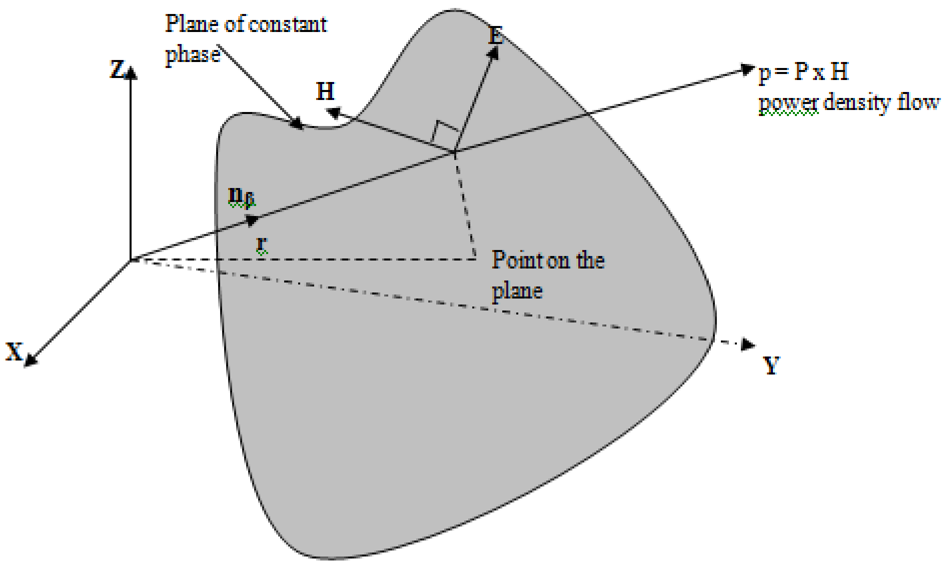


Fig.10.14 Orthogonal relations between $\hat{E}$ *and* $\hat{H}$ and the direction of propagation

10.5 Reflection and refraction at normal and oblique incidence of electric vectors parallel to boundary:

(i) Electric Field Normal to the Plane of Incidence:

The entire electric field is (out of the paper) in the y direction and the magnetic field will have both x and z components. See Figure.

The incident electric and magnetic fields are

$$\hat{E}^i = \hat{E}^i_m e^{-j\beta_i \cdot r}$$

$$\hat{H}^i = \frac{n_{\beta_i} \wedge \hat{E}^i}{\eta} = \frac{\hat{E}^i_m}{\eta}\left(-\cos\theta_i a_x + \sin\theta_i a_z\right) e^{-j\beta_i \cdot r}$$

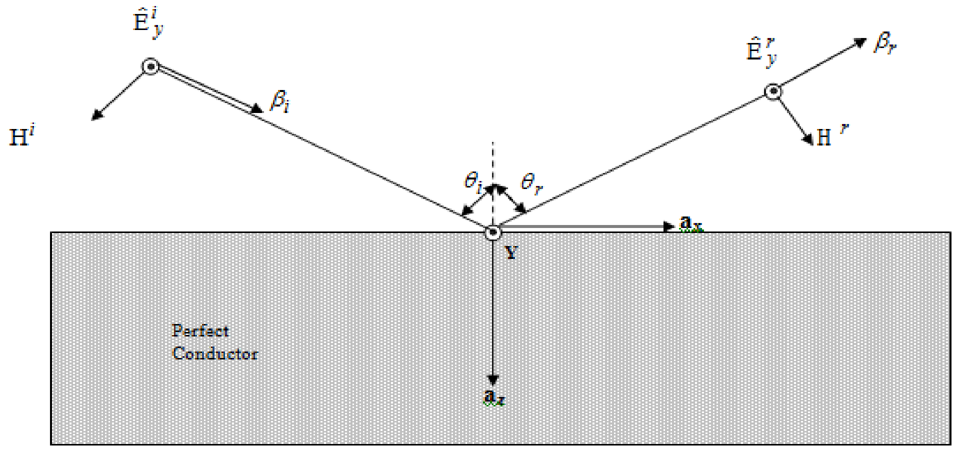


Fig.10.15 Electric Field Normal to the Plane of Incidence

where $\beta_i \cdot r = \beta\left(\sin\theta_i\, x - \cos\theta_i\, z\right)$.

Assume that the reflected field is also in the y direction so the magnetic field must be perpendicular to both E and the Poynting Vector P = E ^ H,

$$\hat{E}^r = \hat{E}_m^r e^{-j\beta_r \cdot r} a_y$$

$$\hat{H}^r = \frac{n\,\beta_r \wedge \hat{E}^r}{\eta} = \frac{\hat{E}_m^r}{\eta}\left(\cos\theta_r a_x + \sin\theta_r a_z\right) e^{-j\beta_r \cdot r}$$

Where $\beta_r \cdot r = \beta\left(\sin\theta_r x - \cos\theta_r z\right)$. Determine the angle of reflection θ_r and the amplitude of the reflected electric field $\hat{E}_m^r$ by using the boundary conditions at z = 0. This also includes zero values of the tangential electrical field E and the normal component of the magnetic field H.

$$\hat{E}_y(x,z) = \hat{E}_y^i + \hat{E}_y^r = 0 \qquad \text{at } z = 0$$

Therefore,

$$\hat{E}_y(x,0) = \hat{E}_m^i \, e^{-j\beta x \sin\theta_i} + \hat{E}_m^r \, e^{-j\beta x \sin\theta_r} = 0$$

And

$$\hat{H}_z(x,0) = \frac{1}{\eta} \hat{E}_m^i \sin\theta_i \, e^{-j\beta x \sin\theta_i} + \frac{1}{\eta} \hat{E}_m^r \sin\theta_i \, e^{-j\beta x \sin\theta_r} = 0$$

Note: These two conditions will provide the same results for the unknowns $\theta_r \, and \hat{E}_m^r$, and be true for every value of x along z = 0 plane, so the phase factors must be equal.

$$\theta_r = \theta_i$$

And

$$\hat{E}_m^r = -\hat{E}_m^i$$

Negative sign indicates the opposite direction of the reflected electric field (i.e. into the paper)

The total E field is

$$\hat{E}_y(x, z) = \hat{E}_m^i \, e^{-j\beta x \sin\theta_i} \left(e^{-j\beta z \cos\theta_i} - e^{j\beta z \cos\theta_i} \right)$$

$$= -2j\hat{E}_m^i \left[\sin\left(\beta z \cos\theta_i\right) \right] e^{-j\beta x \sin\theta_i}$$

The total H field is

$$\hat{H} = \hat{H}^i + \hat{H}^r = \left[\frac{n\beta_i}{\eta} \wedge a_y \hat{E}_m^i \, e^{-j\beta_i \cdot r} \right] - \left[\frac{n\beta_i}{\eta} \wedge a_y \, \hat{E}_m^i \, e^{-j\beta_i \cdot r} \right]$$

And the substitution of $\hat{E}_m^r = \hat{E}_m^i$ has been made. The direction vectors of the incident and reflective wave are

$$n_{\beta i,\ r} = \sin\theta_i\, a_x \pm \cos\theta_i\, a_z$$

And

$$n_{\beta i, r} \wedge a_y = \sin\theta_i\, a_z \mp \cos\theta_i\, a_x$$

The components of the total magnetic field are

$$\hat{\mathrm{H}}_x\,(x, z) = \frac{-2\hat{\mathrm{E}}_m^i}{\eta} \cos\theta_i \cos\left(\beta z \cos\theta_i\right) e^{-j\beta x \sin\theta_i}$$

$$\ddot{\mathrm{H}}_z\,(x,\ z) = \frac{-2j\hat{\mathrm{E}}_m^i}{\eta} \sin\theta_i \sin\left(\beta z \cos\theta_i\right)\ e^{-j\beta x \sin\theta_i}$$

There is a standing-wave in the z direction because the reflected and incident waves travel in the opposite direction along the z-axis. The fields traveling in the x direction and having the only nonzero power flow in the direction parallel to the interface.

The concept can be illustrated by considering the average density flow associated with the wave.

$$\mathrm{P}_{ave}\,(x,\ z) = \frac{1}{2}\ \mathrm{Re}\left[\hat{\mathrm{E}} \wedge \hat{\mathrm{H}}^*\right]$$

$$= \frac{1}{2}\mathrm{Re}\begin{vmatrix} a_x & a_y & az \\ 0 & T & 0 \\ S & 0 & W \end{vmatrix}$$

$$= \frac{2\left|\hat{\mathrm{E}}_m^i\right|}{\eta} \sin\theta_i \sin^2\left(\beta z \cos\theta_i\right) a_x$$

$\Rightarrow$This indicates that the power flow is in the x direction.

$$S = \frac{-2\hat{\mathrm{E}}_m^i}{\eta} \cos\theta_i\ \cos\left(\beta z \cos\theta_i\right) e^{+j\beta x \sin\theta_i}\ ,\quad T = -2\,\hat{\mathrm{E}}_m^i \sin\left(\beta z \cos\theta_i\right) e^{-j\beta x \sin\theta_i}\ ,$$

$$W = \frac{+2\hat{E}_m^{i*}}{\eta} \sin\theta_i \sin(\beta z \cos\theta_i) e^{+j\beta x \sin\theta_i}$$

Parallel Polarization Case – E is in Plane of Incidence:

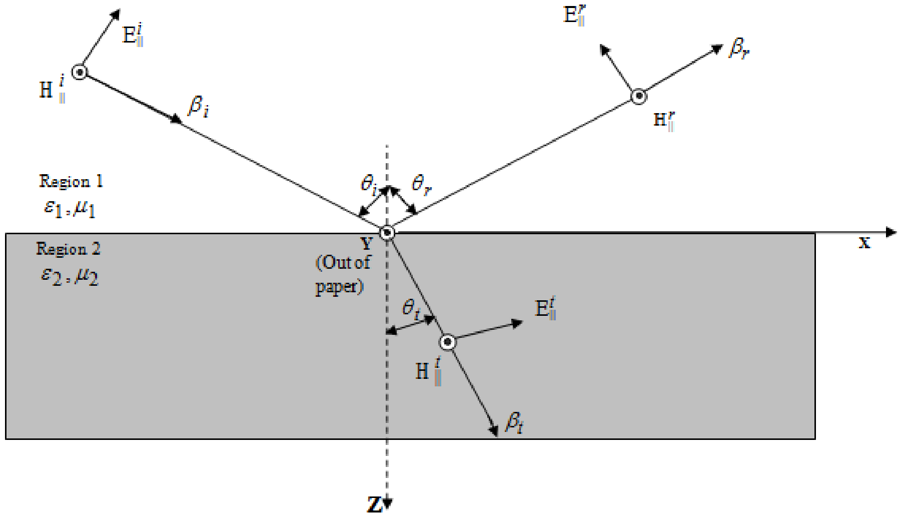


Fig.10.16 E is in Plane of Incidence

The unknown amplitudes of the reflected and transmitted electric fields $E_{\|}^r$ *and* $E_{\|}^t$ can be determined by simply applying the boundary conditions at the dielectric interface. The electric fields $E_{\|}^r$ *and* $E_{\|}^t$ will now be used in the analysis to emphasize the case of parallel polarization, instead of using the electric fields E_m^r *and* E_m^t.

The tangential component of H should be continuous across the boundary. Therefore,

$$\hat{H}_{\|}^i e^{-j\beta_i \cdot r} a_y + \hat{H}_{\|}^r e^{-j\beta_i \cdot r} a_y = \hat{H}_{\|}^t e^{-j\beta_i \cdot r} a_y$$

There is no need to carry the a_y vector, because the magnetic fields only have one component in the y direction. Recall that this relation is valid at z = 0,

$$\hat{H}_{\parallel}^{i}\, e^{-j\beta_1 (\sin\theta_i x)} + \hat{H}_{\parallel}^{r}\, e^{-j\beta_1 (\sin\theta_r x)} = \hat{H}_{\parallel}^{t}\, e^{-j\beta_i (\sin\theta_t x)}$$

β_1 & β_{2_1} are the magnitudes of β in regions 1 & 2, respectively. In order for this to be valid at any value of x at any point on the interface, and knowing $\theta_i = \theta_r$:

$$\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$

Or

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{\beta_2}{\beta_1} = \frac{\frac{\omega}{V_2}}{\frac{\omega}{V_1}} = \frac{V_1}{V_2}$$

* This is the same relation that was determined earlier from Snell's Law. Substitute

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2}$$

$$\hat{H}_{\parallel}^{i} + \hat{H}_{\parallel}^{r} = \hat{H}_{\parallel}^{t} \qquad \text{At } z = 0$$

E and H are related by η, so equation can be rewritten as

$$\hat{E}_{\parallel}^{i} + \hat{E}_{\parallel}^{r} = \frac{\eta_1}{\eta_2}\hat{E}_{\parallel}^{t}$$

Tangential components of E must be continuous across the boundary, therefore

$$\hat{E}_{\|}^{i}\cos\theta_i - \hat{E}_{\|}^{r}\cos\theta_r = \hat{E}_{\|}^{t}\cos\theta_t \quad \text{At } z = 0$$

*Remember the exponential terms cancel out z = 0, (Snell's Law).

Equations 6.15 & 6.16 are solved by –

$$\hat{E}_{\|}^{r} = \hat{E}_{\|}^{i}\,\frac{\eta_1\cos\theta_i - \eta_2\cos\theta_t}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

And

$$\hat{E}_{\|}^{t} = \hat{E}_{\|}^{i}\,\frac{2\eta_2\cos\theta_i}{\eta_1\cos\theta_i + \eta_2\cos\theta_t}$$

*Making use of the fact that $\theta_i = \theta_r$. Define the reflection coefficient $\hat{\Gamma}_{\|}$ and the transmission $\hat{T}_{\|}$:

$$\hat{\Gamma}_{\|} = \frac{\hat{E}_{\|}^{r}}{E_{\|}^{i}} = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i} = \left.\frac{\cos\theta_t - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i}{\cos\theta_t + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i}\right|_{\mu_1=\mu_2=\mu_\circ}$$

And

$$\hat{T}_{\|} = \frac{\hat{E}_{\|}^{t}}{E_{\|}^{i}} = \frac{2\eta_2\cos\theta_t - \eta_1\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_t} = \left.\frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i}\right|_{\mu_1=\mu_2=\mu_\circ}$$

The total electric field in region 1 is

$$\hat{E}_{\|}^{tot} = \hat{E}_{\|}^{i} + \hat{E}_{\|}^{r} = \hat{E}_{m}^{i}(\cos\theta_i\,a_x - \sin\theta_i\,a_z)\,e^{-j\beta_i\cdot r} + \hat{E}_{m}^{r}(-\cos\theta_r\,a_x - \sin\theta_r\,a_z)\,e^{-j\beta_r\cdot r}$$

$$= \cos\theta_i\,\hat{E}_{m}^{i}\,e^{-j\beta x\sin\theta_i}\,(e^{-j\beta z\cos\theta_i} + \hat{\Gamma}_{\|}e^{j\beta z\cos\theta_i})\,a_x$$

$$+\underbrace{\sin\theta_i \hat{E}_m^i e^{-j\beta x\sin\theta_i}}_{\substack{Traveling-wave\\ part}} \underbrace{\left(-e^{-j\beta z\cos\theta_i}+\hat{\Gamma}_{||} e^{j\beta z\cos\theta_i}\right)}_{\substack{Standing\ plus\\ traveling waves}} a_z$$

Substituted $\beta_i \cdot r, \beta_r \cdot r$ from expressions derived earlier, and $\hat{E}_m^r / \hat{E}_m^i = \hat{\Gamma}_{||}$.

Equation states that there is a traveling-wave field in the x direction, and a traveling and standing wave field in the z direction. The difference is that $\hat{\Gamma}_{||} \neq 1$, but that $\hat{\Gamma}_{||} = -\hat{E}_m^r / \hat{E}_m^i$. By rearranging the second term in a_x component of the total field –

$$\left[\left(1-\hat{\Gamma}_{||}\right) e^{-j\beta z\cos\theta_i} + 2\hat{\Gamma}_{||}\left(\beta z\cos\,\theta_i\right)\right]$$

This expression indicates that a wave of amplitude $\left(1-\hat{\Gamma}_{||}\right)$ is propagating in the z direction and another wave of amplitude $\left(2\hat{\Gamma}_{||}\right)$ has the characteristics of a standing wave along the z axis. The characteristic of the wave along the z axis is a combination of a traveling and standing wave. If $\hat{\Gamma}_{||} = 1$ the amplitude of the traveling wave will be zero, and the wave characteristic along the z axis will be a totally standing wave. If $\hat{\Gamma}_{||} = 0$, the amplitude of the standing wave will be zero and the wave characteristic in the z direction would be a totally traveling wave.

The magnetic field in region 1 is

$$\hat{H}_{||}^{tot} = \hat{H}_{||}^i + \hat{H}_{||}^r = \hat{H}_m^i e^{-j\beta_i \cdot r} a_y \quad + \hat{H}_m^r e^{-j\beta_r \cdot r} a_y$$

$$= \frac{\hat{E}_m^i}{\eta_1} e^{-j\beta\, x\sin\theta_i}\left(e^{-j\beta\, z\,\cos\theta_i} + \frac{\hat{E}_m^i}{\hat{E}_m^r} e^{-j\beta x\cos\theta_i}\right) a_y$$

$$= \frac{\hat{E}_m^i}{\eta_1} e^{-j\beta x\sin\theta_i}\left(e^{-j\beta\, z\,\cos\theta_i} - \hat{\Gamma}_{||}\, e^{j\beta z\cos\theta_i}\right) a_y$$

The transmitted fields in medium 2 are

$$\hat{E}^{i}_{\parallel} = \hat{E}^{t}_{m}\left(\cos\theta_t\, a_x - \sin\theta_t\, a_z\right)e^{-j\beta_t \cdot r}$$

$$= \hat{T}_{\parallel}\, \hat{E}^{i}_{m}\left(\cos\theta_t\, a_x - \sin\theta_t\, a_z\right)e^{-j\beta_t \cdot r}$$

And

$$\hat{H}^{t}_{\parallel} = \hat{H}^{t}_{m}\, a_y\, e^{-j\beta_t \cdot r} = \frac{\hat{T}_{\parallel}\, \hat{E}^{i}_{m}}{\eta_2} e^{-j\beta_t \cdot r}\, a_y$$

Where $\beta_t \cdot r = \beta_2\left(x\sin\theta_t + z\cos\theta_t\right)$

And $\hat{E}^{t}_{m} / \hat{E}^{i}_{m} = T_{\parallel}$.

E Field Parallel to Plane of Incidence:

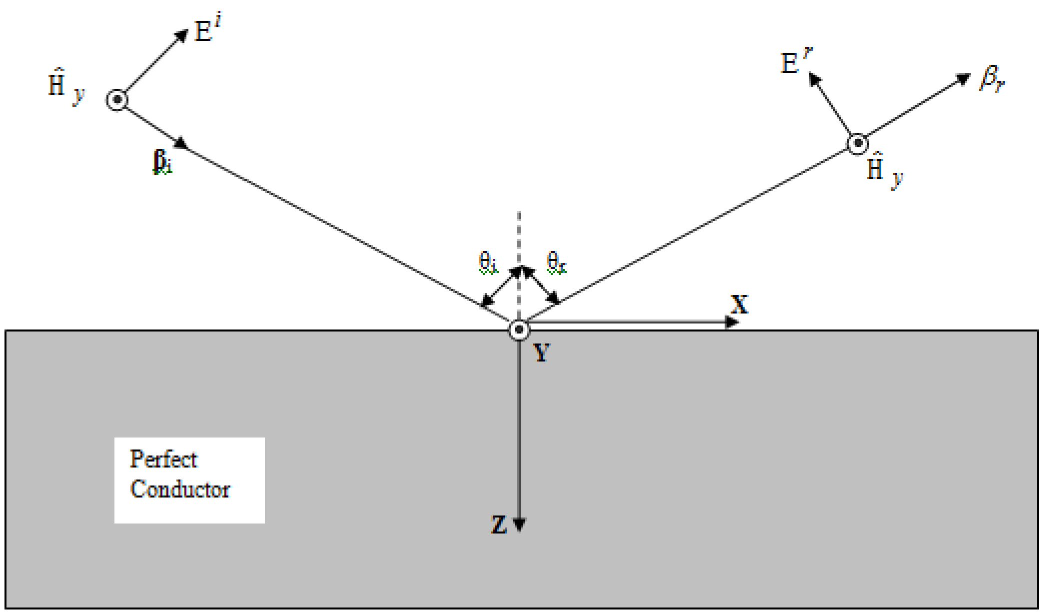


Fig.10.17 E Field Parallel to Plane of Incidence

The figure shows an incident wave polarized with the E field in the plane of incidence and the power flow in the direction of β_i at angle θ_i with respect to the normal to the surface of the perfect conductor.

The direction of propagation is given by the Poynting vector and the β_i, E, and H fields need to be arranged so that β_i is in the same direction as $\mathrm{E}^i \wedge \mathrm{H}^i$ at any time. The magnetic field is out of the plane of the paper, $\mathrm{H} = \hat{\mathrm{H}}_y a_y$ for the direction of the electric field shown. There is no transmitted field within the perfect conductor; however there will be a reflected field with power flow at the angle θ_r with respect to the normal to the interface. To maintain the power density flow $\mathrm{E}^r \wedge \mathrm{H}^r$ will be in the same direction β_r as. The expression for the total electric field in free space is

$$\hat{\mathrm{E}} = \hat{\mathrm{E}}^i + \hat{\mathrm{E}}^r = \hat{\mathrm{E}}_m^i e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r e^{-j\beta_r \cdot r}$$

$$\beta_i \cdot r = \beta \left(\cos\theta_i a_z + \sin\theta_i a_x\right) \cdot \left(x a_x + y a_y + z a_z\right)$$

$$= \beta\left(x \sin\theta_i + z\cos\theta_i\right)$$

$$\beta_r \cdot r = \beta\left(x \sin\theta_r - z\cos\theta_r\right)$$

The total electric field has x and z components:

$$\hat{\mathrm{E}}_x(x,z) = \hat{\mathrm{E}}_m^i \cos\theta_i e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r \cos\theta_r e^{-j\beta_r \cdot r}$$

$$\hat{\mathrm{E}}_z(x,z) = -\hat{\mathrm{E}}_m^i \sin\theta_i e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r \sin\theta_r e^{-j\beta_r \cdot r}$$

$$\hat{\mathrm{E}}_x\Big|_{at\, z=0} = \hat{\mathrm{E}}_m^i \cos\theta_i e^{-j\beta_i \cdot r} - \hat{\mathrm{E}}_m^r \cos\theta_r e^{-j\beta_r \cdot r} = 0$$

$$= \hat{E}_m^i \cos\theta_i \, e^{-j\beta x \cdot \sin\theta_i} - \hat{E}_m^r \cos\theta_r e^{-j\beta x \cdot \sin\theta_r} = 0$$

Equation shows the relationship between the incident and reflected amplitudes for a perfect conductor the total tangential E field at the surface must be zero which satisfies the boundary condition. To be zero at all values of x along the surface of the conducting plane, the phase terms must be equal to each other –

$$\theta_{i} = \theta_r$$

Equation is known as Snell's law of reflection.

Definition: Snell's Law is a rule of Physics that applies to visible light passing from air (or vacuum) to some medium with an index of refraction different from air.

From above equations–

$$\hat{E}_m^i = \hat{E}_m^r$$

Therefore, the total electric field in free space is

$$\hat{E}(x,z) = \hat{E}_x(x,z) a_x + \hat{E}_z(x,z) a_z$$

$$= \hat{E}_{mi} \cos\theta_i \, e^{-j\beta x \sin\theta_i} \left(e^{-j\beta z \cos\theta_i} - e^{j\beta z \cos\theta_i} \right) a_x$$

$$- \hat{E}_m^i \sin\theta_i \, e^{-j\beta x \sin\theta_i} \left(e^{-j\beta z \cos\theta_i} + e^{j\beta z \cos\theta_i} \right) a_z$$

$$= 2j\hat{E}_m^i \cos\theta_i \sin\left(\beta z \cos\theta_i\right) e^{-j\beta x \sin\theta_i} a_x$$

$$- 2j\hat{E}_m^i \sin\theta_i \cos\left(\beta z \cos\theta_i\right) e^{-j\beta x \sin\theta_i} a_z$$

$$= 2\hat{E}_m^i[-j\cos\theta_i \sin\left(\beta z \cos\theta_i\right) \; a_x$$

$$-\sin\theta_i \cos\left(\beta z \cos\theta_i\right) a_z] \, e^{-j\beta x \sin\theta_i}$$

Take equation and recover the time-domain form of the total electric field

$$E(r,t) = \mathrm{Re}\left(\hat{E}\,(r)\, e^{j\varpi t}\right)$$

Observe the variation of the total field with the x variable indicating there is a traveling wave in the x direction with a phase constant

$$\beta_x = \beta \sin\theta_i$$

And in the z direction the field forms a standing wave.

The total magnetic field is

$$\hat{H}(x,z) = \hat{H}_y\,(x,z)\, a_y = \hat{H}_y^i\,(x,z)\, a_y + \hat{H}_y^r\,(x,z)\, a_y$$

Use the relation $H = \dfrac{n_\beta \wedge E}{\eta}$ for each of the incident and reflected fields to employ the expressions x and z components of the incident and reflected electric fields.

$$\hat{H}^i = \frac{n_{\beta i}}{\eta} \wedge \hat{E}^i$$

$$= \frac{1}{\eta}\begin{vmatrix} a_x & a_y & a_z \\ \sin\theta_i & 0 & \cos\theta_i \\ \hat{E}_m^i \cos\theta_i \, e^{-j\beta(\sin\theta_i x + \cos\theta_i z)} & 0 & -\hat{E}_m^i \sin\theta_{i_i} \, e^{-j\beta(\sin\theta_i x + \cos\theta_i z} \end{vmatrix}$$

The solution of the determinant, the only nonzero component of $\hat{H}^i$ is the a_y component given by

$$\hat{H}^i = \frac{1}{\eta} a_y \left[\hat{E}_m^i \cos^2\theta_i \, e^{-j\beta(\sin\theta_i x + \cos\theta_i z)} + \hat{E}_m^i \sin^2\theta_i \, e^{-j\beta(\sin\theta_i x + \cos\theta_i z)} \right]$$

$$= \frac{\hat{E}_m^i}{\eta} e^{-j\beta(\sin\theta_i x + \cos\theta_i z)} a_y$$

The reflected magnetic fields is given by

$$\hat{H}^r = \frac{\hat{E}_m^i}{\eta} e^{-j\beta(\sin\theta_i x - \cos\theta_i z)} a_y$$

The total magnetic field $\hat{H}$ (x, z) is

$$\hat{H}(x,z) = a_y \frac{2\hat{E}_m^i}{\eta} \cos(\beta z \cos\theta_i) \, e^{-j\beta x \sin\theta_i}$$

The average power flow parallel to the conducting surface is

$$P_{ave}(x,z) = \frac{1}{2} \operatorname{Re}\left[\hat{E} \wedge \hat{H}\right]$$

$$= \frac{1}{2} \operatorname{Re} \begin{vmatrix} a_x & a_y & a_z \\ \hat{E}_x & 0 & \hat{E}_z \\ 0 & \hat{H}_y^* & 0 \end{vmatrix}$$

The cross product yields two components:

- One in the x direction
- One in the z direction

$$P_{ave} = \frac{1}{2} \operatorname{Re}\left[-\hat{E}_z \hat{H}_y^* a_x + \hat{E}_x \hat{H}_y^* a_z \right]$$

The expression of P_{ave} will reduce to

$$P_{ave}(x,z)=\frac{1}{2}\mathrm{Re}\left[-\hat{E}_z\hat{H}_y^*\right]a_x\ \frac{2\left|\hat{E}_m^i\right|}{\eta}\ \sin\theta_i\cos^2\left[\beta\ z\cos\theta_i\right]a_x$$

Glancing Incident:

$\left(\theta_i\to 90°\right)$, $P_{ave}=\dfrac{\left(2\left(\hat{E}_m^i\right)^2\right)}{\eta}a_x$, the power flow is at maximum.

Normal Incident:

$\theta_i=0,\ \ P_{x,ave}=0$ (Power flow in the x direction is zero)

Average power flow perpendicular to the conducting surface is zero, because the average Poynting Vector is zero in that direction

$$P_{z,ave}=\frac{1}{2}\mathrm{Re}\left(\hat{E}_x\ \hat{H}_y^*\right)=0$$

Why? Because $\hat{E}_x$ is multiplied by j, therefore $\hat{E}_x$ *and* $\hat{H}_y$ are out of phase by 90°. Therefore, a traveling-wave pattern occurs in the x direction, because the incident and reflected waves travel in the same direction, the standing-wave pattern will be observed in the z direction, because the incident and reflected waves travel in the opposite directions.

The location of zeros (nodes) of the $\hat{E}_x$ field can be found by letting sin $\left(\beta z\cos\theta_i\right)$ = 0. At a distance z from the conducting plane given by

$$\beta z\cos\theta_i=n\pi$$

Or

$$z = n\ \frac{\lambda}{2\cos\ \theta_i}\qquad n=0,1,2,\ldots$$

The zeros will occur at distances larger than integer multiples of $\lambda/2$. So, for normal incidence, $\theta_i = 0$, $\cos\theta_i = 1$, and the positions of the zeros will are the same as those discussed in chapter 5. For the oblique incidence, the locations of the standing-wave nodes are $\lambda/2$ apart along the direction of propagation. The wavelength measured along the z-axis is greater than the wavelength of the incident waves along the direction of propagation. As shown in fig the relation between these wavelengths is $\lambda_z = \dfrac{\lambda}{\cos\theta_i}$.

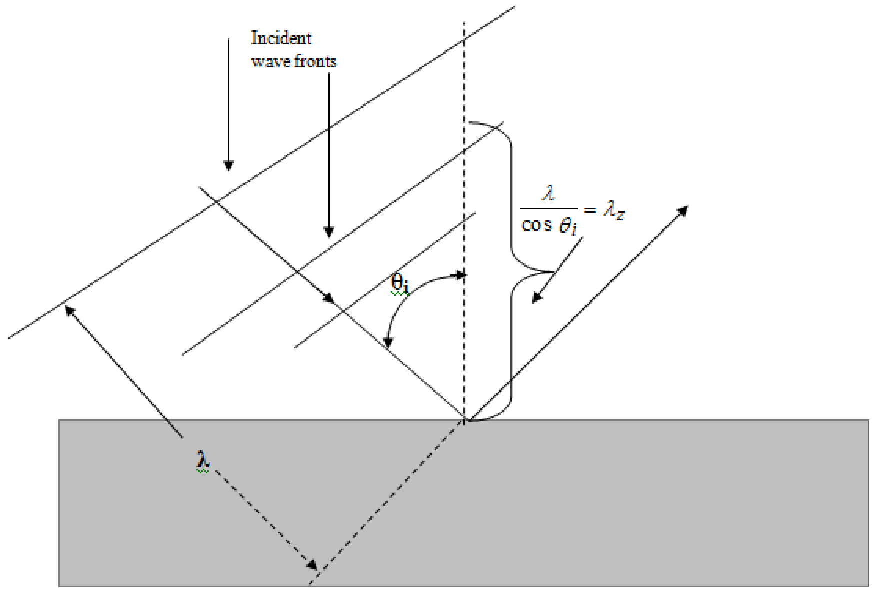


Fig.10.18 Wavelength measured along the z-axis is greater than the wavelength of the incident waves along the direction of propagation

The plane of the zero $\hat{E}_x$ field occur at multiples of $\lambda/2$ along the direction of propagation, and they are located at integer multiples of $\lambda_z/2$ along the z-axis which appear separated by larger distances. Also note that the standing-wave pattern associated with the $\hat{E}_z$ component may appear as if there is no zero value of the electric field at z = 0, but the $\hat{E}_z$ component is normal to the reflecting surface, therefore the boundary condition is not in violation.

SAQ.1

a) Define the Boundary conditions at discontinuity for D and E.
b) Define the Boundary conditions at discontinuity for B and H.
c) What do you mean by Reflection and refraction at normal incidence of electric vectors perpendicular to boundary?
d) Discuss the Reflection and refraction at oblique incidence of electric vectors parallel to boundary.

10.6 Total internal reflection:

A good-quality mirror may reflect more than 90% of the light that falls on it, absorbing the rest. But it would be useful to have a mirror that reflects all of the light that falls on it. Interestingly, we can produce *total reflection* using an aspect of *refraction*.

Consider what happens when a ray of light strikes the surface between two materials, such as is shown in Figure 1a. Part of the light crosses the boundary and is refracted; the rest is reflected. If, as shown in the figure, the index of refraction for the second medium is less than for the first, the ray bends away from the perpendicular. (Since $n_1 > n_2$, the angle of refraction is greater than the angle of incidence—that is, $\theta_1 > \theta_2$.) Now imagine what happens as the incident angle is increased. This causes θ_2 to increase also. The largest the angle of refraction θ_2 can be is *90º*, as shown in Figure 1b.The *critical angle* θc for a combination of materials is defined to be the incident angle θ_1 that produces an angle of refraction of 90º. That is, θc is the incident angle for which θ_2 = 90º. If the incident angle $\theta_1$is greater than the critical angle, as shown in Figure 1c, then all of the light is reflected back into medium 1, a condition called *total internal reflection*.

Critical Angle:

The incident angle θ_1 that produces an angle of refraction of 90^o is called the critical angle, θ_c.

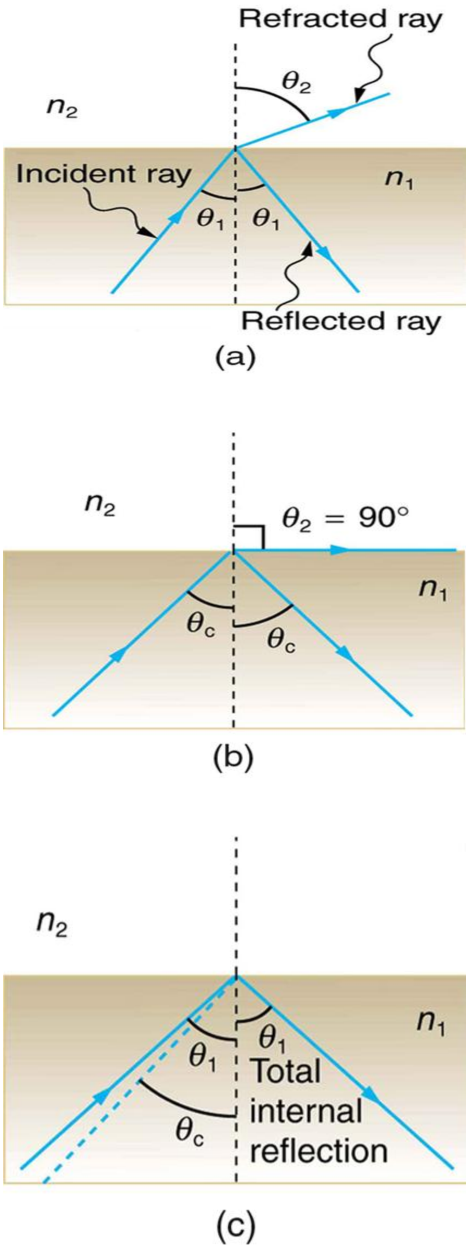


Fig.10.19 (a) A ray of light crosses a boundary where the speed of light increase and the index of refraction decreases. That is, $n_2 < n_1$. The ray

bends away from the perpendicular. (b) The critical angle θc is the one for which the angle of refraction is. (c) Total internal reflection occurs when the incident angle is greater than the critical angle.

Snell's law states the relationship between angles and indices of refraction. It is given by

$$n_1 \sin \theta_1 = n_2 \sin \theta_2.$$

When the incident angle equals the critical angle ($\theta_1 = \theta_c$), the angle of refraction is *90º* ($\theta_2 = 90º$). Noting that sin 90º = 1, Snell's law in this case becomes

$$n_1 \sin \theta_1 = n_2.$$

The critical angle θc for a given combination of materials is thus

$$\theta c = \sin^{-1}(n_2/n_1) \quad \text{for } n_1 > n_2.$$

Total internal reflection occurs for any incident angle greater than the critical angle θc, and it can only occur when the second medium has an index of refraction less than the first. Note the above equation is written for a light ray that travels in medium 1 and reflects from medium 2, as shown in the figure.

Brewster's Law:

Brewster's law is a relationship of light waves at the maximum polarization angle of light. This law is named after Sir David Brewster, a Scottish physicist, who proposed the law in the year 1811. The law states that the p-polarized rays vanish completely on different glasses at a particular angle.

Further, the polarization angle is also called as Brewster's angle. It is an angle of incidence where the ray of light having a p-polarization transmitted through a dielectric surface that is transparent without any reflection. While, the un-polarized light at this angle is transmitted, the light is reflected from the surface.

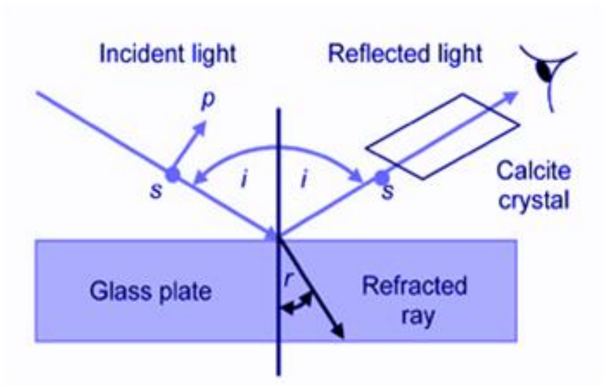


Fig.10.20 Relationship of light waves at the maximum polarization angle of light

Brewster was able to determine that the refractive index of the medium is numerically equal to the tangent angle of polarization. Know more about the Brewster's Law Formula.

$\mu = \tan i$

where,

μ = Refractive index of the medium,

i = Polarization angle.

From Snell's Law:

$$\mu = \frac{\sin i}{\sin r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

From Brewster's Law:

$$\mu = \tan i = \frac{\sin i}{\cos i} \quad \ldots\ldots\ldots\ldots\ldots (2)$$

Comparing both formulas (1) and (2)

$$\cos i = \sin r = \cos\left(\frac{\pi}{2} - r\right)$$

$$i = \frac{\pi}{2} - r, or\ i + r = \frac{\pi}{2}$$

As, i + r = (π/2) < ABC is also equal to the (π/2).

Therefore, the reflected and the refracted rays are at right angles to each other.

Degree of polarization:

Degree of polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized. A perfectly polarized wave has a DOP of 100%, whereas an un-polarized wave has a DOP of 0%. A wave which is partially polarized, and therefore can be represented by a superposition of a polarized and un-polarized component, will have a DOP somewhere in between 0 and 100%. DOP is calculated as the fraction of the total power that is carried by the polarized component of the wave.

DOP can be used to map the strain field in materials when considering the DOP of the photoluminescence. The polarization of the photoluminescence is related to the strain in a material by way of the given material's photo elasticity tensor.

DOP is also visualized using the Poincare sphere representation of a polarized beam. In this representation, DOP is equal to the length of the vector measured from the center of the sphere.

10.7 Plane wave propagation in plasma and its properties (qualitative):

Plasma waves the physical description of an electromagnetic wave propagating in a given medium necessitates a self-consistent handling of the particles comprising the medium (and their mutual interactions) on one hand, and of the electromagnetic field on the other hand. In the case of plasma, the problem is summarized on Fig. 1.1.

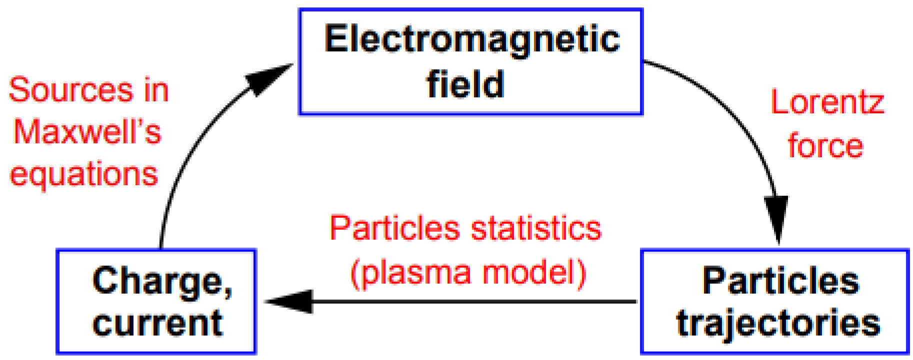


Fig.10.21 Self-consistent description of the electromagnetic field in a plasma

The electromagnetic field, given by the Maxwell's equations, influences the particles trajectories. Since the handling of all individual particles is largely beyond the computational capabilities of available computers in the present, but also in any foreseeable future, a plasma model is needed to derive statistical quantities, such as the charge and current density. In turn, these quantities enter as sources in the Maxwell's equations, and influence the field. Depending on the problem under study, various approximations are introduced to close this loop. In the present lecture, we will be developing a linear theory of plasma waves, by introducing a clear separation between the "equilibrium" fields and the wave perturbation. Maxwell's equations the electromagnetic field in the plasma is described by the Maxwell's equations, which we write in the form:

$$\nabla \cdot \mathbf{D} = \rho_{free} + \rho_{ext} \quad \text{...............} \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0 \quad \text{...................} \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{...............} \quad (3)$$

$$\nabla \times \mathbf{H} = \mathbf{j}_{free} + \mathbf{j}_{ext} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{..................} (4)$$

In these relations, E is the electric field, D is the electric displacement, H is the magnetic intensity, B is the magnetic induction (which we shall refer to as the magnetic field). j_{free} is the current carried by the free charges flowing in the medium, and ρ_{free} is the corresponding charge density. j_{ext} and ρ_{ext} are the current and charge densities from external sources, such as antennas. It is important to notice that in this form, the polarization and magnetization currents are included in D. Formally, it is possible to solve these equations as long as we are able to describe the medium response to a given electromagnetic excitation. In other words, we need to establish the constitutive relations of the medium:

$$\mathbf{D} \stackrel{?}{=} \mathbf{D}(\mathbf{E}), \qquad \ldots\ldots\ldots\ldots\ldots\ldots 5$$

And

$$\mathbf{B} \stackrel{?}{=} \mathbf{B}(\mathbf{H}). \qquad \ldots\ldots\ldots\ldots\ldots\ldots 6$$

In a classical electromagnetism problem [1], it is usual to introduce a polarization vector P, and also a magnetization vector M to write

$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}, \qquad \ldots\ldots\ldots\ldots 7$$

And

$$\mathbf{B} \equiv \mu_0 \mathbf{H} + \mathbf{M}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots 8$$

with $\varepsilon_0 = 1/36\pi \times 10^9$F/m the vacuum dielectric permittivity and $\mu_0 = 4\pi \times 10^7$H/m the vacuum magnetic permeability. We can then manipulate Eq. 1.4 to obtain the more familiar form

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j}_{free} + \mathbf{j}_{mag} + \mathbf{j}_{pol} + \mathbf{j}_{ext}) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \qquad \ldots\ldots\ldots\ldots\ldots\ldots 9$$

with $\partial_t P \equiv j_{pol}$, and $\nabla \times M \equiv \mu_0 j_{mag}$. j_{pol} and j_{mag} are respectively the polarization and magnetization currents. So far, we have followed the exact same method that is employed, e.g., in solid state physics. However, in plasma physics, it is impractical to separate the polarization, the magnetization and the free charges currents. Indeed, all charges are free (at least in a fully ionized plasma), yet all do contribute to the polarization of the medium. Therefore, we rewrite Eq. 1.9 in the form

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \mathbf{j}_{ext}) + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \quad \ldots\ldots\ldots\ldots..10$$

where j is the total current flowing in the plasma in response to the wave perturbation. It is now straightforward to deduce the wave equation from Eqs. 1.3 and 1.10

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial(\mathbf{j} + \mathbf{j}_{ext})}{\partial t}. \quad \ldots\ldots\ldots\ldots11$$

Despite its apparent simplicity, this relation is extremely complicated, because of its non-linear nature: j is a function of E and the properties of the plasma make this relation far from being trivial, as discussed in the next section. In this lecture, we will always assume that this relation is linear in essence, which restricts us to waves of moderate amplitude. This will allow us to retain the self-consistent nature of the problem. Moreover, it has been shown numerous times that the linear theory of waves was well suited to describe a large class of problems, such as high power heating and current drive in magnetic fusion devices.

Properties of plasma:

In an isotropic “standard” medium, the fact that j is a linear function of E can be written as

$$\mathbf{j}(\mathbf{r}, t) = \sigma(\mathbf{r}, t)\mathbf{E}(\mathbf{r}, t),$$

σ is the linear conductivity. Eq. 1.12 is local, both spatially (i.e., the response at location r only depends on the excitation at location r) and temporally (i.e., the response at instant t depends only on the excitation at instant t). Unfortunately, several properties of the plasma make the description more complicated than in this ideal dielectric medium.

Anisotropy: In many situations, plasmas are confined by strong magnetic fields (magnetic fusion plasmas, space plasmas). In this case, the response will obviously be different depending on the direction of the excitation (Fig. 1.2). The relation between j and E thus becomes tensorial in essence, as in a crystal, for instance. Hence, we write

$$\mathbf{j}(\mathbf{r},t) = \bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{E}(\mathbf{r},t).$$

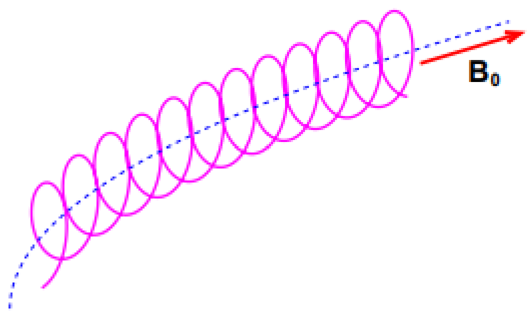


Fig.10.22 Anisotropy in a plasma confined by a magnetic field.

Time dispersion: The plasma is comprised of an assembly of electrons and ions, with various weights. Depending on the wave frequency, due to their inertia, the heavy ions may respond to the excitation with a delay.

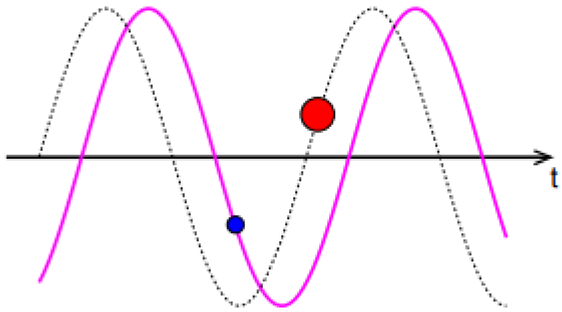


Fig.10.23 Time dispersion in a plasma. The various species respond differently to the wave depending on its frequency and on their respective masses.

In this case, the response of the plasma at instant t is determined by the excitation at all previous instants t'. Taking into account the causality principle which imposes to perform the integral only on times prior to t, we obtain a relation which is non local in time:

$$\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \, \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},t,t') \cdot \mathbf{E}(\mathbf{r},t').$$

Due to the non-local character of the relation between j and E, it is usual to refer $\bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},t,t')$ as the conductivity kernel.

Space dispersion: In a plasma, the finite temperature of the species induces a thermal agitation, and the particles have erratic motions superimposed to their integrable displacement (if any). This means that the particles at position r are influenced by the electromagnetic field in the domain they explore due to this non-deterministic part of their motion. Space dispersion is therefore a consequence of thermal effects (Fig. 1.4). We can thus expect a cold plasma to be non-dispersive in space (but not in time). We will find out later that this is indeed the case.

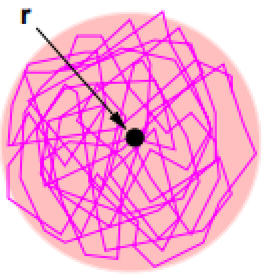


Fig.10.24 Space dispersion in a plasma. The thermal agitation causes the particles located at position r to actually "experience" the field in a region around this position.

The relation between j and E must thus be written in a spatially non local form:

$$\mathbf{j}(\mathbf{r},t) = \int d^3\mathbf{r}'\, \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t)\cdot \mathbf{E}(\mathbf{r}',t).$$

Gathering these three essential properties, it is clear that the functional j(E) must be written in the form

$$\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \int d^3\mathbf{r}'\, \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t,t')\cdot \mathbf{E}(\mathbf{r}',t').$$

This relation is linear (assuming $\bar{\bar{\boldsymbol{\sigma}}}$ is independent E) but retains the fundamental properties of the plasma medium.

Metallic Reflection:

When a light beam encounters a material, radiation can be absorbed or reflected by the surface. Metals are known for having high reflectivity, which explains their shiny appearance. Since the reflectance of light by metals is high, their absorption must also be high, because a high reflectance implies that light cannot penetrate the metal with considerable efficiency. The absorption of light can happen due to lattice vibrations and excitation of electrons to higher energy levels. Also, high reflectance of light in lower frequencies is associated to high conductivity of the metal, according to **Hagens-Ruben relation**.

Absorption Phenomena: If a light beam of certain wavelength is focused on a metal, the radiation is attenuated due to energy loss from lattice vibrations (heat) and excitation of electrons from the valence band to conduction band. In metals, there is an overlap between valence band and conduction band or a partially-filled valence band, which leads to conduction of electrons to energy levels above the Fermi level. This phenomenon is shown in Figure 11.

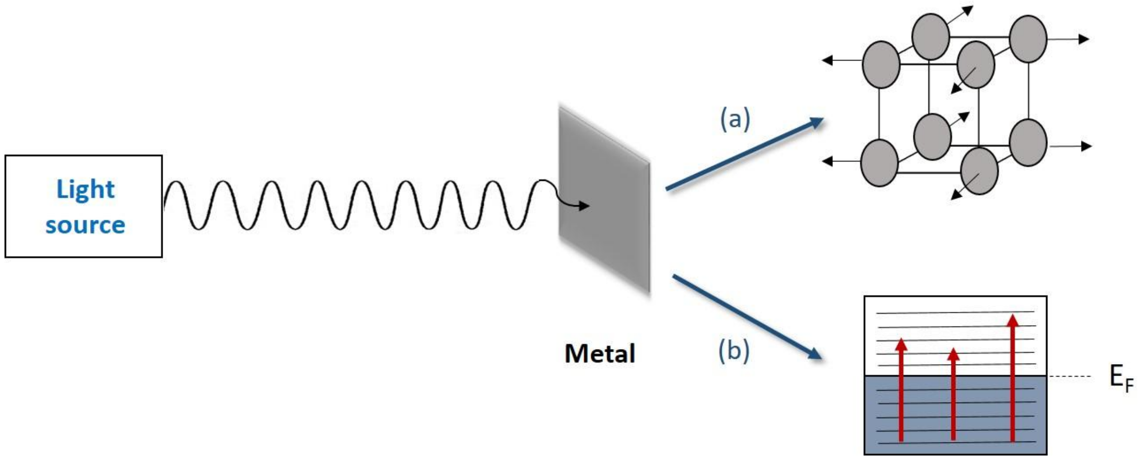


Fig.10.25 Scheme of the absorption of light by a metal, occurring lattice vibrations (a) and electron promotion to higher energy levels (b).

When electromagnetic radiation encounters the metallic surface, the intensity of the incident light (I_0) decreases exponentially while it travels through the metal, leading to a transmitted light (I) of lower intensity (Figure 2). This happens because metals can damp the initial intensity of light (I_0), and the decrease of light intensity is related to thickness of the metal (z), the incident wavelength, and the damping constant (k), or extinction coefficient. Where k describes the efficacy of a metal for light damping. This relationship is shown in Equation 1.

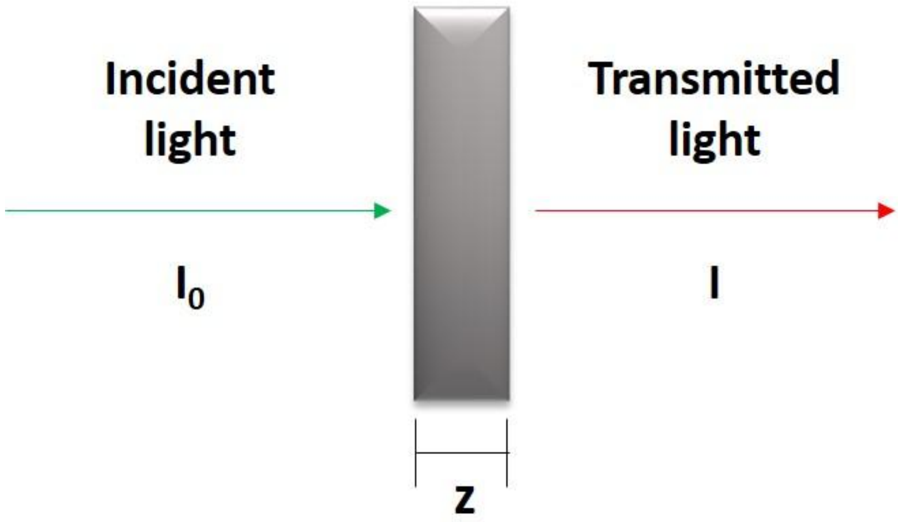


Fig.10.26 Scheme of the initial intensity of light (I_0) changing to transmitted intensity (I) when the radiation passes through a metal with thickness (z)

$$I = I_0 \exp(\frac{-4\pi kz}{\lambda}) \quad \ldots\ldots\ldots\ldots\ldots 1$$

The ratio between the transmitted intensity (I) and the initial intensity (I_0) is defined as transmittance (T), shown in Equation 2.

$$T = \frac{I}{I_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots 2$$

Also, the change in light intensity is related to the penetration depth (W), which is the distance required for the intensity of light (I_0) to be diminished to 1/e or 37% of its initial value. The reciprocal of the penetration depth is defined as absorbance (α), which is the amount of energy absorbed by the metal when radiation passes through (Equation).

$$W = \frac{1}{\alpha} = \frac{\lambda}{4\pi k} \quad \ldots\ldots\ldots\ldots\ldots 3$$

Metals have high reflectivity, reflecting almost all wavelengths in the visible region of the spectrum. This is related to their high damping constant, which leads to a short distance crossed by the light. In addition, some metals have low refractive index and, according to Snell's Law, when light passes through a medium of higher refractive index to a medium of low refractive index, the refracted ray will have a large deflection in relation to the normal.

These features explain the behavior of some metals such as silver, gold and copper towards incidence of electromagnetic radiation. A schematic representation of this process is shown in Figure.

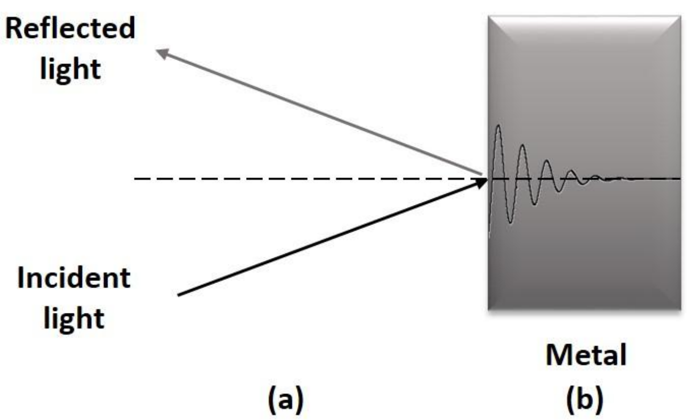


Fig.10.27 Reflectance phenomena where the incidence of light in metal leads to metallic reflection (a) and light attenuation or absorption (b).

Therefore, the Reflectance (R) of a material can be defined as the efficiency of a material to reflect incident light. This value depends only on the complex refractive index (n) and the damping constant (k), which is shown in Equation 4:

$$R = \frac{(n-1)^2 + k^2}{(n+1)^2 + k^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots 4$$

The study of metallic reflectance can be applied on metallic coatings, which is expected that the metal reflect light in a wide range of wavelengths. Also, it can explain the colors displayed by the metals. Silver, for instance, has high reflectivity over the visible range of the spectrum, which makes it colorless when white light is focused on the metal. Gold, however, absorbs the blue and violet regions of the spectrum, leading to a yellow color when illuminated with white light. The reflectance spectra of silver, gold, copper and aluminum is represented in Figure 4, where it can be observed that those metals have high reflectance in a wide range of wavelengths, specially in the visible region of the spectrum. However, if the frequency is large (lower wavelength values), silver, copper and gold have a drop in reflectance.

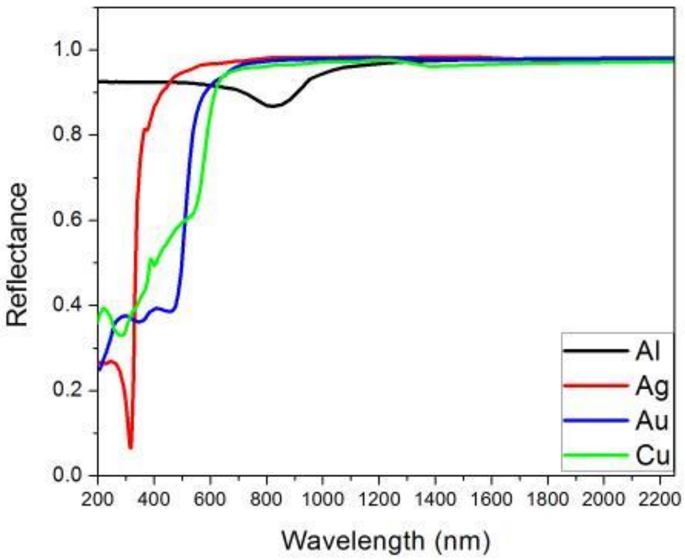


Fig.10.28 Reflectance spectra of the metals: aluminum (black line), silver (red line), gold (blue line) and copper (green line).

Reflectance and conductivity:

The metallic reflectance can be related to the conductivity by the Hagens-Ruben equation (Equation 5), where ν is the light frequency, ε_0 is the vacuum permittivity (8.85 x 10^{-12} F/m), and σ is the conductivity. In the infrared region (small frequencies), this equation shows that metals with high reflectance also are good conductors.

$$R = 1 - 4\sqrt{\frac{\nu\pi\epsilon_0}{\sigma}} \quad \ldots\ldots\ldots\ldots 5$$

This conclusion was derived by Drude and confirmed experimentally by Hagens-Ruben. It was observed that, at higher wavelengths (lower frequencies), the optical constants of metals are similar to the values of Drude`s function, where the complex refractive index is much smaller then the damping constant, or extinction coefficient. This leads to high reflectance. However, in higher frequencies, deviations of Drude`s approach start to appear, because bound electrons of the metal start to respond to the incidence of light, instead of just valence band electrons response. This leads to a decrease in reflectivity, which depends on the metal's characteristics.

10.8 Elementary theory of dispersion:

Dispersion occurs when pure plane waves of different wavelengths have different propagation velocities, so that a wave packet of mixed wavelengths tends to spread out in space. The speed of a plane wave, *v*, is a function of the wave's wavelength:

$$v = v(\lambda).$$

The wave's speed, wavelength, and frequency, *f*, are related by the identity

$$v(\lambda) = \lambda\ f(\lambda).$$

The function f(λ) expresses the dispersion relation of the given medium. Dispersion relations are more commonly expressed in terms of the angular frequency $\omega = 2\pi f$ and wave number $k=2\pi/\lambda$.

Rewriting the relation above in these variables gives

$$\omega(k) = v(k) \cdot k.$$

where we now view *f* as a function of *k*. The use of ω(*k*) to describe the dispersion relation has become standard because both the phase velocity ω/*k* and the group velocity dω/d*k* have convenient representations via this function.

The plane waves being considered can be described by

$$A(x,t) = A_0 e^{2\pi i \frac{x-vt}{\lambda}} = A_0 e^{i(kx-\omega t)},$$

where

A is the amplitude of the wave,

$A_0 = A(0,0)$,

x is a position along the wave's direction of travel, and

t is the time at which the wave is described.

SAQ.2

a) Define the Total internal reflection, Brewster's law and degree of polarization.
b) What do you mean by Plane wave propagation in plasma and its properties?
c) Define the working of Elementary theory of dispersion.

Examples:-

Q.1. If the refractive index of a polarizer is 1.9218. What will be the polarization angle and angle of refraction?

Solution: Looking at the above figures, we will see that we already know the refractive index of the polarizer that means μ is 1.9218. In order to find the polarization angle and angle of refraction, we will apply Brewster's law:

μ = tan ip

Or, ip = tan−1tan−1 (1.9128)

Or, ip = 62^o 24'

Now we will see that our angle of refraction:

It is specified that ip + ir = 90 degrees

Thus, angle of refraction or ir = 90 – 62^o 24'

Therefore, our angle of refraction comes as 27.6 ^{o}

Q.2. Find out Brewster's angle of light which travels from water (n = 1.33) into the air?

Solution: Looking at the question, we see we have already got our n1n1 as 1.33. Thus, by applying the formula we will get:

Brewster's angle = $tan^{-1}\left(\frac{n^2}{n^1}\right)$

Brewster's angle = $tan^{-1}\left(\frac{1.5}{1.33}\right)$

So, Brewster's angle = 48.4°

Therefore, the brewster's angle is 48.4°.

Q.3. A certain polarizer has a refractive index of 1.33. Find the polarization angle and angle of refraction?

Solution: Refractive index of the polarizer = 1.33

The Brewster's law is $\mu = \tan ip$

$ip = \tan{-1}\ (1.33)$

ip = 53.06

Now, Angle of refraction

It is given that ip + ir = 90 degrees

Thus, angle of refraction or ir = 90 – 53.06

Angle of refraction = 36.94

Summary:

1) In this chapter discuss the Boundary conditions at discontinuity for D, E, B and H.

2) Define and explain the Reflection and refraction at normal and oblique incidence of electric vectors perpendicular to boundary.

3) Define and explain the Reflection and refraction at normal and oblique incidence of electric vectors parallel to boundary.

4) Total internal reflection, in physics, complete reflection of a ray of light within a medium such as water or glass from the surrounding surfaces back into the medium. The phenomenon occurs if the angle of incidence is greater than a certain limiting angle, called the critical angle.

5) Brewster's law, relationship for light waves stating that the maximum polarization (vibration in one plane only) of a ray of light may be achieved by letting the ray fall on a surface of a transparent medium in such a way that the refracted ray makes an angle of 90° with the reflected ray.

6) Degree of polarization (DOP) is a quantity used to describe the portion of an electromagnetic wave which is polarized. A perfectly polarized wave has a DOP of 100%, whereas an un-polarized wave has a DOP of 0%.

7) Plasma waves the physical description of an electromagnetic wave propagating in a given medium necessitates a self-consistent handling of the particles comprising the medium (and their mutual interactions) on one hand, and of the electromagnetic field on the other hand.

8) When a light beam encounters a material, radiation can be absorbed or reflected by the surface. Metals are known for having high reflectivity, which explains their shiny appearance.

9) Dispersion occurs when pure plane waves of different wavelengths have different propagation velocities, so that a wave packet of mixed wavelengths tends to spread out in space.

Terminal Questions:

1) Explain the Boundary conditions at discontinuity for D, E, B and H in details.
2) Explain the working of the Reflection and refraction at normal and oblique incidence of electric vectors perpendicular to boundary.
3) Explain the working of the Reflection and refraction at normal and oblique incidence of electric vectors parallel to boundary.

4) Write the short notes of the following: (i) Total internal reflection, (ii) Brewster's law, (iii) Degree of polarization.

5) Explain the working principle of the Plane wave propagation in plasma and its properties.

6) What do you mean by Metallic reflection?

7) Discuss and explain Elementary theory of dispersion.

8) If the refractive index of a polarizer is 1.112. What will be the polarization angle and angle of refraction?

9) A certain polarizer has a refractive index of 1.63. Find the polarization angle and angle of refraction?

10) A light of 632.8 nm is focused on a gold sample of damping constant (k) of 3.068. Calculate the penetration depth and absorbance of the sample.

11) Find out Brewster's angle of light which travels from water (n = 1.33) into the air?