Uttar Pradesh Rajarshi Tandon Open University

Bachelor of Science

DCEMM -112

Advance Analysis

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Advance Analysis

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Block

1 Metric space: Continuity, Compactness and completeness

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Block-1

Metric space: Continuity, Compactness and completeness

In this we introduce the notion of metric spaces. The metric spaces arose from extending the notions of continuity and convergence on the real line to more abstract spaces. A metric space is just a set which is equipped with a function called metric which measures the distance between the elements of various pairs from the set. We shall study various properties of these spaces, open and closed sets. The structure on a metric space allows us to extend the notion of continuity of these spaces. We will see that the notion of continuity is one of the most important notions for further study of Analysis. Here we talk about two important results about continuous functions which are called Urysohn's lemma and glueing lemma. Then we explain the notion of uniform continuity through some examples. The definition of continuity and uniform continuity for metric spaces are similar for Euclidean spaces R^n.

In second unit we shall study about the concepts of a limit and Continuity for the functions of a single variable. we shall discuss the notion of compactness in a metric space.

In the third unit we shall define compact sets and discuss the examples of these sets in different metric spaces. Firstly we give a characterization in terms of convergence of sequences and then in terms of completeness. In this connection, we introduce the concept of "totally bounded sets" which is a stronger version of bounded sets. We show that a set is compact if and only if it is complete and totally bounded. We also discuss the analogue of the famous "Heine Bore1 theorem" in R which characterises compact sets in terms of closed and bounded sets. Here we discuss relationship between continuity and compactness.

In the fourth unit is to study one of the properties of metric space. The notion of distance between points of an abstract set leads naturally to the discussion of uniform continuity and Cauchy sequences in the set. Unlike the situation of real numbers, where each Cauchy sequence is convergent, there are metric spaces in which Cauchy sequences fail to converge. A metric space in which every Cauchy sequence converges is called a 'complete metric space'. This property plays a vital role in analysis when one wishes to make an existence statement. We shall see that a metric space need not be complete and hence we shall find conditions under which such a property can be ensured.

UNIT-1 Metric Space

Structure

1.1 Introduction

In this unit, we introduce the notion of metric spaces. As we already pointed out in the course introduction and block introduction, the metric spaces arose from extending the notions of continuity and convergence on the real line to more abstract spaces. A metric space is just a set which is equipped with a function called metric which measures the distance between the elements of various pairs from the set. We shall first give the definition of a metric and a metric space and consider various examples Then we shall study various properties of these spaces. we shall consider open and closed sets. The structure on a metric space allows us to extend the notion of continuity to functions in the context of these spaces. We shall define this notion and discuss several examples of continuous functions. Later you will see that the notion of continuity is one of the most important notions for further study of Analysis. Here we talk about two important results about continuous functions which are called Urysohn's lemma and glueing lemma. Then we explain the notion of uniform continuity through some examples. You will see that the definition of continuity and uniform continuity for metric spaces are similar to those for Euclidean spaces R_n. But extending these notions to metric spaces, provide not only a new perspective but also a deeper insight into their structure and properties.

1.2 Objectives

After studying this unit, we should be able to:

- state the properties that define a metric and apply them.
- give examples of different metrics on R^n ; $n \geq 1$
- explain a discrete metric space and other metric spaces such as function spaces;
- we try to check whether

  i) a subset of a metric space is open;

  ii) a subset of a metric space is closed;

  iii) a function defined on a metric space is continuous;

  iv) a function defined on a metric space is uniformly continuous.

1.3 Metric Space

A Metric Space is a set equipped with a distance function, also called a metric, which enables us to measure the distance between two elements in the set.

Definition: A Metric Space is a non-empty set M together with a function

$d: M \times M \rightarrow R$ satisfying the following conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in M$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x)$ for all $x, y \in M$

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$ [Triangle Inequality]

d is called a metric or distance function on M and d (x, y) is called the distance between x and y in M. The metric space M with the metric d is denoted by (M, d) or simply by M when the underlying metric is clear from the context.

Example 1. Let **R** be the set of all real numbers. Define a function $d: M \times M \rightarrow R$ by $d(x, y) = | x - y |$. Then d is a metric on **R** called the usual metric on **R.**

Proof: Let $x, y \in R$.

Clearly $d(x, y) = | x - y |$
≥ 0. Moreover,
$d(x,y)=0 \Leftrightarrow |x-y|=0.$

$$\Leftrightarrow x - y = 0.$$

$$\Leftrightarrow x = y$$

$d(x,y)=|x-y|$

$=|y-x|$

$= d(y,x).$

$\therefore d(x , y) = d(y , x).$

Let $x , y , z \in$ **R.**

$d(x , z) = |x - z |$

$= |x - y + y - z |$

$\leq | x - y |+| y - z |$

$= d(x , y) + d(y , z).$

$\therefore d(x , z) \leq d(x , y) + d(y , z)$. Hence d is a metric on**R.**

Note. When R is considered as a metric space without specifying its metric, it is the usual metric.

Example 2. Let M be any non-empty set. Define a function $d : M \times M \to R$ by $d(x , y) = 0$ if $x=y$ and 1 if $x \neq y$

Then d is a metric on M called the **discrete metric** or **trivial metric** on M.

Proof.

Let x , y ε M.

Clearly d(x , y) ≥ 0 and d(x , y) = 0 ⇔ x = y .

0

if x =y Also, d(x

, y) = 1 if x≠y

= d(y , x) .

Let x , y , z ε M.

We shall prove that d(x , z) ≤ d(x , y) + d(y , z).

Case (i) Suppose x = y = z.

Then d(x , z) = 0 , d(x , y) = 0 , d(y , z) = 0 .

∴d(x , z) ≤ d(x , y) + d(y , z).

Case (ii) Suppose x = y and z distinct.

Then d(x , z) = 1 , d(x , y) = 0 , d(y , z) = 1 .

∴d(x , z) ≤ d(x , y) + d(y , z).

Case (iii) Suppose x = z and y distinct. Then d(x , z) = 0 , d(x , y) = 1 , d(y ,z)=1 .

∴d(x , z) ≤ d(x , y) + d(y , z).

Case (iv) Suppose y = z and x distinct. Then d(x , z) = 1 , d(x , y) = 1 , d(y , z) = 0.

∴d(x , z) ≤ d(x , y) + d(y , z).

Case (v) Suppose x ≠ y ≠ z.

Then d(x , z) = 1 , d(x , y) = 1 , d(y , z) = 1.

∴d(x , z) ≤ d(x , y) + d(y , z).

In all the cases, d(x , z) ≤ d(x ,y)+d(y,z). Hence d is a metric on M.

1.4 Open and Closed Ball

Definition: Let (X, d) be a metric space, $x \in X$ and $r > 0$. The set $B(x, r) = \{y \in X \,|d(x, y) < r\}$ is called the open ball with Centre x and radius r.

The set $\bar{B}(x, r) = \{y \in X \,|d(x, y) \leq r\}$ is called the closed ball with Centre x and radius r.

1.5OPEN SETS IN A METRICSPACE

Definition: Let (M , d) be a metric space. Let a ∈ M and r be a positive real number. The open ball or the open sphere with center a and radius r is denoted by B_d (a , r) and is the subset of M defined by B_d (a , r) = {x ∈ M ⁄d(a , x) < r}. We write B(a , r) for B_d (a , r) if the metric d under consideration isclear.

Note. Since d(a, a) = 0 < r, a ∈ B_d (a , r).

Examples

1. In **R** with usual metric B(a , r) = (a - r , a +r).
2. In $\mathbf{R}^2$with usual metric B(a , r) is the interior of the circle with center aand radiusr.
3. In a discrete metric space M, B(a , r) = M ifr>1

   aif r≤1

Definition: Let (M , d) be a metric space. A subset A of M is said to be open in M if for each x ∈ A there exists a real number r > 0 such that B(x , r) ⊆ A.

Note. By the definition of open set, it is clear that ∅ and M are open sets.

Examples

1. Any open interval (a , b) is an open set in **R** with usual metric. For,

   Let x ∈ (a , b).

   Choose a real number r such that 0 < r ≤ min { x-a , b-x }. Then B(x , r) ⊆ (a , b).

   ∴ (a , b) is open in R.
2. Every subset of a discrete metric space M isopen. For,

   Let A be a subset of M. If

$A = \emptyset$, then A is

open.

Otherwise, let x

$\in A$.

Choose a real number r such that 0

$< r \leq 1$. Then $B(x, r) = \{ x \} \subseteq A$

and hence A is open.

3. Set of all rational numbers **Q** is

not open in **R**. For,

Let $x \in \mathbf{Q}$.

For any real number $r > 0$, $B(x, r) = (x - r, x + r)$ contains both rational and irrational numbers.

$\therefore B(x, r) \not\subseteq \mathbf{Q}$ and hence **Q** is not open.

Theorem 1. Let (M , d) be a metric space. Then each open ball in M is an open set.

Proof: Let $B(a, r)$ be an open ball in M. Let $x \in B(a, r)$.

Then $d(a, x) < r$.

Take $r_1 = r - d(a, x)$.

Then $r_1 > 0$. We claim

that $B(x, r_1) \subseteq B(a, r)$.

Let $y \in B(x, r_1)$. Then $d(x, y)$

$< r_1$. Now, $d(a, y) \leq d(a,$

$x) + d(x, y)$

$< d(a, x) + r_1$

$= d(a , x) + r - d(a , x)$

$= r.$

$\therefore d(a , y) < r.$

$\therefore y \in B(a , r).$

$\therefore B(x , r_1) \subseteq B(a , r).$

Hence $B(a , r)$ is an open ball.

Theorem 3. In any metric space M, the union of open sets is open.

Proof: Let A_α be a family of open sets in M. We have to prove $A = \cup A_\alpha$ is open in M. Let $x \in A$.

Then $x \in A_\alpha$ for some α.

Since A_α is open, there exists an open ball $B(x , r)$ such that $B(x , r) \subseteq A_\alpha$.

$\therefore B(x , r) \subseteq A.$

Hence A is open in M.

Theorem 4. In any metric space M, the intersection of a finite number of open sets is open.

Proof: Let $A_1, A_2, \ldots, A_n$ be open sets in M.

We have to prove $A = A_1 \cap A_2 \cap \ldots \cap A_n$ is open in M. Let $x \in A$.

Then $x \in A_i \forall i = 1, 2, \ldots , n.$

Since each A_i is open, there exists an open ball $B(x, r_i)$ such that $B(x, r_i) \subseteq A_i$. Take $r = \min\{ r_1, r_2, \dots, r_n \}$.

Clearly $r > 0$ and $B(x, r) \subseteq B(x, r_i)\ \forall i = 1, 2, \dots, n$. Hence $B(x, r) \subseteq A_i \forall i = 1, 2, \dots, n$.

$\therefore B(x, r) \subseteq A$.

$\therefore$ A is open in M.

Theorem 2. Let (M , d) be a metric space and $A \subseteq M$. Then A is open in M if and only if A can be expressed as union of open balls.

Proof: Suppose that A is open in M.

Then for each $x \in A$ there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \subseteq A$.

$\therefore A =_{x \in A} B(x, r_x)$.

Thus A is expressed as union of open balls.

Conversely, assume that A can be expressed as union of open balls. Since open balls are open and union of open sets is open, A is open.

1.6 Interior of aset

Definition: Let (M , d) be a metric space and $A \subseteq M$. A point $x \in A$ is said to be an interior point of A if there exists a real number $r > 0$ such that $B(x, r) \subseteq A$. The set of all interior points is called as interior of A and is denoted by **Int** A.

Note: Int A ⊆ A.

Example: In **R** with usual metric, let A = [1 , 2]. 1 is not an interior points of A, since for any real number r >0 , B(1 , r) = (1 – r , 1 + r) contains real numbers less than 1. Similarly, 2 is also not an interior point of A.

In fact every point of (1 , 2) is a limit point of A. Hence **Int**A = (1 ,2).

Note: (1)**Int∅= ∅and Int** M = M.

(2) A is open ⇔**Int** A =A.

(3) A ⊆ B ⇒**Int** A⊆**Int**B

Theorem5. Let (M , d) be a metric space and A ⊆ M. Then **Int** A = Union of all open sets contained in A.

Proof: Let G = ∪{ B / B is an open set contained in A } We have to prove Int A = G.

Let x ∈**Int** A .

Then x is an interior point of A.

∴ there exists a real number r > 0 such that B(x , r) ⊆ A.

Since open balls are open, B(x , r) is an open set contained in A.

$\therefore B(x , r) \subseteq G$.

$\therefore x \in G$.

$\therefore$**Int** $A \subseteq G$ .... (1)

Let $x \in G$.

Then there exists an open se B such that $B \subseteq A$ and $x \in B$.

Since B is open and $x \in B$, there exists a real number $r > 0$ such that $B(x , r) \subseteq B \subseteq A$.

$\therefore$ x is an interior point of A.

$\therefore x \in$ **Int** A .

$\therefore G \subseteq$ **Int**A ....(2)

From (1) and (2), we get **Int** A = G.

Note: Int A is an open set and it is the largest open set contained in A.

Theorem7. Let M be a metric space and A , B $\subseteq$ M. Then

(1) **Int** $(A \cap B)$ = (**Int** A) $\cap$ (**Int**A)

(2) **Int** $(A \cup B) \supseteq$ (**Int** A) $\cup$ (**Int**A)

Proof: (1) $A\cap B\subseteq A \Rightarrow \text{Int}(A\cap B)\subseteq \text{Int}A$. Similarly, $\text{Int}(A \cap B) \subseteq \text{Int } B$.

$\therefore \text{Int}(A \cap B) \subseteq (\text{Int } A) \cap (\text{Int}A)$ ....(a)

IntA⊆ A and **Int** B ⊆B .

∴ (**Int** A) ∩ (**Int** A) ⊆ A ∩ B

Now, (**Int** A) ∩ (**Int** A) is an open set contained in A∩B . But, **Int** (A ∩ B) is the largest open set contained in A ∩B .

∴(**Int** A) ∩ (**Int** A) ⊆**Int** (A∩B)..(b)

From (a) and (b) , we get **Int** (A ∩ B) = (**Int** A) ∩ (**Int** A)

(2) A⊆A∪B⇒**Int**A⊆**I**
nt(A∪B) Similarly,
Int B⊆**Int** (A ∪B)
∴**Int** (A ∪ B) ⊇ (**Int** A) ∪ (**Int** A)

Note: Int (A ∪B)need not be equal to(**Int** A) ∪ (**Int** A) For,

In **R** with usual metric, let A = (0 , 1]andB=(1,2).
A ∪ B = (0 , 2).
∴**Int** (A ∪ B) = (0 , 2)

Now, **Int** A (0 , 1) and **Int** B = (1 , 2) and hence (**Int** A) ∪ (**Int** A) = (0 , 2) –{2}.

$\therefore$**Int** $(A \cup B) \neq$ (**Int** $A) \cup$ (**Int** A)

1.5 Subspace

Definition: Let (M , d) be a metric space. Let M_1 be a nonempty subset of M. Then M_1 is also a metric space under the same metric d. We call (M_1 , d) is a subspace of (M, d).

Theorem 8. Let M be a metric space and M_1 a subspace of M. Let $A \subseteq M_1$. Then A is open in M_1 if and only if $A = G \cap M_1$ where G is open in M.

Proof: Let $B_1(a , r)$ be the open ball in M_1 with center a and radius r.

Then $B_1(a , r) = B(a , r) \cap M_1$ where $B(a , r)$ is the open ball in M with center a and radius r.

Let A be an open
set in M_1. Then

$$A={}_{x\in A}B_1(x,r(x))$$

$$={}_{x\in A}[B(x , r(x)) \cap M_1)]$$

$$= [{}_{x\in A}B(x , r(x))] \cap M_1$$

$= G \cap M_1$ where $G ={}_{x\in A}B(x , r(x))$ which is
open in M. Conversely, let $A = G \cap M_1$ where G is
open in M.
We shall prove that A is
open in M_1. Let $x \in A$.
Then $x \in G$ and $x \in M_1$.

Since G is open in M, there exists an open ball B(x , r) such that B(x , r) $\subseteq$ G.

$\therefore B(x , r) \cap M_1 \subseteq G \cap M_1$.

i.e. $B_1(a , r) \subseteq A$.

$\therefore$ A is open in M_1.

Example. Consider the subspace $M_1 = [0 , 1] \cup [2 , 3]$ of **R**.

A = [0 , 1] is open in M

since $A = (-\frac{1}{2} , \frac{3}{2}) \subseteq M$ where $(-\frac{1}{2} , \frac{3}{2})$ is open in **R**.

Similarly, B = [2 , 3], $C = [0 , \frac{1}{2}]$, $D = (\frac{1}{2} , 1]$ are open in M.

Note that A, B, C, D are not open in **R**.

1.6 Closed Sets.

Definition: A subset A of a metric space M is said to be closed in M if its complement is open in M.

Examples

1. In **R** with usual metric any closed interval

[a , b] is closed. For,

$[a , b]^c = \mathbf{R} - [a , b] = (-\infty , a) \cup (b , \infty)$.

$(-\infty , a)$ and (b , ∞) are open sets in R and hence $(-\infty , a) \cup (b , \infty)$ is open in **R**.

i.e. $[a , b]^c$ is open in **R**.

∴ [a , b] is open in **R**.

2. Any subset A of a discrete metric space M is closed since A^c is open as every subset of M isopen.

Note. In any metric space M, ∅ and M are closed sets since $\emptyset^c = M$ and $M^c = \emptyset$ which are open in M. Thus ∅ and M are both open and closed in M.

Theorem 9. In any metric space M, the union of a finite number of closed sets is closed.

Proof: Let $A_1, A_2, \dots , A_n$ be closed sets in a metric space M. Let $A = A_1 \cup A_2 \cup \dots \cup A_n$.

We have to prove A is open in M.

Now, $A^c = [A_1 \cup A_2 \cup \dots \cup A_n]^c$

$= A_1^c \cap A_2^c \cap \dots \cap A_n^c$ [By De Morgan's law.]

Since A_i is closed in M, A_i^c is open in M.

Since finite intersection of open sets is open, $A_1^c \cap A_2^c \cap \dots \cap A_n^c$ is open in M.

i.e. A^c is open in M.

$\therefore$ A is closed in M.

Theorem 10. In any metric space M, the intersection of closed sets is closed.

Proof: Let A_α be a family of closed sets in M. We have to prove $A = \cap A_\alpha$ is open in M. Now, $A^c = (\cap A_\alpha)^c$

$= \cup A^c_\alpha$ [By De Morgan's law.] Since A_α is closed in M, A^c_α is open in M. Since union of open sets is open, $\cup A^c_\alpha$ is open. i.e. A^c is open in M. $\therefore$ A is closed in M.

Theorem 11. Let M_1 be a subspace of a metric space M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 if and only if $F_1 = F \cap M_1$ where F is a closed set in M.

Proof: Suppose that F_1 is closed in M_1. Then $M_1 - F_1$ is open in M_1.

$\therefore M_1 - F_1 = A \cap M_1$ where A is

open in M. Now, $F_1 = A^c \cap M_1$.

Since A is open in M, A^c is closed in M.

Thus, $F_1 = F \cap M_1$ where $F = A^c$ is closed in M.

Conversely, assume that $F_1 = F \cap M_1$ where F is closed in M.

Since F is closed in M, F^c is open in M.

$\therefore F^c \cap M_1$ is open in M_1.

Now, $M_1 - F_1 = F^c \cap M_1$ which is open in M_1.

$\therefore F_1$ is closed in M_1.

1.7 Closure

Definition: Let A be a subset of a metric space (M , d). The closure of A, denoted by $\bar{A}$ is defined as the intersection of all closed sets which contain A.

i.e. $\bar{A} = \cap B$ B is closed in M and $B \supseteq A$

Note :

(1) Since intersection of closed sets is closed, $\bar{A}$ is a closedset.

(2) $\bar{A} \supseteq A$.

(3) $\bar{A}$ is the smallest closed set containing A.

(4) A is closed $\Leftrightarrow$

$A = \bar{A}$ (5) $\bar{\bar{A}} = \bar{A}$

Theorem 12. Let (M, d) be a metric space. Let $A, B \subseteq M$. Then

(1) $A \subseteq B$

$\Rightarrow \bar{A} \subseteq \bar{B}$

(2) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(3) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

Proof: Let $A \subseteq B$. $\bar{B} \supseteq B \supseteq A$.

Thus $\bar{B}$ is a closed set containing A.

But $\bar{A}$ is the smallest closed set containing A.

$\therefore \bar{A} \subseteq \bar{B}$

(1) $A \subseteq A \cup B$.

$\therefore$ by (1),

$\bar{A} \subseteq \overline{A \cup B}$. Similarly, $\bar{B} \subseteq \overline{A \cup B}$

$\therefore \bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$ ....(a)

$\bar{A}$ is a closed set containing A and $\bar{B}$ is a closed set containing B.

$\therefore \bar{A} \cup \bar{B}$ is a closed set containing $A \cup B$.

But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.

$\therefore \overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$ ....(b)

From (a) and (b) we get $\overline{A \cup B} = \bar{A} \cup \bar{B}$

(2) $A \cap B \subseteq A$.

$\therefore \overline{A \cap B} \subseteq \bar{A}$

Similarly, $\overline{A \cap B} \subseteq \overline{B}$

$\therefore \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Note: $\overline{A \cap B}$ need not be equal to $\overline{A} \cap \overline{B}$

For example, in **R** with usual metric take A = (0 , 1) and B = (1 , 2) . A ∩ B = ∅ ⇒ $\overline{A \cap B} = \emptyset$.

But $\overline{A} \cap \overline{B}$ = [0 , 1] ∩ [1 , 2] = { 1 }.

$\therefore \overline{A \cap B} \neq \overline{A} \cap \overline{B}$

1.8 Limit Point

Definition: Let (M , d) be a metric space and A ⊆ M. A point x ∈ M is said to be a limit point of A if every open ball with center x contains a point of A other than x.

i.e. $B(x , r) \cap (A - \{ x \}) \neq \emptyset$ for all r > 0.

The set of all limit points of A is denoted by A'.

Example. In **R** with usual metric let A = (0 , 1).

Every open ball with center 0, B(0 , r) = (-r , r) contains points of (0 , 1) other than 0.

∴ 0 is a limit point of A.

Similarly, 1 is a limit point of A and in fact every point of A is also a limit point of A.

For each real number $x < 0$, if we choose r such that $0 < r \leq -\frac{x}{2}$, then $B(x , r)$

contains no point of $(0 , 1)$, and hence x is not a limit point of limit point of A. Similarly, every real number $x > 0$ is not a limit point of A.

Hence $A' = [0 , 1]$.

Example. In **R** with usual metric, **Z** has no limit point. For,

Let x be any real number.

If x is an integer, then $B(x , \frac{1}{2}) = (x - \frac{1}{2} , x + \frac{1}{2})$ has no integer other than x.

$\therefore$ x is not a limit point of **Z** .

If x is not an integer, choose r such that $0 < r < x-n$ where n is the integer closest to x. Then $B(x , r) = (x – r , x + r)$ contains no integer. Hence x is not a limit point of **Z.**

Thus no real number x is a limit point of **Z.**

$\therefore \mathbf{Z}' = \emptyset$.

Example. In **R** with usual metric, every real number is a limit point of **Q** . For,

Let x be any real number.

Every open ball $B(x , r) = (x - r , x + r)$ contains infinite number of rational numbers.

$\therefore$ x is a limit point of **Q**.

$\therefore$ $\mathbf{Q}^{l} = \mathbf{R}$.

Theorem 13. Let (M , d) be a metric space and $A \subseteq M$. Then x is a limit point of A if and only if every open ball with center x contains infinite number of points of A.

Proof: Let x be a limit point of A.

We have to prove every open ball with center x contains infinite number of points of A.

Suppose not.

Then there exists an open ball $B(x , r)$ contains only a finite number of points of A and hence of $(A - \{ x \})$.

Let $B(x , r) \cap (A - \{ x \}) = x_1, x_2, \ldots , x_n$.

Let $r_1 = \min \{ d(x , x_i) / i = 1 , 2 , \ldots , n \}$.

Since $x \neq x_i$, $d(x , x_i) > 0 \ \forall i = 1 , 2 , \ldots , n$ and hence $r_1 > 0$. Moreover, $B(x , r_1) \cap (A - \{ x \}) = \emptyset$.

∴ x is not a limit
point of A. This is a
contradiction.

∴ every open ball with center x contains infinite number of points of A.

Conversely, assume that every open ball with center x contains infinite number of points of A.

Then, every open ball with center x contains infinite number of points of A – { x }.
Hence x is a limit point of A.

Note: Any finite subset of a metric space has no limit points. **Theorem 14.** Let M be a metric space and $A \subseteq M$. Then

$\overline{A} = A \cup A'$.

Proof: Let $x \in A \cup A'$.

We claim that $x \in \overline{A}$

Suppose $x \notin A$. Then, $x \in M - A$.

Since A is closed , M - A is open.
∴ there exists an open ball B(x , r) such that $B(x, r) \subseteq M - A$.

$\therefore B(x, r) \cap A = \emptyset$.

$\therefore B(x, r) \cap A = \emptyset$. [$\because A \subseteq A$].

$\therefore x \notin A \cup A'$, which is a contradiction.

$\therefore x \in A$.

$\therefore A \cup A' \subseteq A$ .... (1)

Let $x \in A$.

We have to prove x
$\in A \cup A'$. If $x \in A$,
then $x \in A \cup A'$.
Suppose $x \notin A$.

We claim that $x \in A'$.

Suppose $x \notin A'$.

Then there exists an open ball $B(x, r)$ such that $B(x, r) \cap (A – \{ x \}) = \emptyset$.

$\therefore B(x, r) \cap A = \emptyset$. [$\because x \notin A$]

$\therefore A \subseteq B(x, r)^c$.

Since $B(x, r)$ is open, $B(x, r)^c$ is

closed. Thus $B(x, r)^c$ is a closed set containing A. But, A is the smallest closed set containing A. Hence $A \subseteq B(x, r)^c$.

Now, $x \notin B(x, r)^c$.

$\therefore x \notin A$, which is a contradiction.

$\therefore x \in A'$ and hence $x \in A \cup A'$.

$A \subseteq A \cup A'$ ………………………………………………(2)

From (1) and (2), we get $A = A \cup A'$.

Corollary 3. A is closed if and only if A contains all its limit points.

Proof: A is closed $\Leftrightarrow A = A$

$\Leftrightarrow A = A \cup A'$.

$\Leftrightarrow A \subseteq A'$.

Corollary 1. $x \in A \Leftrightarrow B(x, r) \cap A \neq \emptyset \,\forall\, r > 0$.

Proof.

$x \in A \Rightarrow x \in A \cup A'$.

$\therefore$ $x \in A$ or $x \in A'$.

If $x \in A$, then $x \in B(x , r) \cap A$.

If $x \in A'$, then $B(x , r) \cap (A - \{ x \}) \neq \emptyset \forall r > 0$. Thus $B(x , r) \cap A \neq \emptyset \forall r > 0$.

Conversely, let $B(x , r) \cap A \neq \emptyset \forall r > 0$. We have to prove $x \in A$.

If $x \in A$, then $x \in A$.

If $x \notin A$, then $A = A - \{ x \}$.

$\therefore B(x , r) \cap (A - \{ x \}) \neq \emptyset \forall r > 0$.

$\therefore$ x is a limit point of A.

$\therefore$ $x \in A'$.

$\therefore$ $x \in A$.

Corollary 2. $x \in A \Leftrightarrow G \cap A \neq \emptyset$ for all open set G containing x.

Proof: Let $x \in A$.

We have to prove $G \cap A \neq \emptyset$ for all open set G containing x. Let G be an open set containing x.

Then there exists an open ball $B(x , r)$ such that

$B(x , r) \subseteq G$. Since $x \in A$, $B(x , r) \cap A \neq \emptyset$ and

hence $G \cap A \neq \emptyset$.

Conversely, assume that $G \cap A \neq \emptyset$ for every open set containing x.

Then $B(x , r) \cap A \neq \emptyset \; \forall \; r > 0$.

$\therefore x \in \overline{A}$.

1.9 Bounded Sets in a Metric space.

Definition: Let (M , d) be a metric space. A subset A of M is said to be bounded if there exists a positive real number k such that $d(x , y) \leq k \; \forall \; x , y \in A$.

Example. Any finite subset A of a metric space (M , d) is bounded. For,

Let A be any finite subset of M.

If $A = \emptyset$ then A is obviously bounded.

Let $A \neq \emptyset$ .Then $\{d(x , y)/x , y \in A\}$ is a finite set of real numbers. Let $k = \max \{d(x , y)/x , y \in A\}$.

Clearly $d(x , y) \leq k$ for all $x , y \in A$.

$\therefore$ A is bounded.

Example. $[0,1]$ is a bounded subset of **R** with usual metric since $d(x , y) \leq 1$ for all $x , y \in [0,1]$.

Example 1. $(0 , \infty)$ is an unbounded subset of **R.**

Example 2. Any subset A of a discrete metric space M is bounded since $d(x , y) \leq 1$ for all $x , y \in A$.

Note: Every open ball $B(x , r)$ in a metric space (M , d) is bounded. For,

Let $s , t \in B(x , r)$.

$d(s , t) \leq d(s , x) + d(x , t) < r + r.$

$\therefore d(s , t) < 2r.$

Hence $B(x , r)$ is bounded.

Definition : Let (M , d) be a metric space and $A \subseteq M$. The diameter of A,denoted by d(A), is defined by d(A)= l.u.b $\{d(x , y)/x , y \in A\}$.

Example. In R with usual metric the diameter of any interval is equal to the length of the interval. The diameter of [0 , 1] is 1.

1.10 Complete Metric Spaces.

Definition: Let (M , d) be a metric space. Let (x_n) be a sequence in M. Let $x \in M$. We say that (x_n) converges to x if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x) < \varepsilon$ for all $n \geq N$. If (x_n) converges to x , then x is called a limit of (x_n) and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Note :(1) $x_n \to x$ if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $x_n \in B(x , \varepsilon)\ \forall\ n \geq N$. Thus, the open ball B(x , r) contains all but a finite number of terms of the sequence.

(2) $x_n \to x$ if and only if $(d(x_n , x)) \to 0$.

Theorem 15. The limit of a convergent sequence in a metric space is unique.

Proof. Let (M , d) be a metric space and let (x_n) be a sequence in M. Suppose that (x_n) has two limits say x and y.
Let $\varepsilon > 0$ be given.

Since $x_n \to x$, there exists a positive integer N_1 such that $d(x_n,x) < \varepsilon/2$ for all $n \geq N_1$.

Since $x_n \to y$, there exists a positive integer N_2 such that $d(x_n, x) < \varepsilon/2$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$.

Then, $d(x, y) \leq d(x, x_N) + d(x_N, y)$

$$< \varepsilon/2 + \varepsilon/2$$

$\therefore d(x, y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, $d(x, y) = 0$.

$\therefore x = y$.

Theorem16. Let (M, d) be a metric space and A ⊆ B. Then

(i) X is a limit point of A ⇔ there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

(ii) X ∈ A ⇔ there exists a sequence (x_n) in A such that $x_n \to x$.

Proof.

(i) Let x be a limit point ofA.

(ii) Then every open ball B(x , r) contains infinite number of points of A.

Thus, for each natural number n , we can choose x_n

$x_n \neq x_1, x_2, x_3, \ldots, x_{n-1} \cdot \in B(x, {}^1)$ such that Now, (x_n) is a sequence of distinct points in A and

$d(x_n$

$\therefore (d(x_n, x)) \to 0.$

$\therefore x_n \to x$. $, x) <^1 \forall$ n. n

Conversely, assume that there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

We have to prove x is a limit point of A.

Let it be given an open ball $B(x, \varepsilon)$.

Since $x_n \to x$, there exists a positive integer N such that $d(x_n, x) < \varepsilon \ \forall \ n \geq N$.

$\therefore x_n \in B(x, \varepsilon) \ \forall \ n \geq N$.

Since x_n are distinct points of A, $B(x, \varepsilon)$ contains infinite number of points of A.

Thus, every open ball with center x contains infinite number of points of A.

Hence x is a limit point of A.

(iii) Let $x \in A$.

Then $x \in A \cup A^{l}$.

If $x \in A$ then the constant sequence x, x, x, ….. is a sequence in A converges to x.

If $x \notin A$, then $x \in A'$.

∴ x is a limit point of A.

∴ by (i), there exists a sequence (x_n) in A converges to x. Conversely, assume that there exists a sequence (x_n) in A such that $x_n \to x$.

Then every open ball $B(x , \varepsilon)$ contains points in the sequence and hence points of A.

∴ $x \in A$.

Definition: Let (M , d) be a metric space. Let (x_n) be a sequence in M. Then (x_n) is said to be a Cauchy sequence in M if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x_m) < \varepsilon$ for all $n , m \geq N$.

Theorem 17. Every convergent sequence in a metric space (M , d) is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence in M converges to $x \in M$. We have to prove (x_n) is Cauchy.

Let $\varepsilon > 0$ be given.

Since $x_n \to x$, there exists a positive integer N such that $d(x_n, x) < \varepsilon/2$ for all $n \geq N$.

$\therefore d(x_n , x_m) \leq d(x_n , x) + d(x , x_m)$

$< \varepsilon/2 + \varepsilon/2$ for all $n , m \geq N$.

$\therefore d(x_n , x_m) < \varepsilon$ for all n
, $m \geq N$. Hence (x_n) is a
Cauchy sequence.

Definition:A metric space M is said to be complete if every Cauchy sequence in M converges to a point in M.

Example. R with usual metric is complete.

Theorem 18. A subset A of a complete metric space M is complete if and only if A is closed.

Proof: Suppose that A is complete. We have to prove A is closed.
For that it is enough to prove A contains all its
limitpoints. Let x be a limit point ofA.
Then there exists a sequence(x_n) in A such that
$x_n \rightarrow x$. Since A is complete $x \in A$.
$\therefore$ A contains all its limit
points. Hence A is closed.
Conversely, assume that A is a closed
subset of M. Let (x_n) be a Cauchy sequence
in A.
Then (x_n) be a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that
$x_n \rightarrow x$. Thus (x_n) is a sequence in A such that x_n
$\rightarrow \bar{x}$.
$\therefore x \in A$.

Since A is closed $\bar{A} = A$ and hence x ∈ A.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

∴ A is complete.

Note: Every closed interval [a, b] with usual metric is complete since it is a closed subset of the complete metric space **R**.

Limit of a Sequence: if a sequence is convergent, the unique number to which it converges is the limit of the sequence.

1.13 Cauchy Sequences:

Definition: A sequence (S_n) is called Cauchy sequence if, given any $\in > 0$, there exists an $N \in \mathbb{N}$ such that $|S_n - S_m| < \in$ for all $n, m \geq N$.

Symbolically, $(\forall \in > 0)(\exists N \in \mathbb{N})(\forall n, m \in \mathbb{N})[(n \geq N) \wedge (m \geq N) \Rightarrow |x_n - x_m| < \in]$.

Equivalently (S_n) is a Cauchy sequence if $\lim_{n,m \to \infty} |S_n - S_m| = 0$

Example: Show that the sequence (S_n), where $S_n = \frac{n+1}{n}$, is a Cauchy sequence.

Solution: for all $n, m \in N$,

$$|S_n - S_m| = \left|\left(\frac{n+1}{n}\right) - \left(\frac{m+1}{m}\right)\right| = \left|\frac{mn + m - nm - n}{nm}\right|$$

$$\left|\frac{m-n}{nm}\right| \leq \frac{m+n}{nm}$$

Therefore, if $m \geq n$,then

$$|S_n - S_m| \leq \frac{m+n}{nm} \leq \frac{2m}{mn} = \frac{2}{n}$$

Let $\in > 0$ be given then there is an $N \in \mathbb{N}$,

Such that $\frac{1}{N} < \frac{\in}{2}$.thus for all $n \geq N$,

We have $|S_n - S_m| = \left|\left(\frac{n+1}{n}\right) - \left(\frac{m+1}{m}\right)\right| < \frac{2}{n} \leq \frac{2}{N} \in \varepsilon$

Hence the sequence (S_n) is a Cauchy sequence.

Example: Show that the sequence (S_n), where $S_n = 1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}$ is a Cauchy sequence.

Solution:for all $n, m \in \mathbb{N}$,with $m \geq n$, we have that

$$|S_n - S_m| = \left|\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}\right) - \left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{m+1}}{m!}\right)\right|$$

$$= \left|\frac{(-1)^{n+2}}{(n+1)!} + \frac{(-1)^{n+3}}{(n+2)!} + \cdots + \frac{(-1)^{m+1}}{m!}\right|$$

$$\leq \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}$$

$$\leq \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} = \frac{1}{2^n}\left[1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right]$$

$$= \frac{2}{2^n}\left[1 - \left(\frac{1}{2}\right)^{m-n}\right] \leq \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

Since $\frac{1}{2^{n-1}} \to 0 \; as \; n \to \infty$, given any $\varepsilon > 0$ there is an $N \in \mathbb{N}$ Such that

$\frac{1}{2^{n-1}} = \left|\frac{1}{2^{n-1}} - 0\right| < \varepsilon$ for all $n \in N$.

Thus

$$|S_n - S_m| = \left|\left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{n+1}}{n!}\right) - \left(1 - \frac{1}{2!} + \cdots + \frac{(-1)^{m+1}}{m!}\right)\right|$$

$$< \frac{1}{2^{n-1}} < \varepsilon$$

For all $m \geq n \geq N$. That is (S_n) is Cauchy sequence.

Theorem 19. Every Cauchy sequence (S_n) is bounded.

Proof: Suppose that $\varepsilon = 1$ then there exists an $N \in \mathbb{N}$ such that

$|S_n - S_m| < 1$ for all $n, m \geq N$.

Choose $a_k \geq N$ and observe that

$$|S_n| = |S_n - S_k + S_k| \leq |S_n - S_k| + |S_k|$$

$< 1 + |S_k|$ for all $n \geq N$.

Let $M = Max\{|S_1||S_2| \dots \dots \dots . . |S_N|, |S_k| + 1\}$.

Then $|S_n| \leq M$ for all$n \in \mathbb{N}$, and therefore (S_n) is bounded.

Theorem 20. Every Cauchy sequence (S_n) of real Numbers converges.

Proof: We know that (S_n) is bounded, and therefore, by the Bolzono-Weierstras theorem(S_n) has a subsequence $\left(S_{n_k}\right)$ which converges to some real number ⅄. We claim that the sequence (S_n) converges to ⅄ .

Let $\varepsilon > 0$ be given, then there exist natural numbers N_1 and N_2 such that

$|\mathrm{S_n} - \mathrm{S_m}| < \frac{\varepsilon}{2}$ for all $\mathrm{k} \geq \mathrm{N_2}$.

Let $N = max\{N_1, N_2\}$.then for all $n \geq N$

We have $|S_n - ⅄| \leq \left|S_n - S_{n_k}\right| + \left|S_{n_k} - ⅄\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

Therefore $\lim_{n\to\infty} S_n = ⅄$.

Combining theorem, we get Cauchy's Convergence Criterion for sequence.

A sequence (S_n) of real numbers converges if and only if it is a Cauchy sequence.

1.14 Summary

Metric spaces provide a notion of distance and a framework with which to formally study mathematical concepts such as continuity and convergence, and other related ideas. Many metrics can be chosen for a given set, and our most common notions of distance satisfy the conditions to be a metric. Any norm on a vector space induces a metric on that vector space and it is in these types of metric spaces that we are often most interested for study of signals and systems.

1.15 Terminal Questions

1. Show that if (x_n) is a Cauchy sequence, then so is $\{|x_n|\}$.
2. Let (X, d) be a metric space and let a E X and r > 0. Can B[a, r] be an open set? Justify your answer.
3. Show that Int A is an open set.
4. Show that any finite subset of a metric space is closed.

5. If X is a metric space and A is a non-empty subset of X, then show that $\bar{A}$ = {x: d(x,A) = 0).

6. Let (X, d_l), (Y, d_2) and (Z, d_3) be three metric spaces. Let f : X + Y be . continuous at x ∈ X and g. Y + Z be continuous at y = f(x). Then composite map gof: X + Z is continuous at x ∈ X.

7. Let (X_l, d_l) and (X_2, d_2) be two discrete metric spaces. Then verify that the product metric on X_1 x X_2 is discrete.

8. Check whether the function d: R^2 x $R^2 \rightarrow$ R given by d(P_1,P_2) =$|x_1 - x_2||y_1 - y_2|$ where p_l = (x_l,y_2) and p_2 = (x_2,y_2) isa metric or not.

9. Let (X, d) be a metric space. Show that the following functions give metrics on X. $D(x,y) = \frac{d(x,y)}{1+d(x,y)}$

10. Which of the following functions d : R x R → R are metrics on R?

i) $d(x,y) = 5|x-y|$

ii) $d(x,y) = x^2 + y^2$

UNIT-2: Limit and Continuity of Functions

Structure

2.1 Introduction

In this unit we shall study about the concepts of a limit and Continuity for the functions of a single variable.The natural of a surface is defined by an equation between the coordinates of its points, which we represent by $f(x, y, z) = 0$ generally speaking, on passing through the surface the value of change its sign, so that, as long as the continuity is not interreupted, the values are positive on one side and negative on the other, In the extend these concepts for the functions of two variables.

2.2 Objectives

After reading this unit, we should be able to

▪ Define Bounded and Unbounded of a functions

▪ Define Limit of a function

▪Define Algebra of limits

▪ Check Continuity of a Function

▪ Use theCharacterization of Continuity

▪ State and use theorems on Continuity of functions with the help of examples.

2.3Domain and Range of a function:

A function consists of two non-empty sets X and Y and a rule which assigns to each element of the set X one and only one element of the set Y.

The set X is called the domain of the function. If x is an element of X, then the element of Y which corresponds to it is called the value of the function at x (or the image of x) and is denoted by f(x).

The range of a function is the set of all those elements of Y which are the values of the function.

Range of f(x) = {f(x) : x ∈ X}, clearly range of $f \sqsubseteq Y$

2.4Bounded and Unbounded of a functions:

A function is said to be bounded if its range is bounded, otherwise it is unbounded. Thus, a function f(x) is bounded in the domain D. if there exist two real numbers k and K such that

$$k \leq f(x) \leq K for\ all\ x \in D$$

Again, the bounds of the range of a bounded function are called the bounds of the function.

Example 1. The function f defined by f(x) = sin x for all $x \in R$ is a bounded function, because its range is the closed interval [-1, 1] which is a bounded set. Clarly supremum or l.u.b. of f is 1 and infimum or g.l.b. of f is -1.

Example 2. The function f(x) = log x for all $x \in (0, \infty)$has its range $(-\infty, \infty)$ which is not bounded. Thus the function f is unbounded in the domain (0, ∞).

Note: Let f : X → Y (i.e., f is a function whose domain is X and range $f(X) \subseteq Y$, the co-domain)

(i) f is called a monotonically increasing function if $x_1, x_2 \in X \text{ with } x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

(ii) f is called a monotonically decreasing function if $x_1, x_2 \in X \text{ with } x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

(iii) f is called a one-one function if $x_1, x_2 \in X \text{ with } x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

(iv) f is called an onto function if to each $y \in Y, \exists$ at least one $x \in X \text{ s.t. } f(x) = y$.

2.5 Limit of a function:

A function f(x) is said to tend to a limit l as x tends to a if to each given $\varepsilon > 0$, there exists a positive number δ (depending on ε) such that

$|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$

i.e., $f(x) \in (l - \varepsilon, l + \varepsilon)$ for all those values of x (except at x = a) which belong to $(a - \delta, a + \delta)$. This is denoted by $\lim_{x \to a} f(x) = l$.

Left hand and right hand limits

f(x) is said to tend to l and x tends to a through values less than a, if to each $\varepsilon > 0, \exists\ \delta > 0$, such that

$$|f(x) - l| < \varepsilon \text{ when } a - \delta < x < a$$

So that $f(x) \in (l - \varepsilon, l + \varepsilon)$ whenever $x \in (a - \delta, a)$

The limit in this case is called the left hand limit (L.H.L.) and is denoted by f(a – 0).

Thus

$$f(a-0) = \lim_{x \to a-0} f(x)$$

Similarly, if f(x) tends to l and x tends to a through values which are greater than a i.e., if given $\varepsilon > 0, \exists \delta > 0$ such that

$$|f(x) - l| < \varepsilon \; when \; a < x < a + \delta$$

Then f(x) is said to tend to l from the right and the limit so obtained is called the right hand limit (R.H.L.) and is denoted by f(a + 0)

We write

$$f(a+0) = \lim_{x \to a+0} f(x)$$

Existence of a limit at a point. F(x) is said to tend to a limit as x tends to 'a' if both the left and right hand limits exist and are equal, and their common value is called the limit of the function.

Note. How to find the left hand and right-hand limits?

(i) To find f(a – 0) or $\lim_{x \to a-0} f(x)$, we first put x = a – h, h > 0 in f(x) and then take the limit as h → 0+. Thus

$$\lim_{x \to a-0} f(x) = \lim_{h \to 0+} f(a-h)$$

(ii) To find f(a + 0) or $\lim_{x \to a+0} f(x)$, we first put x = a + h, h > 0 in f(x) and then take the limit as h → 0+. Thus

$$\lim_{x \to a+0} f(x) = \lim_{h \to 0+} f(a+h)$$

2.6 Limit at infinity and infinite limits:

(i) $\lim_{x \to \infty} f(x) = l$

A function f(x) is said to tend to l as $x \to \infty$ if given $\varepsilon > 0$ however small, $\exists a + ve$ number k (depending on ε) s.t.

$$|f(x) - l| < \varepsilon \, \forall x \geq k \quad i.e., l - \varepsilon < f(x) < l + \varepsilon \;\; \forall x \geq k$$

(ii) $\lim_{x \to -\infty} f(x) = l$

A function f(x) is said to tend to l as $x \to -\infty$ if given $\varepsilon > 0$ however small, $\exists a + ve$ number k (depending on ε) s.t.

$$|f(x - l)| < \varepsilon \, \forall x \geq -k \quad i.e., l - \varepsilon < f(x) < l + \varepsilon \;\; \forall x \geq -k$$

(iii) $\lim_{x \to a} f(x) = \infty$

A function f(x) is said to tend to ∞ as x tend to a, if given k > 0, however large $\exists a + ve$ number δ.

$$f(x) > k \; for \; 0 < |x - a| < \delta$$

(iv) $\lim_{x \to a} f(x) = -\infty$

A function f(x) is said to tend to $-\infty$ as x tend to a, if given k > 0, however large $\exists a + ve$ number δ.

$$f(x) > -k \; for \; 0 < |x - a| < \delta$$

(v) $\lim_{x \to \infty} f(x) = -\infty$

A function f(x) is said to tend to $-\infty$ as $x \to \infty$, if given k > 0, however large $\exists a$ number $k' > 0$s.t..

$$f(x) < -k \;\; \forall x \geq k'$$

(vi) $\lim_{x \to -\infty} f(x) = \infty$

A function f(x) is said to tend to ∞ as $x \to -\infty$, if given k > 0, however large $\exists a$ number $k' > 0$s.t..

$$f(x) > k \;\; \forall x \geq -k'$$

(vii) $\lim_{x\to -\infty} f(x) = -\infty$

A function f(x) is said to tend to $-\infty$ as $x \to -\infty$, if given k > 0, however large $\exists a$ number $k' > 0$ s.t..

$$f(x) < -k \ \ \forall x \le -k'$$

2.7 The limit of a function at a point, when it exists, is unique

Suppose $\lim_{x\to a} f(x)$ exists and is not unique.

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} f(x) = l'$, where $l \neq l'$

Now $l \neq l' \Rightarrow |l - l'| > 0.$

If we take $\varepsilon = \frac{1}{2}|l - l'| > 0$, then

$\lim_{x\to a} f(x) = l \Rightarrow$ given $\varepsilon > 0, \exists \delta_1 > 0$s.t.

$|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta_1$ ………………….. (i)

Again $\lim_{x\to a} f(x) = l' \Rightarrow$ given $\varepsilon > 0, \exists \delta_2 > 0$s.t.

$|f(x) - l'| < \varepsilon$ whenever $0 < |x - a| < \delta_2$ ………………….. (ii)

Let $\delta = \min\,(\delta_1, \delta_2)$, then from (i) and (ii), we have

$|f(x) - l| < \varepsilon$ and $|f(x) - l'| < \varepsilon$ whenever $0 < |x - a| < \delta$

Now $|l - l'| = |l - f(x) + f(x) - l'|$

$$\le |l - f(x)| + |f(x) - l'| = |f(x) - l| + |f(x) - l'|$$

$< \varepsilon + \varepsilon = 2\varepsilon$ whenever $0 < |x - a| < \delta$

Or $|l - l'| < |l - l'|$ whenever $0 < |x - a| < \delta$

Which is absurd, therefore, our supposition is wrong. Hence $l = l'$ which proves that $\lim_{x \to a} f(x)$, if it exists, is unique.

2.8 Algebra of limits

Let f and g be two functions and a be a point of their common domain.

If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

(i) $\lim_{x \to a} |f(x) + g(x)| = l + m$ (ii) $\lim_{x \to a} |f(x) - g(x)| = l - m$

(iii) $\lim_{x \to a} |f(x). g(x)| = l. m$ (iv) $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$, provided $m \neq 0$

Proof. (i) $\lim_{x \to a} f(x) = l \to$ given $\varepsilon > 0, \exists \delta_1 > 0$s.t.

$|f(x) - l| < \frac{\varepsilon}{2}$ for $0 < |x - a| < \delta_1$ ....(I)

$\lim_{x \to a} g(x) = m \to$ given $\varepsilon > 0, \exists \delta_2 > 0$s.t.

$|g(x) - m| < \frac{\varepsilon}{2}$ for $0 < |x - a| < \delta_2$ ....(II)

Let $\delta =$ min (δ_1, δ_2), then from (i) and (ii), we have

$|f(x) - l| < \frac{\varepsilon}{2}$ and $|g(x) - m| < \frac{\varepsilon}{2}$ for $0 < |x - a| < \delta$

Now $|f(x) + g(x) - (l + m)| = |f(x) - l + g(x) - m| \leq |f(x) - l| + |g(x) - m|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $0 < |x - a| < \delta$

$$\lim_{x \to a} [f(x) + g(x)] = l + m$$

(ii) Proceeding as in (I) above

$$|f(x) - g(x) - (l - m)| = |f(x) - l + m - g(x)|$$

$$\leq |f(x) - l| + |m - g(x)|$$

$$= |f(x) - l| + |g(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ for } 0 < |x - a| < \delta$$

$$\lim_{x \to a} |f(x) - g(x)| = l = m$$

(iii) $\lim_{x \to a} f(x) = l \rightarrow$ given $\varepsilon_1 > 0, \exists \delta_1 > 0 \; s.t.$

$|f(x) - l| < \varepsilon_1 \; for \; 0 < |x - a| < \delta_1$ …………….. (i)

$\lim_{x \to a} g(x) = l \rightarrow$ given $\varepsilon_2 > 0, \exists \delta_2 > 0 \; s.t.$

$|g(x) - m| < \varepsilon_2 \; for \; 0 < |x - a| < \delta_2$ …………….. (ii)

Let $\delta = min.(\delta_1, \delta_2)$, then from (i) and (ii), we get

$|f(x) - l| < \varepsilon_1$ and $|g(x) - m| < \varepsilon_2$ for $0 < |x - a| < \delta$

Also $|g(x)| = |g(x) - m + m| \leq |g(x) - m| + |m|$

$< \varepsilon_2 + |m| < 1 + |m|$ □ $\varepsilon_2 < 1$

Now $|f(x)g(x) - lm| = |f(x)g(x) - \text{lg}(x) + \text{lg}(x) - lm| = |g(x)(f(x) - l) + l(g(x) - m)|$

$\leq |g(x)(f(x) - l)| + |l(g(x) - m)| = |g(x)||f(x) - l| + |l||g(x) - m|$

$< (1 + |m|)\varepsilon_1 + |l|\varepsilon_2$ for $0 < |x - a| < \delta$

Taking $\varepsilon_1 = \frac{\varepsilon}{2(1+|m|)}$ and $\varepsilon_2 = \frac{\varepsilon}{2(1+|l|)}$, we have

$$|f(x)g(x) - lm| < (1 + |m|)\varepsilon_1 + |l|\varepsilon_2$$

$$= (1 + |m|).\frac{\varepsilon}{2(1 + |m|)} + |l|\frac{\varepsilon}{2(1 + |l|)}$$

$= \frac{\varepsilon}{2} + \frac{|l|}{1+|l|}.\frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \geq \varepsilon \; for \; 0 < |x - a| < \delta$ □ $\frac{|l|}{1+|l|} < 1$

$$\rightarrow \lim_{x \to a} f(x)g(x) = lm$$

(iv) Let us first prove that

$$\lim_{x \to a} g(x) = m(\neq 0) \quad \rightarrow \lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$$

$$\lim_{x \to a} g(x) = m \qquad \rightarrow given\ \varepsilon > 0, \exists \delta > 0 s.t.$$

$$|g(x) - m| < \varepsilon\ for\ 0 < |x - a| < \delta$$

$$|m| = |m - g(x) + g(x)| \leq |m - g(x)| + |g(x)|$$

$$= |g(x) - m| + |g(x)| < \varepsilon + |g(x)|\ for\ 0 < |x - a| < \delta$$

$$|g(x)| > |m| - \varepsilon\ for\ 0 < |x - a| < \delta$$

Taking $\varepsilon > \frac{|m|}{2}$, we get $|g(x)| > \frac{|m|}{2}$ for $0 < |x - a| < \delta$

$\frac{1}{|g(x)|} < \frac{2}{|m|}$ for $0 < |x - a| < \delta$ …………………..(i)

Again $\lim_{x \to a} g(x) = m \rightarrow given\ \varepsilon_1 = \frac{|m|^2}{2}\varepsilon > 0, \exists \delta > 0 s.t.$

$|g(x) - m| < \varepsilon_1$ for $0 < |x - a| < \delta$ …………………..(ii)

□ $\left|\frac{1}{g(x)} - \frac{1}{m}\right| = \left|\frac{m - g(x)}{mg(x)}\right| \frac{|g(x) - m|}{|m||g(x)|}$

$< \frac{\varepsilon_1}{|m|} \cdot \frac{2}{|m|}$ for $0 < |x - a| < \delta$ using (i) and (ii)

$< \frac{2}{|m|^2} \cdot \frac{|m|^2}{2}\varepsilon = \varepsilon\ i.e., \left|\frac{1}{g(x)} - \frac{1}{m}\right| < \varepsilon$ for $0 < |x - a| < \delta$

□ $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{m}$

Now $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \cdot \frac{1}{g(x)} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{1}{g(x)} = l \cdot \frac{1}{m} = \frac{l}{m}.$

Example 1. Do the following limits exist? If yes, find them:

(a) $\lim_{x\to l} \sin\frac{1}{x-1}$ (b) $\lim_{x\to 0} x\sin\frac{1}{x}$ (c) $\lim_{x\to 1} 2^{\frac{1}{x-1}}$

(d) $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$ (e) $\lim_{x\to 0} \frac{1}{1+e^{1/x}}$

(f) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} 3x-2 & where\ x<1 \\ 4x^2-3x & where\ x>1 \end{cases}$

Solution. (a) $\lim_{x\to 1} \sin\frac{1}{x-1}$

L.H.L $= \lim_{x\to 1-0} \sin\frac{1}{x-1}$ [Put x = 1 – h, h > 0]

$= \lim_{h\to 0} \sin\frac{1}{1-h-1} = \lim\limits_{h\to 0} -\sin\frac{1}{h}$.

Now as h → 0, $\sin\frac{1}{h}$ is finite and oscillates between -1 and 1; so it does not tend to any unique and definite value as h → 0. Hence L.H.L. does not exist.

Similarly the right hand limit also does not exist as x →1.

Thus $\lim_{x\to 0} x\sin\frac{1}{x-1}$ does not exist.

(b) $\lim_{x\to 0} x\sin\frac{1}{x}$

L.H.L. $= \lim_{x\to 0-0} x\sin\frac{1}{x}$ [Put x = 0 – h, h > 0]

$$= \lim_{h\to 0}(0-h)\sin\frac{1}{0-h} = \lim_{h\to 0} h\sin\frac{1}{h}$$

$= 0 \times a$ finite quantity between -1 and 1 = 0

Similarly, R.H.L.$= \lim_{x\to 0-0} x\sin\frac{1}{x}$ [Put x = 0 + h, h > 0]

$$= \lim_{h\to 0}(0+h)\sin\frac{1}{0+h} = \lim_{h\to 0} h\sin\frac{1}{h}$$

$= 0 \times a$ finite quantity between -1 and 1 = 0

Thus L.H.L and R.H.L. both exist and are equal and hence $\lim_{x\to 0} x\sin\frac{1}{x}$ exists and is equal to zero.

□ $\lim_{x\to 0} x\sin\frac{1}{x} = 0$

(c) Let $f(x) = 2^{\frac{1}{x-1}}$

L.H.L. $= f(1-0) = \lim_{x\to 1-0} f(x) = \lim\limits_{x\to 1-0} 2^{\frac{1}{x-1}}$ [Put x = 1 – h, h > 0]

$$= \lim_{h\to 0} 2^{\frac{1}{1-h-1}} = \lim_{h\to 0} 2^{\frac{-1}{h}} = 2^{-\infty} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0$$

R.H.L. = $\lim\limits_{x\to 1+0} 2^{\frac{1}{x-1}}$ [Put x = 1 + h, h > 0]

$$= \lim_{h\to 0} 2^{\frac{1}{1+h+1}} = \lim_{h\to 0} 2^{\frac{1}{h}} = 2^{\infty} = \infty$$

Since, L.H.L. ≠ R.H.L. □ $\lim\limits_{x\to 1} 2^{\frac{1}{x-1}}$ does not exist.

(d) $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$

L.H.L. $= \lim_{x\to 0-0} \frac{e^{1/x}}{e^{1/x}+1}$ [Put x = 0 – h, h > 0]

$$= \lim_{h\to 0} \frac{e^{\frac{1}{0-h}}}{e^{\frac{1}{0-h}}+1} = \lim_{h\to 0} \frac{e^{\frac{-1}{h}}}{e^{\frac{-1}{h}}+1} = \frac{0}{0+1} = 0 \left[\lim_{h\to 0} e^{-\frac{1}{h}} = e^{-\infty} = \frac{1}{\infty} = 0\right]$$

R.H.L. $= \lim_{x\to 0+0} \frac{e^{1/x}}{e^{1/x}+1}$ [Put x = 0 + h, h > 0]

$= \lim_{h\to 0} \frac{e^{\frac{1}{0+h}}}{e^{\frac{1}{0+h}}+1} = \lim_{h\to 0} \frac{e^{\frac{1}{h}}}{e^{\frac{1}{h}}+1}$ [divide the numerator and denominator by $e^{\frac{1}{h}}$]

$$= \lim_{h\to 0} \frac{1}{1+e^{\frac{-1}{h}}} = \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$$

Since, L.H.L ≠ R.H.L □ $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$ does not exist.

(e) Please try yourself.

(f) L.H.L. $= \lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}(3x-2)$ [Put x = 1 – h, h > 0]

$$= \lim_{h\to 0}(3-3h-2) = \lim_{h\to 0}(1-3h) = 1-0 = 1$$

R.H.L. $= \lim_{h\to 1+0} f(x) = \lim_{h\to 1+0}(4x^2-3x)$ [Put x = 1 + h, h > 0]

$$= \lim_{h\to 0}[4(1+h)^2 - 3(1+h)] = \lim_{h\to 0}(1+5h+4h^2) = 1$$

L.H.L. = R.H.L.=1.

Hence $\lim_{x\to 0} f(x) = 1$

Example 2. Using the definition of limit, prove that

(i) $\lim_{x\to 0} \frac{x^2-a^2}{x-a} = 2a$ (ii) $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

(iii) $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Solution. (i) Here $f(x) = \frac{x^2-a^2}{x-a}$, x ≠ a

We must show that for any $\varepsilon > 0, \exists \delta > 0 s.t.$

$$|f(x) - 2a| < \varepsilon \; for \; 0 < |x-a| < \delta$$

Now $|f(x) - 2a| = \left|\frac{x^2-a^2}{x-a} - 2a\right| = \left|\frac{x^2-a^2-2ax+2a^2}{x-a}\right|$

$$= \left|\frac{x^2-2ax+a^2}{x-a}\right| = \left|\frac{(x-a)^2}{x-a}\right| = |x-a|$$

$$|f(x) - 2a| < \varepsilon \; whenever \; 0 < |x - a| < \varepsilon$$

Choosing $\delta = \varepsilon$

$$|f(x) - 2a| < \varepsilon \; whenever \; 0 < |x - a| < \delta$$

Hence $\lim_{x \to a} \frac{x^2 - a^2}{x - a} = 2a.$

(ii) Here $f(x) = xsin \frac{1}{x}$

$$|f(x) - 0| = \left| xsin \frac{1}{x} \right| = |x| \left| \sin \frac{1}{x} \right| \leq |x| \left(\left| \sin \frac{1}{x} \right| \leq 1 \right)$$

$|f(x) - 0| < \varepsilon$ whenever $0 < |x| < \varepsilon$

Choosing $\delta = \varepsilon$

$$|f(x) - 0| < \varepsilon \; whenever \; 0 < |x| < \delta$$

Hence $f(x) = x \sin \frac{1}{x} = 0$

(iii) Here $f(x) = x^2 \sin \frac{1}{x}$

$$|f(x) - 0| = \left| x^2 sin \frac{1}{x} \right| = |x^2| \left| \sin \frac{1}{x} \right| \leq |x|^2 \left(\left| \sin \frac{1}{x} \right| \leq 1 \right)$$

$|f(x) - 0| < \varepsilon$ whenever $0 < |x|^2 < \varepsilon$ i.e., whenever $0 < |x| < \sqrt{\varepsilon}$

Choosing $\delta = \sqrt{\varepsilon}$

$$|f(x) - 0| < \varepsilon \; whenever \; 0 < |x| < \delta$$

Hence $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$

Example 3. If $f(x) = [x]$ where [x] denotes the greatest integer not greater than x, show that $\lim_{x \to 1} f(x)$ does not exist.

Solution. We have L.H.L = $\lim_{x\to 1-0} f(x) = \lim_{x\to 1-0}[x]$ [Put x = 1 – h, h > 0]

$$= \lim_{h\to 0}[1-h] = \lim_{h\to 0}(0) = 0$$

R.H.L. = $\lim_{x\to 1+0} f(x) = \lim_{x\to 1+0}[x]$ [Put x = 1+ h, h > 0]

$$= \lim_{h\to 0}[1+h] = \lim_{h\to 0}(1) = 1$$

□ $\lim_{x\to 1-0} f(x) \neq \lim_{x\to 1+0} f(x)$

□ $\lim_{x\to 1} f(x)$ does not exist.

Example 4. Find $\lim_{x\to a} f(x)$ where $f(x) = \begin{cases} \frac{x^2}{a} - a & for\ 0 < x < a \\ 0 & for\ x = a \\ a - \frac{a^3}{x^2} & for\ x > a \end{cases}$

Sol. L.H.L = $\lim_{x\to a-0} f(x) = \lim_{x\to a-0}\left(\frac{x^2}{a} - a\right)$ [Put x = a – h, h > 0]

$$= \lim_{h\to 0}\left[\frac{(a-h)^2}{a} - a\right] = \frac{a^2}{a} - a = a - a = 0$$

R.H.L. = $\lim_{x\to a+0} f(x) = \lim_{x\to a+0}\left(a - \frac{a^3}{x^2}\right)$ [Put x = a + h, h > 0]

$$= \lim_{h\to 0}\left[a - \frac{a^3}{(a+h)^2}\right] = a - \frac{a^3}{a^2} = a - a = 0$$

□ $\lim_{x\to a-0} f(x)$ and $\lim_{x\to a+0} f(x)$ both exist and each is equal to 0.

□ $\lim_{x\to a} f(x) = 0$

Example 5. Let $f(x) = \frac{x^2+2}{x^2+1}$, then given $\varepsilon > 0$, find a real number $\delta > 0$ such that

$$|f(x) - 2| < \varepsilon \; for \; 0 < |x| < \delta$$

Solution. $|f(x) - 2| = \left|\frac{x^2+2}{x^2+1} - 2\right| < \varepsilon$

If $\left|\frac{x^2+2-x^2-2}{x^2+1}\right| < \varepsilon$ or if $\left|\frac{-x^2}{x^2+1}\right| < \varepsilon$

Or if $\left|\frac{x^2}{x^2+1}\right| < \varepsilon(|-x| = |x|)$

Or if $\frac{x^2}{x^2+1} < \varepsilon\left(\frac{x^2}{x^2+1} \geq 0\right)$

Or if $x^2 < \varepsilon(x^2 + 1) \quad (x^2 + 1 > 0)$

Or if $(1 - \varepsilon)\, x^2 < \varepsilon$ or if $x^2 < \frac{\varepsilon}{1-\varepsilon}$ (if $1 - \varepsilon > 0$ i.e., $\varepsilon < 1$)

Or if $|x| < \sqrt{\frac{\varepsilon}{1-\varepsilon}}$

$\therefore$ Choosing $\delta = \sqrt{\frac{\varepsilon}{1-\varepsilon}}, 0 < \varepsilon < 1$, we have

$$|f(x) - 2| < \varepsilon \; for \; 0 < |x| < \delta$$

Example 6. Let $f(x) = \frac{1}{x}, \; x \neq 0$. Prove from definition (ϵ, δ method) that $\lim_{x\to 2} f(x) = \frac{1}{2}$.

Solution. To prove that $\lim_{x\to 2} f(x) = \frac{1}{2}$, we have to show that for any $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$s.t.

$$\left|f(x)-\frac{1}{2}\right| < \varepsilon \; when \; 0 < |x-2| < \delta$$

Now $\left|f(x)-\frac{1}{2}\right| = \left|\frac{1}{x}.\frac{1}{x}\right| = \left|\frac{2-x}{2x}\right| = \frac{|2-x|}{2|x|}$ …………(i)

Choosing $\delta \leq 1$ and $0 < |x-2| < \delta$, we have

$0 < |x-2| < 1 \Rightarrow |x-2| > 0$ and $|x-2| < 1$

$\Rightarrow x \neq 2$ and $2-1 < x < 2+1 \Rightarrow x \neq 2$ and $1 < x < 3$

$\Rightarrow x \neq 2$ and $1 > \frac{1}{x} > \frac{1}{3} \Rightarrow x \neq 2$ and $\frac{1}{3} < \frac{1}{x} < 1$

$\Rightarrow x \neq 2$ and $\frac{1}{|x|} < 1 \left[\because \frac{1}{x} > \frac{1}{3} > 0 \;\therefore \frac{1}{x} = \frac{1}{|x|}\right]$

$\therefore$ From (i), $\left|f(x)-\frac{1}{2}\right| = \frac{|x-2|}{2}.\frac{1}{|x|} < \frac{\delta}{2}.1$

Let us choose $\delta \; s.t. \frac{\delta}{2} < \varepsilon \; i.e., \delta < 2\varepsilon$

Also $\delta \leq 1 \therefore$ Choosing δ= min. (1, 2ε), we have

$\left|f(x)-\frac{1}{2}\right| < \frac{\delta}{2} < \varepsilon$ when $0 < |x-2| < \delta$

$\therefore \lim_{x\to 2} f(x) = \frac{1}{2}$

Example 7. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, can $\lim_{x\to a}[f(x)+g(x)]$ exist? Prove your assertion.

Solution. $\because \lim_{x\to a} f(x)$ exists, let $\lim_{x\to a} f(x) = l$

$\Rightarrow \lim_{x\to a-0} f(x) = l = \lim_{x\to a+0} f(x)$

$\because \lim_{x\to a} g(x)$ does not exist, let $\lim_{x\to a-0} g(x) = m_1$ and $\lim_{x\to a+0} g(x) = m_2$ where $m_1 \neq m_2$

Now $\lim_{x\to a-0}[f(x)+g(x)] = \lim_{x\to a-0} f(x) + \lim_{x\to a-0} g(x) = l + m_1$

$$\lim_{x\to a+0}[f(x)+g(x)] = \lim_{x\to a+0} f(x) + \lim_{x\to a+0} g(x) = l + m_2$$

Since $\quad l + m_1 \neq l + m_2 \quad [\because m_1 \neq m_2]$

$\therefore \lim_{x\to a}[f(x)+g(x)]$ does not exist.

Example 8. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x)\, g(x)$ both exist, then does it follow that $\lim_{x\to a} g(x)$ exists?

Solution. Let $\qquad f(x) = x, g(x) = \frac{|x|}{x}, x \neq 0 = \begin{cases} 1 & if \quad x > 0 \\ -1 & if \quad x < 0 \end{cases}$

$$f(x)g(x) = x.\frac{|x|}{x} = |x|$$

$\lim_{x\to a} f(x) = 0$ exists; $\lim_{x\to a} f(x)\, g(x) = 0$ exists

But $\lim_{x\to 0-0} g(x) = \lim_{x\to 0-0} -1 = -1$

$$\lim_{x\to 0+0} g(x) = \lim_{x\to 0+0} 1 = 1$$

$\Rightarrow \lim_{x\to a} g(x)$ does not exists.

Thus $\lim_{x\to a} f(x)$ and $\lim_{x\to a} f(x)\, g(x)$ both exist does not necessarily imply that $\lim_{x\to a} g(x)$ also exists.

Example 9. If $\lim_{x\to a} f(x) = l$, then show that $\lim_{x\to a}|f(x)| = |l|$. Is its converse true?

Solution. We have$\lim_{x\to a} f(x) = l \Rightarrow$ for any given $\varepsilon > 0, \exists \delta > 0\ s.t.$

$|f(x) - l| < \varepsilon$ when $0 < |x - a| < \delta$ $\qquad\qquad$ …………… (1)

Since $\quad |a - b| \geq \big||a| - |b|\big|$

$\therefore |f(x) - l| \geq \big||f(x)| - |l|\big|$

$\Rightarrow \big||f(x)| - |l|\big| \leq |f(x) - l| < \varepsilon$ when $0 < |x - a| < \delta$

using (i)

$\Rightarrow \lim_{x\to 0}|f(x)| = |l|$

The converse of this statement is not always true.

For example, consider $f(x) = \begin{cases} -1 & if \quad x < a \\ 1 & if \quad x \geq a \end{cases}$

Then $\lim_{x\to a-0} f(x) = \lim_{x\to a-0} -1 = -1$

$$\lim_{x\to a+0} f(x) = \lim_{x\to a+0} 1 = 1$$

$\Rightarrow \lim_{x\to a} f(x)$ does not exists.

But $|f(x)| = 1 \quad \forall x \Rightarrow \lim_{x\to a}|f(x)| = 1 \ exists.$

Example 10. If $f(x) \leq g(x) \leq h(x)$ and $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$, then prove that $\lim\limits_{x\to a} g(x)$ exists and is equal to l.

Solution. We have $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$

$\Rightarrow$ Given $\varepsilon > 0, \exists \delta_1, \delta_2 > 0 s.t.$

$|f(x) - l| < \varepsilon$ for $0 < |x - a| < \delta_1$

And $|h(x) - l| < \varepsilon$ for $0 < |x - a| < \delta_2$

$\Rightarrow l - \varepsilon < f(x) < l + \varepsilon$ for $0 < |x - a| < \delta_1$

And $l - \varepsilon < h(x) < l + \varepsilon$ for $0 < |x - a| < \delta_2$

Let $\delta = min.(\delta_1, \ \delta_2)$, then

$l - \varepsilon < f(x) < l + \varepsilon$ and $l - \varepsilon < h(x) < l + \varepsilon$ for $0 < |x - a| < \delta$ ……(i)

Also $f(x) \leq g(x) \leq h(x)$ given ……………………………………………….. (ii)

From (i) and (ii) $l - \varepsilon < f(x) \leq g(x) \leq h(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow l - \varepsilon < g(x) < l + \varepsilon$ for $0 < |x - a| < \delta$

$\Rightarrow |g(x) - l| < \varepsilon \; for \; 0 < |x - a| < \delta \;\Rightarrow\; \lim_{x\to a} g(x) = l$

2.9 Characterization of Continuity:

Definitions: (i) Continuity at a point

A function f : A → R is said to be continuous at the point a ∈ A if given $\varepsilon > 0$, however small, ∃a real number $\delta > 0$, such that

$|f(x) - f(a)| < \varepsilon$ whenever $x \in A \; and \; |x - a| < \delta$

i.e., $f(x) \in (f(a) - \varepsilon, f(a) + \varepsilon)$ whenever $x \in (a - \delta, a + \delta) \cap A.$

Equivalently, a function f is continuous at x = aiff$\lim_{x\to a} f(x) = f(a)$

i.e., iff$\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a)$

(ii) Continuity from the left at a point

A function f : A → R is said to be continuous from the left (or left continuous) at the point $a \in A$ given $\varepsilon > 0$ however small, ∃ a real number $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ whenever $x \in A \; and \; a - \delta < x \leq a$

Equivalently, a function f is continuous from the left (or left continuous) at x = a if $\lim_{x \to a} f(x) = f(a)$

(iii) Continuity from the right at a point

A function f : A → R is said to be continuous from the right (or right continuous) at the point $a \in A$ if given $\varepsilon > 0$ however small, ∃ a real number $\delta > 0$ such that

$|f(x) - f(a)| < \varepsilon$ whenever $x \in A \; and \; a \leq x < a + \delta.$

Equivalently, a function f is continuous from the right (or right continuous) at x = a if $\lim_{x \to a} f(x) = f(a)$

Note. Clearly, f is continuous at x = aiff f is left as well as right continuous at x = a.

(iv) A function f is said to be continuous in an open interval (a, b) if f is continuous at every point of (a, b).

Thus, f is continuous in the open interval (a, b) iff for every $c \in (a, b), \lim_{x \to c} f(x) = f(c)$.

(v) Continuity in a closed interval

A function f is said to be continuous in a closed interval [a, b] if it is

(i) Right continuous at a i.e. $\lim_{x \to a+} f(x) = f(a)$

(ii) Continuous in the open interval (a, b) i.e.$\lim_{x \to c} f(x) = f(c)$ for every $c \in (a, b)$

(iii) Left continuous at b i.e.$\lim_{x \to b-} f(x) = f(b)$

(vi) Continuity in a semi closed interval

I. A function f is said to be continuous in semi closed interval (a, b) if it is

(i) Continuous in the open interval (a, b) i.e., $\lim_{x\to c} f(x) = f(c)$ for every $c \in (a, b)$

(ii) Left continuous at b i.e., $\lim_{x\to b-} f(x) = f(b)$

II. A function f is said to be continuous in semi closed interval [a, b) if it is

(i) Right continuous at a i.e.$\lim_{x\to a+} f(x) = f(a)$

(ii) Continuous in the open interval (a, b) i.e., $\lim_{x\to c} f(x) = f(c)$ for every $c \in (a, b)$

(vii) Continuity on a set

A function f is said to be continuous on an arbitrary set $S(\subset R)$ if for each $\varepsilon < 0$ and for every $a \in S, \exists$ a real number $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in S \; and \; |x - a| < \delta$.

Equivalently, a function f is said to be continuous on a set S if it continuous at every point of
S, i.e., if for every $a \in S$, $\lim_{x\to a} f(x) = f(a)$.

(viii) Continuous Function

A function f : A → R is said to be continuous iff it is continuous on A.

Thus f is continuous if it is continuous at every point of its domain.

(ix) Discontinuity of a function

A function f which is not continuous at a point 'a' is said to be discontinuous at the point 'a'.

‘a’ is called a point of discontinuity of f or f is said to have a discontinuity at ‘a’.

A function which is discontinuous even at a single point of an interval is said to be discontinuous in the interval.

A function f can be discontinuous at a point x = a because of any one of the following reasons:

(i) f is not defined at ‘a’

(ii) $\lim_{x\to a} f(x)$ does not exist i.e., $\lim_{x\to a-} f(x) \neq \lim_{x\to a+} f(x)$

(iii) $\lim_{x\to a} f(x)$ and f(a) both exist but are not equal

(x) Types of Discontinuity

Let f be a function defined on an interval I. Let f be discontinuous at a point $a \in I$.

(1)Removable Discontinuity

If $\lim_{x\to a} f(x)$ exists but is not equal to f(a), then f is said to have a removable discontinuity at ‘a’.

This type of discontinuity can be removed by defining a new function g as

$$g(x) = \begin{cases} f(x) & if\ x \neq a \\ \lim\limits_{x\to a} f(x) & if\ x = a \end{cases}$$

Then g is continuous at ‘a’

Note. If $\lim_{x\to a} f(x)$ does not exist, then the function cannot be made continuous, no matter how we define f(a).

(2)Discontinuity of First Kind (or Jump Discontinuity)

If $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$ both exist but are unequal then f is said to have a discontinuity of first kind at ‘a’ or jump discontinuity at ‘a’.

F is said to have a discontinuity of the first kind from the left at ‘a’ if $\lim_{x\to a-} f(x)$ exists but is not equal to f(a).

F is said to have a discontinuity of the first kind from the right at ‘a’ if $\lim_{x\to a+} f(x)$ exists but is not equal to f(a).

(3)Discontinuity of Second Kind

If neither $\lim_{x\to a-} f(x)$ nor $\lim_{x\to a+} f(x)$ exist, then f is said to have a discontinuity of second kind at ‘a’.

f is said to have a discontinuity of the second kind from the left at ‘a’ if $\lim_{x\to a-} f(x)$ does not exist.

f is said to have a discontinuity of the second kind from the right at ‘a’ if $\lim_{x\to a+} f(x)$ does not exist.

(4)Mixed Discontinuity

If a function f has a discontinuity of the second kind on one side of a and on the other side, a discontinuity of the first kind or may be continuous, then f is said to have a mixed discontinuity at ‘a’.

Thus f has a mixed discontinuity at ‘a’ if either

(i) $\lim_{x\to a-} f(x)$ does not exist and $\lim_{x\to a+} f(x)$ exists, however $\lim_{x\to a+} f(x)$ may or may not equal f(a).

(ii) $\lim_{x\to a+} f(x)$ does not exist and $\lim_{x\to a-} f(x)$ exists, however $\lim_{x\to a-} f(x)$ may or may not equal f(a).

(xi) Piecewise Continuous Function

A function f : A → R is said to be piecewise continuous on A if A can be divided into a finite number of parts so that f is continuous on each part.

Clearly, in such a case f has a finite number of discontinuities and the set A is divided at the points of discontinuities.

For example, consider f : (0, 5) → R defined by f(x) = $|x|$, then f is discontinuous at 1, 2, 3 and 4. If the interval (0. 5) is divided at 1, 2, 3 and 4, then f is continuous in (0, 1), (1, 2), (2, 3), (3, 4) and (4, 5).

∴ f is piecewise continuous.

Example 1. Using $\varepsilon - \delta$ definition, prove that

(i) $f(x) = 3x + 1$ is continuous at x = 2.

(ii) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & if \quad x \neq 2 \\ 4 & if \quad x = 2 \end{cases}$ is continuous at x = 2

(iii) $f(x) = \begin{cases} \frac{x^3-1}{x^2-1}, & if \quad x \neq 1 \\ 3/2, & if \quad x - 1 \end{cases}$ is continuous at x = 1

Solution. (i) Here $f(x) = 3x + 1, f(2) = 3 \times 2 + 1 = 7$

Let $\varepsilon > 0$ be given

Now $|f(x) - f(2)| = |(3x + 1) - 7| = |3(x - 2)|$

$= 3|x - 2| < \varepsilon$ whenever $3|x - 2| < \varepsilon$ i.e., $|x - 2| < \frac{\varepsilon}{3}$

∴ if we choose $\delta = \frac{\varepsilon}{3}$, then $|f(x) - f(2)| < \varepsilon$ whenever $|x - 2| < \delta$

⇒ f is continuous at x = 2.

(ii) Here $f(x) = \frac{x^2-4}{x-2}, \ x \neq 2$

f(2) = 4

let $\varepsilon > 0$

Now $|f(x) - f(2)| = \left|\frac{x^2-4}{x-2} - 4\right| = \left|\frac{(x+2)(x-2)}{x-2} - 4\right|$

$= |(x+2) - 4| = |x-2| < \varepsilon$ whenever $|x-2| > \varepsilon$

$\therefore$ if we choose $\delta = \varepsilon$, then $|f(x) - f(2)| < \varepsilon$ whenever $|x-2| < \delta$

⇒ f is continuous at x = 2.

(iii) Here $f(x) = \frac{x^3-1}{x^2-1}, x \neq 1$

f(1) = 3/2

Let $\varepsilon > 0$ be given

Now $|f(x) - f(1)| = \left|\frac{x^3-1}{x^2-1} - \frac{3}{2}\right| = \left|\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} - \frac{3}{2}\right|$

$$= \left|\frac{x^2+x+1}{x^2+1} - \frac{3}{2}\right| = \left|\frac{2x^2-x-1}{2(x+1)}\right|$$

$$= \left|\frac{(x-1)(2x+1)}{2x+2}\right| = |x-1|\left|\frac{2x+1}{2x+2}\right|$$

$$= |x-1| \left[\because \left|\frac{2x+1}{2x+2}\right| < 1\right]$$

$< \varepsilon$ whenever $|x-1| < \varepsilon$

$\therefore$ If we choose $\delta = \varepsilon$, then $|f(x) - f(1)| < \varepsilon$ whenever $|x-1| < \varepsilon$

⇒ f is continuous at x = 1

Example 2. Using $\varepsilon - \delta$ definition, prove that

(i) $f(x) = \begin{cases} x\sin\frac{1}{x}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ is continuous at x = 0

(ii) $g(x) = \begin{cases} x^2\cos\frac{1}{x}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ is continuous at x = 0

Solution. (i) Here $f(x) = x\sin\frac{1}{x},\ x \neq 0$

$$f(0) = 0$$

Let $\varepsilon > 0$ be given.

Now $|f(x) - f(0)| = \left|x\sin\frac{1}{x} - 0\right| = \left|x\sin\frac{1}{x}\right| = |x|\left|\sin\frac{1}{x}\right| \leq |x| \left[\because \left|\sin\frac{1}{x}\right| \leq 1\right]$

$< \varepsilon$ whenever $|x| < \varepsilon$

$\therefore$ If we choose $\delta = \varepsilon$, then $|f(x) - f(0)| < \varepsilon$ whenever $|x - 0| < \delta$

$\Rightarrow$ f is continuous at x = 0

(ii) Here $g(x) = x^2\cos\frac{1}{x}, x \neq 0$

$$g(0) = 0$$

Let $\varepsilon > 0$ be given

Now $|g(x) - g(0)| = \left|x^2\cos\frac{1}{x} - 0\right| = \left|x^2\cos\frac{1}{x}\right| = |x^2|\left|\cos\frac{1}{x}\right| \leq |x^2| \left[\because \left|\cos\frac{1}{x}\right| \leq 1\right]$

$= |x|^2 < \varepsilon$ whenever $|x|^2 < \varepsilon$ i.e., whenever $|x| < \sqrt{\varepsilon}$

$\therefore$ If we choose $\delta = \sqrt{\varepsilon}$, then $|g(x) - g(0)| < \varepsilon$ whenever $|x - 0| < \delta$

$\Rightarrow$ g is continuous at x = 0

Example 3. Using $\varepsilon - \delta$ definition, prove that the following functions are continuous:

(i) $|x|$ (ii) $\cos x$ (iii) $\sin x$

(iv) $cos^2 x$ (v) $sin^2 x$

Solution. A function f is said to be continuous if it is continuous at every point of its domain

(i) Let $f(x) = |x|$ domain of f = R

Let a be any real number so that $f(x) = |a|$

Let $\varepsilon > 0$ be given

Now $|f(x) - f(a)| = \big||x| - |a|\big| \le |x - a| \Big| \because \big||a| - |b|\big| \le |a - b| \Big|$

$< \varepsilon$ whenever $|x - a| < \varepsilon$

$\therefore$ if we choose $\delta = \varepsilon$, then $|f(x) - f(a)| << \varepsilon$ whenever $|x - a| < \delta$

$\Rightarrow$ f is continuous at x = 0

$\Rightarrow$ f is continuous at ever $a \in R$

$\Rightarrow$ f is continuous

(ii) Let $f(x) = \cos x$. Domain of f = R

Let a be any real number so that $f(a) = \cos a$

Let $\varepsilon > 0$ be given

Now $|f(x) - f(a)| = |\cos x - \cos a|$

$$= \left|-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}\right| = 2\left|\sin\frac{x+a}{2}\right|\left|\sin\frac{x-a}{2}\right|$$

$$\leq 2\left|\sin\frac{x-a}{2}\right|\left[\because \ \left|\sin\frac{x+a}{2}\right| \leq 1\right]$$

$< 2\left|\frac{x-a}{2}\right| \because |\sin x| \leq |x|$

$= 2.\frac{|x-a|}{2} = |x-a| < \varepsilon$ whenever $|x-a| < \varepsilon$

$\therefore$ if we choose $\delta = \varepsilon$, then

$|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$

$\Rightarrow$ f is continuous at x = 0.

$\Rightarrow$ f is continuous at ever $a \in R$

$\Rightarrow$ f is continuous

(iii) Please try yourself.

(iv) Let $f(x) = cos^2x$ domain of f = R

Let a be any real number so that $f(a) = cos^2a$

Let $\varepsilon > 0$ be given

Now $|f(x) - f(a)| = |cos^2x - cos^2a| = |(1 - sin^2x) - (1 - sin^2a)|$

$= |sin^2a - sin^2x| = |sin^2x - sin^2a| = |\sin(x+a).\sin(x-a)|$

$\leq |\sin(x-a)| \because |\sin(x+a) \leq 1|$

$\leq |x-a| \because |\sin x| \leq |x|$

$< \varepsilon$ whenever $|x-a| < \varepsilon$

$\therefore$ if we choose $\delta = \varepsilon$, then

$|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$

$\Rightarrow$ f is continuous at x = 0.

$\Rightarrow$ f is continuous at ever $a \in R$

$\Rightarrow$ f is continuous

Example 4. Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.

(i) $f(x) = \begin{cases} \frac{x^2-4}{x-2}, & if \quad x \neq 2 \\ 4, & if \quad x = 2 \end{cases}$ at x = 2

(ii) $f(x) = \begin{cases} \frac{x^2-9}{x-3}, & if \quad x \neq 3 \\ 5 & if \quad x = 3 \end{cases}$ at x = 3

(iii) $f(x) = \frac{x^3-8}{x-2}$ at x = 2

(iv) $f(x) = \begin{cases} \frac{\sin 2x}{x} & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$ at x = 0

(v) $f(x) = \begin{cases} \frac{sin^{-1}x}{2x}, & if \quad x \neq 0 \\ \frac{1}{2} & if \quad x = 0 \end{cases}$ at x = 0

Solution. (i) Here $f(2) = 4$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2}$$

[Cancelling (x – 2), since x →2 ⇒ x ≠ 2]

$$= \lim_{x\to 2}(x+2) = 2+2 = 4$$

Since $\lim_{x\to 2} f(x) = f(2)$, f is continuous at x = 2

(ii) Here $f(3) = 5$

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{x-3} = \lim_{x\to 3}(x+3) = 3+3 = 6$$

Thus $\lim_{x\to 3} f(x)$ exists but $\lim_{x\to 3} f(x) \neq f(3)$.

∴ f has a removable discontinuity at x = 3

f can be made continuous at x = 3 be redefining it as follows:

$$f(x) = \begin{cases} \dfrac{x^2-9}{x-3}, & if \quad x \neq 3 \\ 5 & if \quad x = 3 \end{cases}$$

(iii) $f(x) = \frac{x^3-8}{x-2}$ is not defined at x = 2, since f(2) assumes at form 0/0

However$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3-8}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2}$

$$= \lim_{x\to 2}(x^2+2x+4) = 2^2+2\times 2+4 = 12$$

Thus $\lim_{x\to 2} f(x)$ exists. Therefore, f has a removable discontinuity at x = 2

f can be made continuous at x = 2 by redefining it as follows:

$$f(x) = \begin{cases} \dfrac{x^3-8}{x-2} & if \quad x \neq 2 \\ 12 & if \quad x = 2 \end{cases}$$

(iv) Here $f(0) = 1$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} 2 \times \frac{\sin 2x}{2x} = 2\times 1 = 2$$

Thus $\lim_{x\to 0} f(x)$ exists but $\lim_{x\to 0} f(x) \neq f(0)$

⇒ f has a removable discontinuity at x = 2

f can be made continuous at x = 0 by redefining it as follows:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & if \quad x \neq 0 \\ 2 & if \quad x = 0 \end{cases}$$

(v) Here $f(0) = \frac{1}{2}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{sin^{-1}x}{2x}$$

[Put $sin^{-1}x = \theta$ so that $x = \sin\theta$ As x → 0, θ → 0]

$$\lim_{\theta \to 0} \frac{\theta}{2\sin\theta} = \frac{1}{2}\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = \frac{1}{2} \times 1 = \frac{1}{2}$$

Since $\lim_{x \to 0} f(x) = f(0)$, f is continuous at x = 0

Example 5. Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.

(i) $f(x) = \begin{cases} \frac{e^{1/x}-1}{e^{1/x}+1}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ at x = 0

(ii) $f(x) = \begin{cases} \frac{e^{1/x}}{1+e^{1/x}}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ at x = 0

(iii) $f(x) = \begin{cases} \frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & if \quad x \neq 0 \\ 1, & if \quad x = 0 \end{cases}$ at x = 0

(iv) $f(x) = \begin{cases} (x-a)\frac{e^{\frac{1}{x-a}}-1}{e^{\frac{1}{x-a}}+1}, & if \quad x \neq a \\ 0 & if \quad x = a \end{cases}$ at x = a

(v) $f(x) = \begin{cases} \frac{xe^{1/x}}{1+e^{1/x}}, & if \quad x \neq 0 \\ 0, & if \quad x = 0 \end{cases}$ at x = 0

(vi) Show that the function f defined on R as

$f(x) = x.\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}$ $if\ x \neq 0$ and f(0) = 0 is continuous at x = 0.

Solution.(i) Here $f(0) = 0$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}-1}{e^{1/x}+1} = \frac{0-1}{0+1}$$

$$= -1\left(as\ x \to 0-, \frac{1}{x} \to -\infty \quad \therefore \quad e^{\frac{1}{x}} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x}-1}{e^{1/x}+1}$ (dividing the num, and denom by$e^{1/x}$)

$$= \lim_{x\to 0+} \frac{1-e^{-1/x}}{1+e^{-1/x}} = \frac{1-0}{1+0} = 1$$

$$\left(as\ x \to 0-, \frac{1}{x} \to \infty \quad \therefore \quad e^{\frac{1}{x}} \to \infty\ and\ \ e^{\frac{1}{x}} \to 0\right)$$

Thus $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal.

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Also none of the left and right limits is equal to f(0)

∴ f has a discontinuity of the first kind at x = 0

(ii) Here f(0)=0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0}$$

$$= 0\left(as\ x \to 0-, \frac{1}{x} \to -\infty \ \therefore\ e^{1/x} \to 0\right)$$

$$= \lim_{x\to 0+} \frac{1}{e^{-1/x}+1} = \frac{1}{0+1} = 1$$

$$\left(as\ x \to 0+, \frac{1}{x} \to \infty\ \therefore\ e^{1/x} \to \infty\ and\ e^{-1/x} \to 0\right)$$

Thus $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Since $\lim_{x\to 0-} f(x) = f(0) \neq \lim_{x\to 0+} f(x)$

Therefore, f is continuous from the left at x = 0 and has a discontinuity of the first kind from the right at x = 0.

(iii) Here f(0) = 1

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{1/x} - e^{-1/x}}{e^{2/x} + e^{-1/x}}$$

$$= \lim_{x\to 0-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = \frac{0-1}{0+1}$$

$$= -1 \left(as\ x \to 0-, \frac{2}{x} \to -\infty\ \therefore\ e^{2/x} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$

$$= \lim_{x\to 0+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = \frac{1-0}{1+0} = 1$$

$$\left(as\ x \to 0+, \frac{2}{x} \to \infty\ \therefore\ e^{2/x} \to \infty\ and\ e^{-2/x} \to 0\right)$$

Thus $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Since $\lim_{x\to 0-} f(x) \neq f(0) = \lim_{x\to 0+} f(x)$

Therefore, f is continuous from the right at x = 0 and has a discontinuity of the first kind from the left at x = 0.

(iv) Here f(a) = 0

$\lim\limits_{x \to a-} f(x) = \lim\limits_{x \to a-} (x-a) \frac{e^{\frac{1}{x-a}} - 1}{e^{\frac{1}{x-a}} + 1}$[Put x = a – h, h > 0 so that as x →a – h, h → 0+]

$$= \lim_{x \to a+} -h \frac{e^{-\frac{1}{h}} - 1}{e^{-\frac{1}{h}} + 1} = -(0) \times \frac{0-1}{0+1} = 0$$

$$\left[as\ h\ \to 0+, \frac{1}{h} \to \infty\ \ \therefore e^{1/h} \to \infty\ and\ e^{-1/h} = 0\right]$$

And $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} (x-a). \frac{e^{\frac{1}{x-a}} - 1}{e^{\frac{1}{x-a}} + 1}$

[Put x = a + h, h > 0 so that as x → a + h, h → 0+]

$$= \lim_{h \to 0+} h. \frac{e^{1/h} - 1}{e^{1/h} + 1} = \lim_{h \to 0+} h. \frac{1 - e^{-1/h}}{1 + e^{-1/h}} = 0 \times \frac{1-0}{1+0} = 0$$

Since $\lim\limits_{x \to a-} f(x) = 0 = \lim\limits_{x \to a+} f(x)$

$\therefore \lim\limits_{x \to a} f(x) = 0$Also f(a) = 0

Hence f is continuous at x = 0

(v) Here f(0) = 0

$\lim\limits_{x \to 0-} f(x) = \lim\limits_{x \to 0-} \frac{xe^{1/x}}{1+e^{1/x}}$ [Put x = 0 – h, h > 0 so that as x →0-, h → 0+]

$$= \lim_{x \to 0-} \frac{-he^{-1/h}}{1 + e^{-1/h}} = -\frac{0 \times 0}{1+0} = 0$$

$$\left(as\ h \to 0+, \frac{1}{h} \to \infty\ \therefore\ e^{1/h} \to \infty\ and\ e^{-1/h} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{xe^{1/x}}{1+e^{1/x}}$ (dividing the num. and denom by $e^{1/x}$)

$$= \lim_{x\to 0+} \frac{x}{e^{-1/x}+1} = \frac{0}{0+1} = 0$$

Since $\lim_{x\to 0-} f(x) = 0 = \lim_{x\to 0+} f(x)$

$\therefore \lim_{x\to 0} f(x) = 0$. Also f(0) = 0

Hence f is continuous at x = 0

(vi) Please try yourself.

Example 6. Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.

(i) $f(x) = \begin{cases} e^{1/x} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 0 (ii)

$f(x) = \begin{cases} e^{-1/x} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 0

(iii) $f(x) = \begin{cases} \dfrac{e^{\frac{1}{x^2}}}{1-e^{\frac{1}{x^2}}} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 0 (iv)

$f(x) = \begin{cases} \frac{x}{1+e^{1/x}} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 0

(v) $f(x) = \begin{cases} \frac{1}{1-e^{1/x}} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 0

(vi) $f(x) = \begin{cases} \dfrac{x-1}{1+e^{\frac{1}{x}-1}} & when\ x \neq 0 \\ 0 & when\ x = 0 \end{cases}$ at x = 1

Solution. (i) Here f(0) = 0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} e^{1/x} = 0 \left[as\ x \to 0-, \frac{1}{x} \to -\infty\ \therefore\ e^{1/x} \to 0\right]$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} e^{1/x} = \infty$i.e. $\lim_{x\to 0+} f(x)$ does not exist.

∴ f has a discontinuity of the second kind from the right at x = 0.

(ii) Please try yourself. [Ans. Discontinuity of the second kind from the left]

(iii) Here f(0)=0

$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{e^{\frac{1}{x^2}}}{1-e^{\frac{1}{x^2}}}$ [Put x = 0 – h, h > 0, so that as x → 0-, h→ 0+]

$= \lim_{h\to 0+} \frac{e^{\frac{1}{h^2}}}{1-e^{\frac{1}{h^2}}}$ (dividing the num. and denom by $e^{\frac{1}{h^2}}$)

$$= \lim_{h\to 0+} \frac{1}{e^{-\frac{1}{h^2}} - 1} = \frac{1}{0-1} = -1$$

$$\left(as\ h \to 0+, \frac{1}{h^2} \to \infty\ \therefore\ e^{\frac{1}{h^2}} \to \infty\ and\ e^{-\frac{1}{h^2}} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{e^{\frac{1}{x^2}}}{1-e^{\frac{1}{x^2}}} = \lim_{x\to 0+} \frac{1}{e^{-\frac{1}{x^2}}-1} = \frac{1}{0-1} = -1$

Since $\lim_{x\to 0-} f(x) = -1 = \lim_{x\to 0+} f(x)$

∴$\lim_{x\to 0} f(x) = -1$ but f(0) = 0 so that $\lim_{x\to 0} f(x) \neq f(0)$

Thus f has a removable discontinuity at x = 0

(iv) Here f(0) = 0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1}{1+e^{1/x}}$$

[Put x = 0 – h, h > 0, so that as x → 0-, h→ 0+]

$$= \lim_{h\to 0-} \frac{-h}{1+e^{-1/h}} = \frac{0}{1+0} = 0$$

$$\left(as\ h \to 0+, \frac{1}{h} \to \infty\ \therefore\ e^{\frac{1}{h}} \to \infty\ and\ e^{-\frac{1}{h}} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{1}{1+e^{1/x}}$ (dividing the num. and denom by $e^{1/x}$)

$$= \lim_{x\to 0+} \frac{xe^{-1/x}}{e^{-1/x}+1} = \frac{0\times 0}{0+1} = 0$$

Since $\lim_{x\to 0-} f(x) = 0 = \lim_{x\to 0+} f(x)$

$\therefore \lim_{x\to 0} f(x) = 0$. Also f(0) = 0

∴ f is continuous at x = 0

(v) Here f(0) = 0

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1}{1-e^{1/x}} = \frac{1}{1-0} = 1$$

$$\left(as\ x \to 0-, \frac{1}{x} \to -\infty\ \therefore\ e^{1/x} \to 0\right)$$

And $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{1}{1-e^{1/x}} = \lim_{x\to 0+} \frac{e^{-1/x}}{e^{-1/x}-1} = \frac{1}{0-1} = -1$

$$\left(as\ x \to 0+, \frac{1}{x} \to \infty\ \therefore\ e^{\frac{1}{x}} \to \infty\ and\ e^{-\frac{1}{x}} \to 0\right)$$

Thus $\lim_{x\to 0-} f(x)$ and $\lim_{x\to 0+} f(x)$ both exist but are not equal.

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

Since $\lim_{x \to 0-} f(x) \neq f(0) = \lim_{x \to 0+} f(x)$

Therefore, f is continuous from the right at x = 0 and has a discontinuity of the first kind from the left at x = 0.

(vi) Here f(1) = 0

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \frac{x-1}{1+e^{\frac{1}{x-1}}}$$

[Put x = 1 – h, h > 0, so that as x → 1-, h→ 0+]

$$= \lim_{h \to 0+} \frac{-h}{1+e^{-\frac{1}{h}}} = -\frac{0}{1+0} = 0$$

$$\left(as\ h \to 0+, \frac{1}{h} \to \infty\ \therefore\ e^{\frac{1}{h}} \to \infty\ and\ e^{-\frac{1}{h}} \to 0\right)$$

And $\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \frac{x-1}{1+e^{\frac{1}{x-1}}}$

[Put x = 1 + h, h > 0, so that as x → 1+, h→ 0+]

$$= \lim_{h \to 0+} \frac{h}{1+e^{\frac{1}{h}}} = \lim_{h \to 0+} \frac{he^{-1/h}}{e^{-1/h}+1} = \frac{0 \times 0}{0+1} = 0$$

Since $\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = f(0)$

∴ f is continuous at x = 0.

Example 7. Examine the continuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.

(i) $f(x) = \begin{cases} e^{\frac{1}{x-2}} & if\ x \neq 2 \\ 0 & if\ x = 0 \end{cases}$ at x = 2 (ii) $f(x) = \begin{cases} e^{-\frac{1}{(x-2)^2}} & if\ x \neq 2 \\ 0 & if\ x = 0 \end{cases}$ at x = 2

Solution. (i) Here f(2) = 0

$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} e^{\frac{1}{x-2}}$ [Put x = 2 - h, h > 0, so that as x → 2-, h→ 0+]

$$= \lim_{h\to 0+} e^{-1/h} = 0$$

And $\lim_{x\to 2+} f(x) = \lim_{x\to 2+} e^{\frac{1}{x-2}}$ [Put x = 2 + h, h > 0, so that as x → 2+, h→ 0+]

$= \lim_{h\to 0+} e^{1/h} = \infty$ i.e. $\lim_{x\to 2+} f(x)$ does not exist.

∴ f has a discontinuity of the second kind from the right at x = 0.

2.10 Open Set:

A subset G of a metric space (X, d) is said to be open set in X with respect to the metric d, if G is a neighbourhood of each of its points.i.e., if for each $a \in G$, there is an $r > 0$ such that $S_r(a) \subseteq G$.

Example: Prove that every set in a discrete space (X, d) is open.

Solution: Let G be any non-empty subset of the discrete space $(X, d)$and x be any point of G. Then the open sphere $S_r(x)$ with $r \leq 1$ is the singleton set $\{x\}$ which is contained in G i.e., each point of G is the Centre of some open sphere contained in G.It's a particular, each singleton set is open.

2.11 Closed Set:

A Subset F of a metric space (X, d) is said to be closed if F contains all its limit points.

Example: Every closed sphere is a closed set.

Solution: Let $S_r[x]$ be any closed sphere in a metric space (X, d).

If $X - S_r[x] = \emptyset$, Then $\emptyset$ is open.

Assume $X - S_r[x] \neq \emptyset$. Let $y \in X - S_r[x]$. Then $y \notin S_r[x]$.

This implies $d(y, x) > r$. Let $r_1 = d(y, x) - r$.

The open sphere $S_{r_1}(y) \subseteq X - S_r[x]$. For if $z \in S_{r_1}(y)$. Then $d(z, y) < r_1$.

So $d(z, y) < d(y, x) - r$

$i.e., r < d(y, x) - d(z, y) = d(x, z)$ by triangle inequality.

Thus $z \in S_{r_1}(y) \subseteq X - S_r[x]$

This implies $X - S_r[x]$ is open. Hence $S_r[x]$ is closed.

2.12 Closer of a Set:

Let A be any subset of a metric space (X, d). The Closer of A denote by $\bar{A}$ is the set of all adherent points of A. $i.e., \bar{A} = A \cup A'$

Symbolically $\bar{A} = \{x \in X: S_r(x) \cap A \neq \emptyset$, for all $r > 0\}$.

Properties: Let A and B be any two subsets of a metric space (X, d). Then

(1) $\bar{A}$ is a closed set.

(2) $if\ A \subseteq B$, then $\bar{A} \subseteq \bar{B}$.

(3)$\bar{A}$ is the smallest closed superset of A.

(4)$A = \bar{A}$ if and only if A is closed.

(5)$\bar{A}$ is the intersection of all closed sets Containing A.

(6)$\overline{A \cup B} = \bar{A} \cup \bar{B}$.

(7)$\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Proof: (1) We show that$\bar{A}$ is a closed. We shall show that its complement $(\bar{A})^c$ is open.

if$(\bar{A})^c = \emptyset$ then $\emptyset$ is open. Suppose that $(\bar{A})^c \neq \emptyset$.

Let $x \in (\bar{A})^c$, then $x \notin \bar{A} \Rightarrow$ *there exists at least one* $r > 0$ such that $S_r(x) \cap A = \emptyset$, We let us a $y \in S_r(x)$, then $d(y, x) < r$.

Let $r_1 = r - d(y, x)$.

Clearly $r_1 > 0$ and $S_{r_1}(y) \subseteq S_r(x)$

$\Rightarrow S_{r_1}(y) \cap A = \emptyset$, for at least one r_1.$[\because S_{r_1}(y) \cap A \subseteq S_{r_1}(x) \cap A]$

$\Rightarrow y \notin \bar{A}$

Since y is an arbitrary number of $S_{r_1}(x)$,therefore,

$$S_{r_1}(x) \subseteq (\bar{A})^c.$$

This implies $(\bar{A})^c$ is open. Hence $\bar{A}$ is a closed.

Proof: (2) Let $x \in \bar{A}$ then$S_{r_1}(x) \cap A \neq \emptyset,\ for\ all\ r > 0$

this implies $S_{r_1}(x) \cap B \neq \emptyset,\ (\because A \subseteq B)\ i.e., x \in \bar{B}$

Hence $\bar{A} \subseteq \bar{B}$.

Proof: (3)we know that $\bar{A}$ is a closed set, and $A \subseteq \bar{A}$. To show that $\bar{A}$ is the smallest closed set containing A, we suppose that if F is any other closed set containing A, then$A \subseteq F \Rightarrow \bar{A} \subseteq \bar{F} = F[\because$ F is closed].

Since F is arbitrary, so $\bar{A}$ is the smallest closed set containing A.

Proof: (4) If $A = \bar{A}$, Then by (1) $\bar{A}$ is closed, and so A is closed.

Conversely, let A be any closed set.

Since $A \subseteq \bar{A}$. So we need to show that $\bar{A} \subseteq A$.

Let x be any element of $\bar{A}$, then either $x \in A$ *or* $x \notin A$.

If $x \in A$, then the result is proved.

If $x \notin A$, *and* $x \in \bar{A}$, Then for every $r > 0$, the open sphere $S_r(x)$ contains a point of A other than x.

$\Rightarrow x$ is a limit point of A.

But A being closed, therefore x must belong to A. Hence $\bar{A} \subseteq A$.

Proof: (5) Let F be the intersection of all closed sets containing A.

Then F is closed.

$A \subseteq F \Rightarrow \bar{A} \subseteq \bar{F} = F$

i.e., $\bar{A} \subseteq F$, Thus every closed set which contains A, Contains $\bar{A}$.

But $\bar{A}$ is a closed set containing A. F, being the intersection of all closed sets containing A, is contained in $\bar{A}$.

Therefore $\bar{A} = F$.

Proof: (6) We know that $A \subseteq A \cup B$, and $B \subseteq A \cup B$

$\therefore \bar{A} \subseteq \overline{A \cup B}$, and $\bar{B} \subseteq \overline{A \cup B}$

and so $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$.

Nowto show that

$\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$

We proceed as follows:

Let, if possible $x \in \overline{A \cup B}$, *but* $x \notin \bar{A} \cup \bar{B}$.

The x is neither an adherent point of A nor that of B. Consequently, there exist open spheres $S_{r_1}(x), and\ S_{r_2}(x)$ containing no point of A and B respectively.

Let $r = \min\{r_1, r_2\}$, then $S_r(x)$ containing no point of A as well as no point of B, and therefore of $A \cup B$.

$\therefore$ x is not an adherent point of $A \cup B$.

i.e., $x \in \overline{A \cup B}$, thus, we arrive at a contradiction.

Hence $x \in \overline{A \cup B} \Rightarrow x \in \bar{A} \cup \bar{B}$.

Proof: (7) Since $A \cap B \subseteq A$, and $A \cap B \subseteq B$

$\therefore \overline{A \cap B} \subseteq \overline{A}$, and $\overline{A \cap B} \subseteq \overline{B}$,

The result can be extended to the intersection of an arbitrary family $\{A_\alpha\}$ of subsets of X,

$i.e., \overline{\cap_{\alpha \in \Lambda} A_\alpha} \subseteq \cap_{\alpha \in \Lambda} \overline{A_\alpha}$

Note: Let (X, d) be a metric space and $A \subseteq Y \subseteq X$. Then the closure of A in (Y, d_Y) is denoted by $\overline{A^Y}$. It is very simple to verify $\overline{A^Y} = \bar{A} \cap Y$.

Summary

We end this unit by summarising what we have covered in it.

- The limit of a function f at a point p of its domain is L is given $\in > 0, \exists\ \delta > 0$, such that $|f(x) - L| < \in$ Where ever $|x - p| < \delta$.

- $\lim_{x \to p} f(x)$ exists if and only if $\lim_{x \to p^-} f(x)$ and $\lim_{x \to p^+} f(x)$ both exist and are equal.
- A function f is Continuous at a point $x = p$ if $\lim_{x \to p} f(x) = f(p)$

- Let A and B be any two subsets of a metric space (X, d). Then

$\bar{A}$ is a closed set.

Terminal Questions

1. Examine the continuity of the following function at the indicated point. Also point out the type of discontinuity if any

$$f(x) = \begin{cases} \sin\frac{1}{x}, & for\ x \neq 0 \\ 0, & for\ x = 0 \end{cases} \text{ at x = 0}$$

2. Discuss the continuity of the following functions at x = 0. Specify the type of discontinuity, if any.

(i) $f(x) = \begin{cases} x\sin\frac{1}{x} - 1 & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ (ii)

$f(x) = \begin{cases} \sin x \cos\frac{1}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$

3. Examine the discontinuity of the following functions at the indicated point. Also point out the type of discontinuity, if any.

(i) $f(x) = \begin{cases} \frac{|x|}{x}, & when\ x \neq 0 \\ 1, & when\ x = 0 \end{cases}$ at x = 0

(ii) $f(x) = |x| + |x - 1|$ at x = 0 and x = 1

4. Discuss the continuity of the function $f(x) = |x|$ at the point $\frac{1}{2}$ and 1, where $|x|$ denotes the largest integer ≤ x.

5. Discuss the continuity of f at x = 1, where $f(x) = [1 - x] + [x - 1]$

6.Prove that $\lim_{x\to 0} f(x) = 0$,where $f(x) = x\sin\frac{1}{x}, x \neq 0$.

7.Prove that$\lim_{x\to 2}|3x - 1| = 5$

8.Let $\mathrm{f(x)} = \frac{\sin x}{x}$, find the limit of f(x)when $x \to 0$.

9.Prove that $sin\ x$ is continuous for every value of x.

10. Show by example that a set which fails to be closed need not be open.

Unit-3 Compactness

Structure

3.1 Introduction:

In this unit, we shall discuss the notion of compactness in a metric space. we shall define compact sets and discuss the examples of these sets in different metric spaces.We discuss certain theorems which characterise compact sets and give a complete description of compact sets in a metric space. Firstly we give a characterization in terms of convergence of sequences and then in terms of completeness. In this connection, we introduce the concept of "totally bounded sets" which is a stronger version of bounded sets. We show that a set is compact if and only if it is complete and totally bounded. We also discuss the analogue of the famous "Heine Bore1 theorem" in R which characterises compact sets in terms of closed and bounded sets. The deals with special properties of compact sets. Here we discuss relationship between continuity and compactness.

One of the main reasons for studying the compact sets is that they are in some ways very similar to finite sets. In other words, there are many results which are easy to show for finite sets, the formulations as well as the proofs of which carry over with minimal changes to compact sets. It is often said that "compactness is the next best thing to finiteness".

3.2 Objectives:

After studying this unit, we should be able to

- use the definition of compact sets to check whether a given set in a metric space is compact or not;
- explain the connection between compactness and sequential convergence;
- explain the relationship between compact sets and totally bounded sets; and that between compact sets and sets having finite intersection property;
- state and prove Heine-Burel theorem for R^n;
- explain the relationship between continuity and compactness.

3.3 Compactness of Metric Space:

Definition: Let M be a metric space. A collection of open sets G_α is said to be an **open cover** for M if $\cup\, G_\alpha = M$. A sub collection of G_α which itself is an open cover is called a **subcover.**

A metric space M is said to be **compact** if every open cover for M has a finite sub- cover.i.e., for each collection of open sets$G_\alpha$such that $\bigcup_{i=1}^{n} G\alpha = M$, there exists a finite sub-collection $G\alpha_1, G\alpha_2, \ldots, G\alpha_n$ such that $\bigcup_{i=1}^{n} G\,\alpha_i = M$.

Remark:

1. Any closed interval with the usual metric is compact.
2. The discrete space (X, d) when $X$is a finite set , is compact.
3. The space (R, d) when R is the set of real and d is the usual metric is not compact, for the cover $\{]-n, n[: n \in N\}$ is such that $\bigcup_{n=1}^{\infty}]-n, n[= R$, which do not have a finite subcover.

Example 1: Prove that the open interval $]0,1[$ with the usual metric is not compact.

Solution: we the family of open intervals $\left\{]\frac{1}{n}, 1[: n = 2,3 \ldots\right\}$ is such that

$$\bigcup_{n=2}^{\infty}]\frac{1}{n}, 1[=]0,1[.$$

therefore,$\left\{]\frac{1}{n}, 1[: n = 2,3 \ldots\right\}$ is an open cover of $]0,1[$, which has no finite subcover.

Example 2: Let X be an infinite set with the discrete metric. Show that (X, d) is not compact.

Solution: for each $x \in X, \{x\}$ is open in X. Also $\bigcup_{x \in X} \{x\} = X$

Therefore $\{\{x\}: x \in X\}$ is an open cover of X and since X is infinite, this open cover has no finite subcover.

Theorem 1: Let M be a metric space. Let $A \subseteq M$. Then A is compact if and only if for every collection G_α of open sets in M such that $\cup G_\alpha \supseteq A$ there exists a finite subcollection $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$. i.e., A is compact if and only if every open cover for A by sets open in M has a finite subcover.

Proof: Let A be a compact subset of M.

Let G_α be a collection of open sets in M such that $\cup G_\alpha \supseteq A$.

Then $(\cup G_\alpha) \cap A = A$. $\therefore \cup (G_\alpha \cap A) = A$.

Since G_α is open in M, $G_\alpha \cap A$ is open in A. $\therefore G_\alpha \cap A$ is an open cover for A.

Since A is compact, this open cover has a finite subcover say

$\{G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots, G_{\alpha_n} \cap A\}$.

$$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$$

$$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cap A = A$$

Conversely, assume that for every collection G_α of open sets in M

such that $\cup G_\alpha \supseteq A$ there exists a finite sub collection $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$.

such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

We have to prove A is compact.

Let H_α be an open cover for A. Then H_α is open in A ∀ ⍰.

∴ $H_\alpha = G_\alpha \cap A$ where G_α is open in M ∀ ⍰.

Now $\cup H_\alpha = A \Rightarrow \cup (G_\alpha \cap A) = A. \Rightarrow (\cup G_\alpha) \cap A = A. \Rightarrow \cup G_\alpha \supseteq A.$

Hence by our assumption, there exists a finite sub collection

$\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that

$\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A.$

$\bigcup_{i=1}^{n} G\alpha_i \cap A = A.$

$\bigcup_{i=1}^{n}$ Ⱨ$\alpha_i \cap$ ⍰) = A.

$\bigcup_{i=1}^{n}$ $G\alpha_i$ n I=1$H_{\alpha I}$= A.

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}\}$ is a finite subcover of the given open cover $\{H_\alpha\}$ of A.

∴ A is compact.

Theorem 2: Any compact subset A of a metric space (M , d) is closed

Proof: We shall prove that A^c is open. Let $y \in A^c$.

Now, for each $x \in A$, $x \neq y$.

∴ $d(x , y) = r > 0$ and $B\left(x , \frac{rx}{2}\right) \cap B\left(y , \frac{rx}{2}\right) = \emptyset$

Clearly the collection $\{ B(x , \frac{rx}{2}) \mid x \in A \}$ is an open cover for A by setsopen in M.

Since A is compact, there exists $x_1, x_2, \ldots, x_n \in A$ such that

$\bigcup_{i=1}^{n} B\left(x, \frac{rx}{2}\right) \supseteq A$ ....(1)

Let $V_y \bigcap_{i=1}^{n} B(y, \frac{r_{x_i}}{2})$

Then V_y is an open set containing y.

Since, $B(x,\frac{r_{x_i}}{2}) \cap B\left(y, \frac{r_{x_i}}{2}\right) = \emptyset$

$V_y \cap B(x, \frac{r_{x_i}}{2}) = \emptyset \ \forall \ i = 1, 2, \ldots, n$.

$\therefore V_y \cap \left[\bigcup_{i=1}^{n} B\ x, \frac{r_{x_i}}{2}\right] = \emptyset$.

$\therefore V_y \cap A = \emptyset$. [By (1)]

$\therefore V_y \subseteq A^c$.

Thus, for each $y \in A^c$ there exists an open set V_y containing y such that $V_y \subseteq A^c$

$\therefore A^c =_{y \in A} c\ V_y$.

$\therefore A^c$ is open . Hence A is closed.

Theorem 3: Any compact subset A of a metric space M is bounded.

Proof. Let $x \in A$. Now, { B(x , n) / n $\in$**N** } is an open cover for A by sets open in M. Since A iscompact, there exists natural numbers $n_1, n_2, \ldots, n_k$, such that

$\bigcup_{i=1}^{n} B(x, nk) \supseteq A$.Let $N = \max \{ n_1, n_2, \ldots, n_k\}$.

Then $\bigcup_{i=1}^{k} B(x, nk) = B(x, N)$

$\therefore B(x, N) \supseteq A$.

Since B (x, N) is bounded and subset of a bounded set is bounded, A isbounded.

Theorem 4: A closed subset A of a compact metric space M is compact.

Proof. Let$\{G_\alpha\}$be a collection of open sets in M such that $\cup G_\alpha \supseteq A$.

$\therefore A^c \cup \cup G_\alpha = M$.

Since A is closed, A^c is open. $\therefore G_\alpha \cup \{A^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say

$\{G_{\alpha_1}, G_{\alpha_2}, \dots \dots, G_{\alpha_{n,}} A^c\}$

$\therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \cup A^c = M \;\; \therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \supseteq A.$

Hence A is compact.

3.4 Bolzano Weierstrass property:

Definition:A non-empty subset $'A'$ of a metric space (X, d) is said to be totally bounded if for any $\varepsilon > 0$ there exists a finite $\varepsilon - net$ for A, i.e., if for every $\varepsilon > 0$, there is a finite number of open spheres of radius ε whose union is A.

i.e., $A = \bigcup_{x \in B} S_\varepsilon(x)$

where B is a finite $\varepsilon - net$ for A. Clearly total boundedness implies boundedness. Since a totally bounded set is the union of a finite number of bounded sets. But the converse is not always true.

In the case of Euclidean spaces, the converse also holds. In general, this is not so can be seen by the following examples.

Example: Infinite discrete space X is bounded but not totally bounded, for it has no finite $\frac{1}{2} - net$, Since, $S_{\frac{1}{2}}(x) = \{x\}, x \in X$ and X is infinite.

A metric space M has Bolzano – Weierstrass property if every infinite subset of M has a limit point.

Theorem 3.4.4 In a metric space M the following are equivalent.

(i) M is compact.

(ii) M has Bolzano – Weierstrass property

(iii) M is sequentially compact

(iv) M is totally bounded and complete.

Proof.

(i) ⇒(ii). Let M be compact metric space. Let A be an infinite subset of M. Suppose that A has no limit point. Let $x \in M$. Since x is not a limit point if A, there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \cap (A - \{x\}) = \emptyset$.

$B(x, r_x)$ contains at most one point of A (contains x if $x \in A$).

Now, $\{ B(x, r_x) / x \in M \}$ is an open cover for M.

Since M is compact, there exists points $x_1, x_2, \ldots, x_n \in M$

such that $M = B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \ldots \cup B(x_n, r_{x_n})$.

$\therefore A \subseteq B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \ldots\ldots \cup B(x_n, r_{x_n})$.

Since each $B(x_1, r_{x_i})$ has at most one point of A, A must be finite. This is a contradiction to A is infinite. Hence A has a limit point.

(ii) ⇒ (iii)

Suppose that M has Bolzano – Weierstrass property. We ve to prove M is sequentially compact.Let (x_n) be a sequence in M.

If the range of (x_n) is finite, then a term of the sequence is repeated infinitely and hence (x_n) has a constant subsequence which is convergent.

Otherwise (x_n) has infinite number of distinct terms. By hypothesis, this infinite set has a limit point say x.

$\therefore$ for any $r > 0$, the open ball $B(x, r)$ contains infinite number of terms of the sequence (x_n). Choose a positive integer n_1 such that $x_{n_1} \in B(x, 1)$. Now, choose $n_2 > n_1$ such that $x_{n_2} \in B(x, \frac{1}{2})$. In general, for each positive integer k we choose $n_k > nk\text{-}1$ such that $x_{n_k} \in B(x, \frac{1}{k})$. Then (x_{n_k}) is a subsequence of (x_n) and $d(x_{n_k}, x) < \frac{1}{k} \forall k$.

k$\therefore x_{n_k} \to x$.

Thus (x_{n_k}) is a convergent subsequence of (x_n). Hence M is sequentially compact.

(iii) ⇒ (iv)

Suppose that M is sequentially compact. Then every sequence in M has a convergent subsequence. We have every Cauchy sequence is convergent.

Thus, every sequence in M has a Cauchy subsequence. Hence M is totally bounded.

Now, we prove that M is complete. Let (x_n) be a Cauchy sequence in M.

By hypothesis, (x_n) contains a convergent subsequence (x_{n_k}). Let $x_{n_k} \to x$.

Then $x_n \to x$. Thus M is complete.

(iv) ⇒ (i)

Suppose that M is totally bounded and complete. We have to prove M is compact. Suppose it is not. Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover. take $r_n = \frac{1}{2^n}$. Since M is totally bounded, M can be covered by a finite number of open balls of radius r_1 .

Since M is not covered by a finite number of G_α's, at least one of these open balls say $B(x_1 , r_1)$ cannot be covered by finite number of G_α's .

Now, $B(x_1 , r_1)$ is totally bounded. Hence as before we can find $x_2 \in B(x_1 , r_1)$ such that $B(x_2 , r_2)$ cannot be covered by finite number of G_α's.

Proceeding like this we get a sequence (x_n) in M such that $B(x_n , r_n)$ cannot be covered by finite number of G_α's and $x_{n+1} \in B(x_n , r_n)$.

let m and n be positive integers with n < m.

Now, $d(x_n , x_m) \le d(x_n , x_{n+1}) + d(x_{n+1} ,x_{n+2})+\ldots\ldots + d(x_{m-1} ,x_m)<r_n + r_{n+1}+\ldots.+r_{m-1}$

$$< r_n + r_{n+1} + \cdots + r_{m-1}.$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$

$< \frac{1}{2^{n-1}} \left(\frac{1}{2^n} + \frac{1}{2^n} + \cdots\right) < \frac{1}{2^{n-1}}$

∴ (x_n) is a Cauchy sequence in M.

Since M is complete, there $x \in M$ such that $x_n \to x$. Now, $x \in G_\alpha$ for some a.

Since G_α is open, there exists ⍰ > 0 such that $B(x, \varepsilon) \subseteq G_\alpha$ .We have $x_n \to x$ and $r_n = \frac{1}{2^n} \to 0$

∴ there exists a positive integer N such that $d(x_n, x) < \frac{\varepsilon}{2}$ and r $< \frac{\varepsilon}{2}$ $\forall$ n ≥ N.

Fix n ≥ N.

We claim that $B(x_n, r_n) \subseteq B(x, \varepsilon)$.

$y \in B(x_n, r_n) \Rightarrow d(x_n, y) < r < \frac{\varepsilon}{2}$

$\Rightarrow d(x_n, x) + d(x_n, y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow d(x, y) < \varepsilon \Rightarrow y \in B(x, \varepsilon)$.

$\therefore B(x_n, r_n) \subseteq B(x, \varepsilon) \subseteq G_\alpha$.

Thus, $B(x_n, r_n)$ is covered by a single G_α , which is a contradiction.

Hence M iscompact.

Example: Consider the space l_2 consisting of sequences $\{x_n\}$ of complex numbers such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$, and the metric defined by $d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2\right)^{\frac{1}{2}}$,

Where $x = \{x_n\}, y = \{y_n\} \in l_2$.

Solution: Let A be a subset of l_2 consisting of sequences

$\{e_1 = (1,0,0,\ldots), e_2 = (0,1,0,\ldots.0), e_3 = (0,0,1,0,0,\ldots..0)\ldots.e_n = (0,0,0,0,0,\ldots..n)\}$

Since $d(e_i, e_j) = \sqrt{2}, \forall\, i \neq j$, therefore A is bounded, we shall show that A is not totally bounded. Observe that A has no finite $\frac{1}{\sqrt{2}} - net$, for if it has, then there exists a finite set B of X such that

$d(e_i, x) < \frac{1}{\sqrt{2}}$, and $d(e_j, x) < \frac{1}{\sqrt{2}}, i \neq j, and\ x, y\ in\ B.$

Clearly $x \neq y$, for $x = y$ implies by triangle inequality

$$\sqrt{2} = d(e_i, e_j) \leq d(e_i, x) + d(e_j, x) < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$$

So, for each e_i in A there is an x in B with the above property.

Thus, there corresponds an infinite set B, which is a contradiction to the fact that B is finite.

3.5 Heine Borel Theorem:

Theorem 5: Any closed interval [a , b] is a compact subset of **R.**

Proof: Let G_α be a collection of open sets in **R** such that $\cup\, G_\alpha \supseteq$ **R**.

Let S = { x∈ [a , b] / [a , x] can be covered by a finite number of G_α's. }

Clearly a ∈ S and hence S ≠ ∅.

Since S is bounded above by b ,l.u.b of S exists. Let c = l.u.b. of S.

Clearly c ∈ [a , b]. ∴ c ∈ G_{α_1} for some index α_1. Since G_{α_1} is open, there exists ε > 0 such that B(x , ε) ⊆ G_{α_1} .

i.e. $(c - \varepsilon , c + \varepsilon) \subseteq G_{\alpha_1}$

Choose $x_1 \in [a , b]$ such that $x_1 < c$ and $[x_1 , c] \subseteq G_{\alpha_1}$. Since $x_1 < c$, $[a , x_1]$ is

covered by a finite number of G_α's. These finite number of G_α's together with G_{α_1} covers [a , c].

∴ by the definition of S , $c \in S$. Now, we claim that c = b.

Suppose $c \neq b$.

Then choose $x_2 \in [a , b]$ such that $x_2 > c$ and $[c , x_2] \subseteq G_{\alpha_1}$.

Since [a , c] is covered by a finite number of G_α's , these finite number of G_α's together with G_{α_1} covers $[a , x_2]$.

∴ $x_2 \in S$, which is a contradiction to c is l.u.b of S [∵$x_2 > c$]. Hence c = b.

∴ [a , x] can be covered by a finite number of G_α's.

∴ [a , b] is a compact subset of **R** .

Theorem 6: A subset A or **R** is compact if and only if A is closed and bounded.

Proof: If A is compact, then A is closed and bounded.

Conversely, assume that A is closed and bounded subset of R .

Since A is bounded, A has a lower bound and an upper bound say a and b respectively. Then $A \subseteq [a , b]$.

Since A is closed in R, $A \cap [a , b]$ is closed in [a , b]

I.e. A is closed in [a , b]. Thus, A is a closed subset of the compact space [a , b].

Hence A is compact.

3.6Compactness andContinuity:

Theorem 3.6.1: Let M_1 be a compact metric space and M_2 be any metric space.

Let $f : M_1 \rightarrow M_2$ be a continuous function. Then $f(M_1)$ is compact.

i.e. Continuous image of a compact metric space is compact.

Proof: Without loss of generality we assume that $f(M_1) = M_2$.

Let$G_\alpha$be a collection of open sets in M_2 such that $\cup G_\alpha = M_2$.

$\therefore \cup G_\alpha = f(M_2)$.

$\therefore f^{-1}(\cup G_\alpha) = M_1$

$\therefore \cup f^{-1}(G_\alpha) = M_1$.

Since f is continuous, $f^{-1}(G_\alpha)$ is open in $M_1 \forall$

$\therefore \{ f^{-1}(G_\alpha) \}$ is an open cover for M_1.

Since M1 is compact, this open cover has a finite subcover say

$f^{-1} G_\alpha , f^{-1} G_\alpha , \ldots\ldots , f^{-1} G_\alpha$.

$\therefore f^{-1}({}^{n}G_{\alpha i}) = M_1$.

$\left(\cup_{i=1}^{n} G_{\alpha_i}\right)= f(M_1) = M_2$.

Thus $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$ is a finite subcover for the given open cover G_α of M_2. Hence M_2 is compact.

Corollary: Let f be a continuous map from a compact metric space M_1 intoany metric space M_2. Then f(M_1) is closed and bounded.

Proof: Since f is continuous, f(M_1) is compact and hence closed and bounded.

Theorem 3.6.2 Any continuous mapping f defined on a compact metric space (M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof: Let ⍰> 0 be given Let $x \in M_1$.

Since f is continuous at x, for $\varepsilon/2 > 0$, there exists $\delta_x > 0$ such that

$d_1(x, y) < \delta_x \Rightarrow d_2(f(x), f(y)) < \varepsilon/2$ .... (1)

Clearly, $\{ B(x, \frac{\delta_x}{2}) / x \in M_1 \}$ is an open cover for M_1.

Since M_1 is compact, there exists $x_1, x_2, \ldots, x_n \in M_1$ such that

$$\bigcup_{i=1}^{n} B(x_{i,} \frac{\delta_{x_i}}{2}) = M_1$$

Let $\delta = min \left\{ \frac{\delta_{x_1}}{2}, \frac{\delta_{x_1}}{2}, \ldots \frac{\delta_{x_n}}{2} \right\} = M_1$

Now, we shall prove that $d_1(p, q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon \ \forall \ p, q \in M_1$.

Let $p, q \in M_1$ such that $d_1(p, q) < \delta$

$$P \in M \Rightarrow P \in \bigcup_{i=1}^{n} B(x_i, \frac{\delta_{x_i}}{2})$$

$\Rightarrow P \in \bigcup_{i=1}^{n} B\left(x_i, \frac{\delta_{x_i}}{2}\right)$ for some i such that $1 \leq i \leq n$.

$$d_1(p, x_i) < \frac{\delta_{x_i}}{2} < \delta_{x_i}$$

∴ by (1), $d_2(f(p), f(x_i)) < \varepsilon/2$ ....(2)

Similarly, $d_2(f(q), f(x_i)) < \varepsilon/2$ .... (3)

Now, $d_2(f(p), f(q)) \leq d_2(f(p), f(x_i)) + d_2(f(x_i), f(q))$

$$< \varepsilon/2 + \varepsilon/2 \qquad \text{[By (2) and (3)]}$$

∴ $d_2(f(p), f(q)) < \varepsilon$.

Thus, $d_1(p,q) < \delta \Rightarrow d_2(f(p), f(q)) < \varepsilon \ \forall \ p, q \in M_1$. Hence f is uniformly continuous.

3.7Equivalent forms ofCompactness:

Definition: A collection Ƒ of subsets of a set M is said to have finite intersection property if the intersection of any finite number of elements of Ƒ is nonempty.

Theorem: A metric space M is compact if and only if every collection of losed sets in M with finite intersection property has nonempty intersection.

Proof: Suppose that M is compact.

Let F_α be a collection of closed subsets of M with finite intersection property.

We have to prove $F_\alpha \neq \emptyset$. Suppose $F_\alpha = \emptyset$. Then $(F_\alpha)^c = M$.

$\therefore F_\alpha^c = M$. [By De Morgan's laws]

Since each F_α is closed, each F_α^c is open. Thus,{ F_α^c } is an open cover for M.

Since M is compact, this open cover has a finite subcover say

$\left\{F_{\alpha_1}{}^c, F_{\alpha_2}{}^c, F_{\alpha_3}{}^c, \ldots \ldots F_{\alpha_n}{}^c\right\}$

$= \therefore \bigcup_{i=1}^{n} F_{\alpha_i}{}^c\, M \ \therefore \left(\bigcap_{i=1}^{n} F_{\alpha_i}\right)^C = M \ \therefore \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$

This is a contradiction to the collection F_α has finite intersection property.

$\therefore F_\alpha \neq \emptyset$.

Conversely, assume that every collection of closed sets in M with finite intersection property has nonempty intersection.

We have to prove M is compact. Let G_α be an open cover for M.
$\therefore G_\alpha = M$.

$\therefore (G_\alpha)^c = \emptyset$. $\therefore G_\alpha^c = \emptyset$.

Since each G_α is open , each G_α^c is closed.

Hence $\mathbf{F} = \{ G_\alpha^{\mathrm{C}} \}$ is a collection of closed sets whose intersection is empty.

$\therefore$ by hypothesis, this collection does not have finite intersection property.

Hence there exists a finite sub collection $G_{\alpha_1}{}^c, G_{\alpha_2}{}^c, G_{\alpha_3}{}^c \ldots, \ldots. G_n{}^c$ such that

$\bigcap_{i=1}^{n} G_{\alpha_i}{}^{c} = \emptyset \therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)^{c} = \emptyset \therefore \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) = M.$

Thus the given open cover $\{G_\alpha\}$ of M has a finite subcover $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$.

Hence M is compact.

3.8 Total boundedness:

Definition: A metric space M is said to be totally bounded if for every $\varepsilon > 0$,

there exists a finite number of elements $x_1, x_2, \ldots, x_n \in M$ such that $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \ldots B(x_n, \varepsilon) = M$.

A nonempty subset A of a metric space M is said to be totally bounded ifthe subspace A is totally bounded metricspace.

Theorem 3.8.1 Any compact metric space is totally bounded.

Proof: Let M be a compact metric space.

We have to prove M is totally bounded. Let $\varepsilon > 0$ be given.

Now, $\{ B(x, \varepsilon) / x \in M \}$ is an open cover for M.

Since M is compact, there exists points $x_1, x_2, \ldots, x_n \in M$ such that $M = B(x_1, \varepsilon)$

$\cup B(x_2, \varepsilon) \cup \ldots \cup B(x_n, \varepsilon)$. Hence M is totally bounded.

Theorem 3.8.2 Any totally bounded subset A of a metric space M is bounded.

Proof: Let A be a totally bounded subset of a metric space M.

Then for given $\varepsilon>0$, there exists points x_1 , x_2 , ….. , $x_n \in A$ such that
$A = B_1(x_1 ,\varepsilon) \cup B_1(x_2 , \varepsilon) \cup ……. \cup B_1(x_n ,\varepsilon)$ where $B_1(x_i , \varepsilon)$ are open balls in A.

Since open balls are bounded sets and finite union of bounded sets is bounded, A is bounded.

Note: The converse of the above theorem is not true. For,
Let M be an infinite set with discrete metric. Then M is bounded.
Also, $B(x , 1) = \{ x \}$ for all $x \in M$. Since M is infinite, M cannot be expressed as finite union of open balls of radius 1. Hence M is not totally bounded.

Definition: Let (x_n) be a sequence in a metric space M. If $n_1< n_2< …. <n_k<$ ……. is a sequence of positive integers, then (x_{n_k}) is a subsequence of (x_n).

Theorem 3.8.3A metric space M is totally bounded if and only if every sequence in M contains a Cauchy subsequence.

Proof: Suppose that every sequence in M contains a Cauchy subsequence.

We have to prove M is totally bounded. Let $\varepsilon> 0$ be given. Choose $x_1 \in M$.

If $B(x_1 , \varepsilon) = M$, then M is totally bounded.

If $B(x_1 , \varepsilon) \neq M$, Then choose $x_2 \in B(x_1 , \varepsilon) – M$ so that $d(x_1 , x_2) \geq \varepsilon$.

If $B(x_1 , \varepsilon) \cup B(x_2 , \varepsilon) = M$, then M is totally bounded.
Otherwise, choose $x_3 \in [B(x_1 , \varepsilon) \cup B(x_2 , \varepsilon)] – M$.
so that $d(x_3 , x_1) \geq \varepsilon$ and $d(x_3 , x_2) \geq \varepsilon$. We proceed this process and if the process is terminated at a finite stage means M is totally bounded.

Suppose not, then we get a sequence (x_n) in M such that $d(x_n , x_m) \geq$ ⍰ if $n \neq m$.

$\therefore (x_n)$ cannot be a Cauchy sequence, which is a contradiction. Conversely, suppose that M is totally bounded.

Let $S_1 = \{ x_{11}, x_{12} , \ldots, x_{1n} , \ldots \}$ be a sequence in M. If one of the terms in the sequence is repeated infinitely, then S_1 contains a constant subsequence which is in fact a Cauchy sequence. So, we assume that no terms of S_1 is repeated infinitely so that the range of S_1 is infinite.

Since M is totally bounded, M can be covered by a finite number of open balls of radius $\frac{1}{2}$. Hence one of these balls contains infinite number of terms of the sequence S_1.

$\therefore S_1$ contains a subsequence $S_2 = \{x_{21} , x_{22} , \ldots, x_{2n} , \ldots \}$ which lies within an open ball of radius $\frac{1}{2}$. Similarly, S_2 contains a subsequence $S_3 = \{x_{31} , x_{32} , \ldots, x_{3n} , \ldots \}$ which lies within an open ball of radius $\frac{1}{3}$.

We repeat the process of forming successive subsequences and finally we take the diagonal sequence $S = \{ x_{11} , x_{22} , \ldots, x_{nn} , \ldots \}$.

We claim that S is a Cauchy subsequence of S_1. If $m > n$ thenboth$x_{mn}\ and x_{nn}$

$\therefore$ lie within an open ball of radius$\frac{1}{n}$.

$\therefore d(x_{mm}, x_{nn}) < \frac{2}{\varepsilon}$. $\therefore d(x_{mm}, x_{nn}) < \varepsilon\ \forall\ m, n \geq \frac{2}{\varepsilon}$

Hence S is a Cauchy subsequence of S_1.

Thus, every sequence in M has a convergent subsequence.

Corollary: A nonempty subset of a totally bounded set is totally bounded.

Proof: Let A be a totally bounded subset of a metric space M.

Let B be a nonempty subset of A. Let (x_n) be a sequence in B.

Since $B \subseteq A$, (x_n) is a sequence in A. Since A is totally bounded, (x_n) has a Cauchy subsequence. Thus, every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded.

3.9 Sequentially Compact:

Definition: A metric space M is said to be sequentially compact if every sequence in M has a convergent subsequence.

Theorem: Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{n_k}) converges to x , then (x_n) converges to x.

Proof: Suppose that (x_n) has a subsequence (x_{n_k}) which converges to x. We have to prove $x_n \rightarrow x$. Let $\varepsilon > 0$ be given.

Since (x_n) is a Cauchy sequence, there exists a positive integer N

such that$(x, x) < \frac{\epsilon}{2} \forall$ n , m $\geq N_1$..................... (1)

Since $x_{n_k} \rightarrow x$, there exists a positive integer N_2 such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$

$\forall$ $n_k \geq N_2$...........................(2)

Let N = max { N_1 , N_2 }.

Fix $n_k \geq N$. Now. $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x)$

$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \forall\ n \geq N$

$\therefore d(x_n, x) < \varepsilon\ \forall\ n \geq N$. $\therefore x_n \to x$.

Note: in the metric space, totally boundedness is the property that complements completeness to guarantee sequential compactness.

Example: the subspace $X =]0,1[$ of the real line is totally bounded but certainly not sequentially compact, for consider the sequence $\left\{\frac{1}{n}\right\}$, which has no convergent subsequence.

Note than X is not complete, since it is not closed.

Example:A subset$A$ of a metric space (X, d) is totally bounded if and only if $\bar{A}$ is totally bounded.

Solution: Let A be totally bounded. To show that $\bar{A}$ is totally bounded, it is enough to show that every sequence in $\bar{A}$ contains a Cauchy subsequence. Let $\{x_n\}$ be any sequence in $\bar{A}$. Let $\varepsilon > 0$ be given. Then $x_n \in \bar{A}$ implies

$S_{\frac{\varepsilon}{3}}(x_n) \cap A \neq \emptyset$ i.e., $\exists a_n \in A$ such that $d(a_n, x_n) < \frac{\varepsilon}{3}$ (1)

So we obtain a sequence $\{a_n\}$ in A, and A being totally bounded implies $\{a_n\}$ contains a Cauchy subsequence say $\{a_{n_k}\}$. Therefore for $\varepsilon > 0, \exists$ appositive integer m such that

$$d\left(a_{n_j}, a_{n_k}\right) < \frac{\varepsilon}{3}, \forall n_j, n_k \geq m \qquad (2)$$

By using triangle inequality and from (1) and (2), we have

$$d\left(a_{n_j}, a_{n_k}\right) \le d\left(a_{n_j}, a_{n_j}\right) + d\left(a_{n_j}, a_{n_k}\right) + d(a_{n_k}, a_{n_k})$$

$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \forall n_j, n_k \ge m$. Hence $\{a_{n_k}\}$ is a Cauchy sequence of $\{x_n\}$.

Therefore, $\bar{A}$ is totally bounded.

The converse is obvious since A, being a subset of a totally bounded set $\bar{A}$, is itself totally bounded.

.10 Summary:

In this unit, we have covered the following points.

1. We have introduced the notions of open cover, subcover and finite cover in a metric space.
2. We have defined compact sets in a metric and discussed the following properties:
   (i) Every compact set in a metric space is closed and bounded.
   (ii) Closed subsets of compact sets are compact.
   (iii) If A and B are compact sets in a metric space X, then A U B and A ∩ B are compact.
   (iv) An infinite subset of a compact metric space has a limit point.
3. We have shown that Heine-Bore1 theorem holds for R^n.
4. We have obtained the following three characterizations of compact sets.
   (i) X is compact if and only if X is sequentially compact.
   (ii) X is compact if and only if X is complete and totally bounded.
   (iii) X is compact iff every family 3 of closed subsets of X with finite

intersection property, has itself non-empty intersection, that is $\bigcap_{F\in f} F \neq \emptyset$

5. We have explained the relationship between continuity and compactness.
6. We have shown that any continuous function from a compact metric space to any other metric space is uniformly continuous.

3.11 Terminal Questions:

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1. If A and B are two compact subsets of a metric space (X, d). Prove that $A \cup B$ and $A \cap B$ are compact.
2. Show that a subspace of R^n is bounded if and only if it is totally bounded.
3. If A is a subspace of a complete metric space, show that $\bar{A}$ is compact if and only if A is totally bounded.
4. Show that a closed subspace of a complete metric space is compact if and only if it is totally bounded.
5. Prove that boundedness and total boundedness are equivalent in Euclidean spaces.
6. Prove that from any infinite open cover of a separable metric space one can extract a countable open cover.
7. Let $X = \{x: 0 < d(0, x) \leq 1,\}$ and $x \in R^2$, where $0 = (0,0)$, and d is usual metric.Show that X is closed and bounded, but not compact. Also show that X is not totally bounded.

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Advance Analysis

Block

2 Convergence of function of series and Improper Integral

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Block -2

Convergence of function of series and Improper Integral

In the first unit we shall study about point wise convergence and uniform convergence of sequence and series of functions. Necessary and sufficient condition for a uniform convergence, Weierstrass test, Abel's test and Dirichlet's test for uniform convergence.Term by term integration and term by term differentiation. In this unit we shall study convergence of series of functions.

In the second unit we also study about Riemann integrals as developed it requires that the range of integration is finite and the integrand remains bounded in that domain. If either or both of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral. If the integrand of f becomes infinite in the interval $a \le x \le b, i.e., f$ has points of infinite discontinuity in $[a, b]$ or the limits of integration $a\ or\ b$ become infinite, the symbol $\int_a^b f\, dx$ is called an improper integral. Thus $\int_1^\infty \frac{dx}{x^2}, \int_{-\infty}^{\infty} \frac{dx}{1+x^2}, \int_0^1 \frac{dx}{x(1-x)}, \int_{-1}^{\infty} \frac{dx}{x^2}$ are examples of improper integrals.

The integrals which are not improper are called proper integrals. Thus $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral. It will be assumed throughout that the number of singular points in any interval is finite and therefore when the range of integration is infinite, that all the singular points can be included in a finite interval. Further, it is assumed once for all that in a finite interval which encloses no point of infinite discontinuity the integrand is bounded and Integrable.

Unit-4 Complete Metric Spaces

Structure

4.1 Introduction

The aim of this unit is to study one of the properties of metric space. The notion of distance between points of an abstract set leads naturally to the discussion of uniform continuity and Cauchy sequences in the set. Unlike the situation of real numbers, where each Cauchy sequence is convergent, there are metric spaces in which Cauchy sequences fail to converge. A metric space in which every

Cauchy sequence converges is called a 'complete metric space'. This property plays a vital role in analysis when one wishes to make an existence statement. We shall see that a metric space need not be complete and hence we shall find conditions under which such a property can be ensured.

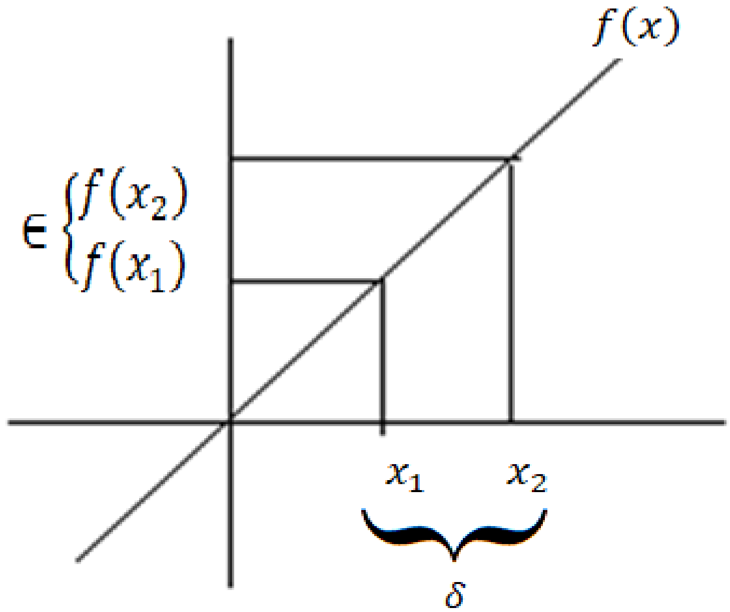
4.2 Objectives

After studying this unit, you should be able to

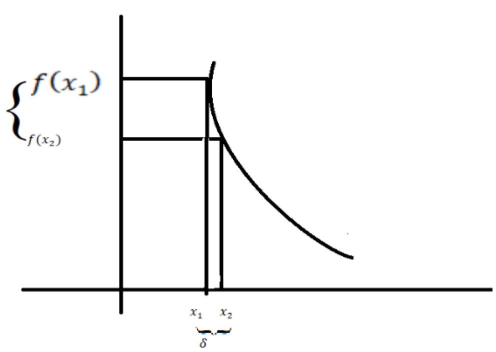
- Obtain the Uniformly Continuity;
- Obtain a Cauchy sequences in the set.
- Obtain the Necessary Condition on Uniformly Continuous
- Learn the results of Lebesgue Number for cover.
- Study the use of Complete and Complete Metric Space.

4.3 Uniform Continuity

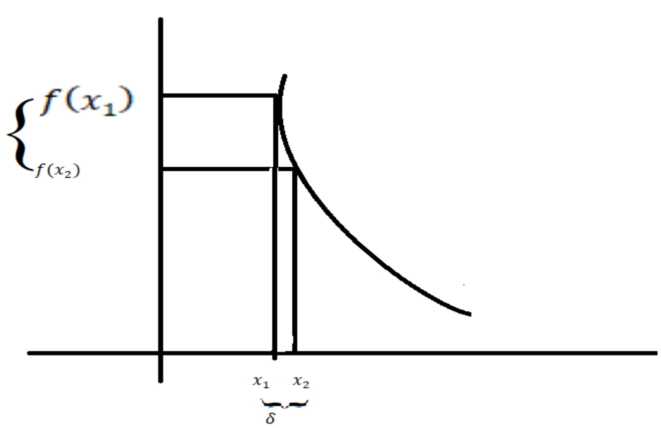
A Function $f(x)$ defined on an interval l if and only if for every $\epsilon > 0, \exists\, \delta > 0 \{i.e. \mathit{depend\ only\ on}\ \epsilon\}$independent of the choice of any point in l such that $|f(x_1) - f(x_2)| < \in$, where $|x_1 - x_2| < \delta$ where $x_1 - x_2$ are any arbitrary point in l.



Example:1 $f(x) = \frac{1}{x}$, $x \in (0,1)$



Solution:



$\Rightarrow \delta$ is not dependent only on $\in$ but it is dependent on point.

$\Rightarrow \frac{1}{x}$ is not Uniformly Continuous in $(0,1)$

Example: 2 Show that the function $f(x) = \sqrt{x}, x \in [0,1]$ is uniformly Continuous on $[0,1]$.

Solution: Let $x_1, x_2 \in [0,1]$ then

$$|f(x_1) - f(x_2)| = \left|\sqrt{x_1} - \sqrt{x_2}\right| \times \frac{\sqrt{x_1} + \sqrt{x_2}}{\sqrt{x_1} + \sqrt{x_2}}$$

$$\Rightarrow \left|\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}}\right|$$

Let $x_1 > x_2 \& \sqrt{x_1}, \sqrt{x_2} \geq 0$

$$\Rightarrow \left|\frac{x_1 - x_2}{\sqrt{x_1} + \sqrt{x_2}}\right| < \frac{|x_1 - x_2|}{\sqrt{x_1}} < \frac{|x_1 - x_2|}{\sqrt{x_1 - x_2}}$$

$$\Rightarrow < \sqrt{x_1 - x_2} < \in \Rightarrow |\sqrt{x_1 - x_2}| < \in^2 = \delta$$

$\Rightarrow$ for every $\epsilon > 0 \Rightarrow |f(x_1) - f(x_2)| < \in$, whenever $|x_1 - x_2| < \in^2$

$\Rightarrow \delta = \epsilon^2 \Rightarrow \sqrt{x}$ is uniformly Continuous on $[0,1]$.

Example:3 Show that $f(x) = x^2, \forall x \in R$ is uniformly Continuous on $[0,1]$.

Solution:we have

$$|f(x) - f(y)| = |x^2 - y^2|$$

$$\Rightarrow |(x + y)(x - y)| < 2|x - y| < \in$$

$\Rightarrow |x-y| < \frac{\in}{2} = \delta \Rightarrow$for every $\in > 0$

$|f(x)-f(y)| <\in$ whenever $|x-y| < \delta(\in)$

$\Rightarrow x^2$is uniformly Continuous on [0,1].

4.4 Necessary Condition on Uniformly Continuous

1.If $f(x)$ is Continuous on (a,b) and limit exist finitely at both ends then$f(x)$is uniformly Continuous on (a,b).

2.If$f(x)$ is Continuous in $[a,b]$then f(x) is uniformly Continuous on $[a,b]$.

3.If$f(x)$ is Continuous in $[a,\infty)$ and $\lim_{x\to\infty} f(x)$ $exist$ then f(x) is uniformly Continuous on $[a,\infty)$.

Example: In which interval $f(x) = \frac{1}{x}$is uniformly Continuous.

Solution: we have $f(x) = \frac{1}{x}$

Then $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x} = \infty$

Example:which of the following are uniformly Continuous on (0,1).

(a)$f(x) = e^x$ (b) $f(x) = x^2$

Solution:we have

(a) $f(x) = e^x$ (b) $f(x) = x^2$

Limit exist of e^x and x^2 $at\ both\ end$ are uniformly Continuous on (0,1).

4.5 Lebesgue Number for cover

Let $\{G_\alpha : \alpha \epsilon A\}$ be an open cover of a metric space (X, d) a real number $\curlywedge > 0$ is said to be Lebesgue number for the open cover $\{G_\alpha\}$ if for each subset A of X with $d(A) < \curlywedge$, there is at least one set G_α with Contains A.

Note: Not Every open Cover of a metric space has a Lebesgue number. For example, let $X =]0,1[$ be a subspace of the real line and $\left\{]\frac{1}{n}, 1[: n = 2,3,4 \ldots \right\}$ be an open cover of $]0,1[$. For arbitrary $\curlywedge > 0$ the set $A =]0, \curlywedge/2[$ is such that $d(A) < \curlywedge$. But A is not contained in any of the members of the cover, note that this space is not sequentially Compact.

Lebesgue Covering lemma:

Theorem: Every open cover of a sequentially compact metric space (X, d) has a Lebesgue number.

Proof: Let $\{G_\alpha : \alpha \epsilon \Lambda\}$ be any open cover of X. Assume that it has no Lebesgue number. then for each natural number n there is a non-empty set $A_n \subseteq X$ With $d(A_n) < \frac{1}{n}$. Such that $A_n \nsubseteq G_\alpha$, for every $\alpha \in A$ for each $n \in N$, choose a point $a_n \in A_n$. Since X is sequentially compact, the sequence $\{a_n\}$ contains a convergent subsequence, say $\{a_{nk}\}$.

Let $\lim_{k \to \infty} a_{nk} = x$ Now since $x \in X = \bigcup_{\alpha \in \Lambda} G_\alpha$ implies $x \in G_\alpha$.

For some $\alpha \in \Lambda$. G_α being open, therefore there is an $\epsilon > 0$,

such that $S_\epsilon(x) \subseteq G_\alpha$

For the above $\epsilon > 0, a_{nk} \to x$, and $d(A_{nk}) \to 0, as\ K \to \infty$ Implies there exists a positive integer k_0, such that $d\left(a_{nk_0}, x\right) < \frac{\epsilon}{2}$, and $d\left(A_{nk_0}\right) < \frac{\epsilon}{2}$……(a)

Let$y$ be any element of A_{nk_0}, then by using tringle inequality, and (a) we get

$d(y, x) \leq d\left(y, a_{nk_0}\right) + (a_{nk_0}, x)$

$\leq d\left(A_{nk_0}\right) + (a_{nk_0}, x)$

$\leq \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$ $[\because a_{nk_0} \in A_{nk_0}]$

This implies that $y \in S_\epsilon(x) \subseteq G_\alpha$.Hence $A_{nk_0} \subseteq G_\alpha$, which contradicts the fact that for each natural number $n, A_n \not\subset G_\alpha$must have a Lebesgue Number.

We are now in a position to prove the converse of the theorem, which will establish the equivalence of compactness and sequential compactness in metric spaces.

Theorem: Every sequentially compact metric space (X, d) is Compact.

Proof: Let$\{G_\alpha\}$ be any open cover of X. Since (X, d) is sequentially compact therefore by above lemma $\{G_\alpha\}$ has a Lebesgue number say λ>0. Also (X, d) being sequentially compact is totally bounded and so it has a finite $\frac{\lambda}{3}$. Net say $\{x_1, x_2, \ldots . x_n\}$. Then $X = \bigcup_{i=1}^{n} S_{\frac{\lambda}{3}}(x_i)$,

Then for each $i, 1 \leq i \leq n$. We have $d(S_{\frac{\lambda}{3}}(x_i) \leq \frac{2\lambda}{3} < \lambda$, and so by definition of Lebesgue number there exists at least one G_{α_i} such that $S_{\frac{\lambda}{3}}(x_i) \subseteq G_{\alpha_i}, i = 1,2 \ldots . n$. This implies $\bigcup_{i=1}^{n} S_{\frac{\lambda}{3}}(x_i) \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$, i.e., $X \subseteq \bigcup_{i=1}^{n} G_{\alpha_i}$

Hence $\{G_{\alpha_1}, G_{\alpha_2}, \dots \dots, G_{\alpha_n}\}$ is a finite subcover of $\{G_\alpha\}$ and so (X, d) is compact.

Corollary: A metric space is compact if and only if it is Sequentially compact.

Theorem: A closed subspace of a complete metric space is compact if and only if it is totally bounded.

Proof: Since a closed subspace of a complete metric space is itself complete, result follows from the above theorem.

We have seen that compactness is another name of Heine -Borel property.

Our results so far establish the following equivalence in a metric space.

1.Bolzano-weirstrass Property

2.Compactness

3.Sequential Compactness

4.Completeness and totally boundedness.

As a consequence of the Lebesgue Covering Lemma and the above corollary, we have the following useful result.

Theorem: Let f be a Continuous Function from a compact metric space (X, d_1) into a metric space (Y, d_2) then f is Uniformly Continuous.

Proof: Let $\in > 0$ be given for each $x \ in \ X$, $f^{-1}(S_{\frac{\in}{2}}(f(x))$ is an open subset of X Containing x, being an inverse image of an open sphere $S_{\frac{\in}{2}}(f(x)$ in Y under the Continuous Function $f: X \to Y$.

Therefore, the collection $f^{-1}(S_{\frac{\in}{2}}(f(x))$ is an open cover of X. Since X is compact, therefore by the Lebesgue covering Lemma and above corollary, this open cover has a Lebesgue number, say, $\delta > 0$. Let x, y be any two elements of X with $d_1(x, y) < \delta$, then the set $\{x, y\}$is a set in X with diameter less than δ and so by the definition of Lebesgue number $x, y \in f^{-1}\left(S_{\frac{\in}{2}}(x')\right)$ for some $x' \in X$. i.e.,$f(x). f(y) \in S_{\frac{\in}{2}}(f(x'))$.

$\Rightarrow d_2(f(x). f(x')) < \frac{\in}{2}$ and $d_2(f(y). f(x')) < \frac{\in}{2}$ By triangle inequality,

$d_2(f(x). f(y)) < d_2(f(x_1). f(x')) + d_2(f(x'). f(y)) < \frac{\in}{2} + \frac{\in}{2} = \in$

Hence f is Uniformly Continuous.

4.6 Complete

Let (X, d) be any metric space. the sequence $\{a_n\}$ of points of X is said to converge to a point 'a' of X, if for each $\in > 0$ there exists a positive integer m, such that $d(a_n, a) < \in, \forall\, n \geq m$.

i.e., $d(a_n, a) \to 0, as\, n \to \infty$ or equivalently, for each open sphere$S_{\in}(a)$ centered at 'a' there exist a positive integer m such that a_n is in $S_{\in}(a)$, for all $n \geq m$. The point 'a' is called the limit of the sequence $\{a_n\}$, and

we write $a_n \to a, as\, n \to \infty$ i.e., $\lim_{n\to\infty} a_n = a$.

Cauchy Sequence: A Sequence $\{a_n\}$ of points of (X, d) is said to be a Cauchy Sequence if for each $\in > 0$ there exists a positive integer n_0, such that

$d(x_n, x_m) < \in, \forall\, n, m \geq n_0$

i.e., $d(x_n, x_m) \to 0, as\ n, m \to \infty$.

Theorem: Every Convergent Sequence is a Cauchy Sequence.

Proof: Let (X, d) be any metric space. Let the sequence $\{a_n\}$ of points of X Converge to a. For every given $\in > 0$ there exists a positive integer n_0 such that $d(a_n, a) < \frac{\in}{2}, \forall\ n \geq n_0$

Then for $n, m \geq n_0$ we have

$$d(a_n, a_m) \leq d\left(a_{n,} a\right) + d(a, a_m) < \frac{\in}{2} + \frac{\in}{2} = \in$$

This implies $\{a_n\}$is a Cauchy Sequence.

Note: The following examples show that the converse of the statement need not be true.

Example: Let Q be the set of rational numbers in which the metric d is defined by $d(x, y) = |x - y|, \forall x, y \in Q$.

(Q, d) is a metric space. The Sequence $\left\{\frac{1}{3^n}\right\}$ is a Cauchy Sequence which converges to the limit 0. But the Sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ is also a Cauchy Sequence in it. Which does not converge to a point of Q.

4.7 Complete Metric Space:

A metric space (X, d) is said to be complete if every Cauchy Sequence Converges to a point of X. The spaces in the examples mentioned above are not complete.

Note: Any metric space which is not already complete can be made so by adjoining additional points to it.

Example:1 The discrete space (X, d) is a complete metric space. For in this space a Cauchy sequence must be a constant sequence (i.e., It must consist of a single point repeated from some place on) and so converges.

Example:2 The space(R, d) is a complete metric space. The convergence in R is the ordinary convergence of numerical sequences.

Example:3 The space R^n of all ordered n-tuples with the metric d,

$d(x, y) = (\sum_{i=1}^{n}(x_i - y_i)^2)^{\frac{1}{2}}$ is a complete metric space. The convergence in this space is Coordinate wise. This space (R^n, d) is called n-dimensional Euclidean space.

Example:4 The space C[0,1] of all bounded continuous real valued functions defined on the closed interval[0,1] with the metric d given by

$d(f, g) = \max_{0\leq x\leq 1}|f(x) - g(x)|$ is a complete metric space.

Solution: Let $\{f_n\}$ be a Cauchy sequence in C[0,1].

Let $\in > 0$ be given. Then there exists a positive integer $n_{0,}$such that $d(f_n, f_m) < \in$ $, \forall\ n, m \geq n_0$.

i.e., $\max_{0\leq x\leq 1}|f_n(x) - f_m(x)| < \in,\ \forall\ n, m \geq n_0$

$$|f_n(x) - f_m(x)| < \in, \forall\ n, m \geq n_0\ and\ \forall\ x \epsilon [0,1].$$

By Cauchy Criterion of uniform Convergence, the Sequence of Function $\{f_n\}$ Converges Uniformly on $[0,1]$. let f be the limit of a uniformly Convergent

Sequence of Continuous Functions so this itself is Continuous on $[0,1]$. Hence the Cauchy Sequence $\{f_n\}$ Converges to a point of C$[0,1]$.

Example:5 Let l_∞ be the set of all bounded numerical Sequences $\{x_n\}$ in which the metric d is defined by $d(x,y) = Sup|x_n - y_n|, \forall\, x = \{x_n\}, y = \{y_n\} \in l_\infty$.

Solution: Let x_n be a Cauchy Sequence of elements of l_∞ and let $x_n = \left\{a_i^{(n)}\right\}$. Since $x_n \in l_\infty$, so $\exists\, M > 0$.

$\left|a_i^{(n)}\right| \leq M, for\ i = 1,2,3 \ldots.$

Therefore for $\in > 0$, there exists an integer n_0 . such that $d(x_n, x_m) < \in, \forall\, n, m \geq n_0$.

i.e., sup$\left|a_i^{(n)} - a_i^{(m)}\right| < \in, \forall\, n, m \geq n_0, and\ for\ all\ i = 1,2,3 \ldots$

Let i be fixed. Then (1) implies that the sequence $\left\{a_i^{(1)}, a_i^{(2)} \ldots \ldots . a_i^{(n)} \ldots .\right\}$ is Cauchy and so convergence. Taking limit in(1) as $m \to \infty, we\ have$

$$\left|a_i^{(n)} - a_i^{(m)}\right| < \in, \forall\, n \geq n_0$$

And this is true for all $i = 1,2,3 \ldots$

Hence $|a_i| \leq \left|a_i^{(n)} - a_i\right| + \left|a_i^{(n)}\right| < \in + M, \forall\, i.$

This implies $\{a_i\}$ is bounded. Let $x = \{a_i\}$. Then $x \in l_\infty$. Hence (l_∞, d) is a complete Space.

Example:6 Let l_p be the set of all real numerical sequences for which $\sum_{i=1}^{\infty} |x_i|^p < \infty$.

Solution:we defined the metric d in l_p by

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p\right)^{1/p}, \forall\, x = \{x_i\}, y = \{y_i\} \in l_p$$

The space (l_p, d) is a complete metric space, and is known as Hilbert Sequence Space. Consider a Cauchy sequence $\{x_n\} = \left\{\left\{x_i^{(n)}\right\}\right\}$ in l_p.

Therefore, for a given $\varepsilon > 0$ there exists an integer n_0,

Such that $d(x_n, y_m) = \left(\sum_{i=1}^{\infty} \left|x_i^{(n)} - x_i^{(m)}\right|^p\right)^{1/p} < \in, \forall\, n, m \geq n_0$ (1)

Hence $\left|x_i^{(n)} - x_i^{(m)}\right| < \in, \forall\, n, m \geq n_0, and\ for\ all\ i \in N$ (2)

Fixing$i$, we see that the sequence $\left\{x_i^{(1)}, x_i^{(2)} \dots \dots . x_i^{(n)} \dots .\right\}$

Converges to a limit x_i, i.e., $\lim_{n\to\infty} x_i^{(n)} = x_i$.

Let $x = \{x_i\}$, then the inequality (1) implies

$\sum_{i=1}^{\infty} \left|x_i^{(n)} - x_i^{(m)}\right|^p < \in^p$, for every k, and for $n, m \geq n_0$

Taking limit as $m \to \infty$, we have

$$\sum_{i=1}^{\infty} \left|x_i^{(n)} - x_i^{(m)}\right|^p < \in^p, for\ n \geq n_0$$

Letting $k \to \infty$, we get

$$\sum_{i=1}^{\infty}\left|x_i^{(n)}-x_i\right|^p <\in^p, for\ n \geq n_0$$

This implies $x_n - x \in l_p$, and so $x = x_n - (x_n - x) \in l_p$.

Also $d(x_n, x) \to 0\ as\ n \to \infty$. Hence l_p is Complete.

Example:7let X be the set of all Continuous real -valued functions defined on [0,1], and let $d(x,y) = \int_0^1 |x(t) - y(t)|dt,\ \ x, y \in X$

Show that (X, d) is not complete.

Solution: let $\{x_n\}$ be a sequence in X defined by

$$x_n(t) = \begin{cases} n, & if\ 0 \leq t \leq \dfrac{1}{n^2} \\ \dfrac{1}{\sqrt{t}}, & if\ \dfrac{1}{n^2} \leq t \leq 1. \end{cases}$$

For $n > m, we\ have\ d(x_n, x_m) = \int_0^1 |x_n(t) - x_m(t)|\ dt$

$$= \int_0^{\frac{1}{n^2}} |n - m|\ dt + \int_0^{\frac{1}{n^2}} \left|\frac{1}{\sqrt{t}} - m\right| dt + \int_0^{\frac{1}{n^2}} \left|\frac{1}{\sqrt{t}} - \frac{1}{\sqrt{t}}\right| dt$$

$$= \frac{(n-m)}{n^2} + \left|2t^{\frac{1}{2}} - mt\right|_{\frac{1}{n^2}}^{\frac{1}{m^2}}$$

$$= \frac{1}{n} - \frac{m}{n^2} + \left(\frac{2}{m} - \frac{1}{m}\right) - \left(\frac{2}{n} - \frac{1}{n^2}\right)$$

$$= \frac{1}{m} - \frac{1}{n} \to 0, as\ m, n\ \to \infty$$

Hence $\{x_n\}$ is a Cauchy Sequence in X.

Now we shall show that this Cauchy sequence does not convergent in X. For every $x \in X$.

$$d(x_n, x) = \int_0^1 |x_n(t) - x_m(t)|\, dt$$

$$\int_0^{\frac{1}{n^2}} |n - x(t)|\, dt + \int_{\frac{1}{n^2}}^1 \left|\frac{1}{\sqrt{t}} - x(t)\right| dt$$

Since integrals are, so is each integral on the right, and hence$d(x_n, x) \to 0$, as n$n \to \infty$ would imply that each integral approaches zero, and since x is in X, SO x is Continuous.

But $x(t) = \begin{cases} t^{-\frac{1}{2}} \\ 0\,, \quad if\ t = 0 \end{cases}, if\ 0 < t \leq 1.$

Which is discontinuous at $t = 0$. Hence $d(x_n, x)$ does not tend to zero for each $x \in X$. i.e., the Sequence $\{x_n\}$ does not converge to the point of the space. This implies that (X, d) is not Complete.

4.8 Lemma:

Let (X, d) be any metric space and A be any non-empty subset of X. Then $x \in \bar{A}$ if and only if there exists a sequence $\{x_n\}$ in A such that $x_n \to x, as\ n \to \infty$. Let $x \in \bar{A}$ Then every open sphere Centered at x intersects A. In particular $S_{\frac{1}{n}}(x) \cap A \neq \emptyset$, For all n. So, we get a sequence $\{x_n\}$ in A such that

$$d(x_n, x) < \frac{1}{x}, \forall\, n \Rightarrow \lim_{n\to\infty} x_n = x.$$

Again, let$\{x_n\}$ be sequence in A which converges to x. to show that $x \in \bar{A}$. We must show that every open sphere centered at x intersects A.

Let $S_r(x)$ be any open sphere. Then for $r > 0, \lim_{n\to\infty} x_n = x$ implies that there exists a positive integer n_0, such that $d(x_n, x) < r, \forall\, n \geq n_0$.

In particular $d(x_{n_0}, x) < r \Rightarrow x_{n_0} \in S_r(x)$

$\Rightarrow S_r(x) \cap A \neq \emptyset \Rightarrow x \in \bar{A}$

Theorem: Let(X, d) be a complete metric space and$Y$ be a subspace of X. Then Y is complete if and only if it is closed in (X, d).

Proof: Let Y be a complete subspace of x. In order to show that Y is closed we need to show that $Y = \bar{Y}$ by definition $Y \subset \bar{Y}$, so we shall show that $\bar{Y} \subset Y$.

Let x be an element of Y. If $x \in \bar{Y}$,the result is proved. If $x \notin Y$,then x is a limit point of Y. By definition of limit point, every neighborhood $S_{\frac{1}{n}}(x)$ contains at least one member of Y other limit than x. Thus for each n we get a sequence$\{y_n\}$ in Y such that

$d(y_n, x) < \frac{1}{n}$. Thus $y_n \to y, as\; n \to \infty$.

Now the Sequence $\{y_n\}$ being a Convergent sequence must be a Cauchy sequence. Since Y is complete, this Cauchy sequence $\{y_n\}$ must converge in Y, hence $x \in Y$. But x is an arbitrary point of $\bar{y}$, therefore $\bar{Y} \subset Y$.

Conversely, we assume that Y is a closed subspace of X, and establish that Y is complete. Let $\{y_n\}$ be a Cauchy Sequence in Y, and since X is given to be a

Complete Space, therefore $\{y_n\}$ must Converge to a point Y in X. But then $y_n \in Y$, for all $n, and\ y_n \to Y, as\ n \to \infty. \Rightarrow y \in \bar{Y} = Y(\because Y\ is\ closed\ )$.

The following is a generalization of Nested Interval theorem.

Next, we define certain sets in general metric spaces. We are discussing these sets here because of their connection with completeness. In fact, these sets arose as an extension of the property of R that the set of rationals Q is dense in R and it is not Complete.

Definition: A Subset A of a metric space (X, d) is said to be dense in X

if $\bar{A} = X$.As we stated above, we have Q is a dense in R.

We will see that in the space $C[0,1]$ with sup metric, the set P consisting of all the real polynomials restricted to $[0,1]$ is dense in $C[0,1]$.

Now, we discuss a theorem.

4.9 Cantor's intersection Theorem

Let $\{F_n\}$ be a sequence of non-empty closed subsets of a Complete metric space X Such that $F_n \supseteq F_{n+1}$ for each positive integer n and $d(F_n) \to 0$. Let $F = \cap_{n=1}^{\infty} F_n$. Then F is a Singleton, i.e., it Contains exactly one element of X.

Proof: Let $F = \cap_{n=1}^{\infty} F_n$, since let$F \subseteq F_n$ we have

Let $d(F) \leq d(F_n)$, for each positive integer n.As $d(F_n) \to 0$, we get $d(F) \leq O$. So F can not contain more than one element.

Thus, the theorem is proved if we show that $F \neq \emptyset$.

Since F_n is not empty, we can choose an element $x_n \in F_n$. We thus, get a sequence $\{x_n\}$ in X. Let $\varepsilon > 0$ be given. Since $d(F_n) \to 0$, there exists a positive integer m such that $d(F_n) < \varepsilon \; if \; n \geq m$.

Let $n \geq k \geq m$. Then $F_n \subseteq F_k \Rightarrow d(x_n, x_k) \leq d(F_k) < \in$.

This show that $\{x_n\}$ is a Cauchy sequence in X. As X is a complete metric space,$\{x_n\}$ converges to some $x \in X$. We shall now show that $x \in F_n$ for every n, which in turn will give $x \in F$. Let, if possible,$x \notin F_k$ for some fixed positive integer positive integer$k$. Then $x \in F_K^C$. Since F_k is a closed subset of X, F_K^C is an open subset of X, so there exists $r > 0$ such that $B(x, r) \subseteq F_K^C. As \; x_n \to x$, there exists a positive integer p such$d(x_n, x) < r \; if \; n \geq p, that \; is, x_n \in B(x, r) \; if \; n \geq p$.

Choose $m = \max(p, k)\,. then \; x_n \in B(x, r) \subseteq F_K^C \; and \;\; x_n \in F_n \subseteq F_k$.

This is a Contradiction.

This proves that $x \in F_{k.}$ Since this is true for all K, we get that $x \in F_n \; for \; all \; n. thus \; x \in F, Hence \; the \; result$.

Example:1 In this example, we show that the set F in the Cantor's intersection theorem may be empty if the hypothesis $d(F_n) \to 0$ is dropped.

Example:2 Let $X = R$ and $F_n = \{x \in R : x \geq n\}$. Then X is a complete metric space and $\{F_n\}$ is a decreasing sequence of non- empty closed subsets of X.

Also $d(F_n) = \infty$ for each n. So that the condition $d(F_n) \to 0$ is not satisfied here. Also, we have $F = \cap_{n=1}^{\infty} F_n = \emptyset$. Hence the claim.

Example:3 Here we give an example to show that the set F in the cantor's intersection theorem may be empty if the hypothesis that each F_n is a closed subset of X is dropped.

Let $X = R$ and $F_n = \{x \in R: 0 < x \leq \frac{1}{n}$

Then x is a complete metric space and $\{F_n\}$ is a decreasing sequence of non-empty subsets such that of X such that $d(F_n) = \frac{1}{n} \to 0$. But F_n is not a closed subset of R. Now we have $F = \cap_{n=1}^{\infty} F_n = \emptyset$. Hence the claim.

Baire's Theorem:

If X is a complete metric space, the intersection of a countable number of dense open subsets is dense in X.

Proof: Let the closed ball cantered at x with radius r by $B[x, r]$:

$B[x, r] = \{y \in X | d(y, x) \leq r\}$.

Note that any open set in a metric space contains a closed ball. indeed, if we shrink the radius of an open ball slightly, we obtain a closed ball contained in that open ball.

Suppose that $V_1, V_{2,}$ ….. are dense and open in X and let W be a nonempty open set in X. We will show that $(\cap_{n=1}^{\infty} V_n) \cap W \neq \emptyset$.

Since V_1 is dense in $X, W \cap V_1$ is a nonempty open set. Hence, we can find $x_1 \in X$ and $0 < r_1 < 1$ such that $B[x_1, r_1] \subseteq W \cap V_1$ (1)

If $n \geq 2$ and $x_{n-1} and\ r_{n-1}$ are chosen, the denseness of V_n show that

$V_n \cap B\ (x_{n-1}, r_{n-1})$ is a nonempty open set, and therefore we can find $x_n \in X$ and $0 < r_n < \frac{1}{n}$ such that $[x_n, r_n] \subseteq V_n \cap B(x_{n-1}, r_{n-1})$ (2)

By induction, this process produces the sequence $\{x_n\}$ in X. If $m, n \geq N$,

Then $x_m\ and x_n$ are in $B(x_N, r_N)$, and thus

$$d(x_m, x_n) \leq d(x_m, x_N) + (x_N, x_n) < 2r_N < \frac{2}{N}.$$

Hence,$\{x_n\}$ is a Cauchy Sequence. Since X is complete,$x_n$converges to some $x \in X. If\ k \geq n$, then $x_k$lies in a closed set $B[x_n, r_n]$. thus $x \in B[x_n, r_n]$ for all $n \geq 1$. By (1), $x \in\ W \cap V_1$, and by (2), we have $x \in V_n for\ all\ n \geq 2$.

Hence $x \in (\bigcap_{n=1}^{\infty} V_n) \cap W$.

We can conclude that the intersection of all V_n is dense in X.

Note: the completeness assumption is necessary in this theorem as the following example.

Let X be a complete metric space. Assume that $X = \mathbb{Q}$. Write $\mathbb{Q} = \{r_n | n \in \mathbb{N}\}$ and let $G_n = \mathbb{Q} - \{r_n\}$ for each $n \in \mathbb{N}$. Then G_n is open and dense in $\mathbb{Q}$ for each n, but $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

Corollary: if a complete metric space is a union of countably many closed sets, then at least one of the closed sets has nonempty interior.

Proof: Let X be a complete metric space. Assume that $X = \bigcup_{n=1}^{\infty} F_n$,where each F_n is closed. For each $n \in \mathbb{N}$, let $G_n = {F_n}^c$.then $\bigcap_{n=1}^{\infty} G_n = \emptyset$.

By Baire's theorem, there exists an open set G_n which is not dense in X. Thus, $\overline{G_n} \neq X$. But then Int $F_n = X - \overline{G_n}$, and hence F_n has nonempty interior.

4.10 Summary

In this unit, we have covered the following points:

1. We defined a sequence in a metric space (X, d) and discussed its convergence.

2. We defined subsequences of a sequence and have shown the relationship between convergence of a sequence and its subsequence's.

3. We have shown the connection between continuity and convergence. "f is continuous iff x, + x implies that f(x,) + f(x)".

4. We defined Cauchy sequences and explained the connection between Cauchy sequences and convergence. A Cauchy sequence is convergent if and only if it has a convergent subsequence.

5. We defined complete metric spaces. A metric space (X, d) is complete if every Cauchy sequence in X is convergent in X.

6. We discussed two important theorems and explained the importance of them. (1). Cantor's Intersection Theorem (2). Baire's Theorem.

4.11 Terminal Questions

1. What are the dense subset? of a discrete metric space?

2.Show that a closed set is nowhere dense if and only if it contains no open set.

3.Give an example of a set which is neither dense nor nowhere dense.

4.Show that a Cauchy sequence is convergent$\Leftrightarrow$ it has a convergent subsequence.

5.Prove that if (X, d) is a complete space, and each $x \in X$ is a limit point of X,then X is uncountable.

6.Given an example of a complete metric space(X, d) and a sequence of non-empty closed sets $\{A_n\}$in X with $A_1 \supseteq A_2 \supseteq A_3 \dots\dots\dots.. \supseteq A_n \dots..$such that $\bigcap_{n=1}^{\infty} A_n = \emptyset$

7.Let X be the real line R with the usual metric, and let $F_n = [n, \infty[$.

8. Let (X, d) be a metric space and $A \subseteq X$.Show that $\bar{A} = \{x \in X: d(x, A) = 0\}$.

9.Define Complete Metric Space. Given an example of a metric space which is not Complete.

10. Let (X, d) be a metric space and let $\langle F_n \rangle$ be a decreasing sequence of non-empty closed subsets of$X$ such that $d\langle F_n \rangle \to 0$ as $n \to \infty$. If $F = \bigcap_{n=1}^{\infty} F_n$contains exactly one point, then show that X is complete.

UNIT 5: Convergence of sequence and series of functions

Structure:

5.1 Introduction

In this unit we shall study about point wise convergence and uniform convergence of sequence and series of functions. Necessary and sufficient condition for a uniform convergence, Weierstrass test, Abel's test and Dirichlet's test for uniform convergence. Term by term integration and term by term differentiation.The term uniform convergence was probably used first time by Christoph Gudermann in a paper on elliptic functions.

Later Gudermann's pupil Karl Weierstrass who attended his course on elliptic functions, He used uniformly convergent in his paper in 1841, so Weierstrass's discovery was the earliest, and he alone fully realized its far reaching importance. It was one of the fundamental ideas of analysis. In this unit we shall study convergence of series of functions.

5.2 Objectives

After reading this unit the learner should be able to deal with:

- Necessary and sufficient condition for a uniform convergence
- Weierstrass test for uniform convergence
- Abel's test for uniform convergence
- Dirichlet's test for uniform convergence.
- Term by term integration and term by term differentiation

Sequences: A sequence is a function from natural numbers to real numbers i.e., $f: N \to R$. Sequences are written in a few different ways like

Way 1: $a_1,\ a_2,\ a_3, \dots \dots \dots \dots$

Way 2: $\{a_n\}_{n=1}^{\infty}$

Way 3: $\{f(n)\}_{n=1}^{\infty}$

Examples: $f(i) = \frac{i}{i+1},\ f(n) = \frac{n}{n+1}, f(n) = \frac{1}{2^n}, f(n) = \sin\frac{n\pi}{6}$.

Series:Series is the sum of a sequence. If $\{a_n\}_{n=1}^{\infty}$ is a sequence then the associated series is

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \; \dots \dots . . \dots \dots ..$$

Meaning of $f_n(x)$: $f_n(x)$ is the function in x and n where x is the real number or variable and n is natural number.

Examples:$f_n(x) = nx$

$f_1(x) = x, f_2(x) = 2x, f_3(x) = 3x \dots \dots \dots, f_n(x) = nx, f_{n+1}(x) = (n+1)x$

Hence $\{f_n(x)\}_{n=1}^{\infty}$ is called the sequence of functions.

$\{f_n(x)\}_{n=1}^{\infty}$is equivalent to $f_1(x),\; f_2(x),\; f_3(x), \dots \dots \dots \dots \dots$.

$\{f_n(x)\}_{n=1}^{\infty}$ is also denoted as $\{f_n\}$ or $\{f_n$(x)$\}$ or $< f_n >$.

Here $f_1(x) + f_2(x) + f_3(x) + \; \dots \dots \dots \dots \dots . = \sum_{n=1}^{\infty} f_n\,(x)$ is known as series of functions.

5.3 Pointwise convergence

Let D be a subset of Real numbers and let $\{f_n(x)\}$ be a sequence of functions define on D we say that $\{f_n(x)\}$ converges point wise on D if $\lim_{n\to\infty} f_n(x)$ exists for each point x in D i.e.

$f_n(x): D \to R$

$\lim_{n\to\infty} f_n(x) = f(x)$ for all x belongs to D.

This means that $\lim_{n\to\infty} f_n(x)$ is a real number that depends only on x.

If $\{f_n\}$ is point wise convergent then the function defined by

f(x) = $\lim_{n\to\infty} f_n(x)$ for every x in D is called the pointwise limit of the sequence.

Example 1: Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = \frac{nx+x^2}{n^2}$ for all x in Real numbers show that $\{f_n\}$ converges pointwise.

Solution: For every real number x , we have:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\frac{nx+x^2}{n^2}\right) = \lim_{n\to\infty}\left(\frac{x}{n} + \frac{x^2}{n^2}\right)$$

$$= x \lim_{n\to\infty}\frac{1}{n} + x^2 \lim_{n\to\infty}\frac{1}{n^2} = 0 + 0 = 0$$

Thus $\{f_n\}$ converges pointwise to the zero on real numbers.

Example 2: : Consider the sequence $\{f_n\}$ of functions defined by $f_n(x) = nx$ for all x in Real numbers show that $\{f_n\}$ does not converge pointwise.

Solution: For every real number x, we have

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx = x \lim_{n\to\infty} n = \infty$ for any x more than zero.

Hence $\{f_n\}$ does not converge pointwise.

Definition of uniform convergence:

Let $\{f_n\}$ be a sequence of functions defined on interval 'D' if for every $\in >$ 0, there can be found a positive integer 'm' such that $|f_n(x) - f(x)| < \in$ $\forall\, n \geq m \;\forall\, x \in D$. The f(x) is called the uniform limit of the sequence on D.

Note:

1 In point wise convergence: one m for each x.

2. In uniform convergence: one m for all x.

Uniform convergence of a series of functions:

Definition: Let $\sum_{n=1}^{\infty} f_n(x)$ be a series of functions define on interval 'D'. The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly if the sequence $\{f_n(\text{x})\}$ or $< f_n >$ is uniformly convergent.

5.4 Necessary and sufficient condition for uniform convergence (Cauchy's general principle of uniform convergence)

Theorem: Let $\{f_n(\text{x})\}$ or $< f_n >$ be a sequence of real valued functions defined on 'D'. Then $\{f_n(\text{x})\}$ or $< f_n >$ is uniformly convergent on 'D' if and only if for every $\in > 0$, there exists a positive integer 'm' such that $\left|f_n(x) - f_p(x)\right| < \in \;\forall\, n,\; p \geq m \;\forall\, x \in D$.

Proof: Necessary condition:

Let the sequence $\{f_n(x)\}$ is uniformly convergent on 'D'. By the definition given $\in > 0$, there exists a positive integer 'm' such that $|f_n(x) - f(x)| < \in /2 \;\forall\, n \geq m \;\forall\, x \in D$.

Here, n, p belongs to natural numbers and , $p \geq m \; \forall \; x \; \in D$, we have

$$\left|f_n(x) - f_p(x)\right| = \left|f_n(x) - f(x) + f(x) - f_p(x)\right|$$

$$= \; \leq |f_n(x) - f(x)| + \left|f(x) - f_p(x)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\left|f_n(x) - f_p(x)\right| < \in \; \forall \; n, p \geq m \; \forall \; x \; \in D \; is \; satisfied.$$

Sufficient condition: Let $< f_n >$ be any sequence of functions on 'D' and the condition

$\left|f_n(x) - f_p(x)\right| < \in \; \forall \; n, p \geq m \; \forall \; x \; \in D$ is satisfied.

According to the given condition, we can say that $< f_n >$ is a Cauchy sequence. Since every Cauchy sequence is convergent so $< f_n >$ is convergent or we can say that

$$\lim_{n\to\infty} f_n(x) = f(x) \; for \; all \; x \; belongs \; to \; D$$

If 'p' is fixed in the condition and let $n \to \infty$, we can say that

$$\left|f(x) - f_p(x)\right| < \; \varepsilon \; \forall \; p \geq m \; and \; \forall \; x \in D.$$

Therefore $< f_n >$ converges uniformly on 'D'.

Note : A series $\sum_{n=1}^{\infty} f_n\,(x)$ will converge uniformly on 'D' if and only if for every ε>0 there exists a positive integer 'm' such that

$$\left|f_{n+1}(x) + f_{n+2}(x) + \cdots \ldots . + f_{n+p}(x)\right| < \varepsilon, \forall \; n \geq m \; and \; \forall \; x \; \in D.$$

Where $p = 1,2,3, \ldots.$

Example 3 : Let $\{f_n(x)\}$ be the sequence of functions on $(0,\ \infty)$ defined by $f_n(x) = \frac{nx}{1+n^2x^2}$ is converges pointwise to zero but does not converge uniformly.

Solution: $\lim_{n\to\infty} f_n(x) = \ \lim_{n\to\infty}(\frac{nx}{1+n^2x^2}) = \lim_{n\to\infty}\left(\frac{x}{\frac{1}{n}+nx^2}\right) = 0$

So given sequence is pointwise convergent to zero

We know that ε>0 but very near to zero

Here when x=1/n then $f_n(x) = \frac{nx}{1+n^2x^2}$ equals to $f_n\left(\frac{1}{n}\right) = \frac{n\left(\frac{1}{n}\right)}{1+n^2\left(\frac{1}{n}\right)^2} = \frac{1}{1+1} = \frac{1}{2}$

, x is taken a fixed real value i.e. 1/n.

Now by the condition $|f_n(x) - f(x)| = \left|\frac{1}{2} - 0\right| = \frac{1}{2} > \varepsilon$

So given sequence is not uniformly convergent.

5.5 Test for uniform convergence

Theorem 1. $(\boldsymbol{M_n - test})$: Let $< f_n >$ be a sequence of functions defined on 'D'.

Let $\lim\ _{n\to\infty} f_n(x) = f(x)\ for\ all\ x\ belons\ to\ D$

Set $M_n = Sup\{|f_n(x) - f(x)| : x\ belons\ to\ D\}$ then $< f_n >$ is uniformly convergent if and only if $M_n \to 0\ when\ n \to \infty$.

Theorem 2. (Weierstrass's M- test): A series $\sum_{n=1}^{\infty} f_n\,(x)$ of functions will be uniformly convergent on D. If there exists convergent series $\sum_{n=1}^{\infty} M_n$of positive constants such that $|f_n(x)| \leq M_n \;\forall\; n \in N \; and \;\forall\; x \in D.$

Abel's lemma: If the sequence $< v_n >$ of positive terms is monotonic decreasing and numbers

$u_1, u_2,\; u_3, \dots \dots \dots \dots . . u_n \; and \; k_1, k_2$ are such that $k_1 < u_1 + u_2 + \dots \dots \dots . u_n < k_2 \quad for \; 1 \leq r \leq n \;\; then \; k_1 v_1 \leq \sum_{r=1}^{\infty} u_r v_r \leq k_2 v_1.$

Theorem 3. (Abel's Test): The series $\sum u_n v_n$ will be uniformly convergent on [a, b] if

(i) $\sum u_n\,(x) \; or \; \sum u_n$ is uniformly convergent on [a, b].

(ii) $The \; sequence \; \langle v_n(x) \rangle is \; monotonic \; for \; every \; x \; in \; [a, b].$

(iii) $\langle v_n(x) \rangle$ is uniformly bounded in [a, b] i.e. there is a positive number k, independent of x and n, such that
$|v_n(x)| < k \;\forall\; x \;\in [a, b] and \; for \; every \; positive \; integer \; n.$

Proof: Let $f_n(x) = u_1 + u_2 + u_3 + \;\dots \dots \dots \dots . +u_n$,$\sum u_n$is uniformly convergent on [a, b].

Cauchy's principle for given $\varepsilon > 0, \exists\; m \; (positive \; integer)$ such that

$$\left|f_q(x) - f_p(x)\right| < \frac{\varepsilon}{k} \;\forall\; q, p \;\in N, q > p > m \; and \;\forall\; x \;\in [a, b]$$

$$\left|\sum_{n=p+1}^{q} u_n(x)\right| < \frac{\varepsilon}{k}, \forall\, q > p > m \; and\; \forall\, x \in [a,b]$$

$\therefore \frac{\varepsilon}{k}$ $is\ an\ upper\ bound\ of$ $\sum_{n=p+1}^{q} u_n(x)$ By hypothesis , the sequence $\langle v_n(x) \rangle$ is monotonic in [a, b]

Hence by Abel's lemma, we have

$$\left|\sum_{n=p+1}^{q} u_n(x).\, v_n(x)\right| \le \frac{\varepsilon}{k} v_{p+1}(x) < \frac{\varepsilon}{k}.\, k = \varepsilon, \forall\, p, q \in N, q > p$$
$$> m \; and \; \forall\, x \in [a,b]$$

Consequently, by Cauchy's principle $\sum u_n v_n$ is uniformly convergent on [a, b].

Example 4: Prove that $\sum a_n n^{-x}$ is uniformly convergent on [0, 1] if $\sum a_n$ converges uniformly on [0, 1].

Solution: Take $\sum v_n(x) = n^{-x} = \frac{1}{n^x}$ $and\ u_n(x) = a_n$.

The sequence $< n^{-x} >$ is monotonic decreasing sequence on [0, 1]

Therefore $\frac{1}{n^x} \le \frac{1}{n^0} = 1\ \forall\, n \ \in N\ and\ x \in [0,1]$.

$\therefore$ $|v_n(x)| = |n^{-x}| \le 1\ \forall\, n \in N\ and\ \forall\, x \ \in [0,1]$.

Thus$\sum v_n(x) = n^{-x}$ is uniformly bounded and monotonic decreasing sequence on [0, 1]. Also $u_n(x) = a_n$ is uniformly convergent on [0, 1].

Hence by Abel's test $\sum v_n(x)u_n(x) = \sum a_n n^{-x}$ is uniformly convergent on [0, 1].

Theorem 4:(Dirichlet's Test) The series $\sum u_n v_n$ is uniformly convergent on

[a, b] if (i) $< v_n(x) >$ is a positive monotonic decreasing sequence converging uniformly to zero on [a, b].

(ii)$|f_n(x)| = |\sum_{r=1}^{n} u_r(x)| < k, \forall\, x \in [a,b] and\ n \in N$ where k is a fixed number independent of x.

Proof: We have $|f_n(x)| < k \quad \forall\, x \in [a,b]\ and\ \forall\, n \in N$

$\therefore \quad for\ all\ x \epsilon [\,a,b]and\ for\ all\ p,q\ \epsilon\ N,$

q > p >m_1, we get

$|f_q(x) - f_p(x)| \le |f_q(x)| + |f_p(x)| < k + k = 2k,$

i.e. $\left|\sum_{n=p+1}^{q} u_n(x)\right| < 2k \ \ \forall\, x \in [a,b]\ and\ \forall\ q > p > m_1$

Therefore, 2k is an upper bound of $\sum_{n=p+1}^{q} u_n(x)$

Also $< v_n(x) >$ is a positive monotonic decreasing so by Abel's lemma.

$\left|\sum_{n=p+1}^{q} u_n(x)v_n(x)\right| < 2k.\, v_{p+1}(x)$. -------------(1)

Again $< v_n(x) >$ converges uniformly to zero on [a, b]. Given ϵ>0 there exists $m_2\ \epsilon$ Nsuch that $|v_n(x)| < \frac{\varepsilon}{2k}\ \forall\, n \ \ge m_2\ and\ \forall\ x \in [a,b]$------------ (2)

Let m = max{ $m_{1,}\ m_2$ } then by (1) and (2) hold for all n > m.

$\left|\sum_{n=p+1}^{q} u_n(x)v_n(x)\right| < 2k.\frac{\varepsilon}{2k} = \epsilon.\ \forall\ x \in [a,b]\ and\ \forall\ q > p > m.$

Hence $\sum u_n v_n$ is uniformly convergent on [a, b].

Examples 5: Prove that if k is any positive number less than unity, the series $\sum\frac{x^n}{n+1}$is uniformly convergent in [-k, k].

Solution: Let $u_n(x) = x^n\ and\ \ v_n(x) = \frac{1}{n+1}$

|x| $\leq$ k < 1, we have $|f_n(x)| = |x + x^2 + x^3 + \ --- \ + \ x^n|$

$|\ f_n(x)| \leq |x| + |x^2| + |x^3| + \ ------ \ + |x^n|$

$|f_n(x)| \leq k + k^2 + k^3 + k^4 + k^5 + \ --------- k^n$

$|f_n(x)| \leq \ \frac{k}{1-k}\ \ as\ n \to \infty$ (sum of infinite GP)

Also $v_n(x) = \frac{1}{n+1}$ is monotonic decreasing sequence converging to zero.

Hence by the Dirichlet's test given series $\sum\frac{x^n}{n+1}$ is uniformly convergent.

Example 6: When $0 < p \leq 1\ the\ series\ \sum\frac{\cos n\theta}{n^p}$converges uniformly in any interval $[\alpha, 2\pi - \alpha]$, α> 0.

Solution: Take $u_n(x) = \frac{1}{n^p}\ and\ v_n(x) = \cos n\theta$

$$\left| \sum_{r=1}^{n} u_r(x) \right| = \left| \cos r\,\theta \right| = \left| cos\theta + cos2\theta + ------ + \cos n\theta \right|$$

$$\left|\sum_{r=1}^{n} u_r(x)\right| = \left|\frac{\left(\cos\frac{(n+1)\theta}{2}.\sin\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}\right| \le cosec\frac{\alpha}{2}\ \forall\ \theta \in\ [\alpha, 2\pi - \alpha]$$

Now by Dirichlet's test the series $\sum \frac{\cos n\theta}{n^p}$converges uniformly.

Example 7: Show that the series $\sum \frac{(-1)^{n-1}}{n+x^2}$ converges uniformly for all values of x.

Solution: $Let\ u_n(x) = \ (-1)^{n-1}\ and\ v_n(x) = \frac{1}{n+x^2}$

Since, $f_n(x) = \sum_{r=1}^{n}(-1)^{r-1} = 0\ 0r - 1\ as\ n\ is\ even\ or\ odd.$

$f_n(x)\ is\ bounded\ for\ all\ x$

Also $v_n(x) = \frac{1}{n+x^2}$ is a positive monotonic decreasing sequence, converging to zero for all value of x.

Hence by Dirichlet's test series$\sum \frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent.

Example 8: Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ $, x \in \boldsymbol{R}$ is uniformly convergent.

Solution: Here $f_n(x) = \frac{x}{n(1+nx^2)}$

To find the maximum value of above function

We differentiate the function(we will apply maxima minima concept)

$$f_n'(x) = \frac{n(1+nx^2).1 - x.n.2nx}{n^2(1+nx^2)^2}$$

$$f_n'(x) = \frac{n + n^2x^2 - 2n^2x^2}{n^2(1+nx^2)^2}$$

Thus $f_n'(x) = 0$, given

$n + n^2x^2 - 2n^2x^2 = 0 \Rightarrow n - n^2x^2 = 0$

$or, x^2 = \frac{1}{n}$ or, x = $\sqrt{\frac{1}{n}}$, here $f_n''(x)$ is negative so it will give maximum value.

The maximum value will be $f_n\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{2n^{\frac{3}{2}}}$ hence $f_n(x) \leq \frac{1}{2n^{3/2}}$

since $\frac{1}{2n^{3/2}}$ is convergent. So by Weierstrass M-test series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$, $x \in \boldsymbol{R}$is uniformly convergent.

Example 9: Show that the series $\sum \frac{(-1)^n}{n} |x|^n \; on \; [-1, 1]$is uniformly convergent.

Solution: Since $|x|^n$ is positive, monotonic decreasing and bounded on [-1, 1].

And series $\sum \frac{(-1)^n}{n}$ $is\ uniformly\ convergent.$

Therefore, by using Abel's test given series$\sum \frac{(-1)^n}{n}|x|^n\ on\ [-1,1]$ converges uniformly.

5.6 Term by term integration and term by term differentiation

Theorem 1 Let <f_n> be a sequence of real valued functions defined on the closed and bounded interval [a, b] and letf_n belongs to R [a, b] for n = 1, 2, 3, …………… if <f_n> converges uniformly to the function f(x) on [a, b] then f belongs to R [a, b] and $\int_a^b f(x)\,dx = \lim_{n\to\infty}\int_a^b f_n(x)\,dx.$

Theorem1':(Term by term integration) Let$u_n\colon [a,b] \to R\ for\ n =$ $1,2,3,\ldots$ …. is integrable on [a, b] and $\sum_{n=1}^{\infty} u_n(x)$ converges uniformly on [a, b]. Then the sum $f(x) = \sum_{n=1}^{\infty} u_n(x)$ is integrable on [a, b] and $\int_a^b f(x)dx =$ $\sum_{k=1}^{\infty}\int_a^b u_n(x)\,dx.$

Example 10: Show that $\int_0^1 \sum_{n=1}^{\infty}\frac{x^n}{n^2}\,dx = \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}$.

Solution: By weierstrass's M-test, the series $\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,dx$ is uniformly convergent on [0, 1]. Therefore, it can be integrated term by term

Hence $\int_0^1 \sum_{n=1}^{\infty}\frac{x^n}{n^2}\,dx = \sum_{n=1}^{\infty}\int_0^1 \frac{x^n}{n^2}\,dx = \sum_{n=1}^{\infty}\left[\frac{x^{n+1}}{n^2(n+1)}\right]_0^1$

$= \sum_{n=1}^{\infty}\left\{\frac{1^{n+1}}{n^2(n+1)} - \frac{0^{n+1}}{n^2(n+1)}\right\} = \sum_{n=1}^{\infty}\frac{1}{n^2(n+1)}.$

Theorem 2:(Term by term differentiation) Let$u_n: [a, b] \to R \ for \ n =$ 1,2,3, ...has continuous derivative on [a, b] and further suppose that

(i) The series$\sum_{n=1}^{\infty} u_n(c)$ converges at some point $c \in [a, b]$ and

(ii) The series of derivatives $\sum_{n=1}^{\infty} u_n'(x)$ converges uniformly on [a, b], to f(x)=$\sum_{n=1}^{\infty} u_n'(x)$ say. Then

(1) The series $\sum_{n=1}^{\infty} u_n(x)$ converges at every $x \in [a, b]$ and the sum F(x)=$\sum_{n=1}^{\infty} u_n(x)$ *is differentiable* with F'(x)= f(x) for each $x \in [a, b]$;

(2) moreover, the convergence of F(x)=$\sum_{n=1}^{\infty} u_n(x)$ is uniform on [a, b].

Example: Show that $\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3} = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$.

Solution: First we shall show that the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is uniformly convergent on [0, k] for any k > 0.

Let $u_n(x) = \frac{1}{n^3+x^3}$ *and* $v_n(x) = nx^2$

Then $|u_n(x)| \leq \frac{1}{n^3} \forall \ x \in [0, k]$.

But $\sum \frac{1}{n^3}$is a convergent series hence by Weierstrass's M-test, the series $u_n(x) = \frac{1}{n^3+x^3}$ is uniformly convergent on [0, k]

Also, for every $x \in [0, k]$, $v_n(x) = nx^2$ is monotonically increasing.

By Abel's test the series $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$converges uniformly on [0, k].

Hence by the term-by-term differentiation

$$\lim_{x\to 1}\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3} = \sum_{n=1}^{\infty}\lim_{x\to 1}\frac{nx^2}{n^3+x^3} = \sum_{n=1}^{\infty}\frac{n}{n^3+1}.$$

5.7 Summary

We conclude with summarizing what we learnt in this unit "point wise convergence and uniform convergence of sequence and series of functions. Necessary and sufficient condition for a uniform convergence, Weierstrass test, Abel's test and Dirichlet's test for uniform convergence. Term by term integration and term by term differentiation".

5.8 Terminal Questions

1. prove that $f_n(x) = \frac{n^2x}{1+n^2x^2}\ on\ [0,1]$ is not uniformly convergent.

2. prove that $f_n(x) = \frac{nx}{1+n^2x^2}\ on\ [0,1]$ is uniformly convergent.

3. prove that $f_n(x) = \frac{1}{1+x^n}\ on\ [2,\infty)$ is uniformly convergent.

4. Prove that $f_n(x) = x^n\ on\ [0,\ k],\ k < 1$ is uniformly convergent.

5. Prove that $f_n(x) = nxe^{-nx^2}$ is not uniformly convergent $on\ [0,\infty)$.

6. Prove that $f_n(x) = \frac{x}{1+nx}$ is uniformly convergent.

7. Prove that $f_n(x) = \frac{x}{1+nx^2}$ is uniformly convergent.

8. Show that the series $\sum(-1)^{n-1}x^n$converges uniformly in $0 \le x \le k \le 1$.

9. Prove that the series $\sum_{n=1}^{\infty} \frac{x}{(n+x^2)^2}, x \in R$ is uniformly convergent.

10. Show that the series for which $i)\ f_n(x) = \frac{1}{1+nx}$ $\quad ii)\ f_n(x) = nx(1-x)^n$

Can be integrated term by term in $0 \le x \le 1$, although series are not convergent in this interval.

11. Show that the function represented by $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is differentiable for every x and its derivative is $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

Unit-6: Improper Integrals

Structure

6.1 Introduction

In this unit we study about Riemann integrals as developed it requires that the range of integration is finite and the integrand remains bounded in that domain. If either or both of these assumptions is not satisfied it is necessary to attach a new interpretation to the integral.

In the integrand of f becomes infinite in the interval $a \leq x \leq b, i.e., f$ has points of infinite discontinuity in $[a, b]$ or the limits of integration $a \text{ or } b$ become infinite, the symbol $\int_a^b f\, dx$ is called an improper integral. Thus $\int_1^\infty \frac{dx}{x^2}, \int_{-\infty}^\infty \frac{dx}{1+x^2}, \int_0^1 \frac{dx}{x(1-x)}, \int_{-1}^\infty \frac{dx}{x^2}$ are examples of improper integrals.

The integrals which are not improper are called proper integrals. Thus $\int_0^1 \frac{\sin x}{x} dx$ is a proper integral. It will be assumed throughout that the number of singular points in any interval is finite and therefore when the range of integration is infinite, that all the singular points can be included in a finite interval. Further, it is assumed once for all that in a finite interval which encloses no point of infinite discontinuity the integrand is bounded and integrable.

6.2 Objectives

After studying this unit, we should be able to

▪ use the definition of finite and infinite intervals;

▪ explain the bounded function;

▪ explain the proper and improper integral;

▪ state and prove convergence of $\int_a^b f(x)dx$

▪ find the convergence the improper integrals.

6.3 Finite and infinite intervals

An interval is said to be finite or infinite according as its length is finite or infinite. Thus, the intervals [a, b], [a, b), (a, b], (a, b) each with length (b – a), are finite (or bounded) if both a and b are finite. The intervals [a, ∞), (a, ∞), (-∞, b], (-∞, b) and (-∞, ∞) are infinite (or unbounded) intervals.

6.4 Bounded Function

A function f is said to be bounded if its range is bounded. Thus, f:(a, b] → R is bounded, if there exist two real numbers m and M, (m ≤ M) such that

$$m \leq f(x) \leq M \; \forall x \in [a, b]$$

f is also bounded if there exists a positive real number K such that

$$|f(x)| \leq M \; \forall x \in [a, b]$$

6.5 Proper integral

The definite integral $\int_a^b f(x)dx$ is called a proper integral if

(i) The interval of integration [a, b] is finite (or bounded)

(ii) The integrand f is bounded on [a, b]

If f(x) is an indefinite integral of f(x), then $\int_a^b f(x)dx = F(b) - F(a)$.

6.6 Improper Integral

The definite integral $\int_a^b f(x)\, dx$ is an improper integral if either the interval of integration [a, b] is not finite or f is not bounded on [a, b] or neither the interval [a, b] is finite nor f is bounded over it.

(i) In the definite integral $\int_a^b f(x)\, dx$, if either a or b or both a and b are infinite so that the interval of integration is unbounded but f is bounded, then $\int_a^b f(x)\, dx$ is called an improper integral of the first kind.

For example, $\int_1^\infty \frac{dx}{\sqrt{x}}, \int_{-\infty}^0 e^{2x} dx, \int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$ are improper integrals of the first kind.

(ii) In the definite integral $\int_a^b f(x)\, dx$ if both a and b are finite so that the interval of integration is finite but f has one or more points of infinite discontinuity i.e. f is not bounded on [a, b], then $\int_a^b f(x)\, dx$ is called an improper integral of the second kind.

For example, $\int_0^1 \frac{dx}{x^2}, \int_1^2 \frac{dx}{2-x}, \int_1^4 \frac{dx}{(x-1)(4-x)}$ are improper integrals of the second kind.

(iii) In the definite integral $\int_a^b f(x)\, dx$, if the interval of integration is unbounded (so that a or b or both are infinite) and f is also unbounded then $\int_a^b f(x)\, dx$ is called an improper integral of the third kind.

For example, $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\, dx$ is an improper integral of the third kind.

6.7 Improper integral as the limit of a proper integral

(a) when the improper integral is of the first kind, either a or b or both a and b are infinite but f is bounded. we define

(i) $\int_a^\infty f(x)\, dx = \lim_{t\to\infty} \int_a^t f(x)\, dx, (t > a)$

The improper integral $\int_a^\infty f(x)\, dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ $or -\infty$.

If the integral is neither convergent nor divergent, then it is said to be oscillating.

(ii) $\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx, (t < b)$

The improper integral $\int_{-\infty}^b f(x) dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty$ $or -\infty$.

(iii) $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x) dx$ where c is any real number

$$= \lim_{t_1\to-\infty} \int_{t_1}^{c} f(x) dx + \lim_{t_2\to\infty} \int_c^{t_2} f(x) dx$$

The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is said to be convergent if both the limits on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note $\int_{-\infty}^{\infty} f(x)dx \neq \lim_{t\to\infty}\left[\int_{-t}^{c} f(x)dx + \int_{c}^{t} f(x)\,dx\right]$

(b) When the improper integral is of the second kind, both a and b are finite but f has one (or more) points of infinite discontinuity on [a, b].

(i) If f(x) becomes infinite at x = b only, we define $\int_{a}^{b} f(x)dx =$ $\lim_{\varepsilon\to 0}\int_{a}^{b-\varepsilon} f(x)dx$

The improper integral $\int_{a}^{b} f(x)dx$ is said to be convergent if the limit on the right hand side exists finitely and the integral is said to be divergent if the limit is $+\infty\ or -\infty$.

(ii) If f(x) becomes infinite at x = a only, we define $\int_{a}^{b} f(x)dx =$ $\lim_{\varepsilon\to 0}\int_{a+\varepsilon}^{c} f(x)dx$

The improper integral $\int_{a}^{b} f(x)dx$ converges if the limit on the right hand side exists finitely, otherwise it is said to be divergent.

(iii) If f(x) becomes infinite at x = c only where a < c < b, we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_a^{c-\varepsilon_1} f(x)dx + \lim_{\varepsilon_2 \to 0+} \int_{c+\varepsilon_2}^b f(x)dx$$

The improper integral $\int_a^b f(x)dx$ is said to be convergent if both the limit on the right hand side exist finitely and independent of each other, otherwise it is said to be divergent.

Note 1. If f has infinite discontinuity at an end point of the interval of integration, then the point of discontinuity is approached from within the interval.

Thus if the interval of integration is [a, b] and

(i) f has infinite discontinuity at 'a', we consider $[a+\varepsilon, b] as\ \varepsilon \to 0+$

(ii) f has infinite discontinuity at 'b', we consider $[a, b-\varepsilon] as\ \varepsilon \to 0+$

Note 2. A proper integral is always convergent.

Note 3. If $\int_a^b f(x)dx$ is convergent then

(i) $\int_a^b kf(x)dx$ is convergent $k \in R$,

(ii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where a < c < b and each integral or right hand side is convergent.

Note 4. For any c between a and b, i.e. a < c < b, we have

$$\int_{a}^{b} f(x)dx = \int_{a}^{c} f(x)dx + \int_{c}^{b} f(x)dx$$

If $\int_{c}^{b} f(x)dx$ is proper integral, then the two integrals $\int_{a}^{b} f(x)dx$ and $\int_{a}^{c} f(x)dx$ converge or disverge together. Thus while testing the integral $\int_{a}^{b} f(x)dx$ for convergence at α, it may be replaced by $\int_{a}^{c} f(x)dx$ for any convenient c such that a < c < b.

6.8 Convergence of Improper integrals

If the limit of an improper integral, as defined above, is a definite finite number, we say that the given definite integral is convergent and the value of definite integral is equal to the value of that limit.

If this limit is $\infty\ or - \infty$, the integral is said to be divergent. In this case , we say that the value of integral does not exist.

In case, the limit is neither a definite finite number nor $\infty\ or - \infty$,the integral is said to be oscillatory. In this case also, the value of integral does not exist.

Convergence of the integral $\int_{a}^{\infty} f(x)dx$ can be defined as follows:

The integral $\int_{a}^{\infty} f(x)dx$ is said to converge to the value I, if for any arbitrarily chosen positive number ε, however small, there exists a corresponding positive integer n_0 such that

$\left|\int_{a}^{b} f(x)dx - I\right| < \varepsilon$, for all values of $b \geq n_0$.

Similarly, we can define the convergence of an integral when the lower limit is infinite or when the integrand becomes infinite at the lower or upper limit.

Note:the sum and difference of two convergent integrals are evidently convergent.

Example 1. Examine the convergence of the improper integrals:

(i) $\int_1^\infty \frac{1}{x} dx$ (ii) $\int_1^\infty \frac{dx}{\sqrt{x}}$ (iii) $\int_1^\infty \frac{dx}{x^{3/2}}$ (iv) $\int_0^\infty \frac{dx}{1+x^2}$

Solution. (i) By definition $\int_0^\infty \frac{dx}{x} = \lim_{t\to\infty} \int_0^t \frac{dx}{x} = \lim_{t\to\infty} [\log x]_1^t = \lim_{t\to\infty} \log t = \infty$. So, $\int_0^\infty \frac{dx}{x}$ is divergent.

(ii) By definition $\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{t\to\infty} \int_1^t \int \frac{dx}{\sqrt{x}}$

$$= \lim_{t\to\infty} \int_1^t x^{-1/2} dx = \lim_{t\to\infty} \left[2\sqrt{x}\right]_1^t = \lim_{t\to\infty} \left(2\sqrt{t} - 2\right) = \infty$$

$\Rightarrow \int_1^\infty \frac{dx}{\sqrt{x}}$ is divergent

(iii) By definition $\int_1^\infty \frac{dx}{x^{3/2}} = \lim_{t\to\infty} \int_1^t x^{-3/2}\, dx = \lim_{t\to\infty} \left[\frac{x^{-1/2}}{-\frac{1}{2}}\right]_1^t$

$= \lim_{t\to\infty} \left[\frac{-2}{\sqrt{x}}\right]_1^t = \lim_{t\to\infty} \left(-\frac{2}{\sqrt{t}} + 2\right) = 0 + 2 = 2$ which is finite

$\Rightarrow \int_1^\infty \frac{dx}{x^{3/2}}$ is convergent and its value is 2.

(iv) By definition $\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{1+x^2} = \lim_{t\to\infty} [tan^{-1}x]_0^t$

$= \lim_{t\to\infty} (tan^{-1}t - tan^{-1}0) = \frac{\pi}{2}$ which is finite.

$\int_0^\infty \frac{dx}{1+x^2}$ is convergent and its value is $\frac{\pi}{2}$

Example 2. Examine for convergence the improper integrals:

(i) $\int_0^{\infty} e^{-mx}dx (m > 0)$ (ii) $\int_0^{\infty} \frac{x}{1+x^2}dx$ (iii) $\int_0^{\infty} \sin x \, dx$

(iv) $\int_0^{\infty} \frac{dx}{(1+x)^3}$ (v) $\int_0^{\infty} \frac{dx}{x^2+4a^2}$

Solution. (i) By definition $\int_0^{\infty} e^{-mx}dx = \lim_{t\to\infty} \int_0^{t} e^{-mx}dx =$

$\lim_{t\to\infty} \left[\frac{e^{-mx}}{-m}\right]_0^t$

$= \lim\limits_{t\to\infty} -\frac{1}{m}(e^{-mt} - 1) = -\frac{1}{m}(0 - 1) = \frac{1}{m}$ which is finite.

$\Rightarrow \int_0^{\infty} e^{-mx}dx$ is convergent and its value is $\frac{1}{m}$

(ii) By definition

$$\int_0^{\infty} \frac{x}{1+x^2}dx = \lim_{t\to\infty} \int_a^{t} \frac{x}{1+x^2}dx = \lim_{t\to\infty} \int_a^{t} \frac{1}{2}\left(\frac{x}{1+x^2}\right)dx$$

$$= \lim_{t\to\infty} \left[\frac{1}{2}\log(1+x^2)\right]_a^t = \lim_{t\to\infty} \frac{1}{2}[\log(1+t^2) - \log(1+a^2)] = \infty$$

$\Rightarrow \int_0^{\infty} \frac{x}{1+x^2}dx$ is divergent

(iii) $\int_0^{\infty} \sin x \, dx = \lim\limits_{t\to\infty} \int_0^{t} \sin x \, dx = \lim\limits_{t\to\infty} [-\cos x]_0^t = \lim\limits_{t\to\infty} (1 - \cos t)$

Which does not exist uniquely since cos t oscillates between -1 and +1 when 1→∞. $\Rightarrow \int_0^{\infty} \sin x \, dx$ oscillates

(iv) $\int_0^{\infty} \frac{dx}{(1+x)^3} = \lim\limits_{t\to\infty} \int_0^{t} (1+x)^{-3}dx = \lim\limits_{t\to\infty} \left[\frac{(1+x)^{-2}}{-2}\right]_0^t$

$\lim\limits_{t\to\infty} -\frac{1}{2}\left[\frac{1}{(1+t)^2} - 1\right] = -\frac{1}{2}(0 - 1) = \frac{1}{2}$ which is finite

$\Rightarrow \int_0^\infty \frac{dx}{(1+x)^3}$ is convergent and its value is $\frac{1}{2}$

(v) $\int_0^\infty \frac{dx}{x^2+4a^2} = \lim_{t\to\infty} \int_0^t \frac{dx}{x^2+(2a)^2} \lim_{t\to\infty} \left[\frac{1}{2a} tan^{-1} \frac{x}{2a}\right]_0^t$

$\lim_{t\to\infty} \frac{1}{2a} \left[tan^{-1} \frac{t}{2a} - tan^{-1} 0\right] = \frac{1}{2a} \left[\frac{\pi}{2}\right] = \frac{\pi}{4a}$ which is finite.

$\Rightarrow \int_0^\infty \frac{dx}{x^2+4a^2}$ is convergent and its value is $\frac{\pi}{4a}$

Example 3. Examine for convergence the improper integrals;

(i) $\int_3^\infty \frac{dx}{(x-2)^2}$ (iii) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$

(iv) $\int_2^\infty \frac{2x^2}{x^4-1} dx$ (v) $\int_1^\infty \frac{x}{(1+2x)^3} dx$

Solution. (i) $\int_3^\infty \frac{dx}{(x-2)^2} = \lim_{t\to\infty} \int_3^t (x-2)^{-2} dx = \lim_{t\to\infty} \left[\frac{(x-2)^{-1}}{-1}\right]_3^t$

$= \lim_{t\to\infty} - \left[\frac{1}{t-2} - 1\right] = -(0-1) = 1$ which is finite.

$\Rightarrow \int_3^\infty \frac{dx}{(x-2)^2}$ is convergent and its value is 1.

(iii) $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} \int_{\sqrt{2}}^t \frac{dx}{x\sqrt{x^2-1}} = \lim_{t\to\infty} [sec^{-1} x]_{\sqrt{2}}^1$

$= \lim_{t\to\infty} \left(sec^{-1} t - sec^{-1}\sqrt{2}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ which is finite.

$\Rightarrow \int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$ is convergent and its value is $\frac{\pi}{4}$

(iv) $\int_2^\infty \frac{2x^2}{x^4-1} dx = \lim_{t\to\infty} \int_2^t \frac{(x^2+1)+(x^2-1)}{(x^2+1)(x^2-1)} dx$

$= \lim_{t\to\infty} \int_2^t \left(\frac{1}{x^2-1} + \frac{1}{x^2+1}\right) dx = \lim_{t\to\infty} \left[\frac{1}{2} \log \frac{x-1}{x+1} + tan^{-1} x\right]_2^t$

$$= \frac{1}{2} \lim_{t \to \infty} \log \frac{1 - \frac{1}{t}}{1 + \frac{1}{t}} + \frac{\pi}{2} + \frac{1}{2} \log 3 - tan^{-1}2$$

$= \frac{1}{2} \log 1 + \frac{\pi}{2} + \frac{1}{2} \log 3 - tan^{-1}2 = \frac{\pi}{2} + \frac{1}{2} \log 3 - tan^{-1}2$ which is finite

$\Rightarrow \int_2^\infty \frac{2x^2}{x^4-1} dx$ is convergent and its value is $\frac{\pi}{2} + \frac{1}{2} \log 3 - tan^{-1}2$

(v) $\int_1^\infty \frac{x}{(1+2x)^3} dx = \lim_{t \to \infty} \int_1^t \frac{x}{(1+2x)^3} dx$

$$= \lim_{t \to \infty} \int_1^t \frac{\frac{1}{2}(1+2x) - \frac{1}{2}}{(1+2x)^3} dx$$

$$= \lim_{t \to \infty} \int_1^t \left[\frac{1}{2}(1+2x)^{-2} - \frac{1}{2}(1+2x)^{-3}\right] dx$$

$$= \lim_{t \to \infty} \left[\frac{1}{2} . \frac{(1+2x)^{-1}}{-1 \times 2} - \frac{1}{2} . \frac{(1+2x)^{-2}}{-2 \times 2}\right]_1^t$$

$$= \lim_{t \to \infty} \left[\frac{-1}{4(1+2x)} + \frac{1}{8(1+2x)^2}\right]_1^t$$

$$= \lim_{t \to \infty} \left[\frac{-1}{4(1+2t)} + \frac{1}{8(1+2t)^2} + \frac{1}{12} - \frac{1}{72}\right]$$

$= 0 + 0 + \frac{1}{12} - \frac{1}{72} = \frac{5}{72}$ which is finite

$\Rightarrow \int_1^\infty \frac{x}{(1+2x)^3} dx$ is convergent and its value is $\frac{5}{72}$.

Example 4. Examine for convergence the integrals:

(i) $\int_1^\infty x e^{-x} dx$ (ii) $\int_0^\infty x^2 e^{-x} dx$ (iii) $\int_0^\infty x e^{-z^2} dx$

(iv) $\int_0^\infty x^3 e^{-z^2} dx$ (v) $\int_0^\infty x sin\, x\, dx$

Solution. (i) $\int_1^\infty xe^{-x} dx = \lim_{t\to\infty} \int_1^t xe^{-x} dx$ (Integrating by parts)

$= \lim_{t\to\infty} [-xe^{-x} - e^{-x}]_1^t = \lim_{t\to\infty} (-te^{-t} - e^{-t} + e^{-1} + e^{-1})$

$= \lim_{t\to\infty} \left(\frac{-t}{e^t}\right) -= \lim_{t\to\infty} e^t + \frac{2}{e}$ (Applying L'Hospital's Rule to first limit)

$= \lim_{t\to\infty} \left(\frac{-1}{e^t}\right) - 0 + \frac{2}{e} = 0 + \frac{2}{e} = \frac{2}{e}$ which is finite

$\Rightarrow \int_1^\infty xe^{-x} dx$ is convergent and its value is $\frac{2}{e}$

(ii) $\int_0^\infty x^2 e^{-x} dx = \lim_{t\to\infty} \int_0^t x^2 e^{-x} dx$

$= \lim_{t\to\infty} [-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}]_0^t = \lim_{t\to\infty} (-t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2)$

$= \lim_{t\to\infty} \left(\frac{-t^2}{e^t}\right) - 2 \lim_{t\to\infty} \left(\frac{t}{e^t}\right) - 0 + 2$ (Applying L'Hospital rule)

$= \lim_{t\to\infty} \left(\frac{-2t}{e^t}\right) - 2 \lim_{t\to\infty} \left(\frac{t}{e^t}\right) + 2$

(Again Applying L'Hospital rule to first limit)

$= \lim_{t\to\infty} \left(\frac{-2}{e^t}\right) - 2 \times 0 + 2 = 0 + 2 = 2$ which is finite

$\Rightarrow \int_0^\infty x^2 e^{-x} dx$ is convergent and its value is 2.

(iii) $\int_0^\infty xe^{-x^2} = \lim_{t\to\infty} \int_0^t xe^{-x^2} dx$ Put $x^2 = z$ so that

$2xdx = dx \quad or\ xdx = \frac{1}{2}dz$

When x = 0, z = 0; when x = t, z = t^2.

$$\therefore \int_0^\infty xe^{-x^2} = \lim_{t\to\infty}\int_0^{t^2}\frac{1}{2}e^{-z}dz = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-z}\right]_0^{t^2}$$

$= \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2}-1\right) = -\frac{1}{2}(0-1) = \frac{1}{2}$ which is finite

$\Rightarrow \int_0^\infty xe^{-x^2}dx$ is convergent and its value is $\frac{1}{2}$

(iv) $\int_0^\infty x^3e^{-x^2}dx = \lim_{t\to\infty}\int_0^t x.x^2e^{-x^2}dx$

Put $x^2 = z$ so that $2xdx = dz$. When x = 0, z = 0; when x = t, z = t^2.

$\int_0^\infty x^3e^{-x^2}dx = \lim_{t\to\infty}\int_0^{t^2}\frac{1}{2}ze^{-z}dz$ (integrating by parts)

$= \lim_{t\to\infty}\frac{1}{2}[-ze^{-z}-e^{-z}]_0^{t^2} = \lim_{t\to\infty}\frac{1}{2}\left[-t^2e^{-t^2}-e^{-t^2}+1\right] =$

$-\frac{1}{2}\lim_{t\to\infty}\left(\frac{t^2}{e^{t^2}}\right) - 0 + \frac{1}{2}$ (Applying L'Hospital rule)

$= -\frac{1}{2}\lim_{t\to\infty}\left(\frac{2t}{2te^{t^2}}\right)+\frac{1}{2} = -\frac{1}{2}\lim_{t\to\infty}\left(-\frac{1}{e^{t^2}}\right)+\frac{1}{2} = 0+\frac{1}{2} = \frac{1}{2}$ which is finite.

$\Rightarrow \int_0^\infty x^3e^{-x^2}dx$ is convergent and its value is $\frac{1}{2}$

(v) $\int_0^\infty x\sin x\,dx = \lim_{t\to\infty}\int_0^t x\sin x\,dx$

$= \lim_{t\to\infty}[-x\cos x + \sin x]_0^t = \lim_{t\to\infty}(-t\cos t + \sin t)$

Which oscillates between $-\infty$ and $+\infty$ since cos t oscillates between -1 and +1 at t $\rightarrow \infty$

$\Rightarrow \int_0^\infty x sin\, x\, dx$ is not convergent. (in fact, it oscillates infinitely)

Example 5. Examine for convergence the integrals:

(i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ (ii) $\int_2^\infty \frac{dx}{x log\, x}$ (iii) $\int_0^\infty e^{-x} \sin x\, dx$ (iv) $\int_0^\infty e^{-ax} \cos bx\, dx$

Solution. (i) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^t \frac{dx}{(1+x)\sqrt{x}}$

Put x = 1, z = 1; when x = t, z = $\sqrt{t}$

$\therefore \int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = \lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{dz}{1+z^2} = \lim_{t\to\infty} [2tan^{-1} z]_0^{\sqrt{t}}$

$= \lim_{t\to\infty} 2\left[tan^{-1}\sqrt{t} - tan^{-1}1\right] = 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{2}$ which is finite

$\Rightarrow \int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent and its value is $\frac{\pi}{2}$

(ii) $\int_2^\infty \frac{dx}{x log\, x} = \lim_{t\to\infty} \int_2^t \frac{1/x}{\log x} dx$

$= \lim_{t\to\infty} [\log(\log x)]_2^t = \lim_{t\to\infty} [\log(\log t) - \log(\log 2)] = \infty$

$\Rightarrow \int_2^\infty \frac{dx}{x log\, x}$ is divergent

(iii) $\int_0^\infty e^{-x} \sin x\, dx = \lim_{t\to\infty} \int_0^t e^{-x} \sin x\, dx$

$$= \lim_{t\to\infty} \left[\frac{e^{-x}}{(-1)^2+1^2}(-1\sin x - 1\cos x)\right]_0^t$$

$$\left[\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - bcos\ bx)\right]$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x}(\sin x + \cos x)\right]_0^t$$

$$= \lim_{t\to\infty} -\frac{1}{2}[e^{-t}(\sin t + \cos t) - 1]$$

$= -\frac{1}{2}[(0 \times a\ finite\ quantity) - 1] = \frac{1}{2}$ which is finite

$\Rightarrow \int_0^\infty e^{-x}\sin x\, dx$ is convergent and its value is $\frac{1}{2}$

(iv) $\int_0^\infty e^{-ax}\cos bx\, dx = \lim_{t\to\infty}\int_0^t e^{-ax}\cos bx\, dx$

$$= \lim_{t\to\infty}\left[\frac{e^{-ax}}{(-a)^2+b^2}(-\mathrm{a}\cos bx + b\sin bx)\right]_0^t$$

$$\left[\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - bsin\ bx)\right]$$

$= \lim_{t\to\infty}\frac{1}{a^2+b^2}[e^{ax}(-\mathrm{a}\cos bt + bsin\ bt) + a] = \frac{a}{a^2+b^2}$ which is finite

$\Rightarrow \int_0^\infty e^{-ax}\cos bx\, dx$ is convergent and its value is $\frac{a}{a^2+b^2}$

Example 6. Examine the convergence of the integrals;

(i) $\int_1^\infty \frac{dx}{x(1+x)}$ (ii) $\int_1^\infty \frac{dx}{x^2(x+1)}$ (iii) $\int_1^\infty \frac{tan^{-1}x}{x^2}dx$ (iv) $\int_0^\infty e^{-\sqrt{x}}dx$

Solution. (i) $\int_1^\infty \frac{dx}{x(1+x)} = \lim_{t\to\infty}\int_1^t \frac{dx}{x(1+x)} = \lim_{t\to\infty}\int_1^t\left(\frac{1}{x}-\frac{1}{x+1}\right)dx$

[Partial Fractions]

$$= \lim_{t\to\infty} [\log x - \log(x+1)]_1^t = \lim_{t\to\infty} \left[\log \frac{x}{x+1}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\log \frac{t}{t+1} - \log \frac{1}{2}\right] = \lim_{t\to\infty} \left(\log \frac{1}{1+\frac{1}{t}}\right) + \log 2$$

$= \log 1 + \log 2 = \log 2$ which is finite.

$\Rightarrow \int_1^\infty \frac{dx}{x(1+x)}$ is convergent and its value is $\log 2$

(ii) $\int_1^\infty \frac{dx}{x^2(x+1)} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2(x+1)} = \lim\limits_{t\to\infty} \int_1^t \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) dx$

[Partial Fractions]

$$= \lim_{t\to\infty} \left[-\log x - \frac{1}{x} + \log(x+1)\right]_1^t = \lim_{t\to\infty} \left[\log \frac{x+1}{x} - \frac{1}{x}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\log\left(1+\frac{1}{t}\right) - \frac{1}{t} - \log 2 + 1\right]$$

$= \log 1 - 0 - \log 2 + 1 = 1 - \log 2$ which is finite

$\Rightarrow \int_1^\infty \frac{dx}{x^2(x+1)}$ is convergent and its value is $1 - \log 2$

(iii) $\int_1^\infty \frac{tan^{-1}x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{tan^{-1}x}{x^2} dx$

Put $x = tan\theta$ so that $dx = sec^2\theta d\theta$

$$\int \frac{tan^{-1}x}{x^2} dx = \int \frac{\theta}{tan^2\theta} sec^2\theta d\theta = \int \theta cosec^2\theta d\theta$$

$$= \theta(-cot\theta) - \int 1(-cot\theta) d\theta$$

$$= -\theta cot\theta + \log\sin\theta = -\frac{tan^{-1}x}{x^2} + \log\frac{x}{\sqrt{1+x^2}}$$

$\therefore \int_1^\infty \frac{tan^{-1}x}{x^2}dx = \lim_{t\to\infty}\left[-\frac{tan^{-1}x}{x^2} + \log\frac{x}{\sqrt{1+x^2}}\right]_1^t$

$$= \lim_{t\to\infty}\left[-\frac{tan^{-1}t}{t^2} + \log\frac{t}{\sqrt{1+t^2}} + tan^{-1}1 - \log\frac{1}{\sqrt{2}}\right]$$

$$= 0 + \lim_{t\to\infty}\log\frac{1}{\sqrt{\frac{1}{t^2}+1}} + \frac{\pi}{2} + \frac{1}{2}\log 2$$

$= \log I + \frac{\pi}{2} + \frac{1}{2}\log 2 = \frac{\pi}{2} + \frac{1}{2}\log 2$ which is finite.

$\Rightarrow \int_1^\infty \frac{tan^{-1}x}{x^2}dx$ is convergent and its value is $\frac{\pi}{2} + \frac{1}{2}\log 2$

(iv) $\int_0^\infty e^{-\sqrt{x}}dx = \lim_{t\to\infty}\int_0^t e^{-\sqrt{x}}\,dx$

Put $\sqrt{x} = z,\ i.e., x = z^2\ so\ that\ dx = 2zdz$

When x = 0, z = 0 when x = t, z = $\sqrt{t}$

$\therefore \int_0^\infty e^{-\sqrt{x}}dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} 2ze^{-z}dz$ [Integrating by parts]

$$= \lim_{t\to\infty} 2[-ze^{-z} - e^{-z}]_o^{\sqrt{t}} = \lim_{t\to\infty} -2\left[\sqrt{t}e^{-\sqrt{t}} + e^{-\sqrt{t}} - 1\right]$$

$= \lim_{t\to\infty}\left(\frac{-2\sqrt{t}}{e^{\sqrt{t}}}\right) - 0 + 2$ (Applying L'Hospital Rule)

$= \lim_{t\to\infty}\left(\frac{-\frac{1}{\sqrt{t}}}{e^{\sqrt{t}}.\frac{1}{2\sqrt{t}}}\right) + 2 = \lim_{t\to\infty}\left(\frac{-2}{e^{\sqrt{t}}}\right) + 2 = 0 + 2 = 2$ which is finite

$\Rightarrow \int_0^\infty e^{-\sqrt{x}} dx$ is convergent and its value is 2.

Example 7. Examine the convergence of the integrals:

(i) $\int_{-\infty}^{0} e^{2x}\, dx$ (ii) $\int_{-\infty}^{0} \frac{dx}{p^2+q^2x^2}$ (iii) $\int_{-\infty}^{0} e^{-x}\, dx$

(iv) $\int_{-\infty}^{0} \sinh x\, dx$

Solution. (i) $\int_{-\infty}^{0} e^{2x}\, dx = \lim_{t\to-\infty} \int_t^0 e^{2x}\, dx$

$= \lim_{t\to-\infty} \left[\frac{e^{2x}}{2}\right]_t^0 = \lim_{t\to-\infty} \frac{1}{2}(1 - e^{2x}) = \frac{1}{2}(1-0) = \frac{1}{2}$ which is finite.

$\Rightarrow \int_{-\infty}^{0} e^{2x}\, dx$ is convergent and its value is ½

(ii) $\int_{-\infty}^{0} \frac{dx}{p^2+q^2x^2} = \lim_{t\to-\infty} \int_t^0 \frac{dx}{q^2\left(\frac{p^2}{q^2}+x^2\right)} = \lim_{t\to-\infty} \left[\frac{1}{q^2} . \frac{1}{p/q} tan^{-1} \frac{x}{p/q}\right]_t^0$

$= \lim_{t\to-\infty} \frac{1}{pq}\left[0 - tan^{-1}\frac{qt}{p}\right] = \frac{1}{pq}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2pq}$ which is finite.

$\Rightarrow \int_{-\infty}^{0} \frac{dx}{p^2+q^2x^2}$ is convergent and its value is $\frac{\pi}{2pq}$

(iii) $\int_{-\infty}^{0} e^{-x}\, dx = \lim_{t\to-\infty} \int_t^0 e^{-x} dx = \lim_{t\to-\infty} [-e^{-x}]_t^0 =$
$\lim_{t\to-\infty} (-1 + e^{-t}) = -1 + \infty = \infty$

$\Rightarrow \int_{-\infty}^{0} e^{-x}\, dx$ diverges to +∞

(iv) $\int_{-\infty}^{0} \sinh x\, dx = \lim_{t\to-\infty} \frac{e^x - e^{-x}}{2} dx$

$$= \lim_{t\to-\infty}\left[\frac{1}{2}(e^x+e^{-x})\right]_t^0 = \lim_{t\to-\infty}\left[1-\frac{1}{2}(e^t+e^{-t})\right] = 1-\frac{1}{2}(0+\infty)$$

$$= -\infty$$

$\Rightarrow \int_{-\infty}^{0} \sinh x\, dx$ diverges to $-\infty$

Example 8. Examine the convergence of the integrals:

(i) $\int_{-\infty}^{\infty} e^{-x} dx$ (ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (iii) $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

(iv) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ (v) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$

Solution. (i)

$\int_{-\infty}^{\infty} e^{-x} dx = \int_{-\infty}^{0} e^{-x} dx + \int_{0}^{\infty} e^{-x} dx = \lim_{t_1\to-\infty} \int_{t_1}^{0} e^{-x}\, dx+=$
$\lim_{t_2\to\infty} \int_{0}^{t_2} e^{-x}\, dx$

$$= \lim_{t_1\to-\infty}[-e^{-x}]_{t_1}^{0} + \lim_{t_2\to-\infty}[-e^{-x}]_{0}^{t_2}$$

$$= \lim_{t_1\to-\infty}(-1+e^{-t_1}) + \lim_{t_2\to-\infty}(-e^{-t_2}+1) = (-1+\infty)+(0+1) = \infty$$

$\Rightarrow \int_{-\infty}^{\infty} e^{-x} dx$ diverges to ∞

(ii) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}$

$$= \lim_{t_1\to-\infty}\int_{t_1}^{0}\frac{dx}{1+x^2} + \lim_{t_2\to-\infty}\int_{0}^{t_2}\frac{dx}{1+x^2}$$

$$= \lim_{t_1\to-\infty}[tan^{-1}x]_{t_1}^{o} + \lim_{t_2\to-\infty}[tan^{-1}x]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} [tan^{-1} t_1] + \lim_{t_2 \to -\infty} [tan^{-1} t_2] = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$ which is finite.

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is convergent and its value is π

(iii) $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} = \int_{-\infty}^{0} \frac{dx}{e^x+e^{-x}} + \int_{0}^{\infty} \frac{dx}{e^x+e^{-x}}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{e^x dx}{e^{2x}+1} += \lim_{t_2 \to -\infty} \int_{0}^{t_2} \frac{e^x dx}{e^{2x}+1}$$

Now $\int \frac{e^x dx}{e^{2x}+1} = \int \frac{dz}{z^2+1}$ where $z = e^x$

$$= tan^{-1} z = tan^{-1} e^x$$

$$\int_{-\infty}^{\infty} \frac{dx}{e^x + e^{-x}} = \lim_{t_1 \to -\infty} [tan^{-1} e^x]_{t_1}^{0} + \lim_{t_2 \to -\infty} [tan^{-1} e^x]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} [tan^{-1} 1 - tan^{-1} e^{t_1}] + \lim_{t_2 \to -\infty} [tan^{-1} e^{t_2} - tan^{-1} 1]$$

$$= \left(\frac{\pi}{4} - tan^{-1} 0\right) + \left(tan^{-1} \infty - \frac{\pi}{4}\right) = \frac{\pi}{2}$$ which is finite.

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$ is convergent and its value is $\frac{\pi}{2}$

(iv) $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \int_{-\infty}^{0} \frac{dx}{(1+x^2)^2} + \int_{0}^{\infty} \frac{dx}{(1+x^2)^2}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{(1+x^2)^2} + \lim_{t_2 \to -\infty} \int_{0}^{t_2} \frac{dx}{(1+x^2)^2}$$

Now putting $x = tan\theta \; so \; that \; dx = sec^2\theta d\theta$, we have

$$\int \frac{dx}{(1+x^2)^2} = \int \frac{sec^2\theta d\theta}{(1+tan^2\theta)^2} = \int \frac{sec^2\theta d\theta}{sec^4\theta} = \int cos^2\theta$$

$$= \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) = \frac{1}{2}(\theta + \sin\theta\cos\theta) = \frac{1}{2}\left(tan^{-1}x + \frac{x}{1+x^2}\right)$$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \lim_{t_1 \to -\infty} \left[\frac{1}{2}\left(tan^{-1}x + \frac{x}{1+x^2}\right)\right]_{t_1}^{0} + \lim_{t_2 \to -\infty} \left[\frac{1}{2}\left(tan^{-1}x + \frac{x}{1+x^2}\right)\right]_{0}^{t_2}$

$$= \lim_{t_1 \to -\infty} \frac{1}{2}\left[-tan^{-1}t_1 + \frac{t_1}{1+{t_1}^2}\right] + \lim_{t_2 \to -\infty} \frac{1}{2}\left[tan^{-1}t_2 + \frac{t_2}{1+{t_2}^2}\right]$$

$$= \frac{1}{2}.\frac{\pi}{2} - \lim_{t_1 \to -\infty} \frac{t_1}{2(1+{t_1}^2)} + \frac{1}{2}.\frac{\pi}{2} + \lim_{t_2 \to -\infty} \frac{t_2}{2(1+{t_2}^2)}$$

$= \frac{\pi}{2} - \lim_{t_1 \to -\infty} \frac{1}{4t_1} + \lim_{t_2 \to -\infty} \frac{1}{4t_2}$ (By L'Hospital rule)

$= \frac{\pi}{2} - 0 + 0 = \frac{\pi}{2}$ which is finite.

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ is convergent and its value is $\frac{\pi}{2}$

(v) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2} = \int_{-\infty}^{0} \frac{dx}{x^2+2x+2} + \int_{0}^{\infty} \frac{dx}{x^2+2x+2}$

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{0} \frac{dx}{(x+1)^2+1} + \lim_{t_2 \to -\infty} \int_{0}^{t_2} \frac{dx}{(x+1)^2+1}$$

$$= \lim_{t_1 \to -\infty} [tan^{-1}(x+1)]_{t_1}^{0} + \lim_{t_2 \to -\infty} [tan^{-1}(x+1)]_{0}^{t_2}$$

$$= \lim_{t_1 \to -\infty} \left[\frac{\pi}{4} - tan^{-1}(t_1 + 1)\right] + \lim_{t_2 \to -\infty} \left[tan^{-1}(t_2 + 1) - \frac{\pi}{4}\right]$$

$= \frac{\pi}{4} - tan^{-1}(-\infty) + tan^{-1}\infty - \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ which is finite.

$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ is convergent and its value is x.

Example 9. Test the convergence of the integrals:

(i) $\int_0^1 \frac{dx}{\sqrt{x}}$ (ii) $\int_0^1 \frac{dx}{x^2}$ (iii) $\int_1^2 \frac{x}{\sqrt{x-1}} dx$

Solution. (i) 0 is the only point of infinite discontinuity of the integrand on [0, 1]

$\therefore \int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 x^{-1/2} dx$

$= \lim_{\varepsilon \to 0+} \left[2\sqrt{x}\right]_\varepsilon^1 = \lim_{\varepsilon \to 0+} 2\left(1 - \sqrt{\varepsilon}\right) = 2$ which is finite.

$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x}}$ is convergent and its value is 2.

(ii) 0 is only point of infinite discontinuity of the integrand on [0, 1]

$$\therefore \int_0^1 \frac{dx}{x^2} = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 x^{-2} dx = \lim_{\varepsilon \to 0+} \left[-\frac{1}{2}\right]_\varepsilon^1 = \lim_{\varepsilon \to 0+} \left(-1 + \frac{1}{\varepsilon}\right) = \infty$$

$\Rightarrow \int_0^1 \frac{dx}{x^2}$ diverges to ∞

(iii) 1 is the only point of infinite discontinuity of the integrand on [1, 2]

$$\therefore \int_1^2 \frac{x}{\sqrt{x-1}} dx = \lim_{\varepsilon \to 0+} \int_{1+\varepsilon}^2 \frac{(x-1)+1}{\sqrt{x-1}} dx$$

$$= \lim_{\varepsilon\to 0+} \int_{1+\varepsilon}^{2} \left(\sqrt{x+1} + \frac{1}{\sqrt{x-1}}\right) dx = \lim_{\varepsilon\to 0+} \left[\frac{2}{3}(x-1)^{3/2} + 2\sqrt{x-1}\right]_{1+\varepsilon}^{2}$$

$= \lim_{\varepsilon\to 0+} \left[\frac{2}{3} + 2 - \frac{2}{3}\varepsilon^{\frac{3}{2}} - 2\sqrt{\varepsilon}\right] = \frac{8}{3}$ which is finite

$\Rightarrow \int_1^2 \frac{x}{\sqrt{x-1}} dx$ is convergent and its value is $\frac{8}{3}$

Example 10. Examine the convergence of the integrals:

(i) $\int_0^1 \log x \, dx$ (ii) $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ (iii) $\int_0^e \frac{dx}{x(\log x)^3}$ (iv) $\int_1^2 \frac{dx}{x log\, x}$

Solution. (i) 0 is the only point of infinte discontinuity of the integrand on [0, 1]

$\therefore \int_0^1 \log x \, dx = \lim_{\varepsilon\to 0+} \int_{0+\varepsilon}^{1} (\log x). 1 \, dx$ (integrating by parts)

$$= \lim_{\varepsilon\to 0+} [x \log x - x]_{\varepsilon}^{1} = \lim_{\varepsilon\to 0+} (-1 - \varepsilon \log \varepsilon + \varepsilon)$$

$= -1$ which is finite. $[\because \lim_{x\to 0} x^n \log x = 0, n > 0]$

$\Rightarrow \int_0^1 \log x \, dx$ is convergent and its value is -1.

(ii) Since $\lim_{x\to 0} x(\log x)^n = 0, n > 0$, therefore 0 is the only point of infinite discontinuity of the integrand on $\left[0, \frac{1}{e}\right]$.

$\therefore \int_0^e \frac{dx}{x(\log x)^3} = \lim_{\varepsilon\to 0+} \int_{0+\varepsilon}^{1/e} (\log x)^{-2} . \frac{1}{x} dx = \lim_{\varepsilon\to 0+} \left[\frac{(\log x)^{-1}}{-1}\right]_{\varepsilon}^{1/e}$

$$= \lim_{\varepsilon\to 0+}\left[\frac{1}{\log\frac{1}{e}} - \frac{1}{\log\varepsilon}\right] = -[-1-0] = 1 \text{ which is finite.}$$

$\therefore \int_0^e \frac{dx}{x(\log x)^3}$ is convergent and its value is 1.

(iii) Please try yourself. [Ans. Converges to -1/2]

(iv) 1 is the only point of infinite discontinuity of the integrand on [1, 2]

$$\therefore \int_1^2 \frac{dx}{x\log x} = \lim_{\varepsilon\to 0+}\int_{1+\varepsilon}^2 \frac{1/x}{\log x}dx$$

$$= \lim_{\varepsilon\to 0+}[\log(\log x)]_{1+\varepsilon}^2 = \lim_{\varepsilon\to 0+}[\log\log 2 - \log\log(1+\varepsilon)]$$

$$= \log\log 1 - \log 0 = \log\log 2 - (-\infty) = \infty$$

$\Rightarrow \int_1^2 \frac{dx}{x\log x}$ diverges to ∞

Example 11. Examine the convergence of the integrals:

(i) $\int_0^a \frac{dx}{\sqrt{a-x}}$ (ii) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ (iii) $\int_1^2 \frac{dx}{2-x}$ (iv) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}dx$

(v) $\int_0^1 \frac{dx}{x^2-3x+2}$ (vi) $\int_0^1 \frac{dx}{x^2-1}$ (vii) $\int_0^{\pi/2} \tan\theta\, d\theta$

Solution. (i) a is the only point of infinite discontinuity of the integrand on [0, a]

$$\therefore \int_0^a \frac{dx}{\sqrt{a-x}} = \lim_{\varepsilon\to 0+}\int_0^{a-\varepsilon}(a-x)^{-1/2}\,dx$$

$$= \lim_{\varepsilon\to 0+}\left[-2\sqrt{a-x}\right]_0^{a-\varepsilon} = \lim_{\varepsilon\to 0+} -2\left[\sqrt{\varepsilon}-\sqrt{a}\right] = 2\sqrt{a} \text{ which is finite.}$$

$\therefore \int_0^a \frac{dx}{\sqrt{a-x}}$ is convergent and its value is $2\sqrt{a}$

(ii) 2 is the only point of infinite discontinuity of the integrand on [0, 2]

$$\therefore \int_0^2 \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{\sqrt{4-x^2}} = \lim_{\varepsilon\to 0+} \left[sin^{-1}\frac{x}{2}\right]_0^{2-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \left[sin^{-1}\frac{2-\varepsilon}{2} - sin^{-1}0\right] = sin^{-1}1 - 0 = \frac{\pi}{2} \text{ which is finite.}$$

$\Rightarrow \int_0^2 \frac{dx}{\sqrt{4-x^2}}$ converges to $\frac{\pi}{2}$

(iii) Please try yourself. [Ans. Diverges to ∞]

(iv) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$

$$\therefore \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}} dx = \lim_{\varepsilon\to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} -(1-\sin x)^{-\frac{1}{2}}(-\cos x)\, dx =$$

$$\lim_{\varepsilon\to 0+} \left[-2\sqrt{1-\sin x}\right]_0^{\frac{\pi}{2}-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} 2\left[\sqrt{1-\sin\left(\frac{\pi}{2}-\varepsilon\right)} - 1\right] = -2\left[\sqrt{1-sin\frac{\pi}{2}} - 1\right] = 2$$

$\Rightarrow \int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}} dx$ converges to 2.

(v) 1 is the only point of infinite discontinuity of the integrand on [0, 1]

$$\therefore \int_0^1 \frac{dx}{x^2-3x+2} = \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \frac{dx}{(1-x)(2-x)}$$

$$= \lim_{\varepsilon\to 0+} \int_0^{1-\varepsilon} \left(\frac{1}{(1-x)} - \frac{1}{(2-x)}\right) dx$$

$$= \lim_{\varepsilon\to 0+} [-\log(1-x) + \log(2-x)]_0^{1-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} \left[\log\frac{2-x}{1-x}\right]_0^{1-\varepsilon} = \lim_{\varepsilon\to 0+} \left[\log\frac{1+\varepsilon}{\varepsilon} - \log 2\right]$$

$$= \lim_{\varepsilon\to 0+} \log\left(1+\frac{1}{\varepsilon}\right) - \log 2 = \log\infty - \log 2 = \infty$$

$\Rightarrow \int_0^1 \frac{dx}{x^2-3x+2}$ diverges to ∞

(vi) $\frac{\pi}{2}$ is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{2}\right]$

$\therefore \int_0^{\pi/2} \tan\theta\, d\theta = \lim_{\varepsilon\to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} tan\theta d\theta$

$$= \lim_{\varepsilon\to 0+} [\log\sec\theta]_0^{\frac{\pi}{2}-\varepsilon} = \lim_{\varepsilon\to 0+} \left[\log\sec\left(\frac{\pi}{2}-\varepsilon\right) - \log 1\right]$$

$$= \lim_{\varepsilon\to 0+} \log cosec\ \varepsilon = \log cosec\ 0 = \log\infty = \infty$$

$\Rightarrow \int_0^{\pi/2} \tan\theta\, d\theta$ diverges to ∞

Example 12. Examine the convergence of the integrals:

(i) $\int_{-1}^{1} \frac{dx}{x^2}$ (ii) $\int_a^{3a} \frac{dx}{(x-2a)^2}$ (iii) $\int_0^{2a} \frac{dx}{(x-a)^2}$

Solution. (i) The integrand becomes infinite at x = 0 and -1 < 0 < 1

$\therefore \int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2} = \lim_{\varepsilon_1 \to 0+} \int_{1}^{0-\varepsilon_1} \frac{dx}{x^2} + \lim_{\varepsilon_2 \to 0+} \int_{0+\varepsilon_2}^{0} \frac{dx}{x^2}$

So that 0 enclosed within $(-\varepsilon_1,\ \varepsilon_2)$ is excluded

$$= \lim_{\varepsilon_1 \to 0+} \left[-\frac{1}{x}\right]_{-1}^{-\varepsilon_1} + \lim_{\varepsilon_2 \to 0+} \left[-\frac{1}{x}\right]_{\varepsilon_2}^{1}$$

$$= \lim_{\varepsilon_1 \to 0+} \left(\frac{1}{\varepsilon_1} - 1\right) + \lim_{\varepsilon_2 \to 0+} \left(-1 + \frac{1}{\varepsilon_2}\right) = (\infty - 1) + (-1 + \infty) = \infty$$

$\Rightarrow \int_{-1}^{1} \frac{dx}{x^2}$ diverges to + ∞

(ii) The integrand becomes infinite at x = 2a and a < 2a < 3a

$\therefore \int_{a}^{3a} \frac{dx}{(x-2a)^2} = \int_{a}^{2a} \frac{dx}{(x-2a)^2} + \int_{2a}^{3a} \frac{dx}{(x-2a)^2}$

$$= \lim_{\varepsilon_1 \to 0+} \int_{a}^{2a-\varepsilon_1} \frac{dx}{(x-2a)^2} + \lim_{\varepsilon_2 \to 0+} \int_{2a+\varepsilon_2}^{3a} \frac{dx}{(x-2a)^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\frac{-1}{x-2a}\right]_{a}^{2a-\varepsilon_1} + \lim_{\varepsilon_2 \to 0+} \left[\frac{-1}{x-2a}\right]_{2a+\varepsilon_2}^{3a}$$

$$= \lim_{\varepsilon_1 \to 0+} \left(\frac{1}{\varepsilon_1} - \frac{1}{a}\right) + \lim_{\varepsilon_2 \to 0+} \left(-\frac{1}{a} + \frac{1}{\varepsilon_2}\right) = \left(\infty - \frac{1}{2}\right) + \left(-\frac{1}{a} + \infty\right) = \infty$$

$\Rightarrow \int_{a}^{3a} \frac{dx}{(x-2a)^2}$ diverges to ∞

(iii) Please try yourself. [Ans. Diverges to ∞]

Example 13. Examine the convergence of the integrals:

(i) $\int_0^4 \frac{dx}{x(4-x)}$ (ii) $\int_0^2 \frac{dx}{2x-x^2}$ (iii) $\int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} dx$

(iv) $\int_0^{\pi} \frac{dx}{\sin x}$ (v) $\int_0^{\pi} \frac{dx}{1+\cos x}$

Solution. (i) Both the end points 0 and 4 are points of infinite discontinuity of the integrand on [0, 4]

$$\therefore \int_0^4 \frac{dx}{x(4-x)} = \int_0^1 \frac{dx}{x(4-x)} + \int_1^4 \frac{dx}{x(4-x)}$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{1} \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right) dx + \lim_{\varepsilon_2 \to 0+} \int_{1}^{4-\varepsilon_2} \frac{1}{4}\left(\frac{1}{x} + \frac{1}{4-x}\right) dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\frac{1}{4}\log\frac{x}{4-x}\right]_{\varepsilon_1}^{1} + \lim_{\varepsilon_2 \to 0+} \left[\frac{1}{4}\log\frac{x}{4-x}\right]_{1}^{4-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{4}\left(\log\frac{1}{3} - \log\frac{\varepsilon_1}{4-\varepsilon_1}\right) + \lim_{\varepsilon_2 \to 0+} \frac{1}{4}\left(\log\frac{4-\varepsilon_2}{\varepsilon_2} - \log\frac{1}{3}\right)$$

$$= \frac{1}{4}\left[\log\frac{1}{3} - (-\infty)\right] + \frac{1}{4}\left[\infty - \log\frac{1}{3}\right] = \infty$$

$\Rightarrow \int_0^4 \frac{dx}{x(4-x)}$ diverges to ∞

(ii) Please try yourself. [Ans. Diverges to ∞]

(iii) Both the end point -a and a are point of infinite discontinuity of the integrand on [-a, a]

$$\therefore \int_{-a}^{a} \frac{x}{\sqrt{a^2-x^2}} dx = \int_{-a}^{0} \frac{x}{\sqrt{a^2-x^2}} dx + \int_{0}^{a} \frac{x}{\sqrt{a^2-x^2}} dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{-a+\varepsilon_1}^{0} -\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x)\, dx$$

$$+ \lim_{\varepsilon_2 \to 0+} \int_{0}^{a+\varepsilon_2} -\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}}(-2x) dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[-\sqrt{a^2 - x^2}\right]_{-a+\varepsilon_1}^{0} + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{a^2 - x^2}\right]_{0}^{a+\varepsilon_2}$$

$= \lim_{\varepsilon_1 \to 0+} \left[-a + \sqrt{\varepsilon_1(2a - \varepsilon_1)}\right] + \lim_{\varepsilon_2 \to 0+} \left[-\sqrt{\varepsilon_2(2a - \varepsilon_2)} + a\right] = -a + a = 0 \therefore \int_{-a}^{a} \frac{x}{\sqrt{a^2 - x^2}} dx$ converges to 0

(iv) Both the end point 0 and π are point of infinite discontinuity of the integrand on [0, π]

$$\therefore \int_0^{\pi} \frac{dx}{\sin x} = \int_0^{\pi/2} cosec\ x\ dx + \int_{\pi/2}^{\pi} cosec\ x\ dx$$

$$= \lim_{\varepsilon_1 \to 0+} \int_{0+\varepsilon_1}^{\pi/2} cosec\ x\ dx + \lim_{\varepsilon_2 \to 0+} \int_{\pi/2}^{\pi-\varepsilon_2} cosecx\ dx$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log \tan \frac{x}{2}\right]_{\varepsilon_1}^{\pi/2} + \lim_{\varepsilon_2 \to 0+} \left[\log \tan \frac{x}{2}\right]_{\pi/2}^{\pi-\varepsilon_2}$$

$$= \lim_{\varepsilon_1 \to 0+} \left[\log \tan \frac{\pi}{4} - \log \tan \frac{\varepsilon_1}{2}\right] + \lim_{\varepsilon_2 \to 0+} \left[\log \tan \left(\frac{\pi}{2} - \frac{\varepsilon_2}{2}\right) - \log \tan \frac{\pi}{4}\right]$$

$= 0 - (\infty) + \infty - 0 = \infty. \Rightarrow \int_0^{\pi} \frac{dx}{\sin x}$ diverges to ∞

(v) π is the only point of infinite discontinuity of the integrand on [0, π]

$$\therefore \int_0^{\pi} \frac{dx}{1+\cos x} = \lim_{\varepsilon\to 0} \int_0^{\pi-\varepsilon} \frac{dx}{2cos^2 x/2} = \lim_{\varepsilon\to 0+} \int_0^{\pi-\varepsilon} \frac{1}{2} sec^2 \frac{x}{2}\, dx$$

$$= \lim_{\varepsilon\to 0+} \left[\tan\frac{x}{2}\right]_0^{\pi-\varepsilon} = \lim_{\varepsilon\to 0+} \tan\left(\frac{\pi}{2}-\frac{\varepsilon}{2}\right) = \infty$$

$\Rightarrow \int_0^{\pi} \frac{dx}{1+\cos x}$ diverges to ∞

Terminal Questions

Examine for convergence the improper integrals:

(i) $\int_0^{\infty} e^{2x}\, dx$

(ii) $\int_0^{\infty} \frac{dx}{(1+x)^{2/3}}$

(iii) $\int_1^{\infty} \frac{x}{(1+x)^3}\, dx$

(iv) $\int_{-\infty}^{0} \cosh x\, dx$

(v) $\int_{-\infty}^{0} \frac{dx}{1+x^2}$

Bachelor Of Science

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Advance Analysis

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Block

3 Convergence test, Riemann integral

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Block -3

Convergence test, Riemann integral

In this unit we study the integrands which admit primitives in terms of elementary functions. In such cases it is easy to test the convergence of integrals. But every function does not possess a primitive in terms of elementary function. Improper integrals of such functions cannot be examined for convergence by the procedure discussed so far. Thus in such situation we need more advanced methods for testing the convergence of such integrals, which has been discussed here.

In the second Unit we discuss about a step function which is defined as a piecewise constant function, that has only a finite number of pieces. In other words, a function on the real numbers can be described as a finite linear combination of indicator functions of given intervals. It is also called a floor function or greatest integer function. The step function is a discontinuous function.

In the third Unit we discuss about mean value theorem of real numbers as well as for Riemann integral, we gave application of this theorem. We also discuss about Intermediate value theorem, fundamental theorem of integral calculus and its several applications, we discuss about Substitution method for integration and

Second mean value theorem and its applications.

Unit-7: Convergence Test

Structure

7.1 Introduction:

In this unit we study the integrands which admit primitives in terms of elementary functions. In such cases it is easy to test the convergence of integrals. But every function does not prosses a primitive in terms of elementary function. Improper integrals of such functions cannot be examined for convergence by the procedure discussed so far. Thus in such situation we need more advanced methods for testing the convergence of such integrals, which has been discussed here.

7.2 Objectives

After studying this unit, we should be able to:

▪ Check the test of convergence of a series.

▪ Find Comparison of test.

▪Check the μ −Test

▪ Check the Absolutely convergence of the series

▪Check the Abel's Test

▪Check the Dirichlet's Test

7.3Tests for convergence of $\int_a^b f(x)dx$ at $x = a$

Let a be the only point of infinite discontinuity of f on [a, b], the case when b is the only point of infinite discontinuity can be dealt with in the same way.

Without any loss of generality, we assume that f is positive (or non-negative) on [a, b]

In case f is negative, we can replace it by (-f) for testing the convergence of $\int_a^b f(x)dx$

Theorem 1: A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)dx$ at a, where f is positive on (a, b], is that there exists a positive number M, independent of $\varepsilon > 0$ $\int_{a+\varepsilon}^b f(x)dx <$ $M \forall \varepsilon \; in \; (0, b-a)$

Proof: Since a is the only point of infinite discontinuity of f on [a, b]. Therefore, f is continuous on (a, b]. Also f is positive on (a, b]

$\Rightarrow$ For $0 < a < \varepsilon < b \; i.e., for \; 0 < \varepsilon < b - a$, f is positive and continuous on $[a + \varepsilon, b]$.

$\Rightarrow \int_{a+\varepsilon}^b f(x)dx = A(\varepsilon)$ represents the area bounded by f on $[a + \varepsilon, b]$ and the x-axis.

$\Rightarrow$ As $\varepsilon \to 0+$ i.e., as ε decreases, A(ε) increases since the length of the interval increases.

$\Rightarrow \lim_{\varepsilon \to 0+} A(\varepsilon) = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b f(x)dx$ will exist finitely if and only if A(ε) is bounded above.

$\Rightarrow\int_a^b f(x)dx$ will converge iff ∃ a real number M > 0 and independent of ε such that

A(ε) < M.

$\Rightarrow\int_a^b f(x)dx$ converges if and only if $\int_{a+\varepsilon}^b f(x)dx < M \ \ \forall\forall\varepsilon \ in \ (0, b-a)$

Note: If for every M > 0 and some ε in (0, b – a). A(ε) > M, then $\int_{a+\varepsilon}^b f(x)dx$ is not bounded above.

$\therefore\int_{a+\varepsilon}^b f(x)dx$ tends to $+\varepsilon \ as\varepsilon \to 0+$ and, hence, the improper integral $\int_a^b f(x)dx$ diverges to $+\varepsilon.$

7.4 Comparison Test

Theorem 2. If f and g are two positive functions with f(x) ≤ g(x) for all x in (a, b] and a is the only point of infinite discontinuity on [a, b], then

(i) $\int_a^b g(x)dx$ is convergent $\Rightarrow\int_a^b f(x)dx$ is convergent

(ii) $\int_a^b f(x)dx$ is divergent $\Rightarrow\int_a^b g(x)dx$ is divergent.

Proof. Since f and g are positive and $f(x) \leq g(x) \forall x \in (a, b]$

$\therefore\int_{a+\varepsilon}^b f(x)dx \leq \int_{a+\varepsilon}^b g(x)dx \ \ for \ 0 < \varepsilon < b-a$

(i) Let $\int_a^b g(x)dx$ be convergent, then there exists a positive number M such that

$$\int_{a+\varepsilon}^{b} g(x)dx < M \; for \; 0 < \varepsilon < b-a$$

$\therefore$ From (1), $\int_{a+\varepsilon}^{b} f(x)dx < M \; for \; 0 < \varepsilon < b-a$

Hence $\int_{a}^{b} f(x)dx$ is convergent.

(ii) Let $\int_{a}^{b} f(x)dx$ be divergent, then for every M > 0, there exists ε in (0, b – a) such that $\int_{a+\varepsilon}^{b} f(x)dx > M$

$\therefore$ From (i), $\int_{a+\varepsilon}^{b} g(x)dx > M$. Hence $\int_{a}^{b} g(x)dx$ is divergent.

Theorem 3: Comparison Test II (Limit Form)

If f and g be two positive functions on (a, b], a being the only point of infinite discontinuity, and $\lim_{x\to 0+} \frac{f(x)}{g(x)} = 1$ where l is non-zero finite number, then two integrals $\int_{a}^{b} f(x)dx$ and $\int_{a}^{b} g(x)dx$ converge or diverge together.

Proof: Since f and g are positive on (a, b]

$\therefore \frac{f(x)}{g(x)} > 0$ on (a, b] $\Rightarrow \lim_{x\to 0+} \frac{f(x)}{g(x)} = l \geq 0$

But $\quad l \neq 0$ (given)

$\therefore l > 0$

Let ε be a positive real number such that $l - \varepsilon > 0$.

Since, $\lim_{x\to a+}\frac{f(x)}{g(x)}=l$, therefore, there exists a neighbourhood (a, c), a < c < b, such that

$$\left|\frac{f(x)}{g(x)}-l\right|<\varepsilon\forall x\in(a,c)$$

$$\Rightarrow l-\varepsilon<\frac{f(x)}{g(x)}<l+\varepsilon\forall x\in(a,c)$$

$\Rightarrow(l-\varepsilon)g(x)<f(x)<(l+\varepsilon)g(x)$ [∵ g(x) > 0]

$\Rightarrow kg(x)<f(x)<Kg(x)$ where k, K > 0 ……………….(1)

Now, $\int_a^b f(x)dx$ converges at a $\Rightarrow\int_a^b f(x)dx$ converges at a

$\because\int_c^b f(x)dx$ is proper integral

Since, $kg(x)<f(x)\forall x\in(a,c)$ [Form (1)]

$\therefore k\int_a^b g(x)dx$ converges at a [by comparison test I]

$\Rightarrow\int_a^c g(x)dx$ converges at a

$\Rightarrow\int_a^b g(x)dx$ converges at a $\because\int_c^b g(x)dx$ is proper integral

(i) $\int_a^b f(x)dx$ diverges at a $\Rightarrow\int_a^c f(x)dx$ diverges at a

$\because\int_c^b f(x)dx$ is proper

Since $Kg(x)>f(x)\forall x\in(a,c)$ [Form I]

$\therefore K\int_a^c g(x)dx$ diverges at a - $\Rightarrow\int_a^c g(x)dx$ diverges at a

$\Rightarrow\int_a^b g(x)dx$ diverges at a $\because\int_c^b g(x)dx$ is proper intgral

It can similarly be shown that

$\int_a^b g(x)dx$ converges at a $\Rightarrow\int_a^b f(x)dx$ diverges at a.

And $\int_a^b g(x)dx$ diverges at a $\Rightarrow\int_a^b f(x)dx$ diverges at a

Theorem 4: Let f and g be two positive functions on (a, b], a being the only point of infinite discontinuity. Then

(i) $\lim_{x\to a+}\frac{f(x)}{g(x)}=0$ and $\int_a^b g(x)dx$ converges $\Rightarrow\int_a^b f(x)dx$ converges

(ii) $\lim_{x\to a+}\frac{f(x)}{g(x)}=\infty$ and $\int_a^b g(x)dx$ diverges $\Rightarrow\int_a^b f(x)dx$ diverges.

Proof: $\lim_{x\to a+}\frac{f(x)}{g(x)}=0$

$\Rightarrow$ Given any $\varepsilon>0, \exists a\ nbd\ (a,c), a<c<b$ such that

$\left|\frac{f(x)}{g(x)}\right|<\varepsilon\ \forall x\in(a,c)\Rightarrow-\varepsilon<\frac{f(x)}{g(x)}<\varepsilon\ \forall x\in(a,c)$

$\Rightarrow 0<\left|\frac{f(x)}{g(x)}\right|<\varepsilon\ \forall x\in(a,c)\Rightarrow f(x)<\varepsilon g(x)\forall x\in(a,c)$

Now $\int_a^b g(x)dx$ converges at a $\Rightarrow\int_a^c g(x)dx$ converges at a

$\Rightarrow\int_a^c \varepsilon g(x)dx$ converges at a

Since $f(x)<\varepsilon g(x)\forall x\in(a,c)$

$\therefore \int_a^c f(x)dx$ converges at a $\Rightarrow \int_a^b f(x)$ converges at a

(ii) $\lim_{x\to a+} \frac{f(x)}{g(x)} = \infty$

$\Rightarrow$ Given a real number $K > 0 \ \exists \ a \ nbd \ (a, c), a < c < b$ such that

$\frac{f(x)}{g(x)} > K \ \forall x \in (a, c) \Rightarrow f(x) < Kg(x) \forall x \in (a, c)$

Now $\int_a^b g(x)dx$ diverges at a $\Rightarrow \int_a^c g(x)dx$ diverges at a

$\Rightarrow \int_a^c Kg(x)dx$ diverges at a

Since, $f(x) < Kg(x) \forall x \in (a, c) \therefore \int_a^c f(x)dx$ diverges at a

$\Rightarrow \int_a^b f(x)dx$ diverges at a

Theorem 5: Useful Comparison Integrals

(i) The improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if n < 1.

(ii) The improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if n < 1

Proof: (i) If $n \le 0$, the integral $\int_a^b \frac{dx}{(x-a)^n}$ is proper

If n > 0, the integral is improper and a is the only point of infinite discontinuity of the integrand on [a, b]

Case I. When n = 1

$$\int_a^b \frac{dx}{(x-a)^n} = \int_a^b \frac{dx}{x-a} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a}$$

$$= \lim_{\varepsilon \to 0+} [\log(x-a)]_{a+\varepsilon}^b = \lim_{\varepsilon \to 0+} [\log(b-a) = \log \varepsilon]$$

$$= \log(b-a) - (-\infty) = \infty$$

$\Rightarrow \int_a^b \frac{dx}{(x-a)^n}$ diverges if n = 1

Case II. When n ≠ 1

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^b (x-a)^{-n} dx = \lim_{\varepsilon \to 0+} \left[\frac{(x-a)^{1-n}}{1-n}\right]_{a+\varepsilon}^b$$

$$= \lim_{\varepsilon \to 0+} \frac{1}{1-n} [(b-a)^{1-n} - \varepsilon^{1-n}]$$

Sub-Case I. When n > 1 so that n – 1 > 0

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{n-1} \left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right] = \frac{1}{n-1} \left[\infty - \frac{1}{(b-a)^{n-1}}\right]$$
$$= \infty$$

$\Rightarrow \int_a^b \frac{dx}{(x-a)^n}$ diverges if n > 1

Sub-Case 2. When 0 < n < 1 so that 1 – n > 0

$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n} [(b-a)^{1-n} - \varepsilon^{1-n}] = \frac{(b-a)^{1-n}}{1-n}$ which is finite

$\Rightarrow \int_a^b \frac{dx}{(x-a)^n}$ converges if n < 1

Hence, $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if and only if n < 1

(ii) If n ≤ 0, the integral $\int_a^b \frac{dx}{(b-x)^n}$

If $n > 0$ the integral is improper and b is the only point of infinite discontinuity of the integrand on [a, b].

Case I. When n = 1

$$\int_a^b \frac{dx}{(b-x)^n} = \int_a^b \frac{dx}{b-x} = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} \frac{dx}{b-x} = \lim_{\varepsilon\to 0+} \left[\frac{\log(b-x)}{-1}\right]_a^{b-\varepsilon}$$

$$= \lim_{\varepsilon\to 0+} [-\log\varepsilon + \log(b-a)] = -(-\infty) + \log(b-a) = \infty$$

$\Rightarrow \int_a^b \frac{dx}{(b-x)^n}$ diverges if n = 1

Case II. When n ≠ 1

$$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} (b-x)^{-n} dx$$

$$= \lim_{\varepsilon\to 0+} \left[\frac{(b-x)^{1-n}}{(1-n)(-1)}\right]_a^{b-\varepsilon} = \lim_{\varepsilon\to 0+} \frac{1}{n-1} [\varepsilon^{1-n} - (b-a)^{1-n}]$$

Sub-Case 1. When n > 1 so that n – 1 > 0

$$\Rightarrow \int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon\to 0+} \frac{1}{n-1}\left[\frac{1}{\varepsilon^{n-1}} - \frac{1}{(b-a)^{n-1}}\right] = \frac{1}{n-1}\left[\infty - \frac{1}{(b-a)^{n-1}}\right] = \infty$$

$\Rightarrow \int_a^b \frac{dx}{(b-x)^n}$ diverges if n > 1

Sub-Case 2. When 0 < n < 1 so that 1 – n > 0

$\int_a^b \frac{dx}{(b-x)^n} = \lim_{\varepsilon \to 0+} \frac{1}{1-n}[(b-a)^{1-n} - \varepsilon^{1-n}] = \frac{(b-a)^{1-n}}{1-n}$ which is finite.

$\Rightarrow \int_a^b \frac{dx}{(b-x)^n}$ converges if n > 1

Hence $\int_a^b \frac{dx}{(b-x)^n}$ is convergent if and only if n < 1

Theorem 6: (i) if a is the only point of infinite discontinuity of f on [a, b] and $\lim_{x \to a+}(x-a)^\mu f(x)$ exists and is non-zero finite, then $\int_a^0 f(x)dx$ converges if and only if $\mu < 1$.

(ii) If b is the only point of infinite discontinuity of f on [a, b] and $\lim_{x \to b+}(b-x)^\mu f(x)$ exists and is non-zero finite, then $\int_a^b f(x)dx$ converges if and only if $\mu < 1$.

Proof: (i) Let $g(x) = \frac{1}{(x-a)^\mu}$ then $\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+}(x-a)^\mu f(x)$

Which exists and is non-zero finite. (given)

∴ By comparison test II, the two integrals $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together.

But $\int_a^b g(x)dx = \int_a^b \frac{dx}{(x-a)^\mu}$ converges iff $\mu < 1$.

∴$\int_a^b f(x)dx$ converges iff $\mu < 1$

Example 1. Examine the convergence of the integrals:

(i) $\int_0^1 \frac{dx}{\sqrt{x^2+x}}$ (ii) $\int_1^2 \frac{dx}{(1+x)\sqrt{1-x}}$ (iii) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ (iv) $\int_0^1 \frac{dx}{x^{1/2}(1+x^2)}$

Solution: (i) Here $f(x) = \frac{1}{\sqrt{x^2+x}} = \frac{1}{\sqrt{x}\sqrt{x+1}}$

0 is the only point of infinite discontinuity of f on [0, 1]

Take $g(x) = \frac{1}{\sqrt{x}}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{\sqrt{x^2+x}} = 1$ which is non-zero and finite.

By comparison test $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge or diverge together.

But $\int_0^1 g(x)dx = \int_0^1 \frac{dx}{\sqrt{x}}$ converges. Form $\int_a^b \frac{dx}{(x-a)^n}$ which a = 0.

$\left(since, n = \frac{1}{2} < 2\right)$

$\therefore \int_0^1 f(x)dx = \int_0^1 \frac{1}{\sqrt{x^2+x}} dx$ is convergent.

(ii) Here $f(x) = \frac{1}{(1+x)\sqrt{x^2+x}}$

2 is the only point of infinite discontinuity of f on [1, 2]

Take $g(x) = \frac{1}{\sqrt{2-x}}$, then $\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \frac{1}{1+x} = \frac{1}{3}$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ converge or diverge together.

But, $\int_1^2 g(x)dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$. Form $\int_a^b \frac{dx}{(b-x)^n}$ with b = 2 Converges $\left(n = \frac{1}{2} < 1\right)$

$\therefore \int_1^2 f(x)dx = \int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ is convergent.

(iii) Here $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{\sqrt{1-x}\sqrt{1+x+x^2}}$

1 is the only point of infinite discontinuity of f on [0, 1]

Take $g(x) = \frac{1}{\sqrt{1-x}}$, then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}}$ which is non-zero and finite.

$\therefore$ By comparison test $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge or diverge together

But $\int_0^1 g(x)dx = \int_0^1 \frac{dx}{\sqrt{1-x}}$ Form $\int_a^b \frac{dx}{(b-x)^n}$ with b = 1 Converges $\left(n = \frac{1}{2} < 1\right)$

$\int_0^1 f(x)dx = \int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent

Example 2. Examine the convergence of the integrals:

(i) $\int_0^1 \frac{dx}{x^3(2+x^2)^5}$ (ii) $\int_0^1 \frac{dx}{\sqrt{x(1+x)^2}}$

(iii) $\int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$ (iv) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$

Solution. (i) Here $f(x) = \frac{1}{x^3(2+x^2)^5}$

0 is the only point of infinite discontinuity of f on [0, 1]

Take $g(x) = \frac{1}{x^3}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{(2+x^2)^5} = \frac{1}{32}$ which is non-zero and finite

∴ By comparison test $\int_0^1 f(x)dx$and $\int_0^1 g(x)dx$ converge or diverge together.

But $\int_0^1 g(x)dx = \int_0^1 \frac{dx}{x^3}$ Form $\int_a^b \frac{dx}{(x-a)^n}$ with a = 0 diverges (∵ n = 3 > 1)

$\therefore \int_0^1 f(x)dx = \int_0^1 \frac{dx}{x^3(2+x^2)^5}$ is divergent

(ii) Please try yourself. [Ans. Convergent]

(iii) Here $f(x) = \frac{dx}{(1+x)^2(1-x)^3}$

1 is the only point of infinite discontinuity of f on [0, 1]

Take $g(x) = \frac{1}{(1-x)^3}$, then $\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{(1+x)^2} = \frac{1}{2}$ which is non-zero and finite.

By comparison test, $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge or diverge together.

But $\int_0^1 g(x)dx = \int_0^1 \frac{1}{(1-x)^3} dx$ Form$\int_a^b \frac{dx}{(x-a)^n}$ with b = 1 diverges ($\because$ n = 3 > 1)

$\therefore \int_0^1 f(x)dx = \int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$ is divergent.

(iv) Here $f(x) = \frac{1}{\sqrt{x(1-x)}}$

Both the end points 0 and 1 are the points of infinite discontinuity of f on [0, 1]

We may write $\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^1 \frac{dx}{\sqrt{x(1-x)}} + \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$ ………………(1)

Where 0 < a < 1

To examine the convergence at x = 0

Let $I_1 = \int_0^a \frac{dx}{\sqrt{x(1-x)}}$

0 is the only point of infinite discontinuity of f on [0, a]

Take $g(x) = \frac{1}{\sqrt{x}}$, then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{\sqrt{(1-x)}} = 1$ which is non-zero and finite.

 By comparison test I_1 and $\int_0^a g(x)dx$ converge or diverge together

But, $\int_0^a g(x)dx = \int_0^a \frac{dx}{\sqrt{x}}$ is convergent $\left(n = \frac{1}{2} < 1\right)$

 I_1 is convergent.

To examine the convergence at x = 1

Let $I_2 = \int_a^1 \frac{dx}{\sqrt{x(1-x)}}$

1 is the only point infinite discontinuity of f on [a, 1].

Take $g(x) = \frac{1}{\sqrt{1-x}}$ then $\lim_{x\to 1} \frac{f(x)}{g(x)} = \lim_{x\to 1} \frac{1}{\sqrt{x}} = 1$ which is non-zero and finite.

□ By comparison text, I_2 and $\int_a^1 g(x)dx$ converge or diverge together.

But $\int_a^1 g(x)dx = \int_a^1 \frac{dx}{\sqrt{1-x}}$ is convergent $\left(\square\ n = \frac{1}{2} < 1\right)$

□ I_2 is convergent.

Since I_1 or I_2 are both convergent, therefore, from (1) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ is convergent.

Note. If I_1 or I_2 is divergent, then $\int_0^1 f(x)dx$ is divergent

Example 3. Examine the convergence of the integrals:

(i) $\int_2^3 \frac{dx}{(x-2)^{1/4}(3-x)^2}$

Solution. (i) Here $f(x) = \frac{dx}{(x-2)^{1/4}(3-x)^2}$

2 and 3 are the only points of infinite discontinuity of f on [2, 3], we may write

$\int_2^3 f(x)dx = \int_2^a f(x)dx + \int_a^3 f(x)dx$, where 2 < a < 3 ….. (1)

To test the convergence of $\int_2^a f(x)dx \ \ at\ x = 2$

Take $g(x) = \frac{1}{(x-2)^{1/4}}$

$\lim_{x\to 2+} \frac{f(x)}{g(x)} = \lim_{x\to 2+} \frac{1}{(3-x)^2} = 1$ which is non zero and finite.

□ By comparison test, the integrals $\int_2^a f(x)dx \ \ and \ \ \int_2^a g(x)dx$ converge or diverge together.

But $\int_2^a g(x)dx = \int_2^a \frac{dx}{(x-2)^{1/4}}$ form $\int_a^b \frac{dx}{(x-a)^n}$ is convergent $\left(□\ n = \frac{1}{4} < 1\right)$

□ $\int_2^a f(x)dx$ is convergent.

To test the convergence of $\int_a^3 f(x)dxat\ x = 3$

Take $g(x) = \frac{1}{(3-x)^2}$

$\lim_{x\to 3-} \frac{f(x)}{g(x)} = \lim_{x\to 3-} \frac{1}{(x-2)^{1/4}} = 1$ which is non zero and finite.

□ By comparison test, the integrals $\int_a^3 f(x)dx \ \ and \ \ \int_a^3 g(x)dx$ converge or diverge together.

But $\int_a^3 g(x)dx = \int_a^3 \frac{dx}{(3-x)^2}$. Form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent

□ $\int_0^3 f(x)dx$ is divergent (□ n = 2 > 1)

Hence from (1) $\int_2^3 f(x)dx$ is divergent

Example 4. Examine the convergence of

(i) $\int_0^1 \frac{x^n}{1-x} dx$ (ii) $\int_0^1 \frac{x^n}{1+x} dx$ (iii) $\int_1^2 \frac{x^\lambda}{x-1} dx$ (iv) $\int_2^3 \frac{x^2+1}{x^2-4} dx$

Solution. (i) Here $f(x) = \frac{x^n}{1-x}$

If $n \geq 0$, then 1 is the only point of infinite discontinuity of f on [0, 1].

Take $g(x) = \frac{1}{1-x}$

$\lim_{x \to 1-} \frac{f(x)}{g(x)} = \lim_{x \to 1-} x^n = 1$ which is non-zero and finite.

□ By comparison test, the integrals $\int_0^1 f(x)dx \;\; and \;\; \int_0^1 g(x)dx$ converge of diverge together.

But $\int_0^1 g(x)dx = \int_0^1 \frac{dx}{1-x}$. Form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent (□ n = 1)

□ $\int_0^1 f(x)dx$ is divergent

If n < 0, let n = -m where m > 0

Then $f(x) = \frac{1}{x^m(1-x)}$

0 and 1 both are the points of infinite discontinuity of f on [0, 1]. We may write

$\int_0^1 f(x)dx = \int_0^a f(x)dx + \int_a^1 f(x)dx$ where 0 < a < 1 …… (1)

To test the convergence of $\int_0^a f(x)dx$ at x = 0

Take, $g(x) = \frac{1}{x^m}$

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{1-x} = 1$ which is finite and non-zero.

□ By comparison test, the integrals $\int_0^a f(x)dx$ and $\int_0^a g(x)dx$ converge or diverge together

But $\int_0^a g(x)dx = \int_0^a \frac{dx}{x^m}$, form $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if 0 < m < 1 and divergent if m ≥ 1.

□ $\int_0^a f(x)dx$ is convergent if -1 < n < 0 and divergent if n ≤ -1

Take $g(x) = \frac{1}{1-x}$

$\lim_{x\to 1-} \frac{f(x)}{g(x)} = \lim_{x\to 1-} \frac{1}{x^m} = 1$ which is finite and non zero.

□ By comparison test, the integrals $\int_a^1 f(x)dx$ and $\int_a^b g(x)dx$ converge or diverge together

But $\int_a^1 g(x)d\,x = \int_a^1 \frac{dx}{1-x}$, form $\int_a^b \frac{dx}{(b-x)^n}$ is divergent (□ n = 1)

□ $\int_a^1 f(x)dx$ is divergent.

From (1), $\int_0^1 f(x)dx$ is divergent

Hence $\int_0^1 f(x)dx$ is divergent for all $n \in R$

Note: After a little practice, there is no need testing the convergence of $\int_0^a f(x)dx$ at x = 0, since divergence of $\int_a^1 f(x)dx$ is sufficient to imply divergence of $\int_0^1 f(x)dx$.

(ii) Here $f(x) = \frac{x^n}{1+x}$

If n ≥ 0, $\int_0^1 f(x)dx$ is proper and, hence convergent

If n < 0, let n = -m where m > 0

Then, $f(x) = \frac{1}{x^m(1+x)}$

0 is the only point of infinite discontinuity of on [0, 1]

Take $g(x) = \frac{1}{x^m}$

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{1}{1+x} = 1$ which is non-zero and finite.

 By comparison test, the integrals $\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converge or diverge together.

But $\int_0^1 g(x)dx = \int_0^1 \frac{dx}{x^m}$ Form $\int_a^b \frac{dx}{(x-a)^n}$ is convergent if 0 < m < 1 i.e.

-1 < n < 0 and divergent if m ≥ 1 i.e., n ≤ -1

□ $\int_0^1 f(x)dx$ is convergent if -1 < n < 0 and divergent if n ≤ -1.

Hence $\int_0^1 f(x)dx$ is convergent if n > -1 and divergent if n ≤ -1

(iii) Hint $f(x) = \frac{x}{x-1}$

For all values of $\lambda \in R$, 1 is the only point of infinite discontinuity of f on [1, 2]

Take $g(x) = \frac{1}{x-1}$ $etc.$ [Ans. Divergent]

(iv) $\int_2^3 \frac{x^2+1}{x^2-4} dx = \int_2^3 \frac{(x^2-4)+5}{x^2-4}$

$= \int_2^3 \left(1 + \frac{5}{x^2-4}\right) dx = [x]_2^3 + 5\int_2^3 \frac{dx}{x^2-4} = 1 + 5\int_3^3 \frac{dx}{x^2-4}$ ………. (1)

Let $f(x) = \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

2 is the only point of infinite discontinuity of f on [2, 3]

Take $g(x) = \frac{1}{x-2}$

$\lim_{x\to 2+} \frac{f(x)}{g(x)} = \lim_{\to 2+} \frac{1}{x+2}$ which is non zero and finite,

□ By comparison test $\int_2^3 f(x)dx$ and $\int_2^3 g(x)dx$ converge or diverge together.

But $\int_2^3 g(x)dx = \int_2^3 \frac{dx}{x-2}$ Form $\int_a^b \frac{dx}{(x-a)^n}$ is divergent (□ n = 1)

□ $\int_2^3 f(x)dx$ is divergent. Hence, from (1) $\int_2^3 \frac{x^2+1}{x^2-4}dx$ is divergent.

Example 5: Examine the convergence of

(i) $\int_0^2 \frac{\log x}{\sqrt{2-x}}dx$ (ii) $\int_0^1 \frac{\log x}{\sqrt{x}}$ (iii) $\int_1^2 \frac{\sqrt{x}}{\log x}$

Solution: (i) Here $f(x) = \frac{\log x}{\sqrt{2-x}}$

Clearly both 0 and 2 are points of infinite discontinuity of f on [0, 2]. We may write

$$\int_0^2 \frac{\log x}{\sqrt{2-x}}dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx \qquad \text{…………..(1)}$$

To test the convergence of $\int_0^1 f(x)dx \; at \; x = 0$

Since f(x) is negative on (0, 1], we consider –f(x).

Take $g(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^n \log x}{\sqrt{2-x}} = 0 \; if \; n > 0 \left[\lim_{x\to 0+} x^n \log x = 0 \; if \; n > 0\right]$$

□ taking n between 0 and 1, the integral $\int_0^1 g(x)dx$ is convergent.

□ By comparison test, $\int_0^1 -f(x)dx$ is also convergent.

To test the convergence of $\int_1^2 f(x)dx \; at \; x = 2$

Take $g(x) = \frac{1}{\sqrt{2-x}}$

$\lim_{x\to 2-}\frac{f(x)}{g(x)} = \lim_{x\to 2-}\log x = \log 2$ which is non zero and finite.

 By comparison test, $\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ convergence or diverge together.

But, $\int_1^2 g(x)dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$ Form $\int_a^b \frac{dx}{(b-x)^n}$ is convergent $\left(n = \frac{1}{2} < 1\right)$

 $\int_1^2 f(x)dx$ is also convergent

Hence, from (1) $\int_0^2 f(x)dx$ is convergent.

(ii) Since $\frac{\log x}{\sqrt{x}}$ is negative on (0, 1], we take $f(x) = -\frac{\log x}{\sqrt{x}}$

Here 0 is the only point of infinite discontinuity of f on [0, 1]

Take $f(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+} -x^{n-\frac{1}{2}}\log x = 0\; if\; n-\frac{1}{2} > 0\; i.e., if\; n > \frac{1}{2}\left[\lim_{x\to 0+} x^n \log x = 0\; if\; n > o\right]$$

Taking n between ½ and 1, the integral $\int_0^1 g(x)dx$ is convergent.

 By comparison test $\int_0^1 f(x)dx$ is also convergent.

Hence $\int_0^1 \frac{\log x}{\sqrt{x}}dx$ is convergent.

(iii) Here $f(x) = \frac{\sqrt{x}}{\log x}$

1 is the only point of infinite discontinuity of f on [1, 2]

Take $g(x) = \frac{1}{(x-1)^n}$

$$\lim_{x\to1+}\frac{f(x)}{g(x)} = \lim_{x\to1+}\frac{(x-1)^n\sqrt{x}}{\log x}$$

$$= \lim_{x\to1+}\frac{n(x-1)^{n-1}\sqrt{x}+\frac{(x-1)^n}{2\sqrt{2}}}{1/x}$$

$$= \lim_{x\to1+}(x-1)^{n-1}\left[nx^{3/2}+\frac{(x-1)}{2}.\sqrt{x}\right] = 1\; if\; n = 1$$

Taking n = 1, $\int_1^2 g(x)dx = \int_1^2 \frac{dx}{x-1}$ Form $\int_a^b \frac{dx}{(x-a)^n}$ is divergent.

(n = 1)

Since $\lim_{x\to1+}\frac{f(x)}{g(x)} = 1$ Which is non zero and finite.

□ By comparison test $\int_1^2 f(x)dx$ is also divergent.

Example 6: Examine the convergence of

(i) $\int_0^1 \frac{\log x}{1+x}dx$

$(ii)\int_0^1 \frac{\log x}{1-x^2}dx$ (iii) $\int_0^1 \frac{\log x}{\sqrt{1-x^2}}dx$

Solution: (i) Since $\frac{\log x}{1+x}$ is negative on (0, 1], we take $f(x) = -\frac{\log x}{1+x}$

Here 0 is the only point of infinite discontinuity of f on [o, 1]

Take $g(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^n \log x}{1+x} = 0 \; if \; n > 0$$

Taking n between 0 and 1, the integral $\int_0^1 g(x)$ is convergent

 By comparison test $\int_0^1 f(x)$ is convergent.

Hence $\int_0^1 \frac{\log x}{1+x} dx$ is convergent

(ii) Since $\frac{\log x}{1-x^2}$ is negative on (0, 1], we take $f(x) = -\frac{\log x}{1-x^2}$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} -\frac{\log x}{1-x^2}$$

$$= limx \to 1, 1/x - 2x = 12$$

 0 is the only point of infinite discontinuity of f on [0, 1]

Take $g(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^n \log x}{1-x^2} = 0 \; if \; n > 0$$

Taking n between 0 and 1, the integral $\int_0^1 g(x)dx$ is convergent

 By comparison test $\int_0^1 f(x)dx$ is convergent.

Hence $\int_0^1 \frac{\log x}{1-x^2} dx$ is convergent.

Example 7: Examine the convergence of

(i) $\int_0^1 \frac{x^n \log x}{(1+x)^2}$ (ii) $\int_0^1 \frac{(x^p+x^{-p})\log(1+x)}{x} dx$ (iii) $\int_0^1 x^{n-1} \log x \, dx$

Solution. (i) $\lim_{x\to 0+} \frac{x^n \log x}{(1+x)^2} = 0 \; if \; n > 0$

□ $\int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is proper and , hence, convergent so long as n > 0

If n = 0, let $f(x) = \frac{\log x}{(1+x)^2}$

0 is the only point of infinite discontinuity

Take $g(x) = \frac{1}{x^p}$

Taking p between 0 and 1, $\int_0^1 g(x)dx$ is convergent.

$\int_0^1 f(x)dx$ is convergent $\int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is convergent

If n < 0, let n = -m where m > 0

Let $f(x) = \frac{x^n \log x}{(1+x)^2} = \frac{\log x}{x^m(1+x)^2}$

Take $g(x) = \frac{1}{x^q}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{x^{q-m} \log x}{(1+x)^2} = 0 \; if \; q - m > 0$$

Taking 0 < q < 1 and also q – m > 0 i.e., q > m

$0 < m < q < 1 \rightarrow m < 1 \rightarrow n > -1$

$\int_0^1 g(x)dx$ is convergent and hence $\int_0^1 f(x)dx$ is convergent.

 $\int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is convergent for all n > -1

Note. N > -1 also converges the cases n = 0 and n > 1

(iii) Let p be positive and $f(x) = \left(x^p + \frac{1}{x^p}\right)\frac{\log(1+x)}{x}$

0 is the only point of infinite discontinuity.

Take $g(x) = \frac{1}{x^p}$

$$\lim_{x\to 0+}\frac{f(x)}{g(x)} = \lim_{x\to 0+}(x^{2p}+1)\frac{\log(1+x)}{x} = 1$$

Since $\lim_{x\to 0+}\frac{\log(1+x)}{x}$

$$= \lim_{x\to 0+}\frac{\frac{1}{1+x}}{1} = 1$$

Since $\int_0^1 g(x)dx$ converges if p < 1

 $\int_0^1 f(x)dx$ is convergent if 0 < p < 1

If p = 0, $f(x) = \frac{2\log(1+x)}{x}$

Since, $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{2\log(1+x)}{x}$

$$= \lim_{x\to 0+} \frac{\frac{2}{1+x}}{1} = 2$$

$\int_0^1 f(x)dx$ is proper and, hence convergent.

If p < 0, let $\qquad g(x) = \frac{1}{x^{-p}}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \left(1 + \frac{1}{x^{-p}}\right) \frac{\log(1+x)}{x}$$

$$= \lim_{x\to 0+} \left(1 + \frac{1}{x^{-p}}\right) \left[\lim_{x\to 0+} \frac{\log(1+x)}{x} = 1\right]$$

= 1 since p < 0 which is non-zero and finite.

Since $\int_0^1 g(x)dx$ is convergent if –p < 1, i.e., if p > -1, therefore, $\int_0^1 f(x)dx$ if p > -1

Hence, $\int_0^1 f(x)dx$ is convergent if -1 < p < 1.

(iv). We know that $\lim_{x\to 0} x^r \log x = 0$ when R > 0

□ The given integral is a proper integral when n – 1 > 0 i.e., when n > 1.

When n = 1, the given integral becomes

$\int_0^1 \log x \; dx = \lim_{\varepsilon\to 0+} \int_\varepsilon^1 \log x \, dx$ [Integrating by parts]

$$= \lim_{\varepsilon\to 0+} [xlog\; x - x]_\varepsilon^1 = \lim_{\varepsilon\to 0+} (0 - 1 - \varepsilon \log \varepsilon + \varepsilon)$$

$= -1 \left[\lim_{\varepsilon \to 0+} \varepsilon \log \varepsilon = 0 \right]$

The given integral is convergent when n = 1

When n < 1 let $f(x) = -x^{n-1} \log x$ $[x^{n-1} \log x \ \ is\ negative\ in\ (0, 1)]$

Taking $g(x) = \frac{1}{x^{\mu}}$, we have $\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -x^{\mu+n+1} \log x =$

$0\ if\ \mu + n - 1 > 0$

$= \infty\ if\ \mu + n - 1 \leq 0$

Taking $0 < \mu < 1$ and also $\mu > 1 - n$ so that $1 - n < \mu < 1\ or 1 - n <$

$1\ or\ n > 0$

□ $\int_0^1 g(x)dx$ is convergent and hence $\int_0^1 f(x)dx$ is convergent.

□ $\int_0^1 x^{n-1} \log x\, dx$ is convergent for all n > 0

Also taking $\mu = 1$ and also $\mu \leq 1 - n$ so that $n \leq 0$

$\int_0^1 g(x)dx$ is divergent and hence $\int_0^1 f(x)dx$ is divergent

□ $\int_0^1 x^{n-1} \log x\, dx$ is divergent for all n ≤ 0

Example 8: Discuss the convergence of

(i) $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$ (ii) $\int_0^{\pi/2} \frac{\cos x}{x^n} dx$ (iii) $\int_0^1 \frac{cosec\ x}{x} dx$ (iv) $\int_0^1 \frac{\sec x}{x} dx$

Solution: (i) If p is negative or zero, the given integral is a proper integral and hence convergent when p ≤ 0. When p > 0, the only point of infinite discontinuity is 0.

Let $f(x) = \frac{\sin x}{x^p}$

Take $g(x) = \frac{1}{x^\mu}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} x^{\mu-p} \sin x = \lim_{x\to 0+} x^{\mu-p+1} \left(\frac{\sin x}{x}\right)$$

$= 1 if\ \mu - p + 1 = 0$

$= 0\ if\ \mu - p + 1 > 0$

$= \infty if\ \mu - p + 1 < 0$

By taking $0 < \mu < 1$ and also $\mu = p - 1$ so that $0 < p - 1 < 1$ i.e., $1 < p < 2$

$\int_0^{\pi/2} g(x)dx$ is convergent and hence $\int_0^{\pi/2} f(x)dx$ is convergent

Bt taking $0 < \mu < 1$ and also $\mu > p - 1$ so that $-1 < p - 1 < \mu < 1$ i.e., $0 < p < 2$

$\int_0^{\pi/2} g(x)dx$ is convergent and hence $\int_0^{\pi/2} f(x)dx$ is convergent

Hence $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$ is convergent if p < 2 and divergent if p ≥ 2.

Second Method

When p > 0 the only point of infinite discontinue is 0.

Also, $\frac{\sin x}{x^p} = \frac{1}{x^{p-1}} \cdot \frac{\sin x}{x} \leq \frac{1}{x^{p-1}} \left[\frac{\sin x}{x} \leq 1\right]$

But $\int_0^{\pi/2} \frac{dx}{x^{p-1}}$ is convergent if $p - 1 < 1$ i.e., if p < 2.

□ By comparison test, $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$ is convergent if p < 2 and divergent if p ≥ 2.

(ii) If n is negative or zero the given integral is a proper integral and hence convergent when n ≤ 0.

When $n > 0$, the only point of infinite discontinuity is 0.

Let $f(x) = \frac{\cos x}{x^\pi}$

Take $g(x) = \frac{1}{x^\mu}$

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} x^{\mu - \pi} \cos x$$

$= 1 \; if \; \mu - n = 0$

$= 0 \; if \; \mu - n > 0$

$= \infty \; if \; \mu - n < 0$

By taking $0 < \mu < 1$ and also $\mu = n$ so that $0 < n < 1$

$\int_0^{\pi/2} g(x)dx$ is convergent and hence $\int_0^{\pi/2} f(x)dx$ is convergent.

From the above discussion, it follows that the given integral is convergent if n < 1 and divergent if n ≥ 1.

(iii) Since $|cosec\ x| \geq 1$ for all values of x, we have

$\left|\frac{cosec\ x}{x}\right| \geq \frac{1}{|x|} = \frac{1}{x}$ for all x in (0, 1]

But, $\int_0^1 \frac{1}{x} dx$ is divergent. Therefore, $\int_0^1 \frac{cosec\ x}{x} dx$ is divergent

Example 9: Show that $\int_0^{\pi/2} x^m cosec^n x$ exists if and only if n < m + 1.

Solution: Here $f(x) = x^m cosec^n x = \frac{x^m}{sin^n x} = \left(\frac{x}{\sin x}\right)^n . x^{m-n} = \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}}$

$$\lim_{x \to 0+} f(x) = \begin{cases} 0 & if \quad m - n > 0 \\ 1 & if \quad m - n = 0 \\ \infty & if \quad m - n < 0 \end{cases}$$

∴ The given integral is a proper integral if m – n ≥ 0 i.e. if m ≥ n and an improper integral if m – n < 0; 0 being the only point of infinite discontinuity of f on $\left[0, \frac{\pi}{2}\right]$.

When m – n < 0, i.e. n – m > 0.

$$f(x) = \left(\frac{x}{\sin x}\right)^n \frac{1}{x^{n-m}}$$

Take $\qquad g(x) = \frac{1}{x^{n-m}}$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \left(\frac{x}{\sin x}\right)^n = 1$ which is non-zero and finite.

Also $\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ is convergent iff n – m < 1 i.e. n < m + 1

∴ By comparison test, the given integral is convergent iff n < m + 1, which also includes the case n ≤ m when the integral is proper.

Example 10: Show that $\int_0^{\pi/2} \frac{sin^m x}{x^n} dx$ exists if and only if n < m + 1.

Solution; Here $f(x) = \frac{sin^m x}{x^n} = \left(\frac{\sin x}{x}\right)^n \frac{1}{x^{n-m}}$

$$\lim_{x \to 0+} f(x) = \begin{cases} 0 & if \quad n - m < 0 \\ 1 & if \quad n - m = 0 \\ \infty & if \quad n - m > 0 \end{cases}$$

∴ The given integral is a proper integral if n – m ≤ 0 i.e. if m ≥ n and an improper integral if n – m > 0; 0 being the only point of infinite discontinuity of f on $\left[0, \frac{\pi}{2}\right]$.

When n – m > 0, .

Take $g(x) = \frac{1}{x^{n-m}}$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \left(\frac{\sin x}{x}\right)^m = 1$ which is non-zero and finite.

Also $\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{dx}{x^{n-m}}$ is convergent iff n – m < 1 i.e. n < m + 1

∴ By comparison test, the given integral is convergent iff n < m + 1.

Example 11: Examine the convergence of

(i) $\int_0^1 \log x \ dx$ (ii) $\int_0^{\pi/4} \frac{1}{\sqrt{\tan x}} dx$ (iii) $\int_0^1 \frac{\sin x}{x} dx$ (iv) $\int_0^1 \left(\log\frac{1}{x}\right)^n dx$

Solution: (i) 0 is the only point of infinite discontinuity and log x is negative on (0, 1]

$$\int_0^1 \log x \, dx = \lim_{\varepsilon \to 0+} \int_{0+\varepsilon}^1 \log x \ dx = \lim_{\varepsilon \to 0+} [x log \ x - x]_\varepsilon^1$$

$$\lim_{\varepsilon \to 0+} [-1 - \varepsilon \log \varepsilon + \varepsilon] = -1 \qquad [\lim_{\varepsilon \to 0+} \varepsilon \log \varepsilon = 0]$$

$\Rightarrow$ the integral is convergent

Second method:

Let $f(x) = -\log x$

Take $g(x) = \frac{1}{x^n}$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} -x^n \log x = 0$ if n > 0

Taking n between 0 and 1, $\int_0^1 f(x) dx = \int_0^1 \frac{dx}{x^n}$ is convergent

$\therefore$ By comparison test, $\int_0^1 f(x) dx$ is convergent. Hence $\int_0^1 \log x \, dx$ is convergent

(ii) 0 is the only point of infinite discontinuity of the integrand on $\left[0, \frac{\pi}{4}\right]$

Let $f(x) = \frac{1}{\sqrt{\tan x}} = \sqrt{\frac{\cos x}{\sin x}}$

Take, $g(x) = \frac{1}{\sqrt{x}}$

$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0+} \sqrt{\frac{x}{\sin x}}.\sqrt{\cos x} = 1$ which is non-zero and finite.

Since $\int_0^{\pi/4} g(x)dx = \int_0^{\pi/4} \frac{dx}{\sqrt{x}}$. Form $\int_0^b \frac{dx}{x^n}$ is convergent $\left(n = \frac{1}{2} < 1\right)$

$\therefore \int_0^{\pi/4} f(x)dx$ is convergent.

(iii) Since $\lim_{x\to 0+} \frac{\sin x}{x} = 1$ the integral is proper and hence convergent.

(iv) $\int_0^1 \left(\log \frac{1}{x}\right)^n dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx + \int_a^1 \left(\log \frac{1}{x}\right)^n dx$

…………..(1)

Where $0 < a < 1$.

0 and 1 are the points of infinite discontinuity of the integrals on the right.

Let $f(x) = \left(\log \frac{1}{x}\right)^n$

Convergence of $\int_0^a \left(\log \frac{1}{x}\right)^n dx$ at 0

$\lim_{x\to 0+} \left(\log \frac{1}{x}\right)^n = 1$ if n = 0

∴ The integral is proper is n ≤ 0

0 is the only point of infinite discontinuity if n > 0.

For n > 0 take $g(x) = \frac{1}{x^p}$ 0 < p < 1

$$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} x^p \left(\log \frac{1}{x}\right)^n = 0$$

Also $\int_0^a g(x)dx$ converges since 0 < p < 1.

$\therefore \int_0^a f(x)dx = \int_0^a \left(\log \frac{1}{x}\right)^n dx$ converges.

Combining all cases $\int_0^a \left(\log \frac{1}{x}\right)^n dx$ converges for all n.

Convergence of $\int_a^1 \left(\log \frac{1}{x}\right)^n dx$ at 1.

The integral is proper if n ≥ 0 and 1 is the only point of infinite discontinuity if n < 0.

For n < 0 take $g(x) = \frac{1}{(1-x)^{-n}}$

$\lim_{x \to 0+} \frac{f(x)}{g(x)} = \lim_{x \to 0+} \left(\frac{\log \frac{1}{x}}{1-x}\right)^n = 1$ which is non-zero and finite.

But $\int_a^1 g(x)dx = \int_0^1 \frac{dx}{(1-x)^{-n}}$ is convergent if -n < 1 i.e., if n > -1

$\therefore$ By comparison test, $\int_a^1 f(x)dx = \int_a^1 \left(\log \frac{1}{x}\right)^n dx$ is convergent if -1 < n < 0.

Hence, from (1), $\int_0^1 \left(\log \frac{1}{x}\right)^n dx$ is convergent if -1 < n < 0.

7.5 The μ −Test

Let $f(x)$ be bounded and integrable in the interval (a, ∞) where $a > 0$.if there is a number $\mu > 1$. Such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists, then $\int_a^{\infty} f(x)dx$ is convergent.

If there is number $\mu \le 1$ such that $\lim_{x\to\infty} x^{\mu} f(x)$ exists and is not zero, then the integral $\int_a^{\infty} f(x)dx$ is divergent and the same is true if $\lim_{x\to\infty} x^{\mu} f(x)$ is $+\infty\ or -\infty$.

The value of μ is usually select to be "the highest power of x in denominator the highest powers of x in numerator''. so that the highest powers of x in numerator and denominator of $x^{\mu} f(x)$ are same.

Example: test the convergence of $\int_0^{\infty} \frac{dx}{x^{\frac{1}{3}}(1+x^{1/2})}$

Solution: we have take $\mu = \frac{5}{6} - 0 = \frac{5}{6}$, then

$$\lim_{x\to\infty} x^{\mu} f(x) = \lim_{x\to\infty} x^{5/6} \frac{1}{x^{\frac{1}{3}}(1 + x^{1/2})}$$

$$\Rightarrow \lim_{x\to\infty} \frac{x^{5/6}}{x^{\frac{1}{3}}(1 + x^{1/2})} = \lim_{x\to\infty} \frac{x^{5/6}}{x^{1/3} + x^{5/6}}$$

$$\Rightarrow \lim_{x\to\infty} \frac{1}{\frac{1}{x^{1/2}} + 1} = 1$$

Thus, is finite and non-zero and since $\mu = \frac{5}{6} < 1$, it follows form $\mu - test$ that an integral is divergent.

Example: Test the convergence of $\int_0^1 x^{n-1} e^{-x} dx$

Solution: When $n \geq 1$, the given integral is a proper integral and hence it is convergent.

Again when $n < 1$, the integrand is unbounded at $x = 0$.

Now let $f(x) = x^{n-1}e^{-x}$. Then

$$\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} x^{\mu+n-1}\, e^{-x}$$

$\lim_{x\to\infty} x^{\mu} f(x) = 1$ if $\mu + n - 1 = 0, i.e. \mu = 1 - n$

We have $0 < \mu < 1$ when $0 < n < 1$

And $\mu \geq 1$ when $n \leq 0$.

Hence by μ –Test the given integral is convergent when $0 < n < 1$ and divergent when $n \leq 0$.

Example: Test the convergence of $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}} dx$, where $m\ and\ n$ positive integers.

Solution: we have $\int_0^{\infty} \frac{x^{2m}}{1+x^{2n}} dx = \int_0^{a} \frac{x^{2m}}{1+x^{2n}} dx + \int_a^{\infty} \frac{x^{2m}}{1+x^{2n}} dx\ , where\ a > 0.$

The first integral on the right-hand side is a proper integral, therefore, the integral will be convergent or divergent according as $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$ is convergent or divergent.

To test the convergent of $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}dx$

Take $\mu = 2n - 2m, then$

$$\lim_{x\to\infty} x^\mu f(x) = \lim_{x\to\infty} \frac{x^{2n-2m}x^{2m}}{1+x^{2n}} = \lim_{x\to\infty} \frac{x^{2n}}{1+x^{2n}} = 1$$

Which is finite and non-zero.

The given integral is convergent if $\mu > 1\ i.e., if\ 2n - 2m > 1$ which is possible if $n > m$ since m and n are positive integers.

And the given integral is divergent if $\mu \leq 1$ i.e., if $n \leq m$.

7.6 Absolute Convergence

If the integral $\int_a^\infty |f(x)|\,dx$ converges, then the infinite integral $\int_a^\infty f(x)dx$ said to converge absolutely.

Note: Absolute Convergenceof an infinite integral gives a sufficients not necessary condition for its convergence i.e., if the infinite integral **Absolute Convergent,** it is necessarily convergent, but conversely if an infinite integral is convergent, it is not necessarily Absolute Convergent.

Example: Show that $\int_1^\infty \frac{\sin x}{x^4}dx$ is absolutely Convergent.

Solution: we have $\int_1^\infty \left|\frac{sin\, x}{x^4}\right| dx = \lim_{x\to\infty} \int_1^x \frac{|sin\, x|}{|x^4|} dx$

$$\leq \lim_{x\to\infty} \int_1^x \frac{dx}{x^4} [\because |sin\, x| \leq 1]$$

$$= \lim_{x\to\infty} \left[-\frac{1}{3x^3}\right]_1^x = \lim_{x\to\infty} \left[\frac{1}{3} - \frac{1}{3x^3}\right] = \frac{1}{3}$$

The limit exists as a finite value and hence $\int_1^\infty |\frac{sin\, x}{x^4}|dx$ is convergent.

Therefore, it follows that the integral $\int_1^\infty \frac{sin\, x}{x^4} dx$ is absolutely convergent.

Example:2 Test the absolute convergence of the integral $\int_0^\infty f(x)dx$ where $f(x)$ is defined by the following:

$$f(x) = \begin{cases} 1 \quad ; & 0 \leq x \leq 1 \\ 0 \quad ; & n-1 \leq x \leq n - \frac{1}{n} \\ (-1)^{n+1} \ ; & n - \frac{1}{n} < x \leq p \ when\ n = 2,3,4 \end{cases}$$

Example: Test the absolute convergence of $\int_0^\infty e^{-a^2x^2} cos\, bx\, dx$

Solution: we have $\int_0^\infty \left|e^{-a^2x^2} \cos bx\right|dx = \lim_{x\to\infty} \int_0^x \left|e^{-a^2x^2}\right| |\cos bx|dx$

$$\leq \lim_{x\to\infty} \int_0^x e^{-a^2x^2}\, dx\, [\because\ |\cos bx| \leq 1]$$

But $\int_0^x e^{-a^2x^2}\, dx$ convergent.

Hence $\int_0^\infty \left|e^{-a^2x^2} \cos bx\right| dx$ is convergent.

It follows that $\int_0^\infty e^{-a^2x^2}\, cos\ bx\ dx$ is absolute convergent.

Example: Show that the integral $\int_0^\infty \frac{\sin x}{x} dx$divergent but not absolutely convergent.

Solution: we know that $\int_0^\infty \frac{\sin x}{x} dx = \int_0^a \frac{\sin x}{x} dx + \int_a^\infty \frac{\sin x}{x} dx$, where $a > 0$.

Therefore, $\lim_{n\to\infty} \int_0^{n\pi} \frac{|\sin x|}{x} dx = \infty$

When $x > n\pi,\ \int_0^x \frac{|\sin x|}{x} dx > \int_0^{n\pi} \frac{|\sin x|}{x} dx$

Therefore, $\lim_{x\to\infty} \int_0^x \frac{|\sin x|}{x} dx = \infty$

So, $\int_0^x \frac{|\sin x|}{x} dx$, i.e., $\int_0^x \frac{|\sin x|}{x} dx$ is divergent.

Hence, $\int_0^\infty \frac{\sin x}{x} dx$ convergent but not absolutely convergent.

7.7Absolute Convergence of the integral of a Product

The integral $\int_0^\infty \emptyset(x). f(x) dx$ is said to be absolutely convergent, when $f(x)$ is needed for $x \geq a,$ and integral in the arbitrary (a, b) and $\int_a^\infty \emptyset(x) dx$ convergence absolutely.

Example: Show that the integral converges absolutely $\int_0^\infty \frac{cos\ mx}{a^2+x^2} dx$

Solution: Let $f(x) = \cos mx$, $\emptyset(x) = \frac{1}{a^2+x^2}$

Clearly $f(x) i.e.,$ $\cos mx$ is bounded and integrable in the interval (a, b) when $b > a$.

Also $\int_0^\infty |\emptyset(x)| dx = \int_0^\infty \left|\frac{1}{a^2+x^2}\right| dx = \int_0^\infty \frac{1}{a^2+x^2} dx$ (1)

Now to test convergence of $\int_0^\infty \frac{1}{a^2+x^2} dx$, we shall apply $\mu - test$.

Let $\mu = 2$, then $\lim_{x\to\infty} x^\mu \emptyset(x) = \lim_{x\to\infty} \frac{x^2}{x^2+a^2} = 1$

Hence $\int_0^\infty \frac{1}{a^2+x^2} dx$ is convergent since $\mu = 2 > 1$.

It follows from (1) that $\int_a^\infty \emptyset(x) dx$ is absolutely convergent.

∴The given integral$\int_0^\infty \frac{cos\, mx}{a^2+x^2} dx$ is absolutely convergent.

7.8Abel's Test:

If $\int_a^\infty f(x) dx$ convergences and $\emptyset(x)$ is bounded and monotonic for $x > a$, then $\int_0^\infty f(x). \emptyset(x) dx$ is convergent.

Example: Test the convergence of $\int_0^\infty e^{-x} \frac{sin\, x}{x^2}\ dx$

Solution: we have $f(x) = \frac{sin\, x}{x^2}$ and $\emptyset(x) = e^{-x}$

Since $\left|\frac{sin\, x}{x^2}\right| < \frac{1}{x^2}$ and $\int_0^\infty \frac{dx}{x^2}$ is convergent.

Follows by comparison test the $\int_0^{\infty}\frac{sin\,x}{x^2}dx$ is also convergent.

Again e^{-x} is monotonic decreasing and bounded function for the value $x > a$.

Hence by Abel's test $\int_0^{\infty} e^{-x}\frac{sin\,x}{x^2}\ dx$ is convergent.

Example: Test the convergence of $\int_0^{\infty}(1-e^{-x})\frac{cos\,x}{x^2}\ dx$, when $a > 0$.

Solution: we have $f(x) = \frac{cos\,x}{x^2}$ and $\emptyset(x) = 1 - e^{-x}$

Since $\left|\frac{cos\,x}{x^2}\right| < \frac{1}{x^2}$ as $\cos x < 1$. Hence by comparison test $\int_0^{\infty}\frac{\cos x}{x^2}dx$ is convergent.

Again $1 - e^{-x}$ is monotonic increasing and bounded function for the value

$a > 0$. Hence by Abel's test $\int_0^{\infty}(1-e^{-x})\frac{cos\,x}{x^2}\ dx$ is convergent.

7.9Dirichlet's Test:

If $f(x)$ is bounded and monotonic in the interval (a,∞) and $\lim_{x\to\infty} f(x) = 0$,then the integral $\int_a^{\infty} f(x).\emptyset(x)dx$ is convergent,

Provided $\left|\int_a^x \emptyset(x)dx\right|$ is bounded as x takes all finite values.

Abel's Test and Dirichlet's Test are applicable, whenever the integrand can be viewed upon suitably as a product of two functions.

Example: Show that integral $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx$ is convergent when $a > 0$.

Solution: we have $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx = \int_0^{a} e^{-ax}\frac{sin\,x}{x}dx + \int_a^{\infty} e^{-ax}\frac{sin\,x}{x}dx,$ when $a > 0$.

Since $\lim_{x\to\infty} e^{-ax}\frac{sin\,x}{x} = 1.$ $[\sin x\; lies\; between -1\; and\; 1\; as\; x \to \infty\;]$

$\therefore$ The integral $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx$ is a proper integral and we need only to test convergence of $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx.$

Let $f(x) = \frac{e^{-ax}}{x}$ and $\emptyset(x) = \sin x$

$$\therefore \lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{e^{-ax}}{x} = \lim_{x\to\infty}\frac{1}{xe^{ax}} = 0$$

Clearly $f(x)$ is bounded and monotonic decreasing function of x for all value of x greater than zero.

Also $\left|\int_a^{x} \emptyset(x)dx\right| = \left|\int_a^{x} \sin x\; dx\right| = |\cos x - \cos a| \leq 2$

i.e., $\left|\int_a^{x} \emptyset(x)dx\right|$ is bounded for all finite values of x. Hence by Dirichlet's Test $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx$ is convergent. Hence $\int_0^{\infty} e^{-ax}\frac{sin\,x}{x}dx$ is convergent.

7.10 Summary

In this unit, we have covered the following points:

- We defined a Convergence.

▪ We defined the Comparison test.

▪We find the μ −test.

▪ We defined the Abel's Test.

▪We defined a Dirichlet's Test.

7.11 Terminal Questions

1. Test for convergence of $\int_0^\infty \frac{dx}{x\sqrt{x^2+1}}$ by use comparison test.
2. Test for convergence of $\int_0^\infty \frac{\cos x}{1+x^2} dx$ by use comparison test.
3. Discuss the convergence of the integral $\int_0^\infty \frac{x^{\alpha-1}}{x+1} dx$
4. Test for convergence of $\int_0^\infty \frac{dx}{(1+x)^3}$ by use μ −test.
5. Show that $\int_0^\infty \frac{sinx}{x^{1+n}} dx$ converges absolutely when n and a integers.
6. Use Abel's Test prove that $\int_a^\infty e^{-x} \frac{\sin x}{x^2} dx$ is convergent, where $a > 0$.
7. Show that the integral $\int_0^\infty e^{-x^2} dx$ is convergent.
8. Discuss the convergence of the integral $\int_0^1 x^{n-1} \log x \, dx$
9. Examine the convergence of $\int_a^\infty \frac{dx}{(x \log x)^{\mu+1}}$ where $a > 0$.
10. Test for convergence of the integral $\int_a^\infty \frac{\sin x}{\sqrt{x}} dx$, where $a > 0$,by use Dirichlet's test.

Unit- 8 Step Functions

Structure:

8.1 Introduction

In this Unit a step function is defined as a piecewise constant function, that has only a finite number of pieces. In other words, a function on the real numbers can be described as a finite linear combination of indicator functions of given intervals. It is also called a floor function or greatest integer function. The step function is a discontinuous function. However, a mathematical definition of a step function.

8.2 Objectives

After studying in this unit, therefore, you should be able to

- Define Step Function
- Discuss a Integration of a Step Function
- Define a Properties of Integrals of step functions
- Define the Upper integral and lower integral
- Define the Riemann Integral of a bounded function.

8.3 Step Function:

The Step Function $h: [1,6] \to E'$ be defined as

$$h(x) = \begin{cases} 2\ if, x \in \left[1, \frac{3}{2}\right] \\ -2if, x \in \left[\frac{3}{2}, 4\right] \\ 4if, x \in [4,6] \end{cases}$$

we see that here the partition $p = \left\{1, \frac{3}{2}, 4,6\right\}$ and $h(1) = 2,$

$$h(3/2) = -2, h(4) = -2, h(6) = 4$$

the step function $h: [1/2,13/2] \to E'$defined by $h(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

We know that $[x]$ has jump discontinuity at all integer points. So, we consider the partition p of $\left[\frac{1}{2}, \frac{13}{2}\right]$ as $p = \left\{\frac{1}{2}, 1,2,3,4,5,6, \frac{13}{2}\right\}$.

Now $h(x) = 0, when\ \frac{1}{2} \leq x \leq 1$

$$h(x) = 1, when\ 1 \leq x \leq 2$$

$$h(x) = 2, when\ 2 \leq x \leq 3$$

$$h(x) = 3, when\ 3 \leq x \leq 4$$

$$h(x) = 4, when\ 4 \leq x \leq 5$$

$$h(x) = 5, when\ 5 \leq x \leq 6$$

$$h(x) = 6, when\ 6 \leq x \leq \frac{13}{2}$$

and $h\left(\frac{1}{2}\right) = 0, h(1) = 1, h(2) = 2, h(3) = 3, h(4) = 4, h(5) = 5,$

$$h(6) = 6, h\left(\frac{13}{2}\right) = 6$$

8.4 Integration of a Step Function:

if $h: [a, b] \to E'$ be a step function defined for the partition $p = \{x_0 = a, x_1, x_2, x_3, \dots \dots . x_{n-1,} x_n = b\}$ by $h(x) = c_k$for $x \in \left(x_{k-1,} x_k\right)$ $for\ k = 1,2,3 \dots \dots n$. Then the integral of step function $h(x)$ from $a\ to\ b$ defined by $\int_a^b h(x)dx$ and defined by

$$\int_a^b h(x)dx = \sum_{k=1}^{n} c_k(x_k - x_{k-1}) = c_1(x_1 - x_0) + \cdots \dots \dots . + c_n(x_n - x_{n-1}).$$

Remark: It should be noted that the letter 'x' used for the independent variable, which may be replaced by any other convenient letter without altering the definition of integral. That is

$$\int_a^b h(x)dx = \int_a^b h(t)dt = \int_a^b h(u)du \quad etc.$$

Example 1. Define a step function h : [1, 5] → E' as

$$h(x) = \begin{cases} 3 & if \quad 1 \le x \le \frac{3}{2} \\ -2 & if \quad \frac{3}{2} \le x \le 4 \\ 6 & if \quad 4 < x \le 5 \end{cases} \qquad p = \left[1, \frac{3}{2}, 4, 5\right]$$

Then $\int_1^5 h(x)dx = 3\left(\frac{3}{2} - 1\right) + (-2)\left(4 - \frac{3}{2}\right) + 6(5 - 4)$

$$= \frac{3}{2} - 2 \times \frac{5}{2} + 6 \times 1 = \frac{3}{2} + 6 * 5 = \frac{5}{2}$$

Example 2. Define a step function h : [1, 6] → E’ as

$$h(x) = \begin{cases} 2 & if \quad 1 \le x < \frac{3}{2} \\ -2 & if \quad \frac{3}{2} \le x \le 4 \\ 4 & if \quad 4 < x \le 6 \end{cases} \qquad p = \left[1, \frac{3}{2}, 4, 6\right] \text{ then,}$$

$$\int_1^6 h(x)dx = 2\left(\frac{3}{2} - 1\right) + (-2)\left(4 - \frac{3}{2}\right) + 4(6-4)$$

$$= 2 \times \frac{1}{2} - 2 \times \frac{5}{2} + 4 \times 2 = 1 - 5 + 8 = 4$$

Example 3. Define the step function h : [-7/2, 1] → E’ as

$$h(x) = \frac{|x|}{x} \;\; if \; x \neq 0, h(0) = 0$$

We consider the partition P = $\left\{-\frac{7}{2}, 0, 1\right\}$

Now, $h(x) = \frac{|x|}{x} = \begin{cases} -1 & if \quad x \in \left[-\frac{7}{2}, 0\right] \\ +1 & if \quad x \in [0, 1] \end{cases}$

□ $\int_{-\frac{7}{2}}^{1} h(x)dx = -1\left(0 - \left(-\frac{7}{2}\right)\right) + 1(1-0) = -1 \times \frac{7}{2} + 1 = -\frac{5}{2}$

Example 4. Define a step function h : $\left[-\frac{3}{2}, 2\right] \rightarrow E'$ as

$$h(x) = 3[x] - 2\frac{|x|}{x}, \; x \neq 0 \text{ and } h(0) = 0$$

the integers of $\left[-\frac{3}{2},\ 2\right]$ are -1, 0, 1, 2 at which the bracket function [x] is discontinuous.

Hence, we consider the partition of $\left[-\frac{3}{2},\ 2\right]$ as

$P = \left\{-\frac{3}{2}, -1, 0, 1, 2\right\}$. Now,

When $x \in \left(-\frac{3}{2}, -1\right),\ [x] = -2, \frac{|x|}{x} = -1$ and hence,

$$h(x) = 3(-2) - 2(-1) = -4, x \in \left(-\frac{3}{2}, -1\right)$$

Similarly, when $x \in (-1,\ 0); h(x) = 3(-1) - 2(-1) = -1$

When $x \in (0,\ 1);\ \ h(x) = 3.0 - 2(1) = -2$

When $x \in (1, ,2); h(x) = 3.1 - 2.1 = 1$

Hence $\int_{-\frac{3}{2}}^{2} h(x)dx = -4\left(-1 + \frac{3}{2}\right) + (-1)(0 + 1) + (-2)(1 - 0) + 1(2 - 1)$

$$= -2 - 1 - 2 + 1 = -4$$

Example 5 Define a step function h : [-1, 2] → E' as

$$h(x) = [x^2]\ \forall x \in [-1, 2]$$

for the required partition, we try to find those values of x which x^2 is an integer in [-1, 2]. We see that x^2 is an integer in [-1, 2] when x = -1, 0, 1, $\sqrt{2}, \sqrt{3}$, 2. Hence partition P={-1, 0, 1, $\sqrt{2}, \sqrt{3}$, 2}.

Now, $h(x) = [x^2] = 0 \ \ when \ -1 \le x < 0; \ \ x^2 < 1$

$= 1 \ \ when \quad 0 \le x < 1; \ \ x^2 < 1$

$= 1 \ \ when \quad 1 \le x < \sqrt{2}; \ \ 1 \le x^2 < 2$

$= 2 \ \ when \ \sqrt{2} \le x < \sqrt{3}; \ \ 2 \le x^2 < 4$

$= 3 \ \ when \ \sqrt{3} \le x < 2; \ \ 3 \le x^2 < 4$

Hence $\int_{-1}^{2} h(x)dx = 0(0+1) + 0(1-0) + 1\left(\sqrt{2}-1\right) + 2\left(\sqrt{3}-\sqrt{2}\right) + 3(2-\sqrt{3})$

$= \sqrt{2} + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3} = 5 - \sqrt{2} - \sqrt{3}$

Example 6. Define a step function h : [0, 2] → E' as

$$h(x) = [3x+4] \forall x \in [0, \ 2]$$

we want to find those points of [0, 2] for which 3x + 4 is an integer. So, partition of [0, 2] is $P = \left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$

Also $\int_0^2 h(x)dx = 13$ (check it)

Example 7. We define a step function $h: [0, b] \to E'$ as

$h(x) = \frac{|(x-1)(x^2-7x+12)|}{(x-1)(x^2-7x+12)}$; $x \ne 1, 3, 4$ and $h(1) = 2, \ h(3) = 2, \ h(4) = 2.$

Since $h(x) = 1$ if $(x-1)(x^2-7x+12) > 0$

Or if $(x-1)(x-3)(x-4) > 0$

Or if $x > 4$, or $1 < x < 3$

$h(x) = -1$ if $(x-1)(x^2 - 7x + 12) < 0$

Or if $(x-1)(x-3)(x-4) < 0$

Or if $0 \leq x < 1$, or $3 < x < 4$

Hence Partition of [0, 6] is P = {0, 1, 3, 4, 6}

$$\square \quad h(x) = \begin{cases} -1 & if \quad x \in (0,1) \\ +1 & if \quad x \in (1,3) \\ -1 & if \quad x \in (3,4) \\ +1 & if \quad x \in (4,6) \end{cases}$$

Hence $\int_0^6 h(x)dx = -1(1-0) + 1(3-1) + (-1)(4-3) + 1(6-4)$

$$= -1 + 2 - 1 + 2 = 2.$$

Note: Geometrical meaning of $\int_a^b h(x)dx$, where $h: [a, b] \to E'$ is a step function.

Suppose that P = {x0 = a, x_1, x_2, x_3, ……x_{b-1}, x_{5-b}} be a partition of [a, b] for h(x). Now,

i.e. $h(x) = C_1 \; if \; a \leq x < x_1$

$= C_2 \; if \; x_1 < x < x_2$

$= C_3 \; if \; x_2 \leq x < x_3$

$= C_4 \; if \; x_3 \leq x < x_4$

$= C_5 \; if \; x_4 \leq x \leq x_5$

□ $\int_a^b h(x)dx = C_1(x_1 - x_0) + C_2(x_2 - x_1) + C_3(x_3 - x_2) + C_4(x_4 - x_3) + C_5(x_5 - x_4)$

= sum of the areas of the rectangles over the sub intervals of [a, b].

Hence $\int_a^b h(x)dx$ represents the area bounded by the graph of h(x), x-axis and the ordinates x = a and x = b. In this we see that some portion of the area is above the x-axis and some portion may be below the x-axis.

So, $\int_a^b h(x)dx$ will be the sum of all areas (above the x-axis or below the x-axis) bounded by the graph of h(x).

8.5 Properties of Integrals of step functions:

Theorem 1: If $h: [a, b] \rightarrow E'$ be a step function associated with a partition P and P' be a refinement of P, then the value of $\int_a^b h(x)dx$ is not altered if P is replaced by P'.

Note: P' is a refinement of a partition P if P' contains all the sub-division points of P together with some additional points.

Proof: Let the partition P = {x_0 = a, x_1, x_2, x_3, ……x_{k-1}, x_k…x_{n-b}} and let the refinement P' of P.

Here the refinement P' contains additional points y_1 and y_2 between x_{k-1} and x_k.

Let $h(x) = C_r$ when $x \in (x_{k-1}, x_k)$, r = 1, 2, 3, …..n with respect to the partition P.

□ $\int_a^b h(x)dx = \sum_{r=1}^{n} C_r(x_r - x_{r-1}) = \sum_{r=1}^{k-1} C_r(x_r - x_{r-1} + C_k(x_k - x_{k-1}) + \sum_{r=k+1}^{n} C_r(x_r - x_{r-1}))$ ……………(1)

Now with respect to the partition P'

$\int_a^b h(x)dx = \sum_{r=1}^{k-1} C_r(x_r - x_{r-1}) + C_k(y_1 - x_{r-1}) + C_k(y_2 - y_1) + C_k(x_k - y_2) + \sum_{r=k+1}^{n} C_r(x_r - x_{r-1})$ ……………. (2)

By comparing equation (1) and (2) we find that the term $C_k(x_k - x_{k-1})$ in (1) is replaced by $C_k(y_1 - x_{r-1}) + C_k(y_2 - y_1) + C_k(x_k - y_2)$.

Hence the value $\int_a^b h(x)dx$ is the same for both partition P and P' (refinement of P)

Theorem 2: Let $h: [a, b] \rightarrow E'$ be a step function if r is a real number such that a < r < b, then

$$\int_a^b h(x)dx = \int_a^r h(x)dx + \int_r^b h(x)dx$$

Proof: We consider a partition P of [a, b] associated with h(x) which contains r as a point of division P = {a = x_0, x_1, x_2, …..x_m, x_{m+1},…..x_{m+n} = b}

Where x_m = r then we have

$\int_a^b h(x)dx = \sum_{k=1}^{m} C_k(x_k - x_{k-1}) + \sum_{k=m+1}^{m+n} C_k(x_k - x_{k-1})$ …………..(1)

Now, we take P_1 = {a = x_0, x_1, x_2, …., x_m = r} be a partition of [a, r] and hence,

$\int_a^r h(x)dx = \sum_{k=1}^{m} C_k(x_k - x_{k-1})$ ……………. (2)

Again we take P_2 = {r = x_m, x_{m+1}, ………x_{m+n} = b} be a partition of [r, b] and hence

$\int_r^b h(x)dx = \sum_{k=m+1}^{m+n} C_k(x_k - x_{k-1})$ ……….. (3)

From equation (1), (2) and (3) we get that

$$\int_a^b h(x)dx = \int_a^r h(x)dx + \int_r^b h(x)dx$$

Theorem 3: Let $s: [a, b] \to E'$ and $t: [a, b] \to E'$ be two step functions such that $s(x) \leq t(x) \forall x \in [a, b]$ then

$$\int_a^b s(x)dx \leq \int_a^b t(x)dx$$

Proof: Suppose that P1 and P2 be two partitions of [a, b] associated with the step functions s(x) and t(x) respectively.

Let $P = P_1 \cup P_2$, then P is a refinement of P_1 and P_2 both. Hence, $\int_a^b s(x)dx$ and $\int_a^b t(x)dx$ will not be altered if P_1 and P_2 are replaced by their refinement P.

Let P = {a = x_0, x_1, x_2, …., x_m = b}

Let $s(x) = C'_k$ when $x \in (x_{k-1}, x_k); 1 \leq k \leq n.$

$t(x) = C"_k$ when $x \in (x_{k-1}, x_k); 1 \leq k \leq n.$

□ $s(x) \leq t(x) \forall x \in [a, b]$, so

$C'_k \leq C"_k$ for k = 1, 2, 3, ….n

Hence, $\int_a^b s(x)dx = \int_{k=1}^n C'_k(x_k - x_{k-1}) \leq \int_{k=1}^n C"_k(x_k - x_{k-1})$

□ $\int_a^b s(x)dx \leq \int_a^b t(x)dx$

Theorem 4: Let $h: [a, b] \to E'$ be a step function and m, M be real numbers such that $m \leq h(x) \leq M \forall x \in [a, b]$ then, $m(b-a) \leq \int_a^b h(x)dx \leq M(b-a)$.

Proof: We define step functions s and t on [a, b] then $s(x) \leq h(x) \leq t(x) \forall x \in [a, b]$. Then we have $\int_a^b s(x)dx \leq \int_a^b h(x)dx \leq \int_a^b t(x)dx.$ ………. (1)

Now, $\int_a^b s(x)dx = \int_a^b m dx = m(b-a)$ and $\int_a^b t(x)dx = \int_a^b M dx = M(b-a)$

Then from equation (1) we have

$$m(b-a) \leq \int_a^b h(x)dx \leq M(b-a)$$

8.6 Upper integral and lower integral of a bounded function:

Let f: [a, b] → E' be a bounded function, so $\{f(x)x \in [a,b]\}$ is a bounded set, let m and M be the lower and upper bound of the set, so that ∃ real numbers m and M such that $m \leq f(x) \leq M \forall x \in [a,b]$. Let $S_f = \{step\ function\ s: [a,b] \to E' \text{I} S(x) \leq f(x) \forall x \in [a,b]\}$. Thus if $S(x) \in S_f$, then $S(x) \leq f(x) \leq M \forall x \in [a,b]$.

So $\int_a^b S(x)dx \leq \int_a^b M dx = M(b-a) \forall S(x) \in S_f$

Hence the set $\left\{\int_a^b S(x)dx \middle| \ S(x) \in S_f\right\}$ is bounded above by M(b – a). therefore

l.u.b. $\left\{\int_a^b S(x)dx \middle| \ S(x) \in S_f\right\}$ exist (by completeness property)

thus l.u.b. is defined as the lower integral of f from a to b, and it is denoted by $\int_a^b f(x)dx$. Thus

$$\int_a^b f(x)dx = l.u.b.\left\{\int_a^b S(x)dx \middle| \ S(x) \in S_f\right\}$$

Again Let $T_f = \{step\ function\ t: [a,b] \to E' |\ f(x) \leq t(x) \forall x \in [a,b]\}$

Thus if, $t(x) \in T_f$ then $m = f(x) \leq t(x) \forall x \in [a,b]$.

So $\int_a^b t(x)dx \geq \int_a^b mdx = m(b-a) \forall t(x) \in T_f$.

Hence the set $\left\{\int_a^b t(x)dx \middle|\ t(x) \in T_f\right\}$ is bounded below by m(b – a). Therefore g.l.b. of this set exists, which is defined as the upper integral of f from a to b. and it is written as $\int_a^{\bar{b}} f(x)dx$. Thus

$$\int_a^{\bar{b}} f(x)dx = g.l.b.\left\{\int_a^b t(x)dx \middle|\ t(x) \in T_f\right\}$$

Example: Let the function $f: (1,2) \to E'$ be defined by $f(x) =$
$\begin{cases} 3 & if\ x\ is \quad rational \\ 1 & if\ x\ is \quad irrational \end{cases}$

Then evaluate $\int_{\underline{1}}^2 f(x)dx$ and $\int_1^{\bar{2}} f(x)\, dx$.

Solution: Here we see that clearly f is bounded on (1, 2). Suppose that P = {1 = x_0, x_1, x_2, ….x_n = 2} be a partition of (1. 2).

Now, open interval $(x_{k-1},\ x_k)$, k = 1, 2, 3, …..n will contains both rational and irrational points.

Hence if $S(x) \in S_f$ then $S(x) \leq f(x) \forall x \in (1,2)$. So $S(x) \leq 1 \forall x \in (1,2)$. Hence

$\int_1^2 s(x)dx \leq 1.(2-1) \forall S(x) \in S_f$.

Therefore, l.u.b. $\left\{\int_1^2 s(x)dx \mid S(x) \in S_f\right\} = 1$

Hence $\int_{\underline{1}}^2 f(x)\,dx = 1$

Again, if $t(x) \in T_f$, then $t(x) \geq f(x) \forall x \in (1,2)$.

$\Rightarrow \quad 3 \leq t(x) \forall x \in (1,2)$

Hence, $\int_1^2 3dx \leq \int_1^2 t(x)dx$ or $3(2-1) \leq \int_1^2 t(x)\,dx \forall t(x) \in T_f$.

$\therefore \int_1^2 t(x)dx = 3$

Therefore, $\left\{\int_1^2 t(x)\,dx \mid t(x) \in T_f\right\} = 3$

Hence, $\int_1^{\overline{2}} t(x)\,dx = 3$.

Note: (i) $\int_{\underline{1}}^2 f(x)\,dx \neq \int_1^{\overline{2}} f(x)\,dx$ (ii) $\int_{\underline{1}}^2 f(x)\,dx \leq \int_1^{\overline{2}} f(x)\,dx$

i.e., $\int_{\underline{a}}^b f(x)\,dx \leq \int_a^{\overline{b}} f(x)\,dx$

Lemma: If $f:[a,b] \to E'$ be a bounded function and given for any real number $\varepsilon > 0$ $\exists$ step function $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that

(i) $\int_{\underline{a}}^b f(x)dx - \int_a^b s_0(x)dx < \frac{\varepsilon}{2}$

(ii) $\int_a^b t_0(x)dx - \int_a^{\overline{b}} f(x)dx < \frac{\varepsilon}{2}$

Proof: We know that $\underline{\int_a^b} f(x)dx = l.u.b.\left\{\int_a^b S(x)dx \mid S(x) \in S_f\right\}$. Hence , $\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2}$ is not an upper bound of the set $\left\{\int_a^b S(x)dx \mid S(x) \in S_f\right\}$, so there exists a step function $s_0(x) \in S_f$ such that $\underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < \int_a^b s_0(x)dx$ or $\underline{\int_a^b} f(x)dx - \int_a^b s_0(x)dx < \frac{\varepsilon}{2}$.

In similar manner, $\overline{\int_a^b} f(x)dx = g.l.b.\left\{\int_a^b t(x)dx \mid t(x) \in T_f\right\}$. Hence $\overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$ is not a lower bound of the set $\left\{\int_a^b t(x)dx \mid t(x) \in T_f\right\}$. So there exists a step function $t_0(x) \in T_f$ such that $\int_a^b t_0(x)dx \geq \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2}$. Or $\int_a^b t_0(x)dx \geq \overline{\int_a^b} f(x)dx < \frac{\varepsilon}{2}$.

Theorem: If $f: [a, b] \rightarrow E'$ be a bounded function, then $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$.

Proof: By using the above lemma (i) and (ii) we get $\left(\underline{\int_a^b} f(x)dx - \int_a^b s_0(x)dx\right) + \left(\int_a^b t_0(x)dx - \overline{\int_a^b} f(x)dx\right) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$ or $\left(\underline{\int_a^b} f(x)dx - \int_a^b s_0(x)dx\right) + \left(\int_a^b t_0(x)dx - \overline{\int_a^b} f(x)dx\right) < \varepsilon$.

Also $s_0(x) \leq f(x) \leq t_0(x) \forall x \in [a, b]$. So $\int_a^b t_0(x)dx - \int_a^b s_0(x)dx \geq 0$

Thus, we get that $\underline{\int_a^b} f(x)dx - \overline{\int_a^b} f(x)dx < \varepsilon \forall \varepsilon > 0$.

Hence $\underline{\int_a^b} f(x)dx - \overline{\int_a^b} f(x)dx \leq 0$. $(\because A \leq \varepsilon, \forall \varepsilon > 0 \Rightarrow A \leq 0)$

Therefore $\underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx$

8.7 Riemann Integral of a bounded function

Definition: Let $f: [a, b] \to E'$ be a bounded function. Then f is called Riemonn integrable on [a, b] iff $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

Note: The common value of the lower and upper integrals will be denoted by $\int_a^b f(x)dx$ and called the Riemann integral of f from a to b. i.e., $\int_a^b f(x)dx$ exists.

Example 1: Let the function $f: (1.2) \to E'$ be defined by $f(x) = \begin{cases} 5 & if\ x\ is \quad rational \\ 2 & if\ x\ is \quad irrational \end{cases}$ then $\int_1^2 f(x)dx = 2$ (lower integral of f(x) on (1, 2))

And $\int_1^2 f(x)dx = 5$ (upper integral of f(x) on (1, 2))

$\because \underline{\int_1^2} f(x)dx = 2 \neq 5 = \overline{\int_1^2} f(x)dx$ i.e., lower integral of f(x) is not equal to upper integral of f(x) on (1. 2). Hence f is not Riemann integrable on (1, 2).

Example 2: If $f: (a, b) \to E'$ be a step function, then prove that f is Riemann integrable on [a, b].

Solution: $\because\ f:(a,b)\to E'$ be a step function, so it is bounded on [a, b]. Also $f(x)\in S_f=\{S(x)\,|\ S(x)\le f(x)\forall x\in[a,b]\}$

So, if $S(x)\in S_f$ then $S(x)\le f(x)\forall x\in[a,b]$

Therefore, $\int_a^b S(x)dx\le\int_a^b f(x)dx\ \ \forall S(x)\in S_f.$

Hence, $\int_a^b f(x)dx=l.u.b.\left\{\int_a^b S(x)\,|\ S(x)\in S_f\right\}$

$$=\int_{\underline{a}}^{b} f(x)dx. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, $f(x)\in T_f$. If $t(x)\in T_f$ then $f(x)\le t(x)\forall x\in[a,b]$.

$\Rightarrow\quad \int_a^b f(x)dx\le\int_a^b t(x)dx\,\forall t(x)\in T_f.$

Hence $\int_a^{\overline{b}} f(x)dx=g.l.b.\left\{\int_a^b t(x)dx\ |\ t(x)\in T_f\right\}=\int_a^b f(x)dx$ ……………(2)

From (1) and (2) we get that

$$\int_{\underline{a}}^{b} f(x)dx=\int_a^b f(x)dx=\int_a^{\overline{b}} f(x)dx\Rightarrow\int_{\underline{a}}^{b} f(x)dx=\int_a^{\overline{b}} f(x)dx$$

Thus the step function f is integrable on [a, b] means it is Riemann integrable on [a, b].

Example 3: If $h_1 : [a, b] \to E'$ and $h_2 : [a, b] \to E'$ be two step function then prove that $(h_1 + h_2) : [a, b] \to E'$ be a step function and $\int_a^b (h_1 + h_2)(x)dx = \int_a^b h_1(x)dx + \int_a^b h_2(x)dx$.

Solution: Suppose that P_1 and P_2 be two partition associated with the step function $h_1(x)$& $h_2(x)$ respectively. Suppose $P = P_1 \cup P_2$ be a refinement of P_1 and P_2. Let $P = \{x_0, x_1, x_2, \ldots\ldots.x_n = b\}$

Let $h_1(x) = C'_k \forall x \in (x_{k-1}, \ x_k)$ and $h_2(x) = C"_k \forall x \in (x_{k-1}, \ x_k), 1 \leq k \leq n$

Then $(h_1 + h_2)(x) = h_1(x) + h_2(x) = C'_k + C"_k \forall x \in (x_{k-1}, \ x_k)$

Thus, $(h_1 + h_2)$ is constant on each $(x_{k-1}, \ x_k)$ k = 1, 2 , 3, ….n

Hence $(h_1 + h_2) : [a, b] \to E'$ is a step function.

Again $\int_a^b (h_1 + h_2)\,(x)dx = \int_a^b (h_1(x) + h_2(x))dx$

$$= \sum_{k=1}^{n} (C'_k + C"_k)(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} C'_k\,(x_k - x_{k-1}) + \sum_{k=1}^{n} C"_k (x_k - x_{k-1})$$

$$= \int_a^b h_1(x)dx + \int_a^b h_2(x)dx.$$

Example 4: Let $f : [a, b] \to E'$ be a bounded function and m and M are constants such that $m \leq f(x) \leq M \forall x \in [a, b]$ then prove that

$$m(b - a) \leq \underline{\int_a^b} f(x)dx \text{ and } \overline{\int_a^b} f(x)dx \leq (b - a)$$

Hence prove that $m(b-a) \leq \underline{\int_a^b} f(x)dx < M(b-a)$

Solution: We define step function $s_0 : [a,b] \to E'$ and $t_0 : [a,b] \to E'$ as $s_0(x) = m$ and $t_0 = M \forall x \in [a,b]$.

$\because \; s_0(x) \leq f(x) \leq t_0(x) \forall x \in [a,b]$.

So, $s_0(x) \in S_f$ and $t_0(x) \in T_f$.

Now, $\underline{\int_a^b} f(x)dx = l.u.b. \left\{ \int_a^b s(x)dx \mid s(x) \in S_f \right\}$

Hence, $\int_a^b s_0(x)dx \leq \underline{\int_a^b} f(x)dx$.

Also, $\int_a^b s_0(x)dx = \int_a^b m dx = m(b-a)$. So we have $m(b-a) \leq \underline{\int_a^b} f(x)dx$.

Similarly, $\overline{\int_a^b} f(x)dx = g.l.b. \left\{ \int_a^b t(x)dx \mid t(x) \in T_f \right\}$

Hence, $\overline{\int_a^b} f(x)dx \leq \int_a^b t_0(x)dx$

Also, $\int_a^b t_0(x)dx = \int_a^b M dx = M(b-a)$

So, we have $\overline{\int_a^b} f(x)dx \leq M(b-a)$

Theorem 1: Riemann integrability condition of f on [a, b]

If $f:[a,b]\to E'$ be a bounded function, then f is Riemann integrable on [a, b] if and only if $\forall\varepsilon>0, \exists$ step function $s_0(x)\in S_f$ and $t_0(x)\in T_f$ such that $\int_a^b t_0(x)dx-\int_a^b s_0(x)dx<\varepsilon.$

Proof: We know that $\forall\varepsilon>0, \exists$ step functions $s_0(x)\in S_f$ and $t_0(x)\in T_f$ such that

$\int_{\underline{a}}^b f(x)dx-\int_a^b s_0(x)dx<\frac{\varepsilon}{2}$ and $\int_a^b t_0(x)dx-\int_a^{\overline{b}} f(x)dx<\frac{\varepsilon}{2}$ by adding these two equation we have $\forall\varepsilon>0$ $\int_a^b t_0(x)dx-\int_a^b s_0(x)dx<\frac{\varepsilon}{2}$

$\because$ f is integrable on [a, b] i.e. $\int_a^b f(x)dx=\int_{\underline{a}}^b f(x)dx=\int_a^{\overline{b}} f(x)dx$

Hence the above condition is necessary for the integrability of f on [a, b].

Sufficient condition: Suppose that $\forall\varepsilon>0, \exists$ step function $s_0(x)\in S_f$ and $t_0(x)\in T_f$ such that

$$\int_a^b t_0(x)dx-\int_a^b s_0(x)dx<\frac{\varepsilon}{2} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

$\because s_0(x)\in S_f$ so, $\int_a^b s_0(x)dx\le\int_{\underline{a}}^b f(x)dx$ (lower integral) …. (2)

Also, upper integral $\int_a^{\overline{b}} f(x)dx\le t_0(x)$, $t_0(x)\in T_f$ ………….. (3)

From equation (2) and (3) we have

$$\int_a^{\overline{b}} f(x)dx-\int_{\underline{a}}^b f(x)dx\le\int_a^b t_0(x)dx-\int_a^b s_0(x)dx<\varepsilon$$

Thus $\int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^b f(x)dx < \varepsilon.\ \forall\varepsilon > 0$ …………….. (4)

$\Rightarrow \int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^b f(x)dx \leq 0$ ………………….. (5)

But we know that $\int_{\underline{a}}^b f(x)dx \leq \int_a^{\overline{b}} f(x)dx$ …………….. (6)

From (5) and (6) we get that

$$\int_{\underline{a}}^b f(x)dx = \int_a^{\overline{b}} f(x)dx$$

Therefore, f is integrable on [a, b], hence it is the sufficient condition for integrability of the function f on [a, b].

Theorem 2: if $f: [a, b] \to E'$ be continuous then f is Riemann integrable on [a, b] i.e. $\int_a^b f(x)dx$ exists.

Proof: Since $f: [a, b] \to E'$ is continuous, so f is bounded on [a, b] hence f is imiformly continuous on [a, b].

We choose any $\varepsilon > 0$ let $\varepsilon' = \frac{\varepsilon\prime}{(b-a)} > 0$

Since f is imiformly continuous on [a, b] so, for any $\varepsilon' > 0 \exists \delta > 0$ depending on ε' only such that $\forall$ pair of points $x' and\ x" \in [a, b]$.

$$|x' - x"| < \delta \Rightarrow |f(x') - f(x")| < \varepsilon' \quad \text{…………….} (1)$$

We take P = {x_0, x_1, x_2, …….x_n = b} be a partition of [a, b] such that $(x_k - x_{k-1}) < \delta$ for k = 1, 2, 3, ……n then if $x', x" \in (x_{k-1}, x_k)$ then

$|x' - x"| < \delta \Rightarrow \ |f(x') - f(x")| < \varepsilon'$ ……(2)

Since f is continuous on every closed subinterval (x_{k-1}, x_k) of P, so f is bounded on (x_{k-1}, x_k) for k = 1, 2, 3 , …..n.

Hence $\exists u_k, v_k \in (x_{k-1}, x_k)$ such that

$f(u_k) \leq f(x) \leq f(v_k) \forall x \in (x_{k-1}, x_k)$, k = 1, 2, 3 …n ……..
(3)

Now $u_k, v_k \in (x_{k-1}, x_k)$ such that

$|f(u_k) - f(v_k)| = \big(f(u_k) - f(v_k)\big) < \varepsilon'$, k = 1, 2, 3, …n …… (4)

Now define step function $s: [a, b] \rightarrow E'$ and $t: [a, b] \rightarrow E'$ as follows

$$s(x) = \begin{cases} f(u_k) & when\ x \in (x_{k-1}, x_k) \\ f(a) & when\ x = x_0 = a \end{cases} \text{ k = 1, 2, 3, …n}$$

And $$t(x) = \begin{cases} f(v_k) & when\ x \in (x_{k-1}, x_k) \\ f(a) & when\ x = x_0 = a \end{cases} \text{ k = 1, 2, 3, ….n}$$

We find that $s(x) = f(u_k) \leq f(x)$ i.e. $s(x) \leq f(x) \forall x \in [a, b]$ and so, $t(x) \in T_f$.

Now,

$\int_a^b t(x)dx - \int_a^b s(x)dx = \sum_{k=1}^{n} f(v_k)\,(x_k - x_{k-1}) - \sum_{k=1}^{n} f(u_k)\,(x_k - x_{k-1})$

$$= \sum_{k=1}^{n} (f(v_k) - f(u_k))(x_k - x_{k-1}) < \sum_{k=1}^{n} \varepsilon'(x_k - x_{k-1})$$

$$= \varepsilon' \sum_{k=1}^{n} (x_k - x_{k-1}) = \varepsilon'(b - a)$$

Put $\varepsilon' = \varepsilon/(b - a)$ $\therefore \int_a^b t(x)dx - \int_a^b s(x)dx < \varepsilon,$

Thus for any choosen $\varepsilon > 0$ ∃ step function $s(x) \in S_f$ and $t(x) \in T_f$ such that $\int_a^b t(x)dx - \int_a^b s(x)dx < \varepsilon.$

Hence f is Riemann integrable on [a, b].

Theorem 3: If $f: [a, b] \to E'$ be integrable on [a, b] and m, M are real numbers such that $m \leq f(x) \leq M \forall x \in [a, b]$, then $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$.

Proof: We define step function s and t on [a, b] such that $s(x) = m$ and $t(x) = M \forall x \in [a, b]$

$\Rightarrow s(x) \leq f(x) \leq t(x) \forall x \in [a, b] \Rightarrow s(x) \in S_f and\ t(x) \in T_f$

Hence $\int_a^b s(x)dx \leq \int_a^b f(x)dx$ or $\int_a^b m\, dx \leq \int_a^b f(x)dx$

Or $m(b - a) \leq \int_a^b f(x)dx = \underline{\int_a^b} f(x)dx$ (f is integrable) …………. (1)

Also $\int_a^b t(x)dx \geq \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx$

So $M(b - a) \geq \int_a^b f(x)dx$ …………. (2)

From equation (1) and (2) we get

$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Theorem 4: Let $f: [a,b] \to E'$ be a bounded function, Let r be a real number such that $a < r < b$. If $\int_a^r f(x)dx$ and $\int_r^b f(x)dx$ exists then $\int_a^b f(x)dx$ exists, and

$$\int_a^b f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$$

Proof: Since $\int_a^r f(x)dx$ exists then for any given $\varepsilon > 0 \exists$ step functions $s_1(x) \in S_f$ and $t_1(x) \in T_f$ defined on [a, r] st.

$\int_a^r f(x)dx - \int_a^r s_1(x)dx < \varepsilon/2$ …… (1) $\because \int_{\underline{a}}^r f(x)dx = \int_a^r f(x)dx$

And $\int_a^r t(x)dx - \int_a^r f(x)dx < \varepsilon/2$ ………(2) $\because \int_a^{\overline{r}} f(x)dx = \int_a^r f(x)dx$

Again $\int_r^b f(x)dx$ exists so, $\exists$ step functions $s_2(x) \in S_f$ and $t_2(x) \in T_f$ defined on [r, b] such that

$\int_r^b f(x)dx - \int_r^b s_2(x)dx < \varepsilon/2$ ……….. (3) and

$\int_r^b t_2(x)dx - \int_r^b f(x)dx < \varepsilon/2$ …………… (4)

Now we define step function and $s_0(x) and\ t_0(x)$ on [a, b] such that.

$$s_0(x) = \begin{cases} s_1(x); & x \in [a, r] \\ s_2(x); & x \in [r, b] \end{cases}$$

$$t_0(x) = \begin{cases} t_1(x); & x \in [a, r] \\ t_2(x); & x \in [a, b] \end{cases}$$

Now, $\int_a^b s_0(x)dx = \int_a^r s_0(x)dx + \int_r^b s_0(x)dx = \int_a^r s_1(x)dx + \int_r^b s_2(x)dx$ …… (5)

Similarly $\int_a^b t_0(x)dx = \int_a^r t_0(x)dx + \int_r^b t_0(x)dx = \int_a^r t_1(x)dx + \int_r^b t_2(x)dx$ …… (6)

Now, we adding equation (1) and (3) we get

$$\int_a^r f(x)dx + \int_r^b f(x)dx < \int_a^r s_1(x)dx + \int_r^b s_2(x)dx + \varepsilon$$

$$= \int_a^b s_0(x)dx + \varepsilon \leq \int_{\underline{a}}^b f(x)dx + \varepsilon$$

Hence $\int_a^r f(x)dx + \int_r^b f(x)dx - \int_{\underline{a}}^b f(x)dx < \varepsilon \forall \varepsilon > 0$

Therefore, $\int_a^r f(x)dx + \int_r^b f(x)dx - \int_{\underline{a}}^b f(x)dx \leq 0$

Or $\int_a^r f(x)dx + \int_r^b f(x)dx \leq \int_{\underline{a}}^b f(x)dx$ ……….(7)

Similarly we adding equation (2) and (4), we get

$$\left[\int_a^r t_1(x)dx + \int_r^b t_2(x)dx\right] - \left[\int_a^r f(x)dx + \int_r^b f(x)dx\right] < \epsilon$$

Or $\int_a^b t_0(x)dx - \left[\int_a^r f(x)dx + \int_r^b f(x)dx\right] < \epsilon$

Or $\int_a^r f(x)dx + \int_r^b f(x)dx > \int_a^b t_0(x)dx - \epsilon \geq \int_a^{\overline{b}} f(x)dx - \epsilon$

Or $\int_a^{\overline{b}} f(x)dx - \left[\int_a^r f(x)dx + \int_r^b f(x)dx\right] < \varepsilon \forall \varepsilon > 0$

Hence $\int_a^{\overline{b}} f(x)dx \leq \int_a^r f(x)dx + \int_r^b f(x)dx$ .... (8)

From equation (7) and (8) we have

$$\int_a^r f(x)dx + \int_r^b f(x)dx \leq \int_{\underline{a}}^b f(x)dx \leq \int_a^{\overline{b}} f(x)dx$$
$$\leq \int_a^r f(x)dx + \int_r^b f(x)dx$$

Consequently $\int_{\underline{a}}^b f(x)dx = \int_a^{\overline{b}} f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$

Hence $\int_a^b f(x)dx$ exists and $\int_a^b f(x)dx = \int_a^r f(x)dx + \int_r^b f(x)dx$

8.7 Summary

In this unit, we have covered the following points:

- We defined a Step Function
- We defined Integration of a Step Function
- We have shown Properties of Integrals of step functions

▪ We defined Riemann Integral of a bounded function.

▪ We discussed two important theorems and explained the importance of them.

8.9 Terminal Questions

Evaluate $\int_a^b h(x)dx$ where $h\colon [a,b] \to E'$ be a step function.

(1) $h\colon [-2,\frac{9}{2}] \to E'$ defined by

$$h(x) = \frac{3|x^2-3x-4|}{x^2-3x-4} + [x];\ x \neq -1, x = 4.$$

$h(-1) = -1$, h(4)=4

(2) $h\colon [-3,8] \to E'$ defined by

$$h(x) = \frac{2|x^2-x-6|}{x^2-x-6};\ x \neq -2,\ x \neq 3.$$

$h(-2) = 5 = h(3)$ Ans. 2

(3) $h\colon [-2,7] \to E'$ defined by

$$h(x) = \begin{cases} 2 & if \quad -2 \leq x \leq 1 \\ -3 & if \quad 1 < x < 2 \\ 4 & if \quad 2 \leq x < 5 \\ -1 & if \quad 5 \leq x \leq 7 \end{cases}$$

(4) $h\colon [0,2] \to E'$ defined by $h(x) = [2x+3]$.

UNIT-9 Mean Value Theorem

Structure

9.1 Introduction

In this Unit we discuss about mean value theorem of real numbers as well as for Riemann integral, we gave application of this theorem. We also discuss about Intermediate value theorem, fundamental theorem of integral calculus and its several applications, we discuss about Substitution method for integration and

Second mean value theorem and its applications.

9.2 Objectives

After studying in this unit, therefore, you should be able to

- Define Mean value theorem
- Discuss Intermediate value theorem
- Define Fundamental theorem of integral calculus
- Define Substitution method for integration
- Second mean value theorem

9.3 Mean value theorem (Application)

Theorem: If $f: [a, b] \to E'$ is integrable function (a < b) then there exists r such that $a < r < b$ then $\int_a^b f(x)dx = (b-a)f(r), a < r < b.$

Theorem: If $f: [a, b] \to E'$ is integrable (a < b) then

(i) $\int_a^a f(x)dx = 0$ (ii) $\int_a^b f(x)dx = -\int_b^a f(x)dx$

Proof: (i) $\because \int_a^b f(x)dx = (b-a)f(r), a < r < b$

$\therefore \int_a^a f(x)dx = (a-a)fer) = 0$

(ii) $\int_b^a f(x)dx = (a-b)fer) = -(b-a)fer) = -\int_a^b f(x)dx$

Theorem: (i) if f is continuous on [a, b] then

$\int_b^a f(x)dx = (a-b)fer)$ where $r \in [a, b]$

(ii) if f :$[a, b] \to E'$ be continuous and $r_1, r_2, r_3 \in [a, b]$

Then $\int_{r_1}^{r_3} f(x)dx = \int_{r_1}^{r_2} f(x)dx + \int_{r_2}^{r_3} f(x)dx$ irrespective of the relative order of r_1, r_2, r_3

Proof: Let $r_2 < r_1 < r_3$ then $\int_{r_2}^{r_3} f(x)dx = \int_{r_2}^{r_1} f(x)dx + \int_{r_1}^{r_3} f(x)dx$

$\because$ f is continuous on $[r_2, \ r_1]$ and also continuous on $[r_1, \ r_3]$

And so $\int_{r_2}^{r_1} f(x)dx$ and $\int_{r_1}^{r_3} f(x)dx$ both exist.

Hence $\int_{r_2}^{r_3} f(x)dx = -\int_{r_1}^{r_2} f(x)dx + \int_{r_1}^{r_3} f(x)dx$

Or $\int_{r_1}^{r_3} f(x)dx = \int_{r_1}^{r_2} f(x)dx + \int_{r_2}^{r_3} f(x)dx$

Similarly the theorem can be proved for other cases of relative order of r_1, r_2, r_3

Example: if $f: [a, b] \to E'$ and $g: [a, b] \to E'$ are integrable on [a, b], prove that $f + g: [a, b] \to E'$ is integrable and $\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.

Proof: For choosen any $\varepsilon > 0$, f is integrable on [a, b] so, ∃ step functions $s_1(x) \in S_f$ and $t_1(x) \in T_f$ such that

$$\int_a^b t_1(x)\,dx - \int_a^b s_1(x)\,dx < \varepsilon/2 \qquad \ldots\ldots. (1)$$

Again g is integrable on [a, b] so, ∃ step functions $s_2(x) \in S_g$ and $t_2(x) \in T_g$ such that

$$\int_a^b t_2(x)\,dx - \int_a^b s_2(x)\,dx < \varepsilon/2 \qquad \ldots\ldots. (2)$$

So $s_0(x) = s_1(x) + s_2(x) \leq f(x) + g(x) = (f + g)(x)$

Hence $s_0(x) \in S_{f+g}$, similarly $t_0(x) = t_1(x) + t_2(x) \in T_{f+g}$

Now, $\int_a^b t_0(x)\,dx - \int_a^b s_0(x)\,dx = \int_a^b \big(t_1(x) + t_2(x)\big)dx - \int_a^b (s_1(x) + s_2(x))\,dx$

$$= \int_a^b t_1(x)dx + \int_a^b t_2(x)dx - \int_a^b s_1(x)\,dx - \int_a^b s_2(x)\,dx$$

$$= \int_a^b t_1(x)dx - \int_a^b s_1(x)\,dx + \int_a^b t_2(x)dx - \int_a^b s_2(x)\,dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, for given any $\varepsilon > 0 \exists$ step functions $s_0(x) \in S_{f+g}$ and $t_0(x) \in T_{f+g}$ such that $\int_a^b t_0(x)dx - \int_a^b s_0(x)\,dx < \varepsilon$

Hence $\int_a^b (f+g)(x)dx = \int_a^b (f(x) + g(x))dx$ exist

Now, $s_0(x) \in S_{f+g}$ and $t_0(x) \in T_{f+g}$ so,

$\int_a^b s_0(x)\,dx \le \int_a^b (f(x) + g(x))dx \le \int_a^b t_0(x)dx$….(3)

Again $\int_a^b s_0(x)\,dx = \int_a^b s_1(x)\,dx + \int_a^b s_2(x)\,dx \le \int_a^b f(x)dx + \int_a^b g(x)dx$

$$\le \int_a^b t_1(x)dx + \int_a^b t_2(x)dx = \int_a^b t_0(x)dx$$

Thus, we get $\int_a^b s_0(x)\,dx \le \int_a^b f(x)dx + \int_a^b g(x)dx \le \int_a^b t_0(x)dx$ ….. (4)

From equation (3) and (4) we get that both

$\int_a^b (f(x) + g(x))dx$ and $\int_a^b f(x)dx + \int_a^b g(x)dx$ lie between $\int_a^b s_0(x)\,dx$ and $\int_a^b t_0(x)dx$

Hence $\left|\int_a^b (f(x) + g(x))dx - \left\{\int_a^b f(x)dx + \int_a^b g(x)dx\right\}\right| \le \left|\int_a^b t_0(x)dx - \int_a^b s_0(x)\,dx\right| < \varepsilon, \forall \varepsilon > 0$

Therefore, $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Since, $|a - b| < \varepsilon\ \forall\ \varepsilon > 0 \Rightarrow a - b = 0$

Theorem: if $f: [a, b] \to E'$ and $g: [a, b] \to E'$ are integrable on [a, b] and $f(x) \leq g(x) \forall x \in [a, b]$ then $\int_a^b f(x) dx \leq \int_a^b g(x)\, dx$.

Corollary: if $\phi: [a, b] \to E'$ is intergrable and $\phi(x) \geq 0 \forall x \in [a, b]$ then $\int_a^b \phi(x) dx \geq 0$.

Proof: Let the step function $t(x) \in T_\phi$, then $t(x) \geq \phi(x) \geq 0$.

$\therefore \int_a^b t(x) dx \geq 0. \forall t(x) \in T_\phi. \quad \forall x \in [a, b]$

So, $g.l.b.\left\{\int_a^b t(x) dx \mid t(x) \in T_\phi\right\} \geq 0$

Or $\int_a^{\overline{b}} \phi(x) dx \geq 0$ or, $\int_a^b \phi(x) dx = \int_a^{\overline{b}} \phi(x) dx \geq 0$

$\because \int_a^b \phi(x) dx = \int_a^{\overline{b}} \phi(x) dx$

Now, we prove the theorem

We put $g(x) - f(x) = \phi(x)$, so $\phi(x) \geq 0 \quad \forall x \in [a, b]$.

Now, $\int_a^b \phi(x) dx = \int_a^b g(x) dx - \int_a^b f(x) dx$

$\because \int_a^b \phi(x) dx \geq 0$ Hence, $\int_a^b g(x) dx - \int_a^b f(x) dx \geq 0$

Therefore, $\int_a^b f(x) dx \leq \int_a^b g(x) dx$

Theorem: Let $f: [a, b] \to E'$ be bounded and $\int_a^b f(x)dx$ exist prove that $\int_{r_1}^{r_2} f(x)dx$ exist for any sub interval $[r_1,\ r_2]$ of [a, b] where $a \leq r_1 \leq r_2 \leq b$.

Proof: since $\int_a^b f(x)dx$ exists, so for any given $\varepsilon > 0 \exists$ step function $s_0(x) \in S_{f+g}$ and $t_0(x) \in T_{f+g}$ such that $\int_a^b t_0(x)dx - \int_a^b s_0(x)\, dx < \varepsilon$ ………. (1)

Now, by property of integrals of step functions, we get

$$\int_a^b t_0(x)dx = \int_a^{r_1} t_0(x)dx + \int_{r_1}^{r_2} t_0(x)dx + \int_{r_2}^b t_0(x)dx \text{ ………. (2)}$$

$$\int_a^b s_0(x)dx = \int_a^{r_1} s_0(x)dx + \int_{r_1}^{r_2} s_0(x)dx + \int_{r_2}^b s_0(x)dx \text{ ………. (3)}$$

$$\therefore \int_a^b t_0(x)dx - \int_a^b s_0(x)dx = \left[\int_a^{r_1} t_0(x)dx - \int_a^{r_1} s_0(x)dx\right] + \left[\int_{r_1}^{r_2} t_0(x)dx - \int_{r_1}^{r_2} s_0(x)dx\right] + \left[\int_{r_2}^b t_0(x)dx - \int_{r_2}^b s_0(x)dx\right] \quad \text{……… (4)}$$

Now, $\int_{r_1}^{r_2} t_0(x)dx - \int_{r_1}^{r_2} s_0(x)dx \leq \left[\int_a^{r_1} t_0(x)dx - \int_a^{r_1} s_0(x)dx\right] + \left[\int_{r_1}^{r_2} t_0(x)dx - \int_{r_1}^{r_2} s_0(x)dx\right] + \left[\int_{r_2}^b t_0(x)dx - \int_{r_2}^b s_0(x)dx\right]$

From equation (4) we get that

$\int_{r_1}^{r_2} t_0(x)dx - \int_{r_1}^{r_2} s_0(x)dx \leq \int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$ from equation 1

Hence, $\int_{r_1}^{r_2} t_0(x)dx - \int_{r_1}^{r_2} s_0(x)dx < \varepsilon$

Hence, $\int_{r_1}^{r_2} f(x)dx$ exists on [a, b] . ($a \leq r_1 \leq r_2 \leq b$)

Example. Let $f: [a, b] \to E$ be R-integrable then $|f|$ is also R-integrable and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

Proof: We define $f^+(x) = \begin{cases} f(x) & if\ f(x) \geq 0 \\ 0 & if\ f(x) < 0 \end{cases}$

$$f^-(x) = \begin{cases} -f(x) & if\ f(x) \leq 0 \\ 0 & if\ f(x) > 0 \end{cases}$$

Case (i) when f(x) > 0

Then $f^+(x) + f^-(x) = f(x)$ and

when f(x) < 0

Then $f^+(x) + f^-(x) = -f(x)$ thus we get

$|f(x)| = f^+(x) + f^-(x)$ …………….. (1)

$= f^+(x) - f^-(x) = f(x)$ ……… (2) if f(x) > 0 or f(x) < 0

∵ f(x) is R-integrable in [a, b] ∴ By R-condition of integrability we have

$\int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$ ……………. (3) Now from (2) we can write $t_0^+(x) - t_0^-(x) = t_0(x)$ and $s_0^+(x) - s_0^-(x) = s_0(x)$

From (3) $\int_a^b \{t_0^+(x) - t_0^-(x)\}dx - \int_a^b \{s_0^+(x) - s_0^-(x)\}dx < \varepsilon$

Or, $\int_a^b \{t_0^+(x) - s_0^+(x)\}dx + \int_a^b \{s_0^-(x) - t_0^-(x)\}dx < \varepsilon$

Now $\int_a^b t_0^+(x)dx - \int_a^b s_0^+(x)dx = \int_a^b \{t_0^-(x) - s_0^-(x)\}dx < \varepsilon$

From (4) Also $\int_a^b t_0^+(x)dx - \int_a^b t_0^-(x)dx < \varepsilon$

$\therefore f^+(x)$ & $f^-(x)$ are integrable

$\therefore f^+(x) + f^-(x)$ is integrable

And hence $|f(x)|$ is integrable.

Theorem: Let $f: [a, b] \to E'$ be a bounded function then f is integrable on [a, b] iff for any $\varepsilon > 0$ ∃ step function $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that $\int_a^b t_0(x)\, dx - \int_a^b s_0(x)\, dx < \varepsilon$.

(this theorem is called Reimann condition of integrability)

Proof: Let f is integrable on [a, b] then we show that $\int_a^b t_0(x)\, dx - \int_a^b s_0(x)\, dx < \varepsilon$.

For f is integrable on [a, b] $\int_{\underline{a}}^b f(x)dx = \int_a^{\overline{b}} f(x)dx$ ……… (1)

Now for $\varepsilon > 0$ ∃ step functions $s_0(x) \in S_f$ and $t_0(x) \in T_f$ such that

$\int_{\underline{a}}^b f(x)dx - \int_a^b s_0(x)dx < \varepsilon/2$ and $\int_a^b t_0(x)dx - \int_a^{\overline{b}} f(x)dx < \varepsilon/2$

$\therefore \int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$

Conversely, $\int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon$

We suppose that $\varepsilon > 0$ $\exists$ $s_0(x) \in S_f$ and $t_0(x) \in T_f$. By definition

$$\int_{\underline{a}}^{b} f(x)dx = lub\left\{\int_a^b s(x)dx : s(x) \in D_f\right\}$$

$\int_a^{\overline{b}} f(x)dx = glb\left\{\int_a^b t(x)dx : t(x) \in T_f\right\}.$

$\exists$ $s_0(x) \in S_f$ and $t_0(x) \in T_f$ st.

$\int_a^b s_0(x)dx \leq \int_{\underline{a}}^b f(x)dx$ …………. (1)

$\int_a^{\overline{b}} f(x)dx \leq \int_a^b t_0(x)dx$ ……………. (2)

We can write $\int_a^{\overline{b}} f(x)dx \leq \int_a^b t_0(x)dx$

We write (1) as $-\int_a^b s_0(x)dx \geq -\int_{\underline{a}}^b f(x)dx$…..(3)

Now $\int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^b f(x)dx \leq \int_a^b t_0(x)dx - \int_a^b s_0(x)dx$

$\because \int_a^b t_0(x)dx - \int_a^b s_0(x)dx < \varepsilon \forall \varepsilon > 0$

$$\int_a^{\overline{b}} f(x)dx - \int_{\underline{a}}^b f(x)dx < \varepsilon \forall \varepsilon > 0$$

$\int_a^{\overline{b}} f(x)dx \leq \int_{\underline{a}}^b f(x)dx$ ……… (4)

But $\int_{\underline{a}}^b f(x)dx \leq \int_a^{\overline{b}} f(x)dx$ ……….. (5)

$\therefore \underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

So f is integrable.

Example. Let $f:[a,b] \rightarrow E'$ be a bounded function and m and M are constant st. $m \leq f(x) \leq M \forall x \in [a,b]$ then prove that $m(b-a) \leq \underline{\int_a^b} f(x)dx$ and $M(b-a) \geq \overline{\int_a^b} f(x)dx$

Solution. We define step function $s:[a,b] \rightarrow E'$ st. $m \leq s(x) \leq f(x)$

$\Rightarrow \int_a^b mdx \leq \int_a^b s(x)dx \, \forall s(x) \in S_f$

$\Rightarrow \int_a^b mdx \leq lub\left\{\int_a^b s(x)dx \; ; \; s(x) \in S_f\right\}$

$\Rightarrow m(b-a) \leq \underline{\int_a^b} f(x)dx$

Now we define step function $t:[a,b] \rightarrow E'$st. $M \geq t(x) \geq f(x) \forall x \in [a,b]$.

$\Rightarrow \int_a^b Mdx \geq \int_a^b t(x)dx \; \forall \; t(x) \in T_f$

$\Rightarrow \int_a^b Mdx \geq glb\left\{\int_a^b t(x)dx \; : \; t(x) \in T_f\right\}$

$\Rightarrow M(b-a) \geq \overline{\int_a^b} f(x)dx$ hence proved

Theorem: Let $f:[a,b] \rightarrow E'$ be integrable on [a, b] and m, M are real numbers st. $m \leq f(x) \leq M \forall \; x \in [a.b]$. then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Proof: We shall show that $m(b-a) \le \int_{\underline{a}}^{b} f(x)dx$ ……… (1)

And $M(b-a) \ge \int_a^{\overline{b}} f(x)dx$. ………(2)

f(x) is integrable so, $\int_{\underline{a}}^{b} f(x)dx = \int_a^{\overline{b}} f(x)dx = \int_a^b f(x)dx.$

From (1) and (2) we get

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

Note: (1) A function is said to be continuous at x = a if for $\varepsilon > 0\ \exists a\ no. \delta > 0\ st.$

$|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$

(2) Uniform continuous in [a, b] if $\varepsilon > 0\ \exists a\ no. \delta > 0\ st.$

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$$

Remark: 1. If the function f is uniformly continuous on [a, b] then f is cont. on [a, b].

(2) A continuous function need not be uniformly continuous ex. f(x) = 1/x 0 < x < 1 then f is cont. in the (0, 1) but f is not uniform continuous.

3. If a function is continuous on [a. b] then it is uniformly continuous on [a, b].

4. If a function is continuous on a [a, b] then st. attains its supremum and infimum in [a, b] i.e. if f is continuous on [a, b] then it attains its supremum and infimum value in [a, b] i.e.∃ points u and v ∈ $[a, b] st. f(u) \leq f(x) \leq f(v) \forall x \in [a, b]$.

Example: $\int_{-1}^{5} |x| dx = \int_{-1}^{0} -x dx + \int_{0}^{5} x dx$

Theorem: Let $f: [a, b] \to E'$ be a continuous function then $\int_a^b f(x) dx$ exists (f is integ.)

Proof: Since f is continuous on [a, b] hence it is uniformly continuous on [a, b]. Let P = {x_0, x_1, x_2, x_3, …..x_n} where a = x0 < x1 < x2 < …… <x_n = b be the partition of [a, b] then for given $\varepsilon > 0$ ∃ a number $\delta > 0$ such that $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon$

$\forall x_1, x_2 \in [a, b]$.

We choose $\delta > 0\ st.\ x_k - x_{k-1} < \delta$

∵ f is cont. on [a, b] and hence it is cont. on $[x_{k-1} - x_k]$ and f will attain its supremum and infimum in $[x_{k-1} - x_k]$ ∃ $u_k,\ v_k \in [x_{k-1},\ x_k]\ st.$

$f(u_k) \leq f(x) \leq f(v_k) \forall x \in [x_{k-1}, \quad x_k]$

$u_k, v_k \in [x_{k-1}, \quad x_k] \& x_k - x_{k-1} < \delta$

$\Rightarrow u_k, -v_k < \delta$ for $|u_k, -v_k| < \delta$

$\because$ f is continuous and for $\varepsilon > 0 |f(v_k) - f(u_k)| < \varepsilon$

$\therefore f(v_k) - f(u_k) < \varepsilon$ ………..(2)

We define step function $s: [a, b] \to E'$, $t: [a, b] \to E' st$.

$$s(x) = \begin{cases} f(u_k) & if\ x \in [x_{k-1}, \quad x_k] \\ f(a) & if\ x = a = x_0 \end{cases}$$

$$t(x) = \begin{cases} f(v_k) & if\ x \in [x_{k-1}, \quad x_k] \\ f(a) & if\ x = a = x_0 \end{cases}$$

Now consider $\int_a^b t(x)dx - \int_a^b s(x)dx$

$$= \sum_{k=1}^{n} f(v_k)(x_k - x_{k-1}) - \sum_{k=1}^{n} f(u_k)\,(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} \{f(v_k) - f(u_k)\}(x_k - x_{k-1})$$

$\therefore \int_a^b t(x)dx - \int_a^b s(x)dx < \sum_{k=1}^{n} \varepsilon\,(x_k - x_{k-1})$

$\because \in (x_k - x_{k-1}) = (x_1 - x_0) + (x_2 - x_1) + \cdots . (x_n - x_{n-1})$

$= x_n - x_0 = (b - a)$

$\therefore \int_a^b t(x)dx - \int_a^b s(x)dx < (b - a). \varepsilon = \varepsilon'$ (say)

$\therefore$ By Riemann condition for integrability the function f is R-integrable.

Note: The converse of this theorem need not be true. The function is R-integrable for [a, b] then it need not be continuous on [a, b].

Example: Step functions are integrable but not continuous.

9.4 Intermediate value theorem:

Let $f: [a, b] \to E'$ be a continuous function. Let k be a number between f(a) and f(b) then ∃ a number $r \in (a, b) st. k = f(r)$.

Mean value theorem: (of integral calculus): Let $f: [a, b] \to E'$ be a continuous function then ∃ a number $r \in (a, b) st. (b - a)f(r) = \int_a^b f(x)dx$.

Proof: f is continuous in [a, b] and so z+ attains its supremum and infimum in [a, b] i.e.$\exists u$ and $v \in [a, b]\ st. f(u) \leq f(x) \leq f(v) \forall x \in [a, b]$.

$(b - a)f(u) \leq \int_a^b f(x)dx \leq (b - a)f(v)$ ….. (2)

Case (i) Let $(b - a)f(u) = \int_a^b f(x)dx$ but u >r ,$(b - a)f(r) \leq \int_a^b f(x)dx$.

Case (ii) Let $(b - a)f(v) = \int_a^b f(x)dx$ we take x = v so we have $(b - a)f(v) = \int_a^b f(x)dx$.

Case (iii) $(b - a)f(u) < \int_a^b f(x)dx < (b - a)f(v)$

$\Rightarrow f(u) < \frac{1}{b-a}\int_a^b f(x)dx < f(v)$.

Put $k = \frac{1}{b-a}\int_a^b f(x)dx.$

$\Rightarrow f(u) < k < f(v)$ …………. (2)

∴ f is continuous on [u, v] and k lies between f(u) and f(v) and therefore by intermediate value theorem $\exists r \in [u, v]$ st. k = f(r). $\frac{1}{b-a}\int_a^b f(x)dx = f(r)$ ∴ $\int_a^b f(x)dx = (b-a)f(r).$

Geometrically the mean value theorem means that area of curve y = f(x) bounded by x-axis and between x = a and x = b is equal to area of the rectangle whose one side is (b – a) and other side is f(r).

9.5 Fundamental theorem of integral calculus:

Note: Generally, we say that integration is the reverse process of differentiation but it is only true when function is continuous in the range of integration.

Theorem: If $f: [a, b] \to E'$ is continuous on [a, b] and $(x) = \int_a^x f(u)du$ .. (1)

Then $Dx\big(G(x)\big) = f(x)$. $Dx = \frac{d}{dx}$

Proof: $\frac{d}{dx}\big(G(x)\big) = \lim_{h\to 0} \frac{G(x+h)-G(x)}{h}$

$$= \lim_{h\to 0} \frac{\int_a^{x+h} f(u)du - \int_a^x f(u)du}{h}$$

$$= \lim_{h\to 0} \frac{\int_a^x f(u)du + \int_a^{x+h} f(u)du - \int_a^x f(u)du}{h}$$

$\therefore \frac{d}{dx}\left(G(x)\right) = \lim_{h\to 0} \frac{\int_a^{x+h} f(u)du}{h}$ …………… (1)

By mean value theorem we have $\int_a^b f(u)du = (b-a)f(r)$ where a < r < b.

$\therefore \int_x^{x+h} f(u)du = (x+h-x)f(r)$ where x < r < x + h

From (1) $\frac{d}{dx}\left(G(x)\right) = \lim_{h\to 0} \frac{hf(r)}{h}$ x < r < x + h

$= \lim_{h\to 0} f(r)$, x < r < x + h

= f(x)

Note: If $G = \int_a^b f(u)du$ then $\frac{d}{dx}(G) = 0$

Primitive of a function: A function $F: [a,b] \to E'$ is called a primitive of $f: [a,b] \to E'$ such that $\frac{d}{dx}\left(F(x)\right) = f(x)$

And $\frac{d}{dx}(\sin x) = \cos x$ i.e. primitive of cos x is sin x. also $\int \cos x\, dx = sinx$

Note: 1. F and f are continuous function.

2. If f(x) is a primitive of F(x) then f(x) + c is also a primitive of F(x).

$\therefore \frac{d}{dx}(F(x) + cg) = \frac{d}{dx}\left(F(x)\right) = f(x).$

3. Let F(x) and G(x) be two primitive of a function f(x) then F(x) – G(x) = k (constant)

Theorem: if $F: [a, b] \to E'$ is any primitive of $f: [a, b] \to E'$ and if F is continuous on [a, b] then $\int_a^b F(x)dx = F(b) - F(a)$.

Proof: F(x) is given to be a primitive of f(x) and by fundamental theorem of integral calculus $G(x) = \int_a^x f(u)du \,.\, F(x) - \int_a^x f(u)du = k$ ………….. (1)

Putting x = a then $F(a) - \int_a^a f(u)du = k$

∴ k = F(a)   putting it in equation (1)

$\therefore F(x) - \int_a^x f(u)du = F(a)$

$\therefore F(x) - F(a) = \int_a^x f(u)du$

Putting x = b, $F(b) - F(a) = \int_a^b f(u)du$

Note: if $F: [a, b] \to E'$ is constant function then its primitive and integral are same. But if F is not continuous then

(i) A function may have primitive but not integral.

(ii) Function F may be integrable without having primitive.

Example. 1. Consider $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$ Then f is continuous at all points in except at x = 0. Then F is continuous on [a, b] not contain zero, and so F is integrable on [a, b] but ∃ no f(x) st. $\frac{d}{dx}F(x) = f(x)$. ∴ F is integrable but F has no primitive.

2. $F(x) = \frac{1}{2}x^2 \sin\frac{1}{x^2}$, $F'(x) = x\sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x^2} = f(x)$

Let $f(x) = x\sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x^2} \qquad x \neq 0$

Then its primitive is $F(x) = \frac{1}{2}x^2 \sin\frac{1}{x^2}$

$\because \frac{d}{dx}F(x) = x\sin\frac{1}{x} - \frac{1}{x}\cos(\frac{1}{x^2})$

But the function f(x) is not integrable in any interval containing the point zero.

∵ f(x) is not bounded in an interval containing 0.

Let $f: [1, 2] \to E'$ defined by $f(x) = \begin{cases} 3 & if \quad x \text{ is rational} \\ 1 & if \quad x \text{ is irrational} \end{cases}$

∵ P = $\{x_0, x_1, \ldots x_n\}$ be a partition of (1, 2) is $s(x) \in S_f$ then $s(x) \leq f(x) \forall x \in (1, 2)$.

$\Rightarrow s(x) \leq 1 \forall x \in (1, 2) \therefore \int_1^2 s(x)dx \leq 1(2-1) = 1$

$\therefore l.u.b. \left\{\int_1^2 s(x)dx \mid s(x) \in S_f\right\} = 1 = \int_{\underline{1}}^2 f(x)dx$

Also $t(x) \in T_f,\ t(x) \geq f(x) \Rightarrow t(x) \geq 3 \forall x \in (1.2)$

$\therefore \int_1^2 t(x)dx \geq 3(2-1) = 3 \therefore g.l.b. \left\{\int_1^2 t(x)dx \mid t(x) \in T_f\right\} = 3$

$\therefore \int_1^2 f(x)dx = 3 \therefore \int_1^2 f(x)dx \neq \int_a^{\bar{2}} f(x)dx.$

Give an example to show that a function $f:[a,b] \to E'$ may be integrable, still if may not have any primitive function F. thus $\int_a^b f(x)dx = F(b) - f(a)$, where F is a primitive of f, should not be regarded as on alternative definition of the integral $\int_a^b f(x)dx$.

Solution. Define $f:[0,2] \to E'$ as follows f(x) = 0, x ≠ 1 and f(1) = 2.

Here f is a step function, partition for this function is P = {0, 1, 2}.

Also f is Riemann integrable on [0, 2] and $\int_0^2 f(x)dx = 0(1-0) + 0(2-1) = 0$

Now suppose that ∃ a primitive F of f such that $\frac{d}{dx}F(x) = f(x)$, then

F(x) = 2x when x = `1 and f(x) = λ (an arbitrary constant) x ≠ 1, thus F cannot be uniquely defined on [0, 2]

Theorem: If $F:[a,b] \to E'$ and $G:[a,b] \to E'$ be two primitives of $f:[a,b] \to E'$, then F(x) – G(x) = k (constant) $\forall x \in [a,b]$.

Proof: Since, F and G are both primitives of f, so $\frac{d}{dx}F(x) = f(x)$; and $\frac{d}{dx}G(x) = f(x)$.

Or, $\frac{d}{dx}\big(F(x) - G(x)\big) = 0$. Hence, F(x) – G(x) = k

Theorem: if $f: [a, b] \to E'$ and $g: [a, b] \to E'$ be continuous on [a, b], then prove by the fundamental theorem on calculus.

(i) $\int_a^b \big(f(x) \mp g(x)\big)dx = \int_a^b f(x)dx \mp \int_a^b g(x)dx$

(ii) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

Proof: We have already proved that if f and g are integrable then (f ± g) is integrable on [a, b]. Now we prove this by using fundamental theorem on calculus:

Since $\int_a^x f(u)du$ and $\int_a^u g(u)du$ exist.

So, Let $F(x) = \int_a^x f(u)du\,,\ G(x) = \int_a^u g(u)du$

When $x \in [a, b]$

$\frac{d}{dx}\big(F(x)\big) = f(x)\frac{d}{dx}\big(G(x)\big) = g(x)$ ……… (1)

Since $\int_a^b (f(x) \mp g(x))dx$ exists. Since f(x) ± g(x) is continuous on [a, b].

S0 let $H(x) = \int_a^x \big(f(u) \mp g(u)\big)du, x \in [a, b]$

Then $\frac{d}{dx}\big(H(x)\big) = f(x) \mp g(x)$ by fundamental theorem on calculus. ……(2)

From (i) we get $\frac{d}{dx}\{F(x) \mp G(x)\} = f(x) \mp g(x)$

Thus $H(x)$ and $F(x) \mp G(x)$ are both primitives of $f(x) \mp g(x)$. Hence

$[F(x) \mp G(x)] - H(x) = C \; \forall x \in [a,b]$ Put x = a, we get

$[F(a) \mp G(a)] - H(a) = C$

□ $F(a) = G(a) = H(a) = 0 \; \rightarrow C = 0$

Now Put x = b we have $[F(b) \mp G(b)] - H(b) = 0$

□ $F(b) \mp G(b) = H(b)$

Or $\int_a^b \{f(u) \mp g(u)\} du = \int_a^b f(u) du \mp \int_a^b g(u) du$

Or $\int_a^b \{f(x) \mp g(x)\} dx = \int_a^b f(x) dx \mp \int_a^b g(x) dx$

Similarly, we can prove part (ii).

9.6 Substitution method for integration:

Theorem: Let A be a subset of E'. Let $g: [a,b] \rightarrow A$ be a function such that $g'(x)$ exisrs and is continuous $\forall x \in [a,b]$. Let $f: A \rightarrow E'$ be a continuous map then $\int_a^b f\big(g(x)\big) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$.

Proof: Since $g: [a,b] \rightarrow A$ and $f: A \rightarrow E'$ be two continuous maps then $fog: [a,b] \rightarrow E'$ is well defined and continuous map also, g is continuous on [a, b]. therefore, $\int_a^b f\big(g(x)\big) g'(x) dx$ exists. Since, $f\big(g(x)\big) g'(x)$ is also continuous on [a, b]. Let $\psi(x)$= $\int_a^x f\big(g(t)\big) g'(t) dt; x \in [a,b]$. Hence by fundamental theorem of calculus we have $\frac{d}{dx}\big(\psi(x)\big) = f\big(g(x)\big) g'(x) \; \forall \; x \in [a,b]$.

Since, [g(a), g(b)] is a subinterval of A and f is integrable on A. so, f is also integrable on [g(a), g(b)], hence $\int_{x(a)}^{x(b)} f(u)du$ exists. Let $\phi(u) = \int_{g(a)}^{u} f(t)dt\,;\ \ u \in [g(a), g(b)]$. Hence by fundamental theorem of calculus

$\frac{d}{dx}\phi(u) = f(u)$ ………… (2)

Put u = g(x) then $\phi\big(g(x)\big) = \int_{g(a)}^{g(x)} f(t)dt$ ………. (3)

Now $\frac{d}{dx}\phi\big(g(x)\big) = \frac{d}{dx}\phi(u)\frac{d}{dx}g(x) = f(u).g'(x)$

Or $\frac{d}{dx}\phi\big(g(x)\big) = f(g(x))g'(x)$ …………….. (4)

We see that from equation (1) and (4) $\phi(g(x))$ and $\psi(x)$ are primitives of $f(g(x))g'(x)$. Hence $\phi\big(g(x)\big) - \psi(x) = c\ \ \forall x \in [a, b]$ Put x = a then $\phi\big(g(a)\big) - \psi(a) = c$

Or $\int_{g(a)}^{g(b)} f(t)dt - \int_{a}^{b} f\big(g(t)\big)g'(t)dt = c\ \Rightarrow c = 0$

Hence, $\phi\big(g(x)\big) = \psi(x)\ \ \forall x \in [a, b]$

Now, we put x = b then $\phi\big(g(b)\big) = \psi(b)$

Or $\int_{g(a)}^{g(b)} f(t)dt = \int_{a}^{b} f\big(g(t)\big)g'(t)dt$

Now, we change the variable t →u on L.H.S. and t → x on R.H.S.. we get

$$\int_{g(a)}^{g(b)} f(u)du = \int_a^b f(g(x))g'(x)dx$$

9.7 Second mean value theorem:

If f is a monotonic function f, f' and g are all continuous functions on [a, b] then there exists $c \in [a, b]\ st.\ \int_a^b f(x)g(x)dx = f(a)\int_a^c g(x)dx + f(b)\int_c^b g(x)dx.$

Proof: Let $G(x) = \int_a^x g(t)dt$. Clearly G(a) = 0. Under given condition G(x) is differentiable and G'(x) = g(x).

$\therefore \int_a^b f(x)g(x)dx = \int_a^b f(x)G'x\,dx = [f(x)G(x)]_a^b - \int_a^b G(x)f'(x)dx$ (by integrating by parts). Since, G being continuous so it is integrable and f is monotonic and continuous on [a, b]. therefore, by First mean value theorem $\exists c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(b)G(b) - G(c)\int_a^b f'(x)dx$

$= f(b)G(b) - G(c)\{f(b) - f(a)\}$

$= f(b)\{G(b) - G(c)\} + f(b)G(c)$

$= f(b)\int_c^b g(x)dx + f(a)\int_a^c g(x)dx.$

9.8 Summary

After studying of this unit, we should be able to define Mean value theorem and its applications, discuss Intermediate value theorem, define Fundamental theorem of integral calculus and its several applications, we can define Substitution method for integration and second mean value theorem.

9.10 Terminal Questions

1. State the Proof first mean value theorem.

2. The function $f(x)$ is defined on [2,5] as follows:

$$\mathrm{f(x)} = \begin{cases} 1 \text{ if } 2 \leq x < 3 \\ 3 \text{ if } 3 \leq x \leq 5 \end{cases}$$

3. If is Continuous and positive on $[a, b]$,then show that $\int_a^b f\, dx$ is also positive.

4. Show That the second mean value theorem does not hold good in $[-1,1], for\ f(x) = g(x) = x^2$.

5. If $f \in R[a, b]$ and $F(x) = \int_a^x f(t)dt$,for all $x \in$,then show that F is of bounded variation on $[a, b]$.

6. Use fundamental theorem of integration to compute $\int_1^2 x^3 dx$.

7. Verify the second mean value theorem for $f(x) = x$ and $g(x) = e^x$ in $[-1,1]$.